

Near Convexity and Generalized Differentiation

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

Received: June 13, 2023

Accepted: January 3, 2024

We introduce the concept of nearly convex set-valued mappings and investigate fundamental properties of these mappings. Additionally, we establish a geometric approach for generalized differentiation of nearly convex set-valued mappings and nearly convex functions. Our contributions expand the current knowledge of nearly convex sets and functions, while providing several new results involving nearly convex set-valued mappings.

Keywords: Relative interior, nearly convex set, nearly convex function, nearly convex set-valued mapping, subdifferential, normal cone, coderivative.

2020 Mathematics Subject Classification: 49J52, 49J53, 90C31.

1. Introduction

Over the past few decades, the classical notion of convexity and convex analysis have been extensively studied by prominent mathematicians and experts in applied fields. However, researchers have also made significant efforts to go beyond convexity by introducing and studying many generalized convexity notions for sets and functions. One such notion is the *nearly convex set*, defined based on convexity by requiring that the set under consideration lies between a convex set and its closure in the Euclidean space $\mathbb{R}^n$; see [17, Theorem 12.41]. The concept of near convexity originated in the work of Minty [10], who defined it in another equivalent form and referred to it as *almost convexity*. Namely, according to [10, p.135], a set in $\mathbb{R}^n$ is called almost convex if it contains the relative interior of its convex hull. A more general concept of *virtual convexity* for sets in Banach spaces was proposed by Rockafellar [16], who proved (see [16, Lemma on p.82]) that, in finite dimensions, a set is

*Research of this author was partly supported by the USA National Science Foundation under grant DMS-2136228.

virtually convex if and only if it is almost convex. The study of maximal monotone operators reveals that the near convexity appears naturally, as their domains are nearly convex sets. In particular, the domain of the subdifferential mapping of a proper lower semicontinuous convex function is nearly convex, as discussed in [11] and the references therein.

Despite being introduced in the early age of convex analysis, the notion of near convexity had not been systematically studied in the literature until recently in the papers [2, 11], which provide basic properties and examples of nearly convex sets, ranges of maximal monotone operators, and ranges and fixed points of convex combinations of firmly nonexpansive mappings. The new development also creates opportunities for further study, from nearly convex sets to nearly convex functions and set-valued mappings. Progress in this direction can be seen in [8], which builds on the work of Boĭ, Kassay, and Wanka from [5] on characterizations of nearly convex sets and functions, useful for investigating strong duality for nearly convex optimization problems. Further recent studies and applications of near convexity can be found in [1, 3, 6, 9, 14] and the references therein.

In the present paper, we introduce the concept of nearly convex set-valued mappings by requiring their graphs to be nearly convex sets. We then examine how near convexity of set-valued mappings and nonsmooth functions is preserved under different operations. Specifically, we present two distinct proofs that demonstrate how the sum of two nearly convex functions maintains near convexity, as long as the relative interiors of their effective domains intersect. Our proofs are direct and allow us to identify an error in [8, Theorem 4.18]. Then we explore generalized differentiation for both nearly convex set-valued mappings and nonsmooth functions.

It should be noted that there are still challenging open questions related to nearly convex sets and functions. For example, to the best of our knowledge, the three questions raised by Moffat, Moursi, and Wang in [11, p.218] on the domains and ranges of subdifferential mappings have not been solved so far.

Our paper is structured as follows. In Section 2, we provide an overview of the fundamental definitions and properties of nearly convex sets, which will be referred to throughout the paper. In Section 3, we establish basic properties of nearly convex functions and set-valued mappings. Section 4 is dedicated to exploring the preservation of near convexity under various operations on set-valued mappings and nonsmooth functions. In Section 5, we develop a geometric approach to generalized differentiation of nearly convex functions and set-valued mappings.

The paper utilizes standard notions and notations of convex analysis in the Euclidean space $\mathbb{R}^n$, which can be found in many sources, such as [4, 7, 13, 15]. In the sequel, we use the notation $B(z; \rho)$ (resp., $\mathbb{B}(z; \rho)$) to represent the open (resp., closed) ball centered at $z \in \mathbb{R}^k$ with a radius of $\rho > 0$. The affine hull of a subset $D \subset \mathbb{R}^k$ is abbreviated to $\text{aff } D$, while $\bar{D}$ represents the closure of D , and $\text{int } D$ denotes the interior of D .

2. Preliminaries

This section provides a brief overview of fundamental definitions and significant properties of nearly convex sets that are utilized in this paper. For more comprehensive information, we recommend referring to [2, 11].

A subset Ω of $\mathbb{R}^n$ is said to be *nearly convex* if there exists a convex set C in $\mathbb{R}^n$ such that $C \subset \Omega \subset \overline{C}$.

Clearly, any convex set is nearly convex and any nearly convex subset of $\mathbb{R}$ is convex. Meanwhile, in $\mathbb{R}^n$ with $n \geq 2$ there are many nearly convex sets which are not convex.

Recall that the *relative interior* of an arbitrary set Ω in $\mathbb{R}^n$ is defined by

$$\text{ri } \Omega = \{a \in \Omega \mid \text{there exists } \delta > 0 \text{ such that } B(a; \delta) \cap \text{aff } \Omega \subset \Omega\}.$$

It follows from the definition that $a \in \text{ri } \Omega$ if and only if $a \in \text{aff } \Omega$ and there exists $\delta > 0$ such that

$$B(a; \delta) \cap \text{aff } \Omega \subset \Omega.$$

It is clear that if $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^p$ are nearly convex, then $\Omega_1 \times \Omega_2$ is also nearly convex.

Although the next proposition can be found in [2], we provide here a detailed proof for the convenience of the reader.

Proposition 2.1. *Let Ω be a nearly convex set with $C \subset \Omega \subset \overline{C}$, where C is a convex set in $\mathbb{R}^n$. Then $\text{aff } \Omega = \text{aff } C$, $\text{ri } C = \text{ri } \Omega$, and $\overline{C} = \overline{\Omega}$.*

Proof. It follows from the definition that

$$\text{aff } C \subset \text{aff } \Omega \subset \text{aff } \overline{C} = \text{aff } C,$$

which implies the equality $\text{aff } \Omega = \text{aff } C$. Fix any $a \in \text{ri } C$. Then $a \in C$ and there exists $\delta > 0$ such that

$$B(a; \delta) \cap \text{aff } \Omega = B(a; \delta) \cap \text{aff } C \subset C \subset \Omega.$$

It follows that $a \in \text{ri } \Omega$, so $\text{ri } C \subset \text{ri } \Omega$. Applying the same argument again shows that $\text{ri } \Omega \subset \text{ri } \overline{C} = \text{ri } C$, where the last equality is valid by the convexity of C . Thus we obtain $\text{ri } \Omega = \text{ri } C$.

Since $C \subset \Omega \subset \overline{C}$, we have $\overline{C} \subset \overline{\Omega} \subset \overline{C}$, which implies $\overline{\Omega} = \overline{C}$ and completes the proof. $\square$

Theorem 2.2. (See [11, Theorem 4.2 and Corollary 4.8]) *Suppose that the sets $\Omega, \Omega_1, \dots, \Omega_m$ are nearly convex sets in $\mathbb{R}^n$, and $A: \mathbb{R}^n \rightarrow \mathbb{R}^p$ is a linear function. Then*

(a) *$A(\Omega)$ is a nearly convex set in $\mathbb{R}^p$ and $\text{ri } A(\Omega) = A(\text{ri } \Omega)$.*

(b) *If $\bigcap_{i=1}^m \text{ri } \Omega_i \neq \emptyset$, then $\bigcap_{i=1}^m \Omega_i$ is nearly convex,*

$$\text{ri} \left(\bigcap_{i=1}^m \Omega_i \right) = \bigcap_{i=1}^m \text{ri } \Omega_i, \quad \text{and} \quad \overline{\bigcap_{i=1}^m \Omega_i} = \bigcap_{i=1}^m \overline{\Omega_i}.$$

For two subsets Ω_1 and Ω_2 of $\mathbb{R}^n$, if $\text{ri } \Omega_1 = \text{ri } \Omega_2$ and $\overline{\Omega_1} = \overline{\Omega_2}$, then we say that they are *nearly equal* and write $\Omega_1 \approx \Omega_2$.

The proposition below characterizes the near convexity using the near equality.

Proposition 2.3. (See [2, Lemma 2.9]) *Let Ω be a subset of $\mathbb{R}^n$. Then the following properties are equivalent:*

- (a) Ω is nearly convex.
- (b) Ω is nearly equal to a convex set.
- (c) Ω is nearly equal to a nearly convex set.
- (d) $\Omega \approx \text{co } \Omega$.

The preservation of the near equality between two nearly convex sets via a linear mapping is stated as follows.

Proposition 2.4. (See [11, Corollary 4.9]) *Let Ω_1 and Ω_2 be nearly convex subsets of $\mathbb{R}^n$ with $\Omega_1 \approx \Omega_2$ and $A: \mathbb{R}^n \rightarrow \mathbb{R}^p$ be a linear function. Then $A(\Omega_1) \approx A(\Omega_2)$.*

Given a function $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}} = [-\infty, \infty]$, the *effective domain* and the *epigraph* of f are given respectively by

$$\begin{aligned}\text{dom } f &= \{x \in \mathbb{R}^n \mid f(x) < \infty\}, \\ \text{epi } f &= \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq \lambda\}.\end{aligned}$$

We say that f is *proper* if $\text{dom } f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$. The function f is said to be *convex* if $\text{epi } f$ is a convex set, and it is said to be *nearly convex* if $\text{epi } f$ is a nearly convex set.

Example 2.5. (a) Consider the function $f: \mathbb{R} \rightarrow \overline{\mathbb{R}}$ defined by

$$f(x) = \begin{cases} -\infty & \text{if } x < 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

Then $\text{dom } f = \mathbb{R}$, $\text{epi } f = ((-\infty, 0) \times \mathbb{R}) \cup ([0, \infty) \times [0, \infty))$. We can see that f is neither convex nor nearly convex.

(b) Consider the function $g: \mathbb{R} \rightarrow \overline{\mathbb{R}}$ defined by

$$g(x) = \begin{cases} -\infty & \text{if } x < 0, \\ 1 & \text{if } x = 0, \\ \infty & \text{if } x > 0. \end{cases}$$

Then $\text{dom } f = (-\infty, 0]$, $\text{epi } f = ((-\infty, 0) \times \mathbb{R}) \cup (\{0\} \times [1, \infty))$. We can see that f is an improper convex function. $\square$

We continue this section with a representation of the affine hull of the epigraph of an arbitrary function; see [13, Exercise 2.6].

Proposition 2.6. *Let $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be an arbitrary function. Then*

$$\text{aff}(\text{epi } f) = \text{aff}(\text{dom } f) \times \mathbb{R}. \quad (1)$$

Proof. Fix any $(x, \gamma) \in \text{aff}(\text{epi } f)$. Find $\lambda_i \in \mathbb{R}$ and $(x_i, \gamma_i) \in \text{epi } f$ for $i = 1, \dots, m$ such that

$$\sum_{i=1}^m \lambda_i = 1 \quad \text{and} \quad (x, \gamma) = \sum_{i=1}^m \lambda_i (x_i, \gamma_i).$$

Since $(x_i, \gamma_i) \in \text{epi } f$, we have $f(x_i) \leq \gamma_i < \infty$, so $x_i \in \text{dom } f$ for $i = 1, \dots, m$. Then $x = \sum_{i=1}^m \lambda_i x_i \in \text{aff}(\text{dom } f)$ and thus $(x, \gamma) \in \text{aff}(\text{dom } f) \times \mathbb{R}$. This justifies the inclusion $\subset$ in (1).

