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Convergence of Slopes in Asplund Spaces

Pedro Pérez-Aros
Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (CNRS UMI 2807), Universidad de Chile, Santiago, Chile
pperez@dim.uchile.cl

Lionel Thibault
Institut Montpelliérain A. Grothendieck, Université de Montpellier, France
lionel.thibault@umontpellier.fr

Dariusz Zagrodny
Faculty of Mathematics and Computer Science, University of Lodz, Poland
dariusz.zagrodny@wmii.uni.lodz.pl

Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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We investigate the epigraphical convergence of slopes of lower semicontinuous convex functions on Asplund spaces. This type of convergence of slopes is compared to the Painlevé-Kuratowski convergence of subdifferetials. Both convergences are shown to be equivalent under mild conditions of compactness type.

Keywords: Convex function, slope, convergence of slopes, subdifferential, convergence of subdifferentials.

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1. Introduction

Our previous paper [31] largely studied in separable Banach spaces various relationships between the epigraphical convergence of slopes (see Section 2 for definition) of lower semicontinuous (lsc, for short) convex functions and the Painlevé-Kuratowski convergence of graphs of their subdifferentials. The first work [18] in this line dealt with the finite-dimensional variant with slopes of Attouch's theorem. In [18] the authors showed that in a finite dimensional space the epigraphical convergence of a sequence of lsc proper convex functions plus a boundedness from below assumption is equivalent to the epigraphical convergence of their slopes plus a normalization condition. Our main result in [31] established in any separable Banach space E that, under a compactness type condition (CTA), the Painlevé-Kuratowski convergence of graphs of subdifferentials ∂f_k of a sequence of lsc proper convex functions $\{f_k\}_k$ on E to the graph of subdifferential of a function f bounded below is equivalent to the epigraphical convergence of their slopes s_{f_k}. Diverse (counter) examples were also provided in [31] illustrating that in infinite dimension the result fails without additional condition in the line of (CTA).

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Here we continue the study in passing to the setting of non-separable Banach spaces which are Asplund. Doing so, we show that our foregoing result still holds for graphs of ∂f_k and slopes s_{f_k} for lsc proper convex functions f_k on a Banach space E provided that E is Asplund. The development is carried out through our previous paper [31] and through the theory of separable reduction/reducibility via rich families of separable subspaces (see, e.g., [7, 12, 21, 22], see also the last section for other references and comments on this concept). Section 2 is devoted to recall definitions and properties of tools and notions needed for our analysis along the paper. In Section 3 we restate our main result in [31] in separable Banach spaces and establish some new others in such spaces. The principal Section 4 reach and develop, under the condition (CTA), the equivalence in Asplund spaces between the epigraphical convergence of slopes and the Painlevé-Kuratowski convergence of graphs of subdifferentials of lsc convex functions.

The situation of bounded Hausdorff convergence of the sequence of graphs of subdifferentials $\{\partial f_k\}_{k\in\mathbb{N}}$ will be examined in another companion paper.

2. Convergence of sets

In all the paper $\mathbb{R}$ will be the real line and $\mathbb{N}$ the set of positive integers. For a real Banach space $(E, \|\cdot\|)$ its topological dual space will be denoted E^* (as usual) and $\|\cdot\|_*$ will be the canonical dual norm on E^* associated to the primal norm $\|\cdot\|$.

The corresponding distances d on E and d^* on E^* are given as usual by

$$d(x,y) := \|x-y\| \quad \text{and} \quad d^*(x^*,y^*) := \|x^*-y^*\|_*.$$

By $d\times d^*$ we will denote the distance on $E\times E^*$ defined by

$$(d\times d^*)((x,y),(x^*,y^*)) := d(x,y) + d^*(x^*,y^*).$$

Consider a sequence $\{S_k\}_{k\in\mathbb{N}}$ of sets in a Hausdorff topological space (X,τ). Its *sequential lower limit* $\mathrm{Li}_\tau\, S_k$ in (X,τ) is defined as the set of $x\in X$ for which there is a sequence $\{x_k\}_{k\in\mathbb{N}}$ in X τ-converging to x such that $x_k\in S_k$ for all large k. The *sequential upper limit* $\mathrm{Ls}_\tau\, S_k$ in (X,τ) is the set of $x\in X$ for which there is an increasing function $j:\mathbb{N}\to\mathbb{N}$ and a sequence $\{x_k\}_{k\in\mathbb{N}}$ in X converging to x such that $x_k\in S_{j(k)}$ for all $k\in\mathbb{N}$. A set S in X is said to be the Painlevé-Kuratowski (or PK, for short) sequential limit or set limit of the sequence $\{S_k\}_{k\in\mathbb{N}}$ if the lower and upper sequential limits of the sequence coincide with S, which is equivalent to

