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Hamilton-Jacobi Theory in the Calculus of Variations Under Partial Convexity Assumptions on the Lagrangian

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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Using general results about subdifferentials of value functions, under partial convexity assumptions on the Lagrangian, we derive the main result about the Hamilton-Jacobi theory in the calculus of variations obtained by R. T. Rockafellar and P. Wolenski [*Convexity in Hamilton-Jacobi theory*, SIAM J. Control Optimization 39/5 (2000) 1323–1372] under full convexity of the Lagrangian. Since a number of results in the calculus of variations are known to be valid under such an assumption, it is tempting to tackle such an aim, even if the nice duality theory presented in the work cited above seems to be out of reach.

Keywords: Calculus of variations, Fréchet subdifferential, Hamilton-Jacobi theory, Hamiltonian, Lagrangian, value function.

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1. Introduction

Our aim in this paper is to prove the following variant of a result of Rockafellar and Wolenski [15, Theorem 2.5] assuming that the Lagrangian

$$L : E \times E \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$$

is convex in its second variable u and not necessarily jointly convex in (e, u). Since the link between L and the Hamiltonian H is through conjugacy with respect to the velocity variable u, and since a number of classical results in the calculus of variations ([4], [5], [6], [19]) rely on partial convexity, it is tempting to adopt such an assumption, even if the nice full picture in [15], [16] cannot be obtained; in particular, the duality results of [15] seem to be out of reach under our assumptions.

As in [15] we use the value function $V : \mathbb{R}_+ \times E \to \mathbb{R}_\infty$ associated to the Lagrangian $L : E \times E \to \mathbb{R}_\infty$ and an initial cost function $g : E \to \mathbb{R}_\infty$ by

$$V(t, z) := \inf\{g(x(0)) + \int_0^t L(x(s), x'(s))ds : x \in W^{1,1}([0, t], E),\ x(t) = z\}$$

with the understanding that $V(0, z) = g(z)$. The fact that g and L may take the value $+\infty$ allows to take constraints into account.

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In the sequel we trivialize the fibration $q : W^{1,1}([0,t],E) \to E$ given by $q(x) := x(t)$ by identifying $x \in W^{1,1}([0,t],E)$ with $(t,v) \in \mathbb{R}_+ \times L_1([0,t],E)$, where $v := x'$ or, preferably, with $(t,u) \in \mathbb{R}_+ \times L_1([0,1],E)$ by setting $s = tr$, $x(s) = w(t^{-1}s)$, $x(s) := x(0) + P_v(s)$, $w(r) = w(0) + P_r(u)$ and we write $P(u) := P_1(u)$.

We note that the map $P_r : u \mapsto w - w(0) = \int_0^r u(p)dp$ is linear and continuous from $L_1([0,1],E)$ into $W^{1,1}([0,1],E)$ and

$$x(s) = x(0) + P_v(s), \quad P_v(s) = P_s(v) = \int_0^s v(q)dq.$$

Our method relies on an elementary result about value functions. We avoid considering Hamiltonian differential inclusions because the Hamiltonian is not convex-concave as in the case L is jointly convex.

We use here the firm (or Fréchet) subdifferential ∂_F which is the simplest subdifferential for nonconvex and nondifferentiable functions. It does not satisfy some useful rules valid for the limiting subdifferential ∂_L or the circa-differential (or Clarke subdifferential) ∂_C, but it disposes of other properties which are precious for our study. Such rules are either obvious or simple, but are not always mentioned in books about variational analysis, even in [12], albeit (or because) they are close to usual rules in differential calculus. It also enjoys a simple rule for integral functionals given in [3], [9] which is the basis of our study. It is another reason to use here the firm subdifferential ∂_F. We gather these rules in the appendix. Among them is an order compatibility property, a chain rule and a decomposition rule. When there is no risk of confusion, for the sake of simplicity, we often write ∂ instead of ∂_F and we take avantage of some obvious rules for ∂_F such as a product rule, a chain rule and a decomposition rule for $f : X \times Y \to \mathbb{R}_\infty$:

$$\partial_F f(x,y) \subset \partial_F f_y(x) \times \partial_F f_x(y)$$

for $f_y(x) = f(x,y)$ and $f_x(y) = f(x,y)$ which seems to be specific. Moreover, if $f := g + h$ with g differentiable at $\overline{x}$, then $\partial_F f(\overline{x}) = g'(\overline{x}) + \partial_F h(\overline{x})$. Such rules are not valid for the limiting subdifferential ∂_L or the circa-subdifferential (or Clarke subdifferential) ∂_C. We also take advantage of a simple rule for the firm subdifferential of integral functionals given in [3], [9]. It is another reason to use here the firm subdifferential ∂_F.