To verify the inclusion $\supset$ in (1), let $(x, \gamma) \in \text{aff}(\text{dom } f) \times \mathbb{R}$ be given arbitrarily. Then $\gamma \in \mathbb{R}$ and there exist $x_i \in \text{dom } f$ and $\lambda_i \in \mathbb{R}$ for $i = 1, \dots, m$ with $\sum_{i=1}^m \lambda_i = 1$ such that $x = \sum_{i=1}^m \lambda_i x_i$. For each $i = 1, \dots, m$, define $\alpha_i = f(x_i)$ if $f(x_i) \in \mathbb{R}$, and choose any $\alpha_i \in \mathbb{R}$ if $f(x_i) = -\infty$. Clearly, $(x_i, \alpha_i) \in \text{epi } f$ and $(x_i, \alpha_i + 1) \in \text{epi } f$. Letting $\alpha = \sum_{i=1}^m \lambda_i \alpha_i \in \mathbb{R}$ gives us

$$\sum_{i=1}^m \lambda_i (x_i, \alpha_i) = (x, \alpha) \in \text{aff}(\text{epi } f),$$

and
$$\sum_{i=1}^m \lambda_i (x_i, \alpha_i + 1) = (x, \alpha + 1) \in \text{aff}(\text{epi } f).$$

Considering the number $\lambda = \alpha - \gamma + 1$, we have

$$(x, \gamma) = \lambda(x, \alpha) + (1 - \lambda)(x, \alpha + 1) \in \text{aff}(\text{epi } f),$$

which justifies the reverse inclusion in (1) and also the proof of the proposition. $\square$

For a set-valued mapping $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$, define the *domain*, the *range*, and the *graph* of F by

$$\begin{aligned} \text{dom } F &= \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}, \quad \text{rge } F = \bigcup_{x \in \mathbb{R}^n} F(x), \\ \text{gph } F &= \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p \mid y \in F(x)\}. \end{aligned}$$

We say that F is *nearly convex* if $\text{gph } F$ is a nearly convex set in $\mathbb{R}^n \times \mathbb{R}^p$.

Given a function $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, define the *epigraphical mapping* $E_f: \mathbb{R}^n \rightrightarrows \mathbb{R}$ by

$$E_f(x) = \{\lambda \in \mathbb{R} \mid f(x) \leq \lambda\}, \quad x \in \mathbb{R}^n. \quad (2)$$

It follows directly from the definition that $\text{dom } E_f = \text{dom } f$ and $\text{gph } E_f = \text{epi } f$. We also define the *epigraphical range* of f by $\text{rge } f = \text{rge } E_f$.

The *indicator function* $\delta_\Omega: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ of a set $\Omega \subset \mathbb{R}^n$ is defined by

$$\delta_\Omega(x) = \begin{cases} 0 & \text{if } x \in \Omega, \\ \infty & \text{otherwise.} \end{cases}$$

Since $\text{epi } \delta_\Omega = \Omega \times [0, \infty)$, we see that δ_Ω is nearly convex if and only if Ω is nearly convex.

3. Nearly convex functions and set-valued mappings

In this section, we study general properties of nearly convex functions and set-valued mappings. In particular, we are able to show that the relative interior of the graph of a nearly convex set-valued mapping can be computed via the relative interiors of its domain and of the mapping values. This result generalizes a well-known theorem by Rockafellar (see [15, Theorem 6.8] and also [17, Proposition 2.43] for another proof) on the relative interiors of convex graphs to the case of nearly convex graphs.

For two elements $a, b \in \mathbb{R}^n$, define

$$\begin{aligned} [a, b] &= \{(1-t)a + tb \mid 0 \leq t \leq 1\}, \quad (a, b) = \{(1-t)a + tb \mid 0 < t < 1\}, \\ [a, b) &= \{(1-t)a + tb \mid 0 \leq t < 1\}, \quad (a, b] = \{(1-t)a + tb \mid 0 < t \leq 1\}. \end{aligned}$$

Note that if $a = b$, then $[a, b] = (a, b) = [a, b) = (a, b] = \{a\} = \{b\}$.

The next simple result will be used in what follows.

Proposition 3.1. *Let Ω be a nearly convex set in $\mathbb{R}^n$. If $a \in \text{ri } \Omega$ and $b \in \overline{\Omega}$, then*

$$[a, b) \subset \text{ri } \Omega.$$

Proof. Take any $a \in \text{ri } \Omega$ and $b \in \overline{\Omega}$. Choose a convex set C such that $C \subset \Omega \subset \overline{C}$. By Proposition 2.1 we have $a \in \text{ri } C$ and $b \in \overline{C}$.

It follows that $[a, b) \subset \text{ri } C = \text{ri } \Omega$, which completes the proof. $\square$

Given two nonempty sets Ω_1 and Ω_2 , we say that Ω_1 and Ω_2 can be *properly separated* (by a hyperplane) if there exists $v \in \mathbb{R}^n$ such that

$$\begin{aligned} \sup\{\langle v, x \rangle \mid x \in \Omega_1\} &\leq \inf\{\langle v, y \rangle \mid y \in \Omega_2\}, \\ \inf\{\langle v, x \rangle \mid x \in \Omega_1\} &< \sup\{\langle v, y \rangle \mid y \in \Omega_2\}. \end{aligned} \tag{3}$$

Note that the first inequality means that $\langle v, x \rangle \leq \langle v, y \rangle$ whenever $x \in \Omega_1$ and $y \in \Omega_2$, while the second inequality means that there exist $\hat{x} \in \Omega_1$ and $\hat{y} \in \Omega_2$ such that $\langle v, \hat{x} \rangle < \langle v, \hat{y} \rangle$.

The theorem below provides necessary and sufficient conditions for proper separation of two nearly convex sets; see [8, Proposition 3.7]. Here we give a new proof for the result.

Theorem 3.2. *Let Ω_1 and Ω_2 be two nonempty nearly convex sets in $\mathbb{R}^n$. Then Ω_1 and Ω_2 can be properly separated if and only if $\text{ri } \Omega_1 \cap \text{ri } \Omega_2 = \emptyset$.*

Proof. Let C_1 and C_2 be two convex sets in $\mathbb{R}^n$ such that

$$C_1 \subset \Omega_1 \subset \overline{C_1} \quad \text{and} \quad C_2 \subset \Omega_2 \subset \overline{C_2}. \tag{4}$$

Suppose that Ω_1 and Ω_2 can be properly separated and find $v \in \mathbb{R}^n$ such that (3) is satisfied. The inclusions in (4) and the continuity of the inner product yield

$$\sup\{\langle v, x \rangle \mid x \in C_1\} \leq \sup\{\langle v, x \rangle \mid x \in \Omega_1\} \leq \inf\{\langle v, y \rangle \mid y \in \Omega_2\} \leq \inf\{\langle v, y \rangle \mid y \in C_2\}$$

and

$$\begin{aligned} \inf\{\langle v, x \rangle \mid x \in C_1\} &= \inf\{\langle v, x \rangle \mid x \in \overline{C_1}\} \leq \inf\{\langle v, x \rangle \mid x \in \Omega_1\} \\ &< \sup\{\langle v, y \rangle \mid y \in \Omega_2\} \leq \sup\{\langle v, y \rangle \mid y \in \overline{C_2}\} = \sup\{\langle v, y \rangle \mid y \in C_2\}. \end{aligned}$$

Hence, C_1 and C_2 can be properly separated. Since C_1 and C_2 are nonempty and convex, we can apply [13, Theorem 2.40] to get that $\text{ri } C_1 \cap \text{ri } C_2 = \emptyset$. By Proposition 2.1 we have

$$\text{ri } \Omega_1 \cap \text{ri } \Omega_2 = \text{ri } C_1 \cap \text{ri } C_2 = \emptyset.$$

For the converse implication suppose that $\text{ri } \Omega_1 \cap \text{ri } \Omega_2 = \emptyset$ and get from Proposition 2.1 that $\text{ri } C_1 \cap \text{ri } C_2 = \emptyset$. Applying [13, Theorem 2.40] again gives us a vector $v \in \mathbb{R}^n$ such that

$$\begin{aligned} \sup\{\langle v, x \rangle \mid x \in C_1\} &\leq \inf\{\langle v, y \rangle \mid y \in C_2\}, \\ \inf\{\langle v, x \rangle \mid x \in C_1\} &< \sup\{\langle v, y \rangle \mid y \in C_2\}. \end{aligned} \quad (5)$$

Then by (4) and (5) we have

$$\begin{aligned} \sup\{\langle v, x \rangle \mid x \in \Omega_1\} &\leq \sup\{\langle v, x \rangle \mid x \in \overline{C_1}\} = \sup\{\langle v, x \rangle \mid x \in C_1\} \\ &\leq \inf\{\langle v, y \rangle \mid y \in C_2\} = \inf\{\langle v, y \rangle \mid y \in \overline{C_2}\} \leq \inf\{\langle v, y \rangle \mid y \in \Omega_2\}. \end{aligned}$$

The verification of the strict inequality $\inf\{\langle v, x \rangle \mid x \in \Omega_1\} < \sup\{\langle v, y \rangle \mid y \in \Omega_2\}$ by using (4) and (5) is similar. Thus (3) is satisfied for the vector v . This completes the proof. $\square$

Proposition 3.3. *Let Ω be a nearly convex set in $\mathbb{R}^n$ with $y_0 \in \Omega$. Then $y_0 \in \text{ri } \Omega$ if and only if for any $x \in \Omega$ there exists $z \in \Omega$ such that $y_0 \in (x, z)$.*

Proof. Suppose that $y_0 \in \text{ri } \Omega$ and take any $x \in \Omega$. By the definition of relative interior, there exists $\delta > 0$ such that $B(y_0; \delta) \cap \text{aff } \Omega \subset \Omega$. Choose $t > 0$ sufficiently small such that

$$y_0 + t(y_0 - x) = (1 + t)y_0 + (-t)x \in B(y_0; \delta).$$

Put $z = y_0 + t(y_0 - x)$. Since z is an affine combination of y_0 and x , we have $z \in B(y_0; \delta) \cap \text{aff } \Omega \subset \Omega$. Then

$$y_0 = \frac{t}{1+t}x + \frac{1}{1+t}z \in (x, z).$$

To prove the converse implication, suppose on the contrary that for any $x \in \Omega$ there exists $z \in \Omega$ such that $y_0 \in (x, z)$, but $y_0 \notin \text{ri } \Omega$. Choose a convex set C such that $C \subset \Omega \subset \overline{C}$. Then $\text{ri } C = \text{ri } \Omega$ and $\overline{C} = \overline{\Omega}$ by Proposition 2.1; thus $y_0 \notin \text{ri } C$. Applying the proper separation theorem (see, e.g., [13, Theorem 2.40]) to the convex sets C and $\{y_0\}$, we can find $v \in \mathbb{R}^n$ such that

$$\langle v, x \rangle \leq \langle v, y_0 \rangle \quad \text{for all } x \in C \quad (6)$$

and there exists $\hat{x} \in C$ such that $\langle v, \hat{x} \rangle < \langle v, y_0 \rangle$. Since $\hat{x} \in \Omega$, by our assumption there exists $\hat{z} \in \Omega$ such that $y_0 \in (\hat{x}, \hat{z})$. So, thanks to the inclusion $\Omega \subset \overline{C}$ and the convexity of $\overline{C}$, we have $y_0 \in (\hat{x}, \hat{z}) \subset [\hat{x}, \hat{z}] \subset \overline{C}$. Passing to a limit shows that the inequality in (6) holds for all $x \in \overline{C}$. Then

$$\langle v, x \rangle \leq \langle v, y_0 \rangle \quad \text{for all } x \in [\hat{x}, \hat{z}] \quad \text{and} \quad \langle v, \hat{x} \rangle < \langle v, y_0 \rangle.$$

This means that the convex sets $[\hat{x}, \hat{z}]$ and $\{y_0\}$ can be properly separated. So, by [13, Theorem 2.40] we get $y_0 \notin \text{ri}([\hat{x}, \hat{z}]) = (\hat{x}, \hat{z})$, a contradiction. $\square$

The next proposition concerns the near convexity of the domains and ranges of nearly convex functions and set-valued mappings.

Proposition 3.4. *If $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is a nearly convex set-valued mapping, then $\text{dom } F$ and $\text{rge } F$ are nearly convex sets. Consequently, if $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is a nearly convex function, then $\text{dom } f$ and $\text{rge } f$ are both nearly convex.*

Proof. First, observe that $\text{dom } F = \mathcal{P}(\text{gph } F)$, where $\mathcal{P}$ is the linear mapping

$$\mathcal{P}(x, y) = x, \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^p. \quad (7)$$

Since $\text{gph } F$ is nearly convex, the set $\text{dom } F$ is nearly convex by Theorem 2.2(a). Similarly, as $\text{rge } F = \mathcal{P}_1(\text{gph } F)$, where $\mathcal{P}_1$ is the linear mapping

$$\mathcal{P}_1(x, y) = y, \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^p,$$

we can conclude that $\text{rge } F$ is nearly convex.

