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Some Remarks on Convexity

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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This paper deals with some properties of convex functions and their applications to problems of the calculus of variations, both in the convex case, where Rockafellar is one of the main experts in the world, and also in the quasiconvex case, a field which still is not completely understood.

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1. Introduction

This article, dedicated to Ralph Tyrell Rockafellar on his 90^{th} birthday, is divided in two parts. To celebrate Rockafellar's relevant work, mainly – but not only – in the context of Convex Analysis, the next section of this manuscript deals with a classical and fundamental property of convex functions: their *local Lipschitz continuity.* Analogously, Rockafellar in [34], [35], [36], found a way to extend this local Lipschitz property to *directionally Lipschitzian functions* and to *epi-Lipschitzian sets*; in particular a set in $\mathbb{R}^n$ is *epi-Lipschitzian* at a boundary point if its boundary can be locally represented near this point as the *epigraph* of a Lipschitz continuos function. In [34], [35], [36] Rockafellar gave several characterizations of the properties which he introduced and he observed that some of them were confined to the finite dimensional setting. Some details and extensions can be found in the more recent article by Correa, Gajardo, and Thibault [7] in general real Banach spaces.

The second part of this article is devoted to some *applications of convexity to the calculus of variations*, and to the natural *extension of the convexity condition to quasiconvexity* in Morrey's sense. Since the celebrated definitions by Morrey in his book [32] and the well known Ball's article [2] (see also the Dacorogna's book [14]), quasiconvexity received attention: many results are known, in particular related to semicontinuity properties of the associated energy integrals, some questions are still open, in particular to characterizations properties of quasiconvexity. In this manuscript we will describe some relevant problems of a class of elliptic differential systems whose existence of solutions strongly depend on the convexity of the associated energy integral, and at the same time on the quasiconvexity of the related integrands with respect to the gradient variable; see details in Section 3.

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2. On growth and local Lipschitz continuity of convex functions

We start this section with a self-contained proof of a useful property of convex functions defined on a Banach space; in the last part of this section we will apply it to the *gradient growth* of convex functions. We refer to the *local Lipschitz continuity* of locally bounded convex functions. Of course this property is well known: details can be found in the books of Rockafellar [33], Ekeland-Temam [24], and in Chapter 3 of [25]; see also [5] and [38]. We do not assume here that the Banach space is finite dimensional.

Let X be a Banach space with norm $|\cdot|$. As usual set $A \subseteq X$ is *convex* if for every $x_1, x_2 \in A$ and $t \in [0,1]$ then $tx_1 + (1-t)x_2 \in A$. A function $f : A \to \mathbb{R} \cup \{+\infty\}$ is *convex* if

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)\,, \tag{1}$$

for any $x_1, x_2 \in A$ and $t \in [0,1]$. It is well known that a convex function may be unbounded on the neighborhood of a boundary point of the set A or it may also be discontinuous on the boundary of A. However, if it is bounded in A, on the interior of A it is necessarily continuous, as stated in the next Theorem 2.1.

Let us start with the usual definitions of *locally bounded* and *locally Lipschitz continuous* function. The function $f : A \to \mathbb{R} \cup \{+\infty\}$, $A \subseteq X$ convex set with non-empty interior, is *locally bounded from above* on A if for any interior point x_0 of A there exists a ball $B_r(x_0)$, whose closure is contained in the interior of A, of centre x_0 and radius $r > 0$, and there exists a constant $M = M(x_0, r)$ such that $f(x) \leq M$ for every $x \in B_r(x_0)$. Similarly f is *locally bounded from below.* Finally f is locally bounded on A if it is locally bounded on A both from below and from above. The function f is *locally Lipschitz continuous* on A if, for any interior point x_0 of A there exist a ball $B_r(x_0)$, whose closure is contained in the interior of A, and a constant $L = L(x_0, r)$ such that

$$|f(x_1) - f(x_2)| \leq L\,|x_1 - x_2|\ , \qquad \forall\, x_1, x_2 \in B_r(x_0)\,. \tag{2}$$

Theorem 2.1. *Let $f : A \to \mathbb{R} \cup \{+\infty\}$ by a convex function on the convex set $A \subseteq X$ with non-empty interior. If f is locally bounded from above on A, then f is also locally bounded from below and locally Lipschitz continuous on A. Moreover the following estimate holds: for every $x_0 \in A$ and balls $B_r(x_0), B_{r_0}(x_0)$, $r < r_0$, with $\overline{B_{r_0}(x_0)}$ contained in the interior of A,*

$$|f(x) - f(y)| \leq \frac{M - m}{r_0 - r}\,|x - y|\ , \qquad \forall\, x, y \in B_r(x_0)\,, \tag{3}$$

where $M = \sup\{f(x) : x \in B_{r_0}(x_0)\}$ and $m = \inf\{f(x) : x \in B_{r_0}(x_0)\}$.