2. Preliminaries about subdifferentials of performance functions

In the sequel we use elementary properties of the firm (or Fréchet) subdifferential ∂_F gathered in the appendix. We also mention a variant of the results using the directional (or Dini-Bouligand) subdifferential ∂_D for which we refer to [12].

Let V, W be open subsets of normed vector spaces E, F respectively, and let the map $q : V \to W$ be Gateaux differentiable at $\overline{v} \in V$, i.e. directionally differentiable with a continuous linear derivative $q'(\overline{v})$ at $\overline{v}$. Given $j : V \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ define the *performance function* associated with j and q by

$$p(w) := \inf\{j(v) : v \in q^{-1}(w)\}, \qquad w \in W.$$

If q is differentiable at $\overline{v}$ and $w^* \in \partial_F p(\overline{w})$, then one has

$$q'(\overline{v})^\top(w^*) \in \partial_F j(\overline{v}).$$

Proposition 2.1. *Let $\overline{v} \in V$ and $\overline{w} \in W$ be such that $\overline{w} := q(\overline{v})$ and $j(\overline{v}) = p(\overline{w})$. Then, for all $w^* \in \partial_D p(\overline{w})$ one has*

$$q'(\overline{v})^\top(w^*) \in \partial_D j(\overline{v}).$$

If q is differentiable at $\overline{v}$ and $w^* \in \partial_F p(\overline{w})$, then one has

$$q'(\overline{v})^\top(w^*) \in \partial_F j(\overline{v}).$$

Proof. Let $v^* := q'(\overline{v})^\top(w^*) := w^* \circ q'(\overline{v})$ and let $u' \in E$. Since

$$\frac{1}{t}(q(\overline{v} + tv') - q(\overline{v})) \to w' := q'(\overline{v})u'$$

when $(t, v') \to (0_+, u')$, and since $w^* \in \partial_D p(\overline{w})$, $p \circ q \leq j$, we have

$$\begin{aligned}
\langle v^*, v' \rangle = \langle w^*, q'(\overline{v})(v') \rangle &\leq \liminf_{(t,z') \to (0_+, q'(\overline{v})(u'))} \frac{1}{t}[p(\overline{w} + tz') - p(\overline{w})] \\
&\leq \liminf_{(t,v') \to (0_+, u')} \frac{1}{t}[p(\overline{w} + t\frac{q(\overline{v} + tv') - q(\overline{v})}{t}) - p(\overline{w})] \\
&= \liminf_{(t,v') \to (0_+, u')} \frac{1}{t}[p(q(\overline{v} + tv')) - p(\overline{w})] \\
&\leq \liminf_{(t,v') \to (0_+, u')} \frac{1}{t}[j(\overline{v} + tv') - j(\overline{v})].
\end{aligned}$$

Thus, by definition of $\partial_D j(\overline{v})$, we get $v^* \in \partial_D j(\overline{v})$.

Now, assume q is (Fréchet-) differentiable at $\overline{v}$ and $w^* \in \partial_F p(\overline{w})$. Let r, s be remainders such that

$$\begin{aligned}
p(\overline{w} + w) &\geq p(\overline{w}) + \langle w^*, w \rangle + r(w), \quad w \in W, \\
q(\overline{v} + v) &= q(\overline{v}) + q'(\overline{v}).v + s(v), \quad v \in V - \overline{v}.
\end{aligned}$$

Then, setting again $v^* := q'(\overline{v})^\top(w^*)$, for $w := q'(\overline{v}).v + s(v)$ with $v \in E - \overline{v}$ we have

$$\begin{aligned}
p(q(\overline{v} + v)) &\geq p(\overline{w}) + \langle w^*, w \rangle - r(w) \\
&= p(\overline{w}) + \langle w^*, p(\overline{w}) + \langle w^*, q'(\overline{v}).v + s(v) \rangle + r(w) \rangle
\end{aligned}$$

and since $w^* \circ s + r$ is a remainder, we get $w^* \circ q'(\overline{v}) \in \partial_F(p \circ q)(\overline{v})$. Now, since $j \geq p \circ q$ and $j(\overline{v}) = (p \circ q)(\overline{v})$, we see that $w^* \circ q'(\overline{v}) \in \partial_F j(\overline{v})$. □

Let us note that introducing a notion of directional remainder, the second part of the proof could be modelled on the first part.