Now, to prove the second assertion, suppose that $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is nearly convex. Then the epigraphical mapping E_f defined in (2) is nearly convex. As $\text{dom } f = \text{dom } E_f$ and $\text{rge } f = \text{rge } E_f$, the sets $\text{dom } f$ and $\text{rge } f$ are both nearly convex by the first assertion. $\square$

Remark 3.5. Given a proper function $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, define

$$\text{im } f = \{f(x) \mid x \in \text{dom } f\}.$$

We can find an example of a proper convex function for which $\text{im } f$ is not nearly convex. Indeed, consider the function $f: \mathbb{R} \rightarrow \overline{\mathbb{R}}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } 0 < x \leq 1, \\ \infty & \text{otherwise.} \end{cases}$$

Then f is convex, $\text{rge } f = [0, \infty)$, but $\text{im } f = \{0, 1\}$. $\square$

The theorem below allows us to represent the relative interior of the graph of a nearly convex set-valued mapping via the relative interiors of its domain and mapping values.

Theorem 3.6. *Let $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a nearly convex set-valued mapping. Then*

$$\text{ri}(\text{gph } F) = \{(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^p \mid \bar{x} \in \text{ri}(\text{dom } F), \bar{y} \in \text{ri } F(\bar{x})\}. \quad (8)$$

Proof. Since $\text{gph } F$ is nearly convex by our assumption, using the projection mapping $\mathcal{P}$ defined in (7) along with Theorem 2.2(a) gives us

$$\mathcal{P}(\text{ri}(\text{gph } F)) = \text{ri}(\mathcal{P}(\text{gph } F)) = \text{ri}(\text{dom } F). \quad (9)$$

To prove the inclusion $\subset$ in (8), fix any $(\bar{x}, \bar{y}) \in \text{ri}(\text{gph } F)$. By (9) we then have $\bar{x} \in \text{ri}(\text{dom } F)$. By the definition of relative interior, there exists $\delta > 0$ such that

$$[\mathbb{B}(\bar{x}; \delta) \times \mathbb{B}(\bar{y}; \delta)] \cap \text{aff}(\text{gph } F) \subset \text{gph } F.$$

Then we have $[\{\bar{x}\} \times \mathbb{B}(\bar{y}; \delta)] \cap \text{aff}(\text{gph } F) \subset \text{gph } F. \quad (10)$

This implies that $\mathbb{B}(\bar{y}; \delta) \cap \text{aff } F(\bar{x}) \subset F(\bar{x}). \quad (11)$

Indeed, taking any $y \in \mathbb{B}(\bar{y}; \delta) \cap \text{aff } F(\bar{x})$ gives us the representation

$$y = \sum_{i=1}^m \lambda_i y_i,$$

where $y_i \in F(\bar{x})$ and $\lambda_i \in \mathbb{R}$ for $i = 1, \dots, m$ with $\sum_{i=1}^m \lambda_i = 1$. Then

$$(\bar{x}, y) = \sum_{i=1}^m \lambda_i (\bar{x}, y_i) \in \text{aff}(\text{gph } F).$$

Thus, by (10) we see that

$$(\bar{x}, y) \in [\{\bar{x}\} \times \mathbb{B}(\bar{y}; \delta)] \cap \text{aff}(\text{gph } F) \subset \text{gph } F,$$

which yields $y \in F(\bar{x})$. This justifies (11), so $\bar{y} \in \text{ri } F(\bar{x})$ by the definition of relative interior.

To prove the inclusion $\supset$ in (8), take any $\bar{x} \in \text{ri}(\text{dom } F)$ and $\bar{y} \in \text{ri } F(\bar{x})$. Using (9), we find $\hat{y} \in F(\bar{x})$ such that $(\bar{x}, \hat{y}) \in \text{ri}(\text{gph } F)$. Choose a convex set $C \subset \mathbb{R}^n \times \mathbb{R}^p$ such that

$$C \subset \text{gph } F \subset \bar{C}.$$

By Proposition 2.1 one has $\text{ri}(\text{gph } F) = \text{ri } C$. We only need to consider the case where $\bar{y} \neq \hat{y}$ because in the other case it holds that $(\bar{x}, \bar{y}) \in \text{ri}(\text{gph } F)$. Since $\bar{y} \in \text{ri } F(\bar{x})$, by [13, Proposition 2.18], we can choose $\gamma > 0$ such that $\bar{y} + \gamma(\bar{y} - \hat{y}) \in F(\bar{x})$. Note that this does not require the convexity or near convexity of $F(\bar{x})$. Then $(\bar{x}, \bar{y} + \gamma(\bar{y} - \hat{y})) \in \text{gph } F \subset \bar{C}$. By [13, Theorem 2.22(ii)] and the convexity of C , we have the inclusion

$$[(\bar{x}, \hat{y}), (\bar{x}, \bar{y} + \gamma(\bar{y} - \hat{y}))] \subset \text{ri } C,$$

where the set on the left-hand side is the half-open interval connecting $(\bar{x}, \hat{y}) \in \text{ri } C$ with $(\bar{x}, \bar{y} + \gamma(\bar{y} - \hat{y}))$. It follows that

$$[(\bar{x}, \hat{y}), (\bar{x}, \bar{y} + \gamma(\bar{y} - \hat{y}))] \subset \text{ri}(\text{gph } F). \quad (12)$$

For $t = 1/(1 + \gamma) \in (0, 1)$, by (12) we obtain

$$(\bar{x}, \bar{y}) = (1 - t)(\bar{x}, \hat{y}) + t(\bar{x}, \bar{y} + \gamma(\bar{y} - \hat{y})) \in [(\bar{x}, \hat{y}), (\bar{x}, \bar{y} + \gamma(\bar{y} - \hat{y}))] \subset \text{ri}(\text{gph } F),$$

which completes the proof of the theorem. $\square$

The following proposition not only improves Proposition 4.4 from [8] but also provides an alternative simple proof for the result.

Proposition 3.7. *Let $f: \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ be an arbitrary function. Then*

$$\text{ri}(\text{epi } f) \subset \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \text{ri}(\text{dom } f), \lambda > f(x)\}. \quad (13)$$

The reverse inclusion of (13) holds if we assume in addition that f is nearly convex.

Proof. To prove inclusion (13), take any $(\bar{x}, \bar{\lambda}) \in \text{ri}(\text{epi } f)$. Then by the definition of relative interior one has $(\bar{x}, \bar{\lambda}) \in \text{epi } f$ and there exists $\delta > 0$ such that

$$(\mathbb{B}(\bar{x}; \delta) \times [\bar{\lambda} - \delta, \bar{\lambda} + \delta]) \cap \text{aff}(\text{epi } f) \subset \text{epi } f.$$

The representation of $\text{aff}(\text{epi } f)$ from Proposition 2.6 gives us

$$(\mathbb{B}(\bar{x}; \delta) \times [\bar{\lambda} - \delta, \bar{\lambda} + \delta]) \cap (\text{aff}(\text{dom } f) \times \mathbb{R}) = (\text{aff}(\text{dom } f) \cap \mathbb{B}(\bar{x}; \delta)) \times [\bar{\lambda} - \delta, \bar{\lambda} + \delta] \subset \text{epi } f.$$

It follows that

$$(\text{aff}(\text{dom } f) \cap \mathbb{B}(\bar{x}; \delta)) \times \{\bar{\lambda}\} \subset \text{epi } f, \quad \text{and} \quad \{\bar{x}\} \times [\bar{\lambda} - \delta, \bar{\lambda} + \delta] \subset \text{epi } f.$$

The first inclusion gives us $\text{aff}(\text{dom } f) \cap \mathbb{B}(\bar{x}; \delta) \subset \text{dom } f$, and so $\bar{x} \in \text{ri}(\text{dom } f)$. From the second inclusion we have $(\bar{x}, \bar{\lambda} - \delta) \in \text{epi } f$, so $f(\bar{x}) \leq \bar{\lambda} - \delta < \bar{\lambda}$. This shows that $(\bar{x}, \bar{\lambda})$ is contained in the set on the right-hand side of inclusion (13). Thus, (13) is valid.

Now, to prove the second assertion, assume that f is nearly convex. Consider the epigraphical mapping $F = E_f$ defined in (2), which is nearly convex. Take $\bar{x} \in \text{ri}(\text{dom } f)$ and $\bar{\lambda} > f(\bar{x})$. We see that

$$\bar{x} \in \text{ri}(\text{dom } f) = \text{ri}(\text{dom } E_f) = \text{ri}(\text{dom } F).$$

Obviously, $\bar{\lambda} \in (f(\bar{x}), \infty) = \text{ri}(E_f(\bar{x})) = \text{ri}(F(\bar{x}))$. Thus, by Theorem (3.6) we have

$$(\bar{x}, \bar{\lambda}) \in \text{ri}(\text{gph } F) = \text{ri}(\text{gph } E_f) = \text{ri}(\text{epi } f).$$

The proof is complete. □

Given a function $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, recall that f is continuous at $\bar{x} \in \mathbb{R}^n$ if $\bar{x} \in \text{int}(\text{dom } f)$ and for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(\bar{x})| < \epsilon \quad \text{whenever} \quad x \in \mathbb{B}(\bar{x}; \delta) \subset \text{dom } f. \quad (14)$$

The next proposition provides representations for the interior of the graph of a nearly convex set-valued mapping and also the interior of the epigraph of a nearly convex function.

Proposition 3.8. *Let $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set-valued mapping with $\text{int}(\text{gph } F) \neq \emptyset$. Then we have the inclusion*

$$\text{int}(\text{gph } F) \subset \{(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^p \mid \bar{x} \in \text{int}(\text{dom } F), \bar{y} \in \text{int } F(\bar{x})\}. \quad (15)$$

The reverse inclusion in (15) holds if we assume in addition that F is nearly convex. Consequently, if a function $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is nearly convex, and continuous at some point $\bar{x} \in \mathbb{R}^n$, then

$$\text{int}(\text{epi } f) = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \text{int}(\text{dom } f), f(x) < \lambda\}. \quad (16)$$

Proof. Fix any $(\bar{x}, \bar{y}) \in \text{int}(\text{gph } F)$ and find $\delta > 0$ such that

$$\mathbb{B}(\bar{x}; \delta) \times \mathbb{B}(\bar{y}; \delta) \subset \text{gph } F.$$

Then it holds that $\mathbb{B}(\bar{x}; \delta) \times \{\bar{y}\} \subset \text{gph } F$ and $\{\bar{x}\} \times \mathbb{B}(\bar{y}; \delta) \subset \text{gph } F$, which imply that $\bar{x} \in \text{int}(\text{dom } F)$ and $\bar{y} \in \text{int } F(\bar{x})$. Thus we have the inclusion $\subset$ in (15). The reverse inclusion follows from Theorem 3.6 under the assumption that F is nearly convex.

Now, let $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be nearly convex, and continuous at some $\bar{x} \in \mathbb{R}^n$. It follows from the definition of continuity that $\bar{x} \in \text{int}(\text{dom } f)$, so $\text{ri}(\text{dom } f) = \text{int}(\text{dom } f)$. Fix $\epsilon > 0$ and choose $\delta > 0$ such that (14) is satisfied. Then

$$\mathbb{B}(\bar{x}; \delta) \times (f(\bar{x}) + \epsilon, \infty) \subset \text{epi } f,$$

so $\text{int}(\text{epi } f) \neq \emptyset$. Thus, representation (16) follows directly from (15) using again the epigraphical mapping $F = E_f$. $\square$

A fundamental property of the values of a nearly convex set-valued mapping is given in the following theorem.

Theorem 3.9. *Let $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a nearly convex set-valued mapping. If $\bar{x} \in \text{ri}(\text{dom } F)$, then $F(\bar{x})$ is nearly convex. In particular, $\text{ri } F(\bar{x})$ is nonempty.*

Proof. Suppose that F is nearly convex and $\bar{x} \in \text{ri}(\text{dom } F)$. Then $\text{gph } F$ is nearly convex. Let $C = \text{co}(\text{gph } F)$ and define the set-valued mapping

$$G(x) = \{y \in \mathbb{R}^p \mid (x, y) \in C\}, \quad x \in \mathbb{R}^n.$$

Then $\text{gph } G = C$. Since F is nearly convex and $C = \text{co}(\text{gph } F)$, one has $C \approx \text{gph } F$ by Proposition 2.3. Let us consider the mapping $\mathcal{P}$ defined in (7). By Proposition 2.4 we get $\mathcal{P}(\text{gph } G) \approx \mathcal{P}(\text{gph } F)$ or, equivalently, $\text{dom } G \approx \text{dom } F$. Thus we obtain $\text{ri}(\text{dom } G) = \text{ri}(\text{dom } F)$. Take any $\bar{y} \in \text{ri } G(\bar{x})$. Then by Theorem 3.6 one has $(\bar{x}, \bar{y}) \in \text{ri}(\text{gph } G) = \text{ri}(\text{gph } F)$, which implies that $\bar{y} \in \text{ri } F(\bar{x}) \subset F(\bar{x})$. Thus

$$\text{ri } G(\bar{x}) \subset F(\bar{x}). \quad (17)$$

In addition, the convexity of G implies that $G(\bar{x})$ is convex. Since $\text{gph } F \subset \text{gph } G$, one has

$$F(\bar{x}) \subset G(\bar{x}) \subset \overline{G(\bar{x})}. \quad (18)$$

By [13, Proposition 2.23], one obtains $\text{ri } \overline{G(\bar{x})} = \overline{G(\bar{x})}$. Then the inclusions (17) and (18) yield that $F(\bar{x})$ is nearly convex. Since $F(\bar{x})$ is nearly convex and nonempty, we can deduce from Proposition 2.1 that $\text{ri } F(\bar{x}) \neq \emptyset$. $\square$

If $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is a nearly convex set-valued mapping, then $F(x)$ is not necessarily nearly convex for all $x \in \text{dom } F$.