If the Banach space X is finite dimensional then any convex function $f : A \subset X \to \mathbb{R} \cup \{+\infty\}$ is automatically *locally bounded* on A. In this case the following more precise result holds.

Corollary 2.2. *Let X be a finite dimensional Banach space and let $f : A \subset X \to \mathbb{R} \cup \{+\infty\}$ be a convex function on the convex set $A \subseteq X$ with non-empty interior. Then f is locally bounded and Lipschitz continuous on A. Moreover the estimate* (3) *holds, for $x_0 \in A$ and $B_r(x_0), B_{r_0}(x_0)$, $r < r_0$, with $\overline{B_{r_0}(x_0)}$ contained in the interior of A, with $M = \max\left\{f(x) : x \in \overline{B_{r_0}(x_0)}\,\right\}$ and $m = \min\left\{f(x) : x \in \overline{B_{r_0}(x_0)}\,\right\}$.*

The proof of Theorem 2.1 and of Corollary 2.2 will be obtained in several steps.

Step 1. In this step we prove that $f : A \subset X \to \mathbb{R} \cup \{+\infty\}$ is continuous on the interior of A. Precisely, for any interior point x_0 of A there exists a ball $B_r(x_0)$, with centre in x_0 and radius $r > 0$, whose closure is contained in A, for which the following estimate holds

$$|f(x) - f(x_0)| \leq \frac{M - f(x_0)}{r} \, |x - x_0| \, , \quad \forall \, x \in B_r(x_0) \, ; \tag{4}$$

here we use the assumption that f is *locally bounded from above* on A and we denote by $M = \sup\{f(x) : x \in B_r(x_0)\} < +\infty$.

First we show that (4) holds for $x_0 = 0$; i.e.,

$$|f(x) - f(0)| \leq \frac{M - f(0)}{r} \, |x| \, , \quad \forall \, x \in B_r(0) \, , \tag{5}$$

where $x_0 = 0$ is interior to A and let $B_r(0)$ be a ball of centre 0, radius $r > 0$, with closure contained in A and for which $\sup\{f(x) : x \in B_r(0)\} = M < +\infty$. Condition (5) hods if $x = 0$. If $x \neq 0$, call x_1, x_2 the two intersections of the boundary of $B_r(0)$ with the line through $x, x_0 = 0$. Let x_1 be the point with the direction $x/|x|$ of x, and x_2 the antipodal point. Since $|x_1| = |x_2| = r$, then

$$x_1 = r \, \frac{x}{|x|} \, , \qquad\qquad x_2 = -r \, \frac{x}{|x|} \, .$$

It is easy to write x as convex combination of x_1 and 0: if $x = tx_1 + (1-t)0 = tx_1$, we have $x = tx_1 = t \, r \frac{x}{|x|}$ and so $t = \frac{|x|}{r}$. Therefore $x = \frac{|x|}{r} x_1 + \big(1 - \frac{|x|}{r}\big) 0$ and by convexity of f

$$f(x) \leq \tfrac{|x|}{r} f(x_1) + \big(1 - \tfrac{|x|}{r}\big) f(0) \, ,$$

and thus

$$f(x) - f(0) \leq \tfrac{|x|}{r} \left(f(x_1) - f(0) \right) \leq \frac{M - f(0)}{r} |x| \, , \tag{6}$$

which proves one part of (5). To estimate $f(0) - f(x)$ we write $x_0 = 0$ as convex combination of x_2 and x. As before we obtain $0 = tx_2 + (1-t)x = \big(-\frac{tr}{|x|} + 1 - t\big)x$. Therefore $t = \frac{|x|}{r+|x|}$, $1 - t = \frac{r}{r+|x|}$ and by convexity

$$f(0) \leq \frac{|x|}{r + |x|} f(x_2) + \frac{r}{r + |x|} f(x),$$

so $\quad (r + |x|) f(0) \leq |x| f(x_2) + r f(x), \quad r \left(f(0) - f(x) \right) \leq |x| \left(f(x_2) - f(0) \right)$

and

$$f(0) - f(x) \leq \tfrac{|x|}{r} \left(f(x_2) - f(0) \right) \leq \frac{M - f(0)}{r} |x|,$$

which together (6) yield (5).

Finally, if x_0 is a generic interior point, apply (5) to $g(x) = f(x + x_0)$, observing that $g(x)$ is defined on a neighborhood of $x = 0$. This gives $|g(x) - g(0)| \leq \frac{M - g(0)}{r} |x|$, i.e.

$$f(x + x_0) - f(x_0) \leq \frac{M - f(x_0)}{r} |x| \, , \qquad \forall \, x \in B_r(0) \, ,$$

leading to (4), replacing $x - x_0$ with x.

Step 2. By the continuity estimate (4) in particular we deduce that

$$f(x_0) - f(x) \leq |f(x) - f(x_0)| \leq \frac{M - f(x_0)}{r} \, |x - x_0| \ ,$$

for every $x \in B_r(x_0)$. Therefore, since $|x - x_0| \leq r$,

$$f(x) \geq f(x_0) - \frac{M - f(x_0)}{r} \, |x - x_0| \geq -M \ , \qquad \forall \, x \in B_r(x_0) \ ,$$

and f is bounded from below by $-M$ in $B_r(x_0)$. This means that f is also *locally bounded from below* on A.