$$\mathrm{Ls}_\tau\, S_k \subset S \subset \mathrm{Li}_\tau\, S_k. \tag{1}$$

When this is fulfilled, the limit set S of the sequence is called the *Painlevé-Kuratowski* (or *PK*) *sequential limit* in (X,τ) of $\{S_k\}_{k\in\mathbb{N}}$, and this is denoted as $S_k \xrightarrow{PK} S$.

When the sequence $\{S_k\}_{k\in\mathbb{N}}$ PK converges to a set S in (X,τ), for any subsequence $\{S_{j(k)}\}_{k\in\mathbb{N}}$ (i.e., the function $j:\mathbb{N}\to\mathbb{N}$ is increasing) one has

$$\mathrm{Ls}_\tau\, S_{j(k)} \subset \mathrm{Ls}_\tau\, S_k \subset S \subset \mathrm{Li}_\tau\, S_k \subset \mathrm{Li}_\tau\, S_{j(k)}, \tag{2}$$

and hence the subsequence $\{S_{j(k)}\}_{k\in\mathbb{N}}$ also PK converges to the set S in (X,τ).

If the topology τ is metrizable by some distance d, one generally omits the index τ and one merely writes $\operatorname{Li} S_k$ and $\operatorname{Ls} S_k$. In this metrizable situation one knows that

$$\begin{aligned} \operatorname{Li} S_k &= \{x \in X \mid \limsup_{k\to\infty} d(x, S_k) = 0\}, \\ \operatorname{Ls} S_k &= \{x \in X \mid \liminf_{k\to\infty} d(x, S_k) = 0\}; \end{aligned} \tag{3}$$

further, if $S_k \xrightarrow{PK} S$, one has

$$\limsup_{k\to\infty} d(x, S_k) \le d(x, S) \quad \text{for all } x \in X. \tag{4}$$

We recall (see, e.g., [39, Proposition 1.7]) a known lemma (also easy to see via (3)).

Lemma 2.1. *If (X, d) is a metric space and $\{S_k\}_{k\in\mathbb{N}}$ is a sequence in X, then both sets $\operatorname{Ls} S_k$ and $\operatorname{Li} S_k$ are closed.*

In addition to (2), we recall a lemma on PK convergence established in the context of metric spaces in Kuratowski's book [29, Chapter II §25, IX, 1].

Lemma 2.2. *Let $\{S_k\}_{k\in\mathbb{N}}$ be a sequence of sets in a metric space (X, d). Then $\{S_k\}_{k\in\mathbb{N}}$ converges to a set S in (X, d) in the PK sense if and only if every subsequence has a further subsequence which converges to S in the PK sense.*

The epiconvergence of functions is defined via their epigraphs. For the connection between Γ-convergence and epiconvergence of functions we refer, e.g., to [14, Theorem 4.16]. Recall that the epigraph of a function $f : E \to \mathbb{R} \cup \{-\infty, +\infty\}$ is given by $\operatorname{epi} f := \{(x, r) \in E \times \mathbb{R} \mid f(x) \le r\}$. Recall also that this function f is proper if it does not take the value $-\infty$ and it is finite at least at one point in E. The (effective) domain of f is the set $\operatorname{dom} f := \{x \in E : f(x) < +\infty\}$.

Definition 2.3. A sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ from the Banach space $(E, \|\cdot\|)$ into the space $\mathbb{R}\cup\{-\infty, +\infty\}$ epigraphically (for short, epi-) converges to a function $f : E \to \mathbb{R} \cup \{-\infty, +\infty\}$, denoted as $f_k \xrightarrow{e} f$, if $\operatorname{epi} f_k \xrightarrow{PK} \operatorname{epi} f$.