Example 3.10. Let $F: \mathbb{R} \rightrightarrows \mathbb{R}$ be given by

$$F(x) = \begin{cases} [0, 2] & \text{if } x \in [0, 1), \\ [0, 1) \cup (1, 2] & \text{if } x = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is clear to see that F is nearly convex, but $F(1)$ is not nearly convex. $\square$

A natural question arises: *Whether under the assumption of Theorem 3.9 can one assert that $F(\bar{x})$ is convex for every $\bar{x} \in \text{ri}(\text{dom } F)$?* As shown by the example below, the answer is negative.

Example 3.11. Consider the constant set-valued mapping $F: \mathbb{R} \rightrightarrows \mathbb{R}^2$ with

$$F(x) = ([0, 1] \times [0, 1]) \setminus \{(1/2, 1)\}, \quad x \in \mathbb{R}.$$

Then F is nearly convex, $\text{dom } F = \mathbb{R}$, and $F(x)$ is not convex for all $x \in \mathbb{R}$. $\square$

The next theorem shows that the notions of convexity and near convexity are the same for functions defined on the real line.

Proposition 3.12. *If a proper function $f: \mathbb{R} \rightarrow \overline{\mathbb{R}}$ is nearly convex, then it is convex.*

Proof. Suppose that $f: \mathbb{R} \rightarrow \overline{\mathbb{R}}$ is proper and nearly convex. It follows from Proposition 3.4 that $\text{dom } f \subset \mathbb{R}$ is nearly convex. It is easy to see that a nearly convex subset of $\mathbb{R}$ is an interval or a singleton, so $\text{dom } f$ is an interval in $\mathbb{R}$ or a singleton. The function f is clearly convex if its domain is a singleton. Next, observe that for any $x \in \text{ri}(\text{dom } f)$, $u \in \text{dom } f$, and $0 < t < 1$ we have

$$f(tx + (1-t)u) \leq tf(x) + (1-t)f(u). \quad (19)$$

Indeed, taking $\epsilon > 0$ yields $(x, f(x) + \epsilon) \in \text{ri}(\text{epi } f)$ by Proposition 3.7. So, applying Proposition 3.1 to the nearly convex set $\Omega = \text{epi } f$ and noting that $(u, f(u)) \in \text{epi } f$, we have $t(x, f(x) + \epsilon) + (1-t)(u, f(u)) \in \text{ri}(\text{epi } f)$. Using Proposition 3.7 again gives us

$$f(tx + (1-t)u) < tf(x) + (1-t)f(u) + t\epsilon,$$

which implies (19) by letting $\epsilon \rightarrow 0^+$. It suffices to consider the case where $\text{dom } f = [\alpha, \beta]$ with $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$ because the conclusion is obvious for other cases. By the observation above, we only need to prove that for any $t \in (0, 1)$ one has

$$f(t\alpha + (1-t)\beta) \leq tf(\alpha) + (1-t)f(\beta).$$

Take any $t \in (0, 1)$ and let $\hat{x} = t\alpha + (1-t)\beta$. Since $\hat{x} \in \text{ri}(\text{dom } f)$, one has $\frac{\alpha + \hat{x}}{2} \in \text{ri}(\text{dom } f)$ and $\frac{\beta + \hat{x}}{2} \in \text{ri}(\text{dom } f)$. We can deduce from (19) that

$$f\left(\frac{\alpha + \hat{x}}{2}\right) \leq \frac{f(\alpha) + f(\hat{x})}{2} \quad \text{and} \quad f\left(\frac{\beta + \hat{x}}{2}\right) \leq \frac{f(\beta) + f(\hat{x})}{2}.$$

So, using (19) again gives us

$$\begin{aligned} f(\hat{x}) &= f\left(t\frac{\alpha + \hat{x}}{2} + (1-t)\frac{\beta + \hat{x}}{2}\right) \leq tf\left(\frac{\alpha + \hat{x}}{2}\right) + (1-t)f\left(\frac{\beta + \hat{x}}{2}\right) \\ &\leq t\frac{f(\alpha) + f(\hat{x})}{2} + (1-t)\frac{f(\beta) + f(\hat{x})}{2} = \frac{tf(\alpha) + (1-t)f(\beta)}{2} + \frac{f(\hat{x})}{2}. \end{aligned}$$

This yields $f(\hat{x}) \leq tf(\alpha) + (1-t)f(\beta)$. Therefore, f is convex as desired. $\square$

A nearly convex function $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, $n \geq 2$, needs not to be convex.

Example 3.13. Consider the function $f: \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}$ defined by

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) \in ([-1, 1] \times [-1, 1]) \setminus \{(1, 0)\}, \\ \infty & \text{otherwise.} \end{cases}$$

Then f is nearly convex but not convex. $\square$

Given a set $\Omega \subset \mathbb{R}^n \times \mathbb{R}$, define $f_\Omega: \mathbb{R}^n \rightarrow [-\infty, \infty]$ by

$$f_\Omega(x) = \inf\{t \in \mathbb{R} \mid (x, t) \in \Omega\}, \quad x \in \mathbb{R}^n.$$

Let $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be a function. Define $\text{co } f: \mathbb{R}^n \rightarrow [-\infty, \infty]$ by $\text{co } f = f_{\text{co}(\text{epi } f)}$, i.e.,

$$(\text{co } f)(x) = \inf\{t \in \mathbb{R} \mid (x, t) \in \text{co}(\text{epi } f)\}, \quad x \in \mathbb{R}^n.$$

The definition implies that $\text{co } f$ is the largest convex function majorized by f , i.e.,

- (1) $\text{co } f$ is convex, (2) $\text{co } f \leq f$, and
- (3) if $g: \mathbb{R}^n \rightarrow [-\infty, \infty]$ is a convex function such that $g \leq f$, then $g \leq \text{co } f$.

Recall that two subsets Ω_1 and Ω_2 of $\mathbb{R}^n$ are nearly equal and we write $\Omega_1 \approx \Omega_2$ if $\text{ri } \Omega_1 = \text{ri } \Omega_2$ and $\overline{\Omega_1} = \overline{\Omega_2}$. We say that two functions $f_1, f_2: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ are *nearly equal* and write $f_1 \approx f_2$ if their epigraphs are nearly equal sets.

Lemma 3.14. *Let $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be an arbitrary function. Then we have*

$$\text{co}(\text{epi } f) \subset \text{epi}(\text{co } f) \subset \overline{\text{co}(\text{epi } f)}.$$

Proof. Since $\text{epi } f \subset \text{epi}(\text{co } f)$, where the latter set is a convex due to the convexity of $\text{co } f$, we see that $\text{co}(\text{epi } f) \subset \text{epi}(\text{co } f)$.

Next, observe that $\text{co}(\text{epi } f)$ is an *epigraphical set* in the sense that if we have $(x_0, t_0) \in \text{co}(\text{epi } f)$, then $(x_0, t_0 + \alpha) \in \text{co}(\text{epi } f)$ for every $\alpha \geq 0$. Indeed, if we have $(x_0, t_0) \in \text{co}(\text{epi } f)$, then we can write

$$(x_0, t_0) = \sum_{i=1}^m \lambda_i (x_i, t_i),$$

where $(x_i, t_i) \in \text{epi } f$, $\lambda_i \geq 0$ for all $i = 1, \dots, m$, and $\sum_{i=1}^m \lambda_i = 1$. It follows that

$$(x_0, t_0 + \alpha) = \sum_{i=1}^m \lambda_i (x_i, t_i + \alpha).$$

Since $(x_i, t_i) \in \text{epi } f$ and $\alpha \geq 0$, one has $(x_i, t_i + \alpha) \in \text{epi } f$ for all $i = 1, \dots, m$.

It follows that $(x_0, t_0 + \alpha) \in \text{co}(\text{epi } f)$ as desired.

Now, take $(\bar{x}, \bar{t}) \in \text{epi}(\text{co } f)$ and suppose on the contrary that $(\bar{x}, \bar{t}) \notin \overline{\text{co}(\text{epi } f)}$.

Choose $\rho > 0$ and $\epsilon > 0$ such that

$$[B(\bar{x}; \rho) \times [\bar{t} - \epsilon, \bar{t} + \epsilon]] \cap \overline{\text{co}(\text{epi } f)} = \emptyset.$$

In particular, $[(\{\bar{x}\} \times [\bar{t} - \epsilon, \bar{t} + \epsilon])] \cap \overline{\text{co}(\text{epi } f)} = \emptyset$.

We have $\bar{t} \geq (\text{co } f)(\bar{x})$, and by the definition there exists a sequence $\{t_k\} \subset \mathbb{R}$ such that $(\bar{x}, t_k) \in \text{co}(\text{epi } f)$ and $t_k \rightarrow (\text{co } f)(\bar{x})$ as $k \rightarrow \infty$. Consider the case where $(\text{co } f)(\bar{x}) \in \mathbb{R}$. In this case, for some $k_0 \in \mathbb{N}$ we have

$$(\text{co } f)(\bar{x}) \leq t_k < (\text{co } f)(\bar{x}) + \epsilon \text{ when } k \geq k_0.$$

Case 1: $(\text{co } f)(\bar{x}) \geq \bar{t} - \epsilon$. In this case $\bar{t} - \epsilon \leq t_k \leq \bar{t} + \epsilon$ for all $k \geq k_0$. Thus

$$[B(\bar{x}; \rho) \times [\bar{t} - \epsilon, \bar{t} + \epsilon]] \cap \overline{\text{co}(\text{epi } f)} \neq \emptyset$$

because $(\bar{x}, t_k) \in \text{co}(\text{epi } f) \subset \overline{\text{co}(\text{epi } f)}$ for all $k \geq k_0$. This yields a contradiction.

Case 2: $(\text{co } f)(\bar{x}) < \bar{t} - \epsilon$. In this case since $t_k \rightarrow (\text{co } f)(\bar{x})$, there exists $\hat{k} \in \mathbb{N}$ with

$$t_k < \bar{t} - \epsilon < \bar{t} \text{ for all } k \geq \hat{k}.$$

Fix $k > \hat{k}$. Since $(\bar{x}, t_k) \in \text{co}(\text{epi } f)$ and $t_k < \bar{t}$, by the claim above we see that $(\bar{x}, \bar{t}) \in \text{co}(\text{epi } f) \subset \overline{\text{co}(\text{epi } f)}$. This again yields a contradiction.

For the case where $(\text{co } f)(\bar{x}) = -\infty$, we have $t_k \rightarrow -\infty$ as $k \rightarrow \infty$. Then there exists $\hat{k} \in \mathbb{N}$ such that

$$t_k < \bar{t} \text{ for all } k \geq \hat{k}.$$

For any $k \geq \hat{k}$, since we have $(\bar{x}, t_k) \in \text{co}(\text{epi } f)$ and $t_k < \bar{t}$, we evidently obtain $(\bar{x}, \bar{t}) \in \text{co}(\text{epi } f) \subset \overline{\text{co}(\text{epi } f)}$ by the epigraphical property of $\text{co}(\text{epi } f)$. This is a contradiction to the choice of $(\bar{x}, \bar{t})$. $\square$

The following proposition can be found in [8, Theorem 4.12] for which we provide here a self-contained proof for the convenience of the reader.

Proposition 3.15. *Let $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be an arbitrary function. Then f is nearly convex if and only if $f \approx \text{co } f$.*

Proof. Suppose that $f \approx \text{co } f$. Since $\text{co } f$ is a convex function, its epigraph $\text{epi}(\text{co } f)$ is a convex set in $\mathbb{R}^n \times \mathbb{R}$. By the definition of near equality we have $\text{epi } f \approx \text{epi}(\text{co } f)$, which means that $\text{epi } f$ is nearly equal to a convex set. Thus, $\text{epi } f$ is nearly convex by Proposition 2.3, and so f is nearly convex.