The preceding proposition generalizes [1] which deals with the case when V is a product $W \times Z$ and q is the projection from $W \times Z$ onto W; in such a case one has $q'(\overline{v})^\top(w^*) = (w^*, 0)$ for $w^* \in W^*$.

Corollary 2.2. *If for $j : W \times Z \to \overline{\mathbb{R}}$ one sets $p(w) := inf\{j(w,z) : z \in Z\}$ for $w \in W$ and if $p(\overline{w}) = j(\overline{w}, \overline{z})$, then one has*

$$w^* \in \partial_F p(\overline{w}) \Rightarrow (w^*, 0) \in \partial_F j(\overline{w}, \overline{z})).$$

If $j(\overline{w}, \cdot)$ is convex, this implication is an equivalence.

3. Hamilton-Jacobi theory with partial convexity assumptions

It is classical to introduce the *Hamiltonian function* H on $E \times E^*$ associated to the Lagrangian L by using the partial conjugate $L^{\ltimes}$ of L given by

$$L^{\ltimes}(e, \cdot) = L(e, \cdot)^*,$$

or
$$H(e, e^*) = \sup_{u \in E} (\langle e^*, u \rangle - L(e, u)) \qquad for \quad (e, e^*) \in E \times E^*.$$

Incidentally, let us note that since L is assumed to be closed convex with respect to its velocity variable u we have

$$L(e, u) = H^{\ltimes}(e, u).$$

Since $-H(e, e^*) = \inf_u K(e, e^*, u)$ with $K(e, e^*, u) := L(e, u) - e^*(u)$ we recall from Corollary 2.2 that whenever $\overline{u} := u_{e,e^*}$ is a minimizer of $L(e, \cdot) - e^*(\cdot)$, or, equivalently $e^* \in \partial_F L_e(\overline{u})$ in view of the convexity of L_e, and for $(z^*, z^{**}) \in \partial_F(-H)(e, e^*)$ we have

$$(z^*, z^{**}, 0) \in \partial_F K(e, e^*, \overline{u}),$$

hence
$$z^* \in \partial_F L_{\overline{u}}(e), \quad z^{**} = -\overline{u}, \quad e^* \in \partial_F L_e(\overline{u}).$$

The following result proved in [15, Theorem 2.5] under full convexity of L is a crucial point of the Hamilton-Jacobi theory. Here we cannot rely on Hamiltonian trajectories.

Theorem 3.1. *Given $(\overline{t}, \overline{z}) \in \mathbb{R}_+ \times E$, $\overline{v} \in L_1([0, \overline{t}], E)$ such that $V(\overline{t}, \overline{z}) = F(\overline{t}, \overline{z}, \overline{v})$, where $F(t, z, v) := g(x(0)) + \int_0^t L(x(s), x'(s))ds$ with $x(s) := P_s(v) + x(0)$, assume $L(e, \cdot)$ is closed proper convex for all $e \in E$ and that g is differentiable at $\overline{x}(\overline{t}) := \overline{z} - P_{\overline{v}}(\overline{t})$. Then, for $(t^*, z^*) \in \partial_F V(\overline{t}, \overline{z})$, one has*

$$z^* \in \partial_F V_{\overline{t}}(\overline{z}), \quad t^* = -H(\overline{z}, z^*).$$

Proof. The inclusion $z^* \in \partial_F V_{\overline{t}},(\overline{z})$ stems from the definition of $\partial_F V(\overline{t}, \overline{z})$.