Since the PK limit is not depending on the choice of equivalent norm in E or in E^*, the epigraphical convergence of given functions does not depend on the choice of equivalent norms. Moreover, the epigraph of the limit function must be a closed set, see Lemma 2.1, so the function must be lower semicontinuous (lsc, for short), see also [9, Proposition 1.28]. It is known that the epigraphical convergence of functions $\{f_k\}_{k\in\mathbb{N}}$ is equivalent to requiring that the following two conditions hold, see, e.g., [9, Definition 1.5]: for every $x \in E$

$$\begin{aligned} &\liminf_{k\to\infty} f_k(x_k) \ge f(x) \quad \text{for every sequence } \{x_k\}_{k\in\mathbb{N}} \text{ in } E \text{ converging to } x \\ \text{and} \quad &\limsup_{k\to\infty} f_k(x_k) \le f(x) \quad \text{for some sequence } \{x_k\}_{k\in\mathbb{N}} \text{ in } E \text{ converging to } x. \end{aligned}$$

The above is equivalent to saying that for every $x \in E$

$$\inf\{\limsup_{k\to\infty} f_k(x_k) \mid x_k \to x\} \le f(x) \le \inf\{\liminf_{k\to\infty} f_k(x_k) \mid x_k \to x\},$$

see, e.g., [9, Section 1.2] or [3, Proposition 12.1.1]. The function $\operatorname{li} f_k$ (resp. $\operatorname{ls} f_k$) whose epigraph coincides with $\operatorname{Ls} \operatorname{epi} f_k$ (resp. $\operatorname{Li} \operatorname{epi} f_k$) is called the *epi-limit inferior* (resp. *superior*) of the sequence $\{f_k\}_{k\in\mathbb{N}}$. One has $\mathrm{li}\, f_k \le \mathrm{ls}\, f_k$ and one knows (see [2, Theorems 1.13 and 1.36] and [9, Definition 1.24]) that

$$\mathrm{li}\, f_k = \min\{\liminf_{k\to\infty} f_k(x_k) \mid x_k \to x\} \ \text{ and } \ \mathrm{ls}\, f_k = \min\{\limsup_{k\to\infty} f_k(x_k) \mid x_k \to x\}.$$

In the setting of a Banach space $(E, \|\cdot\|)$, given $S_k \subset E\times E^*$, we were led to consider in [31] the following other definition of convergence for sets in $E \times E^*$.

Definition 2.4. The sequence $\{S_k\}_{k\in\mathbb{N}}$ of subsets of $E \times E^*$ is said to converge to a set S in $E \times E^*$ in a *bounded mixed way*, denoted as $S_k \xrightarrow{b\mu} S$, provided that

$$\mathcal{A}_i(\{S_k\}_{k\in\mathbb{N}}) = S = \mathcal{A}_s(\{S_k\}_{k\in\mathbb{N}}), \tag{5}$$

where

$$\begin{aligned}\mathcal{A}_s(\{S_k\}_{k\in\mathbb{N}}) := \big\{(a, a^*) \in E \times E^* \mid\ & \text{there is an increasing function} \\ & j : \mathbb{N} \to \mathbb{N} \text{ and a sequence} \{(a_k, a_k^*)\}_{k\in\mathbb{N}} \text{ in } E \times E^* \text{ such that} \\ & \{a_k^*\}_{k\in\mathbb{N}} \text{ is bounded, } \lim_{k\to\infty} a_k = a \ \text{ and } (a_k, a_k^*) \in S_{j(k)} \text{ for all} \\ & k \in \mathbb{N}, \text{ and such that } a^* \in \textstyle\bigcap_{k\in\mathbb{N}} \mathrm{cl}^{w^*} \{a_k^*, a_{k+1}^*, \ldots\}\big\},\end{aligned} \tag{6}$$

and

$$\begin{aligned}\mathcal{A}_i(\{S_k\}_{k\in\mathbb{N}}) := \big\{(a, a^*) \in E \times E^* \mid\ & \text{there is a sequence} \\ & \{(a_k, a_k^*)\}_{k\in\mathbb{N}} \ \text{ such that } a_k \to a \text{ and } (a_k, a_k^*) \in S_k \text{ for large} \\ & k \in \mathbb{N} \text{ and such that } \{a^*\} = \textstyle\bigcap_{k\in\mathbb{N}} \mathrm{cl}^{w^*} \{a_{j(k)}^*, a_{j(k+1)}^*, \ldots\} \\ & \text{for any increasing function } j : \mathbb{N} \to \mathbb{N}\big\}.\end{aligned} \tag{7}$$

Clearly, the sets defined above in (6) and (7) are not depending on the choice of equivalent norm in E or in E^*. Otherwise stated, the bounded mixed way limit does not depend on the choice of equivalent norms. It is also quite obvious that in finite dimensions the PK convergence and the bounded mixed way convergence coincide.