For the converse implication, suppose that f is a nearly convex function. Then $\text{epi } f$ is a nearly convex set in $\mathbb{R}^n \times \mathbb{R}$. By the definition, there exists a convex set C in $\mathbb{R}^{n+1}$ such that $C \subset \text{epi } f \subset \overline{C}$.

Since $\overline{C}$ is closed and convex, we see that

$$C \subset \text{epi } f \subset \overline{\text{co}(\text{epi } f)} \subset \overline{C}.$$

By Lemma 3.14 we have $\text{co}(\text{epi } f) \subset \text{epi}(\text{co } f) \subset \overline{\text{co}(\text{epi } f)}$, which yields

$$C \subset \text{epi } f \subset \overline{\text{co}(\text{epi } f)} \subset \overline{\text{epi}(\text{co } f)} \subset \overline{C}.$$

This implies by Proposition 2.1 that $\text{epi } f \approx \text{epi}(\text{co } f)$ and therefore $f \approx \text{co } f$. $\square$

4. Preservation of near convexity under basic operations

This section focuses on studying the preservation of near convexity in set-valued mappings and nonsmooth functions under basic operations on them. Our research presents significant advancements beyond the recent findings in [8]. Notably, we offer two distinct proofs demonstrating the preservation of near convexity under the summation of two nearly convex functions. We also provide a counterexample to illustrate that this result does not apply unless a relative interior qualification condition is assumed.

Proposition 4.1. *Let $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be a nearly convex function. Then the function*

$$\varphi: \mathbb{R}^n \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$$

defined by setting $\varphi(x, \alpha) = f(x) + \alpha$ for all $(x, \alpha) \in \mathbb{R}^n \times \mathbb{R}$ is also nearly convex.

Proof. Define the sets

$$\Theta_1 = \text{epi } \varphi = \{(x, \alpha, \lambda) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mid f(x) + \alpha \leq \lambda\},$$

$$\Theta_2 = \{(x, \alpha, \lambda) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mid (\text{co } f)(x) + \alpha \leq \lambda\}.$$

Then Θ_2 is a convex set. To verify that φ is nearly convex, it suffices to show that $\Theta_1 = \text{epi } \varphi$ is nearly convex. To accomplish this goal by Proposition 2.3, we will show that $\Theta_1 \approx \Theta_2$.

Step 1. $\overline{\Theta_1} = \overline{\Theta_2}$: Since $\text{co } f \leq f$, we see that $\Theta_1 \subset \Theta_2$ and thus $\overline{\Theta_1} \subset \overline{\Theta_2}$. To verify the reverse inclusion, let us show that $\Theta_2 \subset \overline{\Theta_1}$. Fix any $(x, \alpha, \lambda) \in \Theta_2$. Then $(\text{co } f)(x) + \alpha \leq \lambda$, so $(\text{co } f)(x) \leq \lambda - \alpha$. It follows that

$$(x, \lambda - \alpha) \in \text{epi}(\text{co } f) \subset \overline{\text{epi}(\text{co } f)} = \overline{\text{epi } f}$$

due to the fact that $f \approx \text{co } f$ from Proposition 3.15 and the near convexity of f . Thus there exists a sequence $\{(x_k, \gamma_k)\} \subset \text{epi } f$ such that

$$(x_k, \gamma_k) \rightarrow (x, \lambda - \alpha) \text{ as } k \rightarrow \infty.$$

Then $f(x_k) \leq \gamma_k$, so $f(x_k) + \alpha \leq \gamma_k + \alpha$ and thus $(x_k, \alpha, \gamma_k + \alpha) \in \Theta_1$. Since $(x_k, \alpha, \gamma_k + \alpha)$ converges to (x, α, λ) , this yields $(x, \alpha, \lambda) \in \overline{\Theta_1}$ and completes the proof of the equality $\overline{\Theta_1} = \overline{\Theta_2}$.

Step 2. $\text{ri } \Theta_1 = \text{ri } \Theta_2$: It is easily verified that the equality $\text{aff } \Omega = \text{aff } \overline{\Omega}$ holds for any subset Ω of a finite-dimensional Euclidean space. So, since $\overline{\Theta_1} = \overline{\Theta_2}$, we see that $\text{aff } \Theta_1 = \text{aff } \Theta_2$, which yields $\text{ri } \Theta_1 \subset \text{ri } \Theta_2$ due to the fact that $\Theta_1 \subset \Theta_2$. It follows from Proposition 2.6 that

$$\text{aff } \Theta_1 = \text{aff}(\text{dom } f) \times \mathbb{R} \times \mathbb{R}.$$

By Proposition 3.15 and the near convexity of f one has $\text{epi}(f) \approx \text{epi}(\text{co } f)$. Using the projection mapping, it follows from Proposition 2.4 that $\text{dom } f \approx \text{dom}(\text{co } f)$. Thus, $\text{dom } f = \overline{\text{dom}(\text{co } f)}$ and hence,

$$\text{aff}(\text{dom } f) = \text{aff}(\overline{\text{dom } f}) = \text{aff}(\overline{\text{dom}(\text{co } f)}) = \text{aff}(\text{dom}(\text{co } f)).$$

Then we have $\text{aff } \Theta_2 = \text{aff}(\text{dom } f) \times \mathbb{R} \times \mathbb{R} = \text{aff } \Theta_1$.

By Proposition 3.7 we have

$$\begin{aligned} \text{ri } \Theta_2 &= \{(x, \alpha, \lambda) \mid x \in \text{ri}(\text{dom}(\text{co}(f))), (\text{co } f)(x) + \alpha < \lambda\} \\ &= \{(x, \alpha, \lambda) \mid x \in \text{ri}(\text{dom}(\text{co } f)), (\text{co } f)(x) < \lambda - \alpha\} \\ &= \{(x, \alpha, \lambda) \mid (x, \lambda - \alpha) \in \text{ri}(\text{epi}(\text{co } f))\} \\ &= \{(x, \alpha, \lambda) \mid (x, \lambda - \alpha) \in \text{ri}(\text{epi } f)\} \\ &= \{(x, \alpha, \lambda) \mid x \in \text{ri}(\text{dom } f), f(x) < \lambda - \alpha\}. \end{aligned}$$

Take any $(x_0, \alpha_0, \lambda_0) \in \text{ri } \Theta_2$ and get $(x_0, \lambda_0 - \alpha_0) \in \text{ri}(\text{epi } f)$. Thus by the definition of the relative interior and Proposition 2.6 we find $\gamma > 0$ and $\epsilon > 0$ such that

$$[B(x_0; \gamma) \times (\lambda_0 - \alpha_0 - \epsilon, \lambda_0 - \alpha_0 + \epsilon)] \cap [\text{aff}(\text{dom } f) \times \mathbb{R}] \subset \text{epi } f. \quad (20)$$

Choose $\delta > 0$ such that if we have $(\alpha, \lambda) \in (\alpha_0 - \delta, \alpha_0 + \delta) \times (\lambda_0 - \delta, \lambda_0 + \delta)$, then $\lambda - \alpha \in (\lambda_0 - \alpha_0 - \epsilon, \lambda_0 - \alpha_0 + \epsilon)$ (use the continuity of $g(\alpha, \lambda) = \lambda - \alpha$). We have

$$[B(x_0; \gamma) \times (\alpha_0 - \delta, \alpha_0 + \delta) \times (\lambda_0 - \delta, \lambda_0 + \delta)] \cap [\text{aff}(\text{dom } f) \times \mathbb{R} \times \mathbb{R}] \subset \Theta_1. \quad (21)$$

Indeed, if (x, α, λ) is in the set on the left-hand side of (21), then by (20) one gets

$$(x, \lambda - \alpha) \in [B(x_0; \gamma) \times (\lambda_0 - \alpha_0 - \epsilon, \lambda_0 - \alpha_0 + \epsilon)] \cap [\text{aff}(\text{dom } f) \times \mathbb{R}] \subset \text{epi } f.$$

Thus, $f(x) \leq \lambda - \alpha$, and so (x, α, λ) is in the set on the right-hand side of (21). We have thus proved that (21) is valid. Clearly (21) implies $(x_0, \alpha_0, \lambda_0) \in \text{ri } \Theta_1$. Therefore, $\text{ri } \Theta_2 \subset \text{ri } \Theta_1$. So, one has $\text{ri } \Theta_1 = \text{ri } \Theta_2$.

The relations $\overline{\Theta}_1 = \overline{\Theta}_2$ and $\text{ri } \Theta_1 = \text{ri } \Theta_2$ yield $\Theta_1 \approx \Theta_2$ and complete the proof. $\square$

Theorem 4.2. *Let $F_1, F_2: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be nearly convex set-valued mappings. Then $F_1 + F_2$ is also nearly convex under the qualification condition*

$$\text{ri}(\text{dom } F_1) \cap \text{ri}(\text{dom } F_2) \neq \emptyset. \quad (22)$$

Proof. Define two sets

$$\begin{aligned} \Omega_1 &= \{(x, y_1, y_2) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p \mid y_1 \in F_1(x)\} = (\text{gph } F_1) \times \mathbb{R}^p, \\ \Omega_2 &= \{(x, y_1, y_2) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p \mid y_2 \in F_2(x)\}. \end{aligned} \quad (23)$$

Since F_1 and F_2 are nearly convex, Ω_1 and Ω_2 are nearly convex. It follows from Theorem 3.6 that

$$\begin{aligned} \text{ri } \Omega_1 &= \{(x, y_1, y_2) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p \mid x \in \text{ri}(\text{dom } F_1), y_1 \in \text{ri } F_1(x)\}, \\ \text{ri } \Omega_2 &= \{(x, y_1, y_2) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p \mid x \in \text{ri}(\text{dom } F_2), y_2 \in \text{ri } F_2(x)\}. \end{aligned}$$

We can choose $\hat{x} \in \text{ri}(\text{dom } F_1) \cap \text{ri}(\text{dom } F_2)$ due to (22). By Theorem 3.9 we can choose $\hat{y}_1 \in \text{ri } F_1(\hat{x})$ and $\hat{y}_2 \in \text{ri } F_2(\hat{x})$. Thus, $(\hat{x}, \hat{y}_1, \hat{y}_2) \in \text{ri } \Omega_1 \cap \text{ri } \Omega_2$. Since Ω_1 and Ω_2 are nearly convex, it follows from Theorem 2.2(b) that $\Omega_1 \cap \Omega_2$ is nearly convex.

Define the linear mapping $A: \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}^n \times \mathbb{R}^p$ by

$$A(x, y_1, y_2) = (x, y_1 + y_2), \quad (x, y_1, y_2) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p.$$

Obviously, $\text{gph}(F_1 + F_2) = A(\Omega_1 \cap \Omega_2)$. Therefore, Theorem 2.2(a) assures that $\text{gph}(F_1 + F_2)$ is nearly convex, so $F_1 + F_2$ is a nearly convex set-valued mapping. $\square$

As a consequence, we obtain the corollary below. We not only provide a correct statement for [8, Theorem 4.18] but also give a simple proof for this result.

Corollary 4.3. *Let $f_1, f_2: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be proper nearly convex functions. Suppose that*

$$\text{ri}(\text{dom } f_1) \cap \text{ri}(\text{dom } f_2) \neq \emptyset. \quad (24)$$

Then $f_1 + f_2$ is nearly convex.