Step 3. From the continuity estimate (4) and the local boundedness (both: from above and from below) we arrive to the proof of Theorem 2.1. In fact, as before we consider a ball $B_{r_0}(x_0)$ whose closure $\overline{B_{r_0}(x_0)}$ is made of interior points of A and let $B_r(x_0)$ be a concentric neighborhood of radius $r < r_0$. We use the notation $M = \sup\{f(x) : x \in B_{r_0}(x_0)\}$ and $m = \inf\{f(x) : x \in B_{r_0}(x_0)\}$. For any $x \in B_r(x_0)$, the ball $B_{r_0 - r}(x)$ of centre x and radius $r_0 - r$ is contained in $B_{r_0}(x_0)$. By the continuity estimate (4) at x, for any $x \in B_r(x_0)$ and any $y \in B_{r_0 - r}(x)$ we have

$$|f(y) - f(x)| \leq \frac{M - f(x)}{r_0 - r} \, |y - x| \leq \frac{M - m}{r_0 - r} \, |y - x| \, . \tag{7}$$

This proves (3) for a generic x in $B_r(x_0)$ but with $y \in B_{r_0 - r}(x)$. Let us now address the case in Theorem 2.1 with $x, y \in B_r(x_0)$. Divide the line segment $[x, y]$ in k parts of equal length $|x - y|/k$, using points x_j, $j = 0, 1, \ldots, k$, $x_0 = x$, x_1, x_2, ..., $x_k = y$, defined as $x_j = x + \frac{j}{k}(y - x)$, for all $j = 0, 1, \ldots, k$. Choose $k \in \mathbb{N}$, $k > \frac{|x-y|}{r_0 - r}$, so that

$$|x_{j+1} - x_j| = \tfrac{1}{k} \, |x - y| < r_0 - r \, , \qquad \forall \, j = 0, 1, \ldots, k - 1 \, . \tag{8}$$

Thus $x_{j+1} \in B_{r_0 - r}(x_j)$. We apply (7) with $x = x_j \in B_r(x_0)$ and $y = x_{j+1} \in B_{r_0 - r}(x_j)$:

$$|f(x_{j+1}) - f(x_j)| \leq \frac{M - m}{r_0 - r} \, |x_{j+1} - x_j| = \frac{M - m}{r_0 - r} \cdot \frac{|x - y|}{k} \, . \tag{9}$$

Adding the above relations for all $j = 0, 1, \ldots, k - 1$ produces

$$|f(x) - f(y)| = \left| \sum_{j=0}^{k-1} [f(x_{j+1}) - f(x_j)] \right|$$

$$\leq \sum_{j=0}^{k-1} |f(x_{j+1}) - f(x_j)| \leq \frac{M - m}{r_0 - r} \cdot \frac{|x - y|}{k} \cdot \sum_{j=0}^{k-1} 1 = \frac{M - m}{r_0 - r} \, |x - y| \ ,$$

which conclude the proof of the continuity estimate (3) and of Theorem 2.1.

Step 4. We start the proof of Corollary 2.2. Thus we assume here that the convex function $f : A \to \mathbb{R}$ is defined on a convex set $A \subset X$ and X is finite dimensional; say $X = \mathbb{R}^n$ for some $n \in \mathbb{N}$, $n \geq 1$. Recall the *discrete Jensen inequality* for the convex function $f : A \to \mathbb{R}$ on the convex set $A \subset X$. If $x_1, x_2, \ldots, x_k$ are k points in A, $k \geq 2$, and $t_1, t_2, \ldots, t_k$ are non-negative real numbers, then

$$\sum_{i=1}^{k} t_i x_i \left(\sum_{i=1}^{k} t_i \right)^{-1} \in A \, , \qquad f\left(\sum_{i=1}^{k} t_i x_i \left(\sum_{i=1}^{k} t_i \right)^{-1} \right) \leq \sum_{i=1}^{k} t_i f(x_i) \left(\sum_{i=1}^{k} t_i \right)^{-1} ,$$

$$\frac{\sum_{i=1}^{k} t_i x_i}{\sum_{i=1}^{k} t_i} \in A, \quad f\left(\frac{\sum_{i=1}^{k} t_i x_i}{\sum_{i=1}^{k} t_i}\right) \le \frac{\sum_{i=1}^{k} t_i f(x_i)}{\sum_{i=1}^{k} t_i}\,. \tag{10}$$

This well known property can be proved by induction on $k \ge 2$ (see for instance (3.73), (3.74) in Chapter 3 of [25]).

Step 5. To conclude the proof of Corollary 2.2 we need to show that any convex function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is locally bounded from above on A. In fact by Theorem 2.1 then follows that f is also locally bounded from below and locally Lipschitz continuous on A, with the local Lipschitz estimate (3) in force.