In order to prove the relation $t^* = -H(\overline{z}, z^*)$, for the sake of simplicity, hereafter we often write ∂, P instead of ∂_F, P_1 and t, z instead of $\overline{t}, \overline{z}$ and we assume an optimal arc $\overline{x}$ (noted x when there is no risk of confusion) exists. We set for $z \in E$, $w \in W^{1,1}([0, 1], E)$, $(r, s, t) \in [0, 1] \times \mathbb{R}^2_+$, with $s = rt$, $u := w'$, $x(.) := w(t^{-1}.)$, $v := x'$,

$$w(r) := w(t^{-1}s) = x(s) = P_u(r) + w(0) = P_r(u) + w(0),$$

so that $x'(s) = t^{-1}u(t^{-1}s) = t^{-1}u(r)$,

$$x(0) = w(0) = w(1) - P_1(u) = z - P(u).$$

Since $L(x(s), x'(s)) = L(w(r), t^{-1}u(r))$, we set

$$K_r(t, z, u) := K_{t,z,u}(r) = L(x(s), x'(s)) = L(w(r), t^{-1}u(r))$$
$$= L(z - P(u) + P_r(u), t^{-1}u(r))$$
$$k(t, z, u) := \int_0^1 K_{t,z,u}(r)dr$$
$$F(t, z, u) := g(z - P(u)) + tk(t, z, u) = g(x(0)) + \int_0^t L(x(s), x'(s))ds,$$

and we get $$V(t, z) = \inf\{F(t, z, u) : u \in L_1([0, 1], E)\}.$$

Then, as above, we recall from Corollary 2.2 that whenever $\overline{u}$ is a minimizer of $F(\overline{t}, \overline{z}, .)$ i.e. $F(\overline{t}, \overline{z}, \overline{u}) = V(\overline{t}, \overline{z})$, one has since G given by $G(t, z, u) := g(z - P(u))$ is differentiable, with derivative given by

$$G'(t, z, u) = (0, g'(x(0)), -g'(x(0))oP),$$
$$(t^*, z^*) \in \partial_F V(\overline{t}, \overline{z}) \ \Rightarrow\ (t^*, z^*, 0) \in \partial_F F(\overline{t}, \overline{z}, \overline{u}) = G'(\overline{t}, \overline{z}, \overline{u}) + \partial_F(tk), z, u).$$

In turn, by [3],[9], the relation $(t^*, z^*, 0) \in \partial_F F(\overline{t}, \overline{z}, \overline{u})$ implies $(t, z, u) := (\overline{t}, \overline{z}, \overline{u})$ is a solution to the inclusion

$$(t^*, z^*, 0) - (0, g'(x(0)), -g'(x(0))oP) \in \int_0^1 \partial_F(tK_r)(t, z, u)dr$$

or, using the product rule of the appendix, that it is a solution to the inclusion

$$(t^* - k(t, z, u), z^* - g'(x(0)), g'(x(0))oP) \in t\int_0^1 \partial_F K_r(t, z, u)dr.$$

Using the map $(p_r, q_r) := \mathbb{R} \times E \times E \to E \times E$ such that $K_r = (Lo(p_r, q_r))$:

$$(p_r, q_r)(t, z, u) := (P_r(u) + z - P(u), t^{-1}u),$$

observing that the derivative of (p_r, q_r) at $(\overline{t}, \overline{z}, \overline{u})$ is onto, the chain rule for the firm subdifferential yields some

$$(e^*, u^*) \in \partial L(p_r(\overline{t}, \overline{z}, \overline{u}), q_r(\overline{t}, \overline{z}, \overline{u})) = \partial L(\overline{x}, \overline{v})$$

such that $$\partial_F K_r(\overline{t}, \overline{z}, \overline{u}) = ((e^*, u^*)o(p'_r, q'_r))(\overline{t}, \overline{z}, \overline{u}),$$

we see that $(t, z, u) := (\overline{t}, \overline{z}, \overline{u})$ is a solution to the equation

$$(t^* - k(t, z, u), z^* - g'(x(0)), g'(x(0))oP) = \int_0^1 (te^*, tu^*)o(p'_r, q'_r)(t, z, u)$$

and to the system
$$t^* - k(t, z, u) = t^{-1}u^*(u)$$
$$z^* - g'(x(0)) = te^*$$
$$-g'(x(0))oP = te^*o(P_r - P)$$

which yields the expected relation since $v = t^{-1}u, u^* \in \partial L_{\overline{x}}(\overline{v})$ and

$$t^* = L(\overline{x}, \overline{v}) - u^*(\overline{v}).$$

Example 3.2. Assume that for a function $b : E \to \mathbb{P} :=]0, \infty[$ and a convex function $c : E \to \mathbb{R}$ the Lagrangian is given by

$$L(e, v) = b(e)c(v) \text{ for } (e, v) \in E \times E.$$

Then for $(t^*, z^*) \in \partial_F V(\overline{t}, \overline{z})$, one has

$$z^* \in \partial_F V_{\overline{t}}(\overline{z}), \quad t^* = b(\overline{z})c^*(\frac{z^*}{b(\overline{z})}).$$