Proof. According to (2), consider the epigraphical mappings $F_i = E_{f_i}$, $i = 1, 2$. Then $\text{dom } F_i = \text{dom } f_i$, $\text{gph } F_i = \text{epi } f_i$, and $(F_1 + F_2)(x) = [f_1(x) + f_2(x), \infty)$ for all $x \in \mathbb{R}^n$. By (24), we can choose $\hat{x} \in \text{ri}(\text{dom } f_1) \cap \text{ri}(\text{dom } f_2)$. Then choose a real number $\hat{\lambda}$ such that $\max\{f_1(\hat{x}), f_2(\hat{x})\} < \hat{\lambda}$. It follows from Proposition 3.7 that

$$(\hat{x}, \hat{\lambda}) \in \text{ri}(\text{epi } f_1) \cap \text{ri}(\text{epi } f_2) = \text{ri}(\text{gph } F_1) \cap \text{ri}(\text{gph } F_2).$$

By Theorem 4.2, the mapping $F_1 + F_2$ is nearly convex. Thus

$$\text{gph}(F_1 + F_2) = \text{epi}(f_1 + f_2)$$

is nearly convex. By the definition of nearly convex function, $f_1 + f_2$ is nearly convex. $\square$

Remark 4.4. We can prove Corollary 4.3 by an alternative way as follows. Define the sets

$$\begin{aligned}\Omega_1 &= \{(x, \lambda, \alpha) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mid f_1(x) \leq \alpha\}, \\ \Omega_2 &= \{(x, \lambda, \alpha) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mid \lambda \geq f_2(x) + \alpha\}.\end{aligned}$$

Let us show that $\text{epi}(f_1 + f_2) = \mathcal{P}(\Omega_1 \cap \Omega_2)$, where $\mathcal{P}$ is the projection mapping onto $\mathbb{R}^n \times \mathbb{R}$ (removing the last component). Take any $(x, \lambda, \alpha) \in \Omega_1 \cap \Omega_2$. Then $f_1(x) \leq \alpha \leq \lambda - f_2(x)$ and thus $\lambda \geq f_1(x) + f_2(x)$. It follows that $(x, \lambda) \in \text{epi}(f_1 + f_2)$. This implies that $\mathcal{P}(\Omega_1 \cap \Omega_2) \subset \text{epi}(f_1 + f_2)$. Now fix any $(x, \lambda) \in \text{epi}(f_1 + f_2)$ and get $f_1(x) + f_2(x) \leq \lambda$. This implies $f_1(x) \leq \lambda - f_2(x)$. Choose a real number α such that $f_1(x) \leq \alpha \leq \lambda - f_2(x)$. Then $(x, \lambda, \alpha) \in \Omega_1 \cap \Omega_2$ and so

$$(x, \lambda) = \mathcal{P}(x, \lambda, \alpha) \in \mathcal{P}(\Omega_1 \cap \Omega_2),$$

which justifies the inclusion $\text{epi}(f_1 + f_2) \subset \mathcal{P}(\Omega_1 \cap \Omega_2)$.

Since f_1 is nearly convex, Ω_1 is nearly convex. The set Ω_2 is nearly convex by Proposition 4.1 and the near convexity of f_2 . It follows from Proposition 3.7 and Proposition 4.1 that

$$\begin{aligned}\text{ri } \Omega_1 &= \{(x, \lambda, \alpha) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mid x \in \text{ri}(\text{dom } f_1), f_1(x) < \alpha\}, \\ \text{ri } \Omega_2 &= \{(x, \lambda, \alpha) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mid x \in \text{ri}(\text{dom } f_2), \lambda > f_2(x) + \alpha\}.\end{aligned}$$

(We may think about Ω_2 as the epigraph of the function $g(x, \alpha) = f_2(x) + \alpha$.)

Choose $x_0 \in \text{ri}(\text{dom } f_1) \cap \text{ri}(\text{dom } f_2)$. Then pick $\alpha_0 > f_1(x_0)$ and $\lambda_0 > f_2(x_0) + \alpha_0$. It is obvious that $(x_0, \lambda_0, \alpha_0) \in \text{ri } \Omega_1 \cap \text{ri } \Omega_2$. Thus $\text{ri } \Omega_1 \cap \text{ri } \Omega_2 \neq \emptyset$, so $\Omega_1 \cap \Omega_2$ is nearly convex. Therefore, the equality $\text{epi}(f_1 + f_2) = \mathcal{P}(\Omega_1 \cap \Omega_2)$ assures that $\text{epi}(f_1 + f_2)$ is nearly convex; hence $f_1 + f_2$ is nearly convex. $\square$

The example below shows that the result in [8, Theorem 4.18] is incorrect without assuming that the relative interiors of the domains of the involved functions have a nonempty intersection.

Example 4.5. Let $\Omega_1 = ([-1, 0] \times [-1, 1]) \setminus \{(0, 0)\} \subset \mathbb{R}^2$ and

$$\Omega_2 = ([0, 1] \times [-1, 1]) \setminus \{(0, 0)\} \subset \mathbb{R}^2.$$

Define $f_i(x) = \delta(x; \Omega_i)$ for $x \in \mathbb{R}^2$ and $i = 1, 2$. Then $f_1 + f_2 = \delta(\cdot; \Omega_1 \cap \Omega_2)$. It is clear that both f_1 and f_2 are proper and nearly convex, but their sum is not nearly convex. $\square$

The next theorem shows that the near convexity is preserved under compositions of set-valued mappings.

Theorem 4.6. Let $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ and $G: \mathbb{R}^p \rightrightarrows \mathbb{R}^q$ be nearly convex set-valued mappings. Suppose that

$$\text{ri}(\text{rge } F) \cap \text{ri}(\text{dom } G) \neq \emptyset. \quad (25)$$

Then the composition $G \circ F: \mathbb{R}^n \rightrightarrows \mathbb{R}^q$, where $(G \circ F)(x) = G(F(x))$ for all $x \in X$, is nearly convex.

Proof. Define the sets

$$\begin{aligned}\Omega_1 &= (\text{gph } F) \times \mathbb{R}^q \subset \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q, \\ \Omega_2 &= \mathbb{R}^n \times (\text{gph } G) \subset \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q.\end{aligned}\tag{26}$$

The near convexity of F and G implies that Ω_1 and Ω_2 are nearly convex. By Theorem 3.6 we have

$$\begin{aligned}\text{ri } \Omega_1 &= \{(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \mid x \in \text{ri}(\text{dom } F), y \in \text{ri } F(x)\}, \\ \text{ri } \Omega_2 &= \{(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \mid y \in \text{ri}(\text{dom } G), z \in \text{ri } G(y)\}.\end{aligned}$$

Thanks to (25), we can find a point $\hat{y} \in \text{ri}(\text{rge } F) \cap \text{ri}(\text{dom } G)$. Since $\hat{y} \in \text{ri}(\text{dom } G)$, by Theorem 3.9 the set $\text{ri } G(\hat{y})$ is nonempty, so we can choose $\hat{z} \in \text{ri } G(\hat{y})$. Since

$$\hat{y} \in \text{ri}(\text{rge } F) = \text{ri}(\text{dom } F^{-1})\tag{27}$$

and the near convexity of F assures that F^{-1} is also nearly convex, by Theorem 3.9 we can choose $\hat{x} \in \text{ri } F^{-1}(\hat{y})$. Combining the last inclusion with (27) and using Theorem 3.6, we have $(\hat{y}, \hat{x}) \in \text{ri}(\text{gph } F^{-1})$. By considering the linear mapping $T: \mathbb{R}^p \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^p$, where $T(y, x) = (x, y)$ for $(y, x) \in \mathbb{R}^p \times \mathbb{R}^n$, we can infer that $(\hat{x}, \hat{y}) \in \text{ri}(\text{gph } F)$. It follows that $\hat{x} \in \text{ri}(\text{dom } F)$. Then $(\hat{x}, \hat{y}, \hat{z}) \in \text{ri } \Omega_1 \cap \text{ri } \Omega_2$. So, by Theorem 2.2(b), the intersection $\Omega_1 \cap \Omega_2$ is nearly convex. Define a linear operator $A: \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}^n \times \mathbb{R}^q$ by

$$A(x, y, z) = (x, z), \quad (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q.$$

Obviously, $A(\Omega_1 \cap \Omega_2) = \text{gph}(G \circ F)$. Thus, $\text{gph}(G \circ F)$ is nearly convex. $\square$

Theorem 4.6 allows us to derive a new result on the near convexity of the composition of a nearly convex function and an affine mapping.

Corollary 4.7. *Let $B: \mathbb{R}^n \rightarrow \mathbb{R}^p$ be the affine mapping defined by*

$$B(x) = Ax + b \quad \text{for } x \in \mathbb{R}^n,$$

where $A \in \mathbb{R}^{p \times n}$ and $b \in \mathbb{R}^p$, and let $f: \mathbb{R}^p \rightarrow \overline{\mathbb{R}}$ be a nearly convex function. Then $f \circ B: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is nearly convex under the qualification condition

$$B(\mathbb{R}^n) \cap \text{ri}(\text{dom } f) \neq \emptyset.\tag{28}$$

Proof. Let $F(x) = \{B(x)\}$ for $x \in \mathbb{R}^n$, and let $G(y) = E_f(y)$ for $y \in \mathbb{R}^p$, where the epigraphical mapping is defined by (2). Then, $\text{gph } F = \text{gph } B$, $\text{gph } G = \text{epi } f$, and

$$(G \circ F)(x) = E_{f \circ B}(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Using (28), we can choose $\hat{x} \in \mathbb{R}^n$ such that $\hat{y} = B(\hat{x}) \in \text{ri}(\text{dom } f)$. Then fix a real number $\hat{\lambda} > f(\hat{y})$ (the situation $f(\hat{y}) = -\infty$ is not excluded). Since $\text{gph } B$ is an affine set, one has $(\hat{x}, \hat{y}) \in \text{ri}(\text{gph } F) = \text{gph } B$. In addition, by Proposition 3.7 we can infer that $(\hat{y}, \hat{\lambda}) \in \text{ri}(\text{epi } f) = \text{ri}(\text{gph } G)$. Therefore, $\hat{y} \in \text{ri}(\text{rge } F) \cap \text{ri}(\text{dom } G)$. So, by Theorem 4.6, the composition $G \circ F$ is nearly convex. Thus, $\text{epi}(f \circ B) = \text{gph}(G \circ F)$ is nearly convex, so $f \circ B$ is nearly convex. $\square$

Given a set-valued mapping $G: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ with $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^p$, recall that

$$G(\Omega) = \bigcup_{x \in \Omega} G(x), \quad G^{-1}(\Theta) = \{x \in \mathbb{R}^n \mid G(x) \cap \Theta \neq \emptyset\}.$$

The corollary below shows that the near convexity of sets is preserved under nearly convex set-valued mappings under direct and inverse images.

Corollary 4.8. *Let $G: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a nearly convex set-valued mapping. The following assertions hold:*

(a) *If $\Omega \subset \mathbb{R}^n$ is a nearly convex set and*

$$\text{ri } \Omega \cap \text{ri}(\text{dom } G) \neq \emptyset, \quad (29)$$

then $G(\Omega)$ is nearly convex.

(b) *If $\Theta \subset \mathbb{R}^p$ is a nearly convex set and*

$$\text{ri } \Theta \cap \text{ri}(\text{rge } G) \neq \emptyset, \quad (30)$$

then $G^{-1}(\Theta)$ is nearly convex.

Proof. To prove the assertion (a), consider the set-valued mapping $F: \mathbb{R} \rightrightarrows \mathbb{R}^n$ with $F(x) = \Omega$ if $x = 0$ and $F(x) = \emptyset$ otherwise. Then $\text{gph } F = \{0\} \times \Omega$, and hence F is nearly convex. By our assumption, G is nearly convex. In addition, from (29) it follows that $\text{ri}(\text{rge } F) \cap \text{ri}(\text{dom } G) \neq \emptyset$. So, $G \circ F$ is nearly convex by Theorem 4.6. We see that

$$(G \circ F)(x) = \begin{cases} G(\Omega) & \text{if } x = 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

Therefore, $\text{gph}(F \circ G) = \{0\} \times G(\Omega)$. Define the linear mapping $A: \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ by $A(x, y) = y$ for $(x, y) \in \mathbb{R} \times \mathbb{R}^p$. Since $G(\Omega) = A(\text{gph}(G \circ F))$, it follows from Theorem 2.2(a) that $G(\Omega)$ is nearly convex.

To justify the assertion (b), define the mapping $H: \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ by

$$H(y) = G^{-1}(y) = \{x \in \mathbb{R}^n \mid y \in G(x)\}, \quad y \in \mathbb{R}^p.$$

Then observe H is nearly convex, $G^{-1}(\Theta) = H(\Theta)$ and $\text{dom } H = \text{rge } G$. Since (30) implies that $\text{ri } \Theta \cap \text{ri}(\text{dom } H) \neq \emptyset$, the desired near convexity of $G^{-1}(\Theta)$ follows from the assertion (a). $\square$

If $F_i: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ for $i = 1, \dots, m$ are set-valued mappings, then one defines

$$\left(\bigcap_{i=1}^m F_i\right)(x) = F_1(x) \cap \dots \cap F_m(x), \quad x \in \mathbb{R}^n.$$

In the next theorem we discuss the near convexity of the intersection mapping.

Theorem 4.9. *Let $F_i: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ for $i = 1, \dots, m$ be nearly convex set-valued mappings. Suppose*

$$\bigcap_{i=1}^m \text{ri}(\text{gph } F_i) \neq \emptyset. \quad (31)$$

Then $\bigcap_{i=1}^m F_i$ is nearly convex.