To this aim, for every x_0 in the interior of the convex set $A \subset \mathbb{R}^n$ consider an n-dimensional cube Q of centre x_0 (x_0 is the intersection of the cube's diagonals) whose closure is contained in A. We shall prove f is upper bounded on Q. This will imply that f is locally bounded from above on A. Write x_i, $i = 1, 2, \dots, 2^n$, for the vertices of Q. The first thing we prove is that any point in the cube's interior is a convex combination of the vertices. That is, for any interior x there are non-negative real $t_1, t_2, \dots, t_{2^n}$ such that $\sum_{i=1}^{2^n} t_i = 1$ and $x = \sum_{i=1}^{2^n} t_i x_i$.

The proof works by recurrence on n. We denote the chosen vertices of Q by v_i, $i = 1, 2, \dots, n+1$. In particular, v_1 is any one vertex of Q, say $v_1 = x_1$. Consider the straight line through v_1 and x, by assumption distinct from v_1 (being interior to Q). This line meets the face opposite to v_1 at some point y_1. The points v_1, x and y_1 are collinear by construction, so x is a convex combination of v_1 and y_1, i.e. there exists $t_1 \in (0,1)$ such that $x = t_1 v_1 + (1-t_1)y_1$. Let v_2 be the vertex opposite to v_1. If $y_1 = v_2$ the proof ends and it suffices to choose $t_2 = 1 - t_1$. Otherwise, consider the line through v_2 and y_1, which intersects the cube's boundary (an edge if $n = 3$) at some y_2. The point y_1 is a convex combination of v_2 and y_2, so there is $s \in (0,1)$ such that $y_1 = sv_2 + (1-s)y_2$. Combining with the previous convex combination we obtain

$$x = t_1 v_1 + (1-t_1)[sv_2 + (1-s)y_2]\,,$$

whence, setting $t_2 = (1-t_1)s$, $(1-t_1)(1-s) = (1-t_1) - (1-t_1)s = 1 - t_1 - t_2$, and then $x = t_1 v_1 + t_2 v_2 + (1 - t_1 - t_2)y_2$. Iterating the construction $n-1$ times we find y_{n-1} on an edge, which is then a convex combination of the vertices v_n and v_{n+1}. The process stops after n steps, giving

$$x = t_1 v_1 + t_2 v_2 + \ldots + t_n v_n + \Big(1 - \sum_{i=1}^{n} t_i\Big) v_n\,. \tag{11}$$

So if we set $t_{n+1} = 1 - \sum\limits_{i=1}^{n} t_i$, since $\sum\limits_{i=1}^{n+1} t_i = 1$, Jensen's inequality implies

$$f(x) = f\Big(\sum_{i}^{n+1} t_i x_i\Big) \le \sum_{i}^{n+1} t_i f(x_i) \le M \sum_{i}^{n+1} t_i = M\,, \tag{12}$$

where $M = \max_i\{f(x_i)\}$. This proves f is bounded on Q. Therefore the convex function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is locally bounded from above on A.

Step 6. We are ready for the conclusion of the proof of Corollary 2.2. Since f is locally bounded from above on A, we can use the continuity estimate (4) obtained in Step 1. Therefore the function f is continuous in $\mathbb{R}^n$ and has maximum and minimum values at every ball $\overline{B_{r_0}(x_0)}$ contained in the interior of A, for example, $M = \max\left\{f(x) : x \in \overline{B_{r_0}(x_0)}\right\}$ and $m = \min\left\{f(x) : x \in \overline{B_{r_0}(x_0)}\right\}$. By Theorem 2.1 f is also locally bounded from below in $A \subset \mathbb{R}^n$ and the local Lipschitz estimate (3) holds for f.

We end this section with a relevant application of the *local Lipschitz continuity* of convex functions to their *gradient-growth*. The *local Lipschitz continuity* of any convex function defined in $\mathbb{R}^n$, by Rademacher's theorem, implies its differentiability almost everywhere.

To explain what we mean, we first observe that, even in the unidimensional case, for a differentiable function $f : \mathbb{R} \to \mathbb{R}$ there are not relations between the growth at infinity of f and the growth of its derivative f'. In fact, for instance $f(x) = \sin(|x|^p)$ or $g(x) = \sin e^x$ are bounded functions on $\mathbb{R}$ but their first derivatives f' and g' have "large growth" (precisely $f'(x)$ has "large growth" if $p > 1$; $g'(x)$ has "large growth" as $x \to +\infty$ and "low growth" as $x \to -\infty$), independent of the "growth at infinity" of f and g. On the contrary, *convexity* of f plays a strong role in the *growth of its gradient* Df, as shown by the following result (see Step 2 of Section 2 in Marcellini [29]).