4. Conclusion

We have chosen to work with a trivial fibration (in Proposition 2) i.e. a projection from a product onto one of its factors or with an almost trivial fibration in Proposition 2.1. However, in the case the Lagrangian L is such that $L(e, 0) = 0$ for all $e \in E$, a natural framework would be to take a subset X of a normed space along with a function $j : X \to \overline{\mathbb{R}}$ and a map $q : X \to W$, where W is a normed space, so that one can set $p(w) := inf\{j(x) : x \in q^{-1}(w)\}$ and look for an interpretation of $\partial_F p(w)$. Here one can take for X the set of $x \in W^{1,1}(\mathbb{R}_+, E)$ that are constant on $[\overline{t}, +\infty[$ for some $\overline{t} \in \mathbb{R}_+$. However only another application would justify such an apparatus that lacks the simplicity of the above trivialization.

5. Appendix

By definition, the firm (or Fréchet) subdifferential of a given function $f : E \to \overline{\mathbb{R}}$ finite at $\overline{x} \in E$ is the set $\partial_F f(\overline{x})$ of elements $x^* \in E^*$ such that there exists a remainder $r : E \to \overline{\mathbb{R}}$ satisfying

$$f(\overline{x} + u) \geq f(\overline{x}) + < x^*, u > + r(u), \quad u \in E,$$

a *remainder* being a function $r : E \to \overline{\mathbb{R}}$ such that $r(0) = 0$ and $\frac{r(u)}{\|u\|} \to 0$ as $u \to 0$ with $u \in X - \{0\}$.

The firm subdifferential ∂_F satisfies the following properties:

- *homotonicity*: when $f \leq g$ and $f(\overline{z}) = g(\overline{z})$ then $\partial f(\overline{z}) \subset \partial g(\overline{z})$;
- *decomposition rule*: for $h : X \times Y \to \overline{\mathbb{R}}$, $\partial h(\overline{x}, \overline{y}) \subset \partial h_{\overline{y}}(\overline{x}) \times \partial h_{\overline{x}}(\overline{y})$;
- *sum*: if $f, g : X \to \overline{\mathbb{R}}$ are finite at $\overline{x}$ then $\partial_F f(\overline{x}) + \partial_F g(\overline{x}) \subset \partial_F(f + g)(\overline{x})$ with equality if g is differentiable at $\overline{x}$;
- *product*: if $f(\overline{x}) \geq 0$, if g is nonegative and bounded above around $\overline{x}$, then
$$g(\overline{x})\partial_F f(\overline{x}) + f(\overline{x})\partial_F g(\overline{x}) \subset \partial_F(fg)(\overline{x});$$
- *chain rule*: ([12], Proposition 4.40, 4.42) if $h : W \to X$ is strictly differentiable at $\overline{w}$ with $h'(\overline{w})$ onto then $\partial_F(foh)(\overline{w}) = \partial_F f(h(\overline{w}))oh'(\overline{w})$;
- *integral functional*: ([3], [9]) if (T, μ) is a measured space, if $F : E \times T \to \mathbb{R}$ is measurable and $f(x) := \int_T F_x(t))d\mu(t)$ then $\partial_F f(x) = \int_T \partial_F F_t(x)d\mu(t)$.

Such properties are either classical (see [11], [12]) or easy to establish. Still we provide a proof for the product rule.

Proof of the product property. Without loss of generality we assume that $\overline{x} = 0$. Let $x^* \in \partial f(0)$, $y^* \in \partial g(0)$ and let r, s be remainders such that

$$f \geq f(0) + x^* + r, g \geq g(0) + y^* + s.$$

Then we have

$$\begin{aligned} f(x)g(x) &\geq (f(0) + \langle x^*, x\rangle + r(x))g(x) \\ &\geq f(0)g(0) + f(0)\langle y^*, x\rangle + f(0)s(x) + \langle x^*, x\rangle(\langle y^*, x\rangle + s(x)) + r(x)g(x). \end{aligned}$$

Since $x \rightarrowtail \partial f(0)s(x) + < x^*, x > (< y^*, x > +s(x)) + r(x)g(x)$ is a remainder under our assumptions, we see that $g(0)x^* + f(0)y^* \in \partial(fg)(0)$.

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