Proof. It is obvious that $\text{gph}\left(\bigcap_{i=1}^m F_i\right) = \bigcap_{i=1}^m \text{gph } F_i$.

By Theorem 2.2(b), the set $\bigcap_{i=1}^m \text{gph } F_i$ is nearly convex under the qualification condition (31). Therefore, $\bigcap_{i=1}^m F_i$ is nearly convex. $\square$

To conclude this section, we discuss the near convexity of the maximum function. Given functions $f_i: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ for $i = 1, \dots, m$, define

$$f(x) = \max\{f_i(x) \mid i = 1, \dots, m\}, \quad x \in \mathbb{R}^n. \quad (32)$$

Corollary 4.10. *Let $f_i: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ for $i = 1, \dots, m$ be nearly convex functions. Suppose that*

$$\bigcap_{i=1}^m \text{ri}(\text{dom } f_i) \neq \emptyset.$$

Then the maximum function f defined in (32) is nearly convex.

Proof. We consider the epigraphical mappings $F_i(x) = E_{f_i}$ as given in (2) for $i = 1, \dots, m$, and let $F = \bigcap_{i=1}^m F_i$. Then we see that

$$\text{epi } f = \bigcap_{i=1}^m \text{epi } f_i = \bigcap_{i=1}^m \text{gph } F_i = \text{gph } F.$$

Choose $x_0 \in \bigcap_{i=1}^m \text{ri}(\text{dom } f_i)$ and let $\lambda = f(x_0) + 1$. Proposition 3.7 implies

$$(x_0, \lambda) \in \bigcap_{i=1}^m \text{ri}(\text{epi } f_i) = \bigcap_{i=1}^m \text{ri}(\text{gph } F_i).$$

So, the set-valued mapping F is nearly convex by Theorem 4.9. Thus, $\text{epi } f = \text{gph } F$ is nearly convex and therefore f is nearly convex. $\square$

5. Nearly convex generalized differentiation

In this section, we explore the topic of generalized differentiation for nearly convex set-valued mappings and nearly convex functions using a geometric approach that has proven successful in convex analysis. Previous work on this approach can be found in [12, 13]. We present new calculus rules for the coderivatives of sums, compositions, and maxima of nearly convex set-valued mappings, as well as related refinements for subdifferentials of nearly convex functions.

Given a nearly convex set Ω in $\mathbb{R}^n$ with $\bar{x} \in \Omega$, define the normal cone to Ω at $\bar{x}$ by

$$N(\bar{x}; \Omega) = \{v \in \mathbb{R}^n \mid \langle v, x - \bar{x} \rangle \leq 0 \text{ for all } x \in \Omega\}.$$

Theorem 5.1. *Let Ω_1 and Ω_2 be nearly convex sets in $\mathbb{R}^n$ such that*

$$\text{ri } \Omega_1 \cap \text{ri } \Omega_2 \neq \emptyset. \quad (33)$$

Then $\Omega_1 \cap \Omega_2$ is nearly convex and

$$N(\bar{x}; \Omega_1 \cap \Omega_2) = N(\bar{x}; \Omega_1) + N(\bar{x}; \Omega_2) \text{ for all } \bar{x} \in \Omega_1 \cap \Omega_2. \quad (34)$$

Proof. Take any $\bar{x} \in \Omega_1 \cap \Omega_2$. The condition (33) implies that $\text{ri } \overline{\Omega_1} \cap \text{ri } \overline{\Omega_2} \neq \emptyset$. In addition, $\overline{\Omega_1}$ and $\overline{\Omega_2}$ are convex thanks to the near convexity of Ω_1 and Ω_2 . Then, applying [13, Theorem 2.56] we have

$$N(\bar{x}; \overline{\Omega_1} \cap \overline{\Omega_2}) = N(\bar{x}; \overline{\Omega_1}) + N(\bar{x}; \overline{\Omega_2}). \quad (35)$$

From our assumptions and Theorem 2.2(b) it follows that $\overline{\Omega_1 \cap \Omega_2} = \overline{\Omega_1} \cap \overline{\Omega_2}$.

Therefore, using (35) we get

$$\begin{aligned} N(\bar{x}; \Omega_1 \cap \Omega_2) &= N(\bar{x}; \overline{\Omega_1 \cap \Omega_2}) = N(\bar{x}; \overline{\Omega_1} \cap \overline{\Omega_2}) \\ &= N(\bar{x}; \overline{\Omega_1}) + N(\bar{x}; \overline{\Omega_2}) = N(\bar{x}; \Omega_1) + N(\bar{x}; \Omega_2). \end{aligned}$$

Since $\bar{x} \in \Omega_1 \cap \Omega_2$ was taken arbitrarily, we have thus obtained (34). $\square$

Let $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a nearly convex set-valued mapping and let $(\bar{x}, \bar{y}) \in \text{gph } F$. The *coderivative* of F at $(\bar{x}, \bar{y})$ is the set-valued mapping $D^*F(\bar{x}, \bar{y}): \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ with the values

$$D^*F(\bar{x}, \bar{y})(v) = \{u \in \mathbb{R}^n \mid (u, -v) \in N((\bar{x}, \bar{y}); \text{gph } F)\}, \quad v \in \mathbb{R}^p.$$

Given $(\bar{x}, \bar{y}) \in \text{gph}(F_1 + F_2)$, define the set

$$S(\bar{x}, \bar{y}) = \{(\bar{y}_1, \bar{y}_2) \in \mathbb{R}^p \times \mathbb{R}^p \mid \bar{y} = \bar{y}_1 + \bar{y}_2, \bar{y}_i \in F_i(\bar{x}), i = 1, 2\}. \quad (36)$$

The theorem below provides a coderivative sum rule for nearly convex set-valued mappings.

Theorem 5.2. *Let $F_1, F_2: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be nearly convex set-valued mappings. Suppose that the qualification condition (22) is satisfied. Then the equality*

$$D^*(F_1 + F_2)(\bar{x}, \bar{y})(v) = D^*F_1(\bar{x}, \bar{y}_1)(v) + D^*F_2(\bar{x}, \bar{y}_2)(v)$$

holds for every $v \in \mathbb{R}^p$ and $(\bar{y}_1, \bar{y}_2) \in S(\bar{x}, \bar{y})$, where S is defined in (36).

Proof. By Theorem 4.2, the condition (22) implies that the set-valued mapping $F_1 + F_2$ is nearly convex. Fix any $(\bar{y}_1, \bar{y}_2) \in S(\bar{x}, \bar{y})$ and $v \in \mathbb{R}^p$. Take any

$$u \in D^*(F_1 + F_2)(\bar{x}, \bar{y})(v).$$

Then we have the inclusion $(u, -v) \in N((\bar{x}, \bar{y}); \text{gph}(F_1 + F_2))$. (37)

Consider the sets Ω_i , $i = 1, 2$, defined in (23). By the construction of Ω_1 and Ω_2 , from (37) we can deduce that

$$(u, -v, -v) \in N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_1 \cap \Omega_2).$$

The proof of Theorem 4.2 tells us that $\text{ri } \Omega_1 \cap \text{ri } \Omega_2 \neq \emptyset$. Then we can employ Theorem 5.1 and get

$$(u, -v, -v) \in N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_1 \cap \Omega_2) = N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_1) + N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_2).$$

Therefore, the rest of the proof follows from that of [13, Theorem 3.37]. $\square$

Let $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be a nearly convex function. We define the *subdifferential* of f at $\bar{x} \in \mathbb{R}^n$ with $f(\bar{x}) \in \mathbb{R}$ by

$$\partial f(\bar{x}) = \{v \in \mathbb{R}^n \mid \langle v, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) \text{ for all } x \in \mathbb{R}^n\}.$$

The next proposition allows us to represent the subdifferential of a nearly convex function via the coderivative of the epigraphical mapping (2).

Proposition 5.3. *Suppose that $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is a nearly convex function and $\bar{x} \in \mathbb{R}^n$ is such that $f(\bar{x}) \in \mathbb{R}$. Then $\partial f(\bar{x}) = D^*E_f(\bar{x}, f(\bar{x}))(1)$, where E_f is defined in (2).*

Proof. By the definition we have

$$D^*E_f(\bar{x}, f(\bar{x}))(1) = \{v \in \mathbb{R}^n \mid (v, -1) \in N((\bar{x}, f(\bar{x})); \text{epi } f)\}.$$

If $D^*E_f(\bar{x}, f(\bar{x}))(1)$ is empty, the inclusion $D^*E_f(\bar{x}, f(\bar{x}))(1) \subset \partial f(\bar{x})$ is valid.

Otherwise pick an arbitrary vector $v \in D^*E_f(\bar{x}, f(\bar{x}))(1)$ and get

$$\langle v, x - \bar{x} \rangle - (\lambda - f(\bar{x})) \leq 0 \text{ whenever } (x, \lambda) \in \text{epi } f. \quad (38)$$

It follows from (38) that $-\infty < f(x)$ for all $x \in \mathbb{R}^n$. Using this inequality with $x \in \text{dom } f$ and $\lambda = f(x)$ gives

$$\langle v, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) \text{ for all } x \in \text{dom } f,$$

which implies that $v \in \partial f(\bar{x})$ since $f(x) = \infty$ if $x \notin \text{dom } f$.

If $\partial f(\bar{x}) = \emptyset$, then $\partial f(\bar{x}) \subset D^*E_f(\bar{x}, f(\bar{x}))(1)$. If $\partial f(\bar{x}) \neq \emptyset$, take any $v \in \partial f(\bar{x})$ and get

$$\langle v, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) \text{ for all } x \in \mathbb{R}^n.$$

This implies (38) since $f(x) \leq \lambda$ whenever $(x, \lambda) \in \text{epi } f$. Therefore we obtain $v \in D^*E_f(\bar{x}, f(\bar{x}))(1)$, which justifies the inclusion $\partial f(\bar{x}) \subset D^*E_f(\bar{x}, f(\bar{x}))(1)$. $\square$

Proposition 5.4. *Let $f: \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ be a nearly convex function and $\bar{x} \in \mathbb{R}^n$ be such that $f(\bar{x}) \in \mathbb{R}$. The following assertions hold:*

- (a) *If $(v, -\alpha) \in N((\bar{x}, f(\bar{x})); \text{epi } f)$, then $\alpha \geq 0$.*
- (b) *If $\bar{x} \in \text{ri}(\text{dom } f)$, then $\partial f(\bar{x}) \neq \emptyset$. In particular, if f is continuous at $\bar{x}$, then $\partial f(\bar{x}) \neq \emptyset$.*
- (c) *If f is continuous at $\bar{x}$, then $N((\bar{x}, \bar{\lambda}); \text{epi } f) = \{(0, 0)\}$ whenever $f(\bar{x}) < \bar{\lambda}$.*
- (d) *$(v, 0) \in N((\bar{x}, f(\bar{x})); \text{epi } f)$ if and only if $v \in N(\bar{x}; \text{dom } f)$.*
- (e) *If $\alpha > 0$, then $(v, -\alpha) \in N((\bar{x}, f(\bar{x})); \text{epi } f)$ if and only if $v \in \alpha \partial f(\bar{x})$.*

Proof. (a) It follows from the definition that

$$\langle v, x - \bar{x} \rangle - \alpha(\lambda - f(\bar{x})) \leq 0 \text{ whenever } (x, \lambda) \in \text{epi } f.$$

Using the last inequality with $x = \bar{x}$ and $\lambda = f(\bar{x}) + 1$ gives us the conclusion.

(b) It follows from Proposition 3.7 that $(\bar{x}, f(\bar{x})) \notin \text{ri}(\text{epi } f)$. By the proper separation from Theorem 3.2, there exist $v \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ such that

$$\langle v, x \rangle - \gamma\lambda \leq \langle v, \bar{x} \rangle - \gamma f(\bar{x}) \text{ whenever } (x, \lambda) \in \text{epi } f. \quad (39)$$

In addition, there exists $(x_0, \lambda_0) \in \text{epi } f$ such that

$$\langle v, x_0 \rangle - \gamma\lambda_0 < \langle v, \bar{x} \rangle - \gamma f(\bar{x}).$$

First, we see that $\gamma \geq 0$ by using (39) with $x = \bar{x}$ and $\lambda = f(\bar{x}) + 1$. If $\gamma = 0$, then $\langle v, x \rangle \leq \langle v, \bar{x} \rangle$ for all $x \in \text{dom } f$, and $\langle v, x_0 \rangle < \langle v, \bar{x} \rangle$ with $x_0 \in \text{dom } f$. Thus, the set $\text{dom } f$ and $\{\bar{x}\}$ can be properly separated, so $\bar{x} \notin \text{ri}(\text{dom } f)$, which is a contradiction. Therefore, dividing both sides of (39) by γ yields

$$\langle \frac{v}{\gamma}, x - \bar{x} \rangle \leq \lambda - f(\bar{x}) \text{ whenever } (x, \lambda) \in \text{epi } f. \quad (40)$$

If $f(x) = -\infty$ for some $x \in \mathbb{R}^n$, then the inequality in (40) holds for all $\lambda \in \mathbb{R}$. This is an absurd. Thus, we must have $f(x) \in \mathbb{R}$ for all $x \in \text{dom } f$. Using the inequality in (40) with $x \in \text{dom } f$ and $\lambda = f(x)$, we see that $v/\gamma \in \partial f(\bar{x})$.