Lemma 2.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function in $\mathbb{R}^n$ (or a quasiconvex function in the sense of definition (19) below). If the growth condition*

$$|f(\xi)| \leq M\ (1 + |\xi|^p) \tag{13}$$

holds for some exponent $p \geq 1$, for a constant M and for all $\xi \in \mathbb{R}^n$, then the gradient $Df : \mathbb{R}^n \to \mathbb{R}^n$ of f, which exists almost everywhere in $\mathbb{R}^n$, satisfies the growth

$$|Df(\xi)| \leq M_1 \left(1 + |\xi|^{p-1}\right) \tag{14}$$

for a constant M_1 and for almost all $\xi \in \mathbb{R}^n$. Moreover the following inequality holds

$$|f(\xi_1) - f(\xi_2)| \leq M_2 \left(1 + |\xi_1|^{p-1} + |\xi_2|^{p-1}\right) |\xi_1 - \xi_2| \tag{15}$$

for a constant M_2 and for all $\xi_1, \xi_2 \in \mathbb{R}^n$.

Proof. We consider one coordinate of the vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$; say ξ_i for a fixed $i \in \{1, 2, \ldots, n\}$. The function $g : \mathbb{R} \to \mathbb{R}$ of the variable $\xi_i \in \mathbb{R}$, defined by $g(\xi_i) = f(\xi_1, \xi_2, \ldots, \xi_n)$ when all the other coordinates ξ_j, $j \neq i$ are fixed, is a convex function in $\mathbb{R}$. Its derivative $g'(\xi_i)$ exists almost everywhere in $\mathbb{R}$; by convexity, $g'(\xi_i)$ is less that or equal to the *right difference quotient*, i.e.

$$g'(\xi_i) \leq \frac{g(\xi_i + h) - g(\xi_i)}{h}, \quad \forall\, h > 0, \tag{16}$$

and similarly it is greater that or equal to the *left difference quotient*,

$$g'(\xi_i) \geq \frac{g(\xi_i + h) - g(\xi_i)}{h}, \quad \forall\, h < 0. \tag{17}$$

Instead of going to the limit as h approaches 0, we choose $h = 1 + |\xi|$ in (16) and $h = -(1 + |\xi|)$ in (17).

We obtain

$$f_{\xi_i}(\xi) = g'(\xi_i) \begin{cases} \leq \dfrac{g(\xi_i + 1 + |\xi|) - g(\xi_i)}{1 + |\xi|} \leq \dfrac{|g(\xi_i + 1 + |\xi|)| + |g(\xi_i)|}{1 + |\xi|} \\ \geq \dfrac{g(\xi_i - 1 - |\xi|) - g(\xi_i)}{-(1 + |\xi|)} \geq -\dfrac{|g(\xi_i - 1 - |\xi|)| + |g(\xi_i)|}{1 + |\xi|} \end{cases}$$

and thus, by assumption (13), there exists a constant M_0 such that.

$$|f_{\xi_i}(\xi)| \leq \frac{|g(\xi_i \pm 1 \pm |\xi|)| + |g(\xi_i)|}{1 + |\xi|} \leq M_0 \left(1 + |\xi|^{p-1}\right).$$

Then $|Df(\xi)| \leq \sqrt{n} M_0 \left(1 + |\xi|^{p-1}\right)$ and (14) holds with $M_1 = \sqrt{n} M_0$. Finally (15) is a direct consequence of (14). □

3. Some applications of convexity in the calculus of variations

Let $n, m \in \mathbb{N}$ with $n, m \geq 1$ and let Ω be an open set in $\mathbb{R}^n$. In this section we consider energy integrals of the form

$$F(u) = \int_\Omega f(x, u, Du)\, dx\,, \tag{18}$$

where $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \to \mathbb{R}$ is a *Carathéodory integrand*, *quasiconvex* with respect to the gradient variable. The map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ belongs to the Sobolev space $W^{1,p}(\Omega; \mathbb{R}^m)$ for some $p > 1$. We consider here function $f = f(x, u, \xi)$ with *full dependence* on $x \in \Omega$, on $u \in \mathbb{R}^m$ and on the gradient variable ξ, which in the general case is a *matrix* $m \times n$; i.e., $\xi \in \mathbb{R}^{m \times n}$. The definition of *quasiconvexity*, introduced by Morrey [32, Section 4.4] (see also Ball [2] and Dacorogna [14]), under the growth condition $|f(x, u, \xi)| \leq M\ (1 + |\xi|^p)$ requires that

$$\int_\Omega f(x_0, u_0, \xi + Du)\, dx \geq |\Omega|\ f(x_0, u_0, \xi)\,, \quad \forall\, u \in W_0^{1,p}(\Omega; \mathbb{R}^m)\,, \tag{19}$$

for all $x_0 \in \Omega$ and $u_0 \in \mathbb{R}^m$. This condition (19) is equivalent to the usual convexity (1) when either $n = 1, m \geq 1$ or $n \geq 1, m = 1$. That is, *if one of the two dimensional parameters n, m is equal to* 1 *then quasiconvexity reduces (and is equivalent to) convexity in the gradient variable ξ.*

Convexity and quasiconvexity in the calculus of variations play a fundamental role. We can refer to Morrey [32], Ball [2], Dacorogna [14] and for instance to [15], [16], [27], [28], [30], [31] and to the references therein.

As we said, we consider here functions $f = f(x, u, \xi)$ with *full dependence* on $x \in \Omega$, on $u \in \mathbb{R}^m$ and on the gradient variable ξ. In particular we emphasize the two "opposite" cases when $n = m = 1$ or when $n > 1$ and at the same time $m > 1$. The case $n = m = 1$ a-priori is the simplest one. However still some interesting open problems remain when there is the full dependence on all the variables (x, u, ξ). For instance, we mention a case studied by Botteron-Marcellini [4], with some aspects that deserve to be explored further, for instance the results specific for the "linear growth". We focus here to a result which can be applied to classical examples with linear growth.

As usual we denote by $u' = Du = \frac{du}{dx}$ the derivative of the function $u : (0,1) \to \mathbb{R}$. In many applications, such as for instance geometrical, physical and ecological (see [4]) it is natural to look for minimizers among *nonnegative functions* u. In many of these applications more relevant is the fact that the "solution", which may be observed, sometimes has discontinuity; in particular the solution (minimizer) may "detach" from the given boundary data. For this reason in general it is usual, sometime necessary, to introduce an extended energy integral $\overline{F}(u)$, as in (21) below. See Remark 3.3 for a precise physical and geometrical example. By $\mathbb{R}_+$ we denote the interval of real numbers $\mathbb{R}_+ := [0, +\infty)$.

Theorem 3.1. (Botteron-Marcellini [4]) *We consider the integral of the calculus of variations with linear growth on the gradient variable*

$$F(u) = \int_0^1 a(x,u) \sqrt{1 + |u'|^2}\, dx\,, \tag{20}$$

with (formal) boundary conditions $u(0) = S_0$ *and* $u(1) = S_1$. *Here* $a : (0,1) \times \mathbb{R}_+ \to \mathbb{R}_+$ *is a function of class* $C^2((0,1) \times \mathbb{R}_+)$, *continuous also at* $x = 0$ *and* $x = 1$, *which satisfies at least one of the following conditions*

(i) $a = a(u) > 0$ *is monotone with respect to* u *and independent of* x;

(ii) $a = a(u) \geq 0$ *is independent of* x *and if* $a'(u_1) = 0$ *for some* $u_1 \in \mathbb{R}_+$ *then* $a''(u_1) > 0$;

(iii) $a = a(x) \geq c > 0$ *is independent of* u *and if* $a'(x_1) = 0$ *for some* $x_1 \in [0,1]$ *then* $a''(x_1) < 0$;

(iv) $a = a(x,u) \geq c > 0$ *and* $\frac{\partial a}{\partial x}(x,u) \neq 0$ *for all* $(x,u) \in (0,1) \times \mathbb{R}_+$.

Under these conditions the extended energy integral $\overline{F}(u)$, *defined by*

$$\begin{aligned} \overline{F}(u) &:= \int_0^1 a(x,u) \sqrt{1 + |u'|^2}\, dx \\ &+ |A(0,u(0)) - A(0,S_0)| + |A(1,u(1)) - A(1,S_1)| \end{aligned} \tag{21}$$

where $A(x,u) := \int_0^u a(x,s)\, ds$, *has a minimizer* u *in the Sobolev class*

$$K := \left\{ u \in W^{1,1}_{\mathrm{loc}}(0,1)\,,\quad u \geq 0 : \quad \int_0^1 a(x,u) \sqrt{1 + |u'|^2}\, dx < +\infty \right\}. \tag{22}$$

Moreover u *is locally Lipschitz continuous in* $(0,1)$.

Remark 3.2. (The Weierstrass energy integral) As well known, also in the one-dimensional case $n = m = 1$, existence of minimizers may fail. As shown for instance in [4, Remark 4.6], the Weierstrass energy integral

$$F(u) = \int_{-1}^1 x^2 (u')^2\, dx\,, \tag{23}$$

lacks the minimum in the Sobolev class

$$K := \left\{ u \in W^{1,1}(-1,1) : \int_{-1}^1 x^2 (u')^2\, dx < +\infty,\;\; u(-1) = S_0,\; u(1) = S_1 \right\}. \tag{24}$$

More precisely, $\inf \{F(u) : u \in K\} = 0$ but $F(u) > 0$ for every $u \in K$ when $S_0 < S_1$ (while of course $F(u) = 0$ in the trivial case $S_0 = S_1$ and u constantly equal to this common value).