Now, suppose that f is continuous at x_0 . Then $x_0 \in \text{int}(\text{dom } f)$ and in consequence $x_0 \in \text{ri}(\text{dom } f) = \text{int}(\text{dom } f)$. Therefore, $\partial f(\bar{x}) \neq \emptyset$.

(c) Suppose that f is continuous at $\bar{x}$ and that $f(\bar{x}) < \bar{\lambda}$. By Proposition 3.8, we see that $(\bar{x}, \bar{\lambda}) \in \text{int}(\text{epi } f)$ and thus $N((\bar{x}, \bar{\lambda}); \text{epi } f) = \{(0, 0)\}$.

The proofs of the last two assertions are quite obvious, so we left them for the reader. $\square$

Using Proposition 5.4 and Theorem 5.2 (see also [8, Theorem 4.29]), we can obtain the following sum rule for subdifferentials of proper nearly convex functions.

Corollary 5.5. *Let $f_i: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ for $i = 1, \dots, m$ be proper nearly convex functions.*

Suppose that

$$\bigcap_{i=1}^m \text{ri}(\text{dom } f_i) \neq \emptyset.$$

Then $f_1 + \dots + f_m$ is nearly convex and we have the equality

$$\partial(f_1 + \dots + f_m)(\bar{x}) = \partial f(\bar{x}) + \dots + \partial f_m(\bar{x}) \quad \text{for every } \bar{x} \in \bigcap_{i=1}^m \text{dom } f_i.$$

Proof. We only need to prove the result for the case where $m = 2$. It suffices to apply Theorem 5.2 to the epigraphical mappings $F_i = E_{f_i}$ for $i = 1, 2$ from (2) with the use of Proposition 5.3. $\square$

Now we consider the composition of two set-valued mappings $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ and $G: \mathbb{R}^p \rightrightarrows \mathbb{R}^q$. Given any $\bar{z} \in (G \circ F)(\bar{x})$, we consider the set

$$M(\bar{x}, \bar{z}) = F(\bar{x}) \cap G^{-1}(\bar{z}).$$

The coderivative chain rule for nearly convex set-valued mappings can be formulated as follows.

Theorem 5.6. *Let $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ and $G: \mathbb{R}^p \rightrightarrows \mathbb{R}^q$ be nearly convex set-valued mappings. Suppose that the qualification condition (25) is satisfied. Then for any $(\bar{x}, \bar{z}) \in \text{gph}(G \circ F)$ we have*

$$D^*(G \circ F)(\bar{x}, \bar{z})(w) = (D^*F(\bar{x}, \bar{y}) \circ D^*G(\bar{y}, \bar{z}))(w)$$

whenever $\bar{y} \in M(\bar{x}, \bar{z})$ and $w \in \mathbb{R}^q$.

Proof. Picking $u \in D^*(G \circ F)(\bar{x}, \bar{z})(w)$ and $\bar{y} \in M(\bar{x}, \bar{z})$ gives us the inclusion

$$(u, -w) \in N((\bar{x}, \bar{z}); \text{gph}(G \circ F)).$$

Consider the nearly convex sets Ω_i for $i = 1, 2$ given in (26). We can directly deduce from the definition of the normal cone that

$$(u, 0, -w) \in N((\bar{x}, \bar{y}, \bar{z}); \Omega_1 \cap \Omega_2).$$

By the condition (25) and the proof of Theorem 4.6 we have $\text{ri } \Omega_1 \cap \text{ri } \Omega_2 \neq \emptyset$.

So, applying Theorem 5.1 gives us

$$(u, 0, -w) \in N((\bar{x}, \bar{y}, \bar{z}); \Omega_1 \cap \Omega_2) = N((\bar{x}, \bar{y}, \bar{z}); \Omega_1) + N((\bar{x}, \bar{y}, \bar{z}); \Omega_2).$$

Then the rest of the proof follows that of [13, Theorem 3.38]. $\square$

The next result is a direct consequence of Theorem 5.6.

Corollary 5.7. *Let $B: \mathbb{R}^n \rightarrow \mathbb{R}^p$ be the affine mapping defined by*

$$B(x) = Ax + b \quad \text{for } x \in \mathbb{R}^n,$$

where $A \in \mathbb{R}^{p \times n}$ and $b \in \mathbb{R}^p$, and let $g: \mathbb{R}^p \rightarrow \bar{\mathbb{R}}$ be a proper nearly convex function. Suppose that $B(\mathbb{R}^n) \cap \text{ri}(\text{dom } g) \neq \emptyset$. Then $\partial(g \circ B)(\bar{x}) = A^T \partial g(B(\bar{x}))$ for every $\bar{x} \in \text{dom}(g \circ B)$.

Proof. This follows from Theorem 5.6 with $F(x) = \{B(x)\}$ for $x \in \mathbb{R}^n$, and $G(y) = [g(y), \infty)$ for $y \in \mathbb{R}^p$. $\square$

Arguing similarly as in the proof of [13, Corollary 3.39], we can obtain the following from Theorem 5.6.

Corollary 5.8. *Let $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a nearly convex set-valued mapping and let $\Theta \subset \mathbb{R}^p$ be a nearly convex set. Suppose that the qualification condition (30), where G is replaced by F , holds. Then we have the equality*

$$N(\bar{x}; F^{-1}(\Theta)) = D^*F(\bar{x}, \bar{y})(N(\bar{y}; \Theta))$$

whenever $\bar{x} \in F^{-1}(\Theta)$ and $\bar{y} \in F(\bar{x}) \cap \Theta$.

Next, we discuss coderivatives of the intersection mapping.

Theorem 5.9. *Let $F_i: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ for $i = 1, \dots, m$ be nearly convex set-valued mappings, and let $F = \bigcap_{i=1}^m F_i$. Assume that the qualification condition (31) is satisfied. Then for any $(\bar{x}, \bar{y}) \in \text{gph } F$ we have*

$$D^*F(\bar{x}, \bar{y})(v) = \bigcup \{D^*F_1(\bar{x}, \bar{y})(v_1) + \dots + D^*F_m(\bar{x}, \bar{y})(v_m) \mid v_1 + \dots + v_m = v\}. \quad (41)$$

Proof. By the qualification condition (31) and Theorem 4.9, the set-valued mapping $\bigcap_{i=1}^m F_i$ is nearly convex. Take any $u \in D^*F(\bar{x}, \bar{y})(v)$, where $(\bar{x}, \bar{y}) \in \text{gph } F$. Then $(u, -v) \in N((\bar{x}, \bar{y}); \text{gph } F)$. Since $\text{gph } F = \bigcap_{i=1}^m \text{gph } F_i$, by (31) and Theorem 5.1 we have

$$(u, -v) \in N((\bar{x}, \bar{y}); \text{gph } F) = N((\bar{x}, \bar{y}); \text{gph } F_1) + \dots + N((\bar{x}, \bar{y}); \text{gph } F_m). \quad (42)$$

Thus, there exist $u_1, \dots, u_m \in \mathbb{R}^n$ and $v_1, \dots, v_m \in \mathbb{R}^p$ such that

$$u = \sum_{i=1}^m u_i, \quad v = \sum_{i=1}^m v_i, \quad (u_i, -v_i) \in N((\bar{x}, \bar{y}); \text{gph } F_i).$$

By the definition, $u_i \in D^*F_i(\bar{x}, \bar{y})(v_i)$ and hence

$$u \in D^*F_1(\bar{x}, \bar{y})(v_1) + \dots + D^*F_m(\bar{x}, \bar{y})(v_m).$$

This justifies the inclusion $\subset$ in (41). The reverse inclusion follows directly from the definition of coderivative and the equality in (42). $\square$

Corollary 5.10. *Let $f_i: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ for $i = 1, \dots, m$ be proper nearly convex functions. Suppose that all functions f_i are continuous at $\bar{x} \in \mathbb{R}^n$. Then the maximum function f defined in (32) is nearly convex and we have*

$$\partial f(\bar{x}) = \text{co} \left[\bigcup_{i \in I(\bar{x})} \partial f_i(\bar{x}) \right], \quad (43)$$

where $I(\bar{x}) = \{i = 1, \dots, m \mid f_i(\bar{x}) = f(\bar{x})\}$.

Proof. Since $\bar{x} \in \text{int}(\text{dom } f_i) = \text{ri}(\text{dom } f_i)$ for $i = 1, \dots, m$ under the continuity of f_i at $\bar{x}$, the near convexity of f follows from Corollary 4.10. Consider the epigraphical mappings $F_i = E_{f_i}$ defined in the proof of Corollary 4.10. Then $E_f = \bigcap_{i=1}^m F_i$. Fix any $v \in \partial f(\bar{x})$ and get by Proposition 5.3 that $v \in D^*E_f(\bar{x}, \bar{\lambda})(1)$, where $\bar{\lambda} = f(\bar{x})$. It is easy to verify that the qualification condition (31) is fulfilled. So, by Theorem 5.9 we have

$$D^*E_f(\bar{x}, \bar{\lambda})(1) = \bigcup \{D^*F_1(\bar{x}, \bar{\lambda})(\lambda_1) + \dots + D^*F_m(\bar{x}, \bar{\lambda})(\lambda_m) \mid \lambda_1 + \dots + \lambda_m = 1\}.$$

Hence, there exist $v_i \in \mathbb{R}^n$ and $\lambda_i \in \mathbb{R}$ for $i = 1, \dots, m$ such that

$$v = \sum_{i=1}^m v_i, \quad \sum_{i=1}^m \lambda_i = 1, \quad (v_i, -\lambda_i) \in N((\bar{x}, \bar{\lambda}); \text{epi } f_i).$$

In consequence of Proposition 5.4(c), the continuity of f_i at $\bar{x}$ ensures that we have $N((\bar{x}, \bar{\lambda}); \text{epi } f_i) = \{(0, 0)\}$ if $i \notin I(\bar{x})$, i.e., $\bar{\lambda} > f_i(\bar{x})$. Thus,

$$v = \sum_{i \in I(\bar{x})} v_i \quad \text{and} \quad \sum_{i \in I(\bar{x})} \lambda_i = 1,$$

where $(v_i, -\lambda_i) \in N((\bar{x}, f_i(\bar{x})); \text{epi } f_i)$ for $i \in I(\bar{x})$. Then $\lambda_i \geq 0$ whenever $i \in I(\bar{x})$ by Proposition 5.4(a). In addition, if $\lambda_i = 0$, then by Proposition 5.4(d) and the property $\bar{x} \in \text{int}(\text{dom } f_i)$ we have $v_i \in N(\bar{x}; \text{dom } f_i) = \{0\}$. By Proposition 5.4(b), we see that $\partial f_i(\bar{x}) \neq \emptyset$ for every i under the continuity of f_i at $\bar{x}$. We can also see from Proposition 5.4(e) that if $\lambda_i > 0$, then $v_i \in \lambda_i \partial f_i(\bar{x})$ for $i \in I(\bar{x})$. So, we have $v_i \in \lambda_i \partial f_i(\bar{x})$ for all $i \in I(\bar{x})$. Thus,

$$v \in \sum_{i \in I(\bar{x})} \lambda_i \partial f_i(\bar{x}), \quad \lambda_i \geq 0, \quad \sum_{i \in I(\bar{x})} \lambda_i = 1.$$

This justifies the inclusion $\subset$ in (43). The reverse inclusion follows directly from the definition of subdifferential. $\square$

Acknowledgments. Nguyen Mau Nam would like to thank the Vietnam Institute of Mathematics-VAST (through the IM-Simons program) and the Vietnam Institute for Advanced Study in Mathematics for hospitality. The research of Nguyen Nang Thieu and Nguyen Dong Yen was supported by the Vietnam Academy of Science and Technology under the project number NVCC01.07/24-25. We are indebted to the anonymous referee for the careful reading of our paper and many valuable suggestions.

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