Not even the extended energy integral $\overline{F}(u)$ in (21) has a minimizer. Note that the assumptions of Theorem 3.1 are not satisfied for this Weierstrass example. □

Remark 3.3. (The surface of revolution of minimal area) In the case of the surface of revolution of minimal area we are led to minimize the energy integral

$$F(u) = \int_a^b u \sqrt{1 + |u'|^2}\, dx \tag{25}$$

in the Sobolev class $u \in W^{1,1}_{\mathrm{loc}}(a,b)$ such that $\int_a^b u \sqrt{1+|u'|^2}\, dx < +\infty$, $u(x) \geq 0$ in $[a,b]$, with boundary conditions $u(a) = S_0$, $u(b) = S_1$. Condition (ii) of Theorem 3.1 applies. The solution exists and it may "detach" from the given boundary values $u(a) = S_0$, $u(b) = S_1$ if $b - a$ is too large with respect to S_0 and S_1; in this case the minimizer of the functional $\overline{F}(u)$ in (21) is the discontinuous function

$$u(x) = 0, \quad \forall\, x \in (a,b), \qquad u(a) = S_0, \; u(b) = S_1. \tag{26}$$

This analytic solution in (26) physically corresponds to the surface of revolution made by two disks perpendicular to the x-axis of radius respectively S_0 and S_1. On the contrary, if b is not too far to a (with respect to S_0 and S_1), as well known the minimizer of the area integral (25) is a catenary in the x, u plane, with $u > 0$, passing trough the points of coordinates (a, S_0) and (b, S_1).

Other than the surface of revolution of minimal area, with $a(u) = u$, Theorem 3.1 applies also to the Fermat's principle, where c is the velocity of light in the vacuum and $a(x,u) \geq \frac{1}{c} > 0$; in this case condition (iv) of Theorem 3.1 plays a role. □

Remark 3.4. (p, q-growth conditions) We notice that it is possible to produce a similar result, as stated in Theorem 3.1, by considering more general growth conditions for the energy integrand in (20), such as the so called p, q-growth conditions. As in [4, Remark 5.3] it is possible to obtain similar results when considering energy integrals of the type

$$F(u) = \int_0^1 a(u)\; g(u')\; dx\,, \tag{27}$$

with $|\xi| \leq g(\xi) \leq \mathrm{const}\,(1 + |\xi|^q)$, for all $\xi \in \mathbb{R}$ and for an exponent $q \geq 1$. Details in Remark 5.3 of [4]. Here we have p, q-growth conditions with $p = 1$. Notice that nowadays the research is very active on p, q-growth problems, mainly about regularity, but non only on regularity. Of course the analysis and the thecniques are different for multidimensional integrals, for partial differential equations of elliptic and parabolic types, for differential systems. A brief list of references is [8]–[13] and [17]–[23]; please, see also the references therein. □

For a further large set of results and examples related to the unidimensional case we refer, for instance, to the book by Buttazzo-Giaquinta-Hildebrandt [6]. More recently see also [1], [26], [37].

We consider here the case $n > 1, m > 1$ in the *quasiconvex* context (while, for completeness, the more general condition $n, m \geq 1$ is also stated). We are interested in the *existence of weak solutions* to the *Cauchy-Dirichlet problem* associated to the *parabolic linear system of m partial differential equations* with *continuous coefficients* $a_{ij}^{\alpha\beta} = a_{ij}^{\alpha\beta}(x)$, $c^{\alpha\beta} = c^{\alpha\beta}(x)$, for $\alpha, \beta = 1, 2, \ldots, m$ and $i, j = 1, 2, \ldots, n$,

$$\frac{\partial u^\alpha}{\partial t} - \sum_{i,j,\beta} \frac{\partial}{\partial x_i}\left(a_{ij}^{\alpha\beta}(x)\frac{\partial u^\beta}{\partial x_j}\right) + \sum_\beta c^{\alpha\beta}(x)\, u^\beta = h^\alpha(x), \quad \alpha = 1, 2, \ldots, m, \tag{28}$$

where $t > 0$, $x = (x_i)_{i=1,2,\ldots,n}$ varies on an open bounded set $\Omega \subset \mathbb{R}^n$ and where $u = u(x,t) = (u^\alpha)_{\alpha=1,2,\ldots,m}$ is a map defined on the parabolic cylinder

$$\Omega \times (0, +\infty) \subset \mathbb{R}^n \times (0, +\infty) \to \mathbb{R}^m$$

for some $n, m \in \mathbb{N}$, $n, m \geq 1$; the gradient Du is a matrix $m \times n$; i.e., Du is an element of $\mathbb{R}^{m\times n}$ for $t > 0$, $x \in \mathbb{R}^n$. Finally $h = h(x) = (h^\alpha(x))_{\alpha=1,2,\ldots,m}$ is a given right hand side. The parabolic linear systems (28) can be represented in the compact, but expressive form

$$u_t - \mathrm{div} A(x)\, D_x u + C(x)\, u = h\,. \tag{29}$$

We assume that the system has a *variational structure*; i.e., in the stationary case when $u(x,t)$ is independent of $t > 0$, the differential system (28) is associated to the Euler-Lagrange first variation of the energy integral (18). In the general nonlinear case this first variation takes the form of a differential system of the type

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} f_{\xi_i^\alpha}(x, u, Du) = f_{u^\alpha}(x, u, Du), \quad \alpha = 1, 2, \ldots, m,$$

where, for the specific quadratic case considered above in (28),

$$f(x, u, \xi) = \sum_{i,j,\alpha,\beta} a_{ij}^{\alpha\beta}(x)\, \xi_i^\alpha \xi_j^\beta + \sum_{\alpha,\beta=1}^N c^{\alpha\beta}(x) u^\alpha u^\beta - \sum_\alpha h^\alpha(x)\, u^\alpha\,, \tag{30}$$

reduces to the *elliptic part* in the system in (28). We consider the general *Legendre-Hadamard rank-one condition*

$$\sum_{i,j=1}^n \sum_{\alpha,\beta=1}^m a_{ij}^{\alpha\beta}(x)\, \eta_i \eta_j \theta^\alpha \theta^\beta \geq \Lambda\, |\eta|^2\, |\theta|^2 \tag{31}$$

valid for some $\Lambda > 0$ and for every $x \in \Omega$ and for every $\eta \in \mathbb{R}^n$ and $\theta \in \mathbb{R}^m$.

We say that (28) is a *parabolic linear quasi-elliptic system*, or a *parabolic linear systems under the general Legendre-Hadamard rank-one condition*, since we do not assume here the simpler but less general ellipticity condition

$$\sum_{i,j,\alpha,\beta} a_{ij}^{\alpha\beta}(x)\, \xi_i^\alpha \xi_j^\beta \geq \lambda\, |\xi|^2\,, \quad \text{for every } \xi = (\xi_i^\alpha) \in \mathbb{R}^{m\times n},$$

which guarantee existence of the solution to the Cauchy-Dirichlet problem associated to the parabolic system (28), but which is equivalent and as general as (31) only either in the scalar case $m = 1$ or in the one-dimensional case $n = 1$.

Boegelein-Dacorogna-Duzaar-Marcellini-Scheven [3] recently proved that under the *Legendre-Hadamard condition* (31) and by assuming additionally that the coefficients $a_{ij}^{\alpha\beta}, c^{\alpha\beta} \in C\left(\overline{\Omega}\right)$ are continuous up to the boundary of the bounded open set $\Omega \subset \mathbb{R}^n$, then the Cauchy-Dirichlet problem associated to the parabolic differential system (28) has a unique weak solution if the zero-order terms $c^{\alpha\beta}(x)$ are sufficiently large, in the sense of positive $m \times m$ matrices; i.e., if

$$\sum_{\alpha,\beta=1}^{m} c^{\alpha\beta}(x) u^\alpha u^\beta \geq c\,|u|^2, \quad \forall\, x \in \Omega,\ \ \forall\, u \in \mathbb{R}^m, \tag{32}$$

for some $c \in \mathbb{R}$ sufficiently large.

The existence of weak solutions to the Cauchy-Dirichlet problem associated to the parabolic system (28) has been obtained with a strict and unexpected relation between quasiconvexity and convexity; in fact it turn out that, under the existence assumptions for the described system, quasiconvexity of the integrand with respect to the gradient variable implies the convexity of the integral. The precise statement is the following.

Theorem 3.5. (Boegelein-Dacorogna-Duzaar-Marcellini-Scheven [3]) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $a_{ij}^{\alpha\beta} \in C^0(\overline{\Omega})$ for $i, j = 1, 2, \ldots, n$ and $\alpha, \beta = 1, 2, \ldots, m$ satisfy the Legendre-Hadamard condition* (31). *Let $\left[c^{\alpha\beta}\right]$ be an $m \times m$ matrix of measurable bounded functions, which is positive definite with constant c, as in* (32). *We consider the corresponding energy integral*

$$F(u) := \int_\Omega \left\{ \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{m} a_{ij}^{\alpha\beta}(x) u_{x_i}^\alpha u_{x_j}^\beta + \sum_{\alpha,\beta=1}^{m} c^{\alpha\beta}(x) u^\alpha u^\beta \right\} dx\,. \tag{33}$$

on the Sobolev affine set $u_0 + W_0^{1,2}(\Omega, \mathbb{R}^m)$ where $u_0 \in W^{1,2}(\Omega, \mathbb{R}^m)$ is any fixed boundary datum. Then there exists c_0, depending on $n, m, \Lambda, \Omega, \|a_{ij}^{\alpha\beta}\|_{L^\infty}$ and on the modulus of continuity of the coefficients $a_{ij}^{\alpha\beta}$, such that $F : u_0 + W_0^{1,2}(\Omega, \mathbb{R}^m) \to \mathbb{R}$ is a convex functional with respect to $u \in u_0 + W_0^{1,2}(\Omega, \mathbb{R}^m)$ for all $c > c_0$.

The proof of this results and its application to the existence of weak solutions to the Cauchy-Dirichlet problem associated to the differential system (28) can be found in the quoted paper [3].

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