For the bounded mixed way convergence a variant has been shown in [31] for the famous Zarankiewicz theorem (see [44, 4]).

Proposition 2.5. [31] *Let $(E, \|\cdot\|)$ be a separable Banach space. Every sequence $\{S_k\}_{k\in\mathbb{N}}$ of subsets of $E \times E^*$ contains a subsequence which has a (possibly empty) bounded mixed way limit, that is, there is an increasing function $\gamma : \mathbb{N} \longrightarrow \mathbb{N}$ and a subset S of $E \times E^*$ such that*

$$\mathcal{A}_i(\{S_{\gamma(k)}\}_{k\in\mathbb{N}}) = S = \mathcal{A}_s(\{S_{\gamma(k)}\}_{k\in\mathbb{N}}). \tag{8}$$

The (CTA) condition mentioned in the introduction and already involved in [31] was formulated there in the following form:

Compactness Type Assumption. Let $(E, \|\cdot\|)$ be a Banach space and $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions on E.

(CTA) For any $a \in E$, any $a^* \in E^*$, any increasing function $j : \mathbb{N} \to \mathbb{N}$, and any bounded sequence $\{(b_k, b_k^*)\}_{k\in\mathbb{N}}$, with $(b_k, b_k^*) \in \mathrm{gph}\,\partial f_{j(k)}$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \big(\|b_k^* - a^*\|_*^2 + \|b_k - a\|^2 + 2\langle b_k^* - a^*, b_k - a\rangle\big) = 0, \tag{9}$$

the sequence $\{b_k\}_{k\in\mathbb{N}}$ has a subsequence converging strongly to some $b \in E$.

In (9) the norm $\|\cdot\|$ and its dual norm are involved, so if (9) holds and the related norm $\|\cdot\|$ needs to be emphasized we will say that the sequence $\{f_k\}_{k\in\mathbb{N}}$ fulfills the condition (CTA) with respect to the norm $\|\cdot\|$.

Under the condition (CTA) it was proved in [31] that the properties reformulated in Proposition 2.6 hold true. Before stating the proposition recall first some notions. Given a (set-valued) operator $A : E \rightrightarrows E^*$ between a Banach space E and its dual E^* its graph and (effective) domain are respectively the sets

$$\operatorname{gph} A := \{(x, x^*) \in E \times E^* \mid x^* \in A(x)\} \quad \text{and} \quad \operatorname{Dom} A := \{x \in E \mid A(x) \neq \emptyset\}.$$

The operator A is *monotone* if $\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq 0$ whenever $(x_i, x_i^*) \in \operatorname{gph} A$, $i = 1, 2$, and it is *maximal monotone* if there is no distinct monotone operator whose graph contains $\operatorname{gph} A$. Recall also that A is *cyclically monotone* if

$$\langle x_1^*, x_2 - x_1\rangle + \ldots + \langle x_n^*, x_{n+1} - x_n\rangle \leq 0,$$

for $(x_i, x_i^*) \in \operatorname{gph} A$ with $i = 1, \ldots, n$ and $x_{n+1} = x_1$. The operator A is *maximal cyclically monotone* provided that its graph is not inluded in the graph of any distinct cyclically monotone operator from E into E^*. The operator A on the Banach space E is maximal cyclically monotone if and only if it is the subdifferential of an lsc proper convex function on E (see [33]).

Proposition 2.6. [31] *Let $(E, \|\cdot\|)$ be a Banach space, let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions on E and let $A : E \rightrightarrows E^*$ be an operator.*

(a) *If* (CTA) *is valid, then the following equivalence*

$$\operatorname{gph} A = \mathcal{A}_i(\{\partial f_k\}_{k\in\mathbb{N}}) \Longleftrightarrow \operatorname{gph} A = \operatorname{Li}_{d\times d^*} \operatorname{gph}(\partial f_k) \tag{10}$$

holds true. Further, one has

$$(\operatorname{gph} \partial f_k \xrightarrow{b\mu} \operatorname{gph} A) \Longrightarrow (\operatorname{gph} \partial f_k \xrightarrow{PK} \operatorname{gph} A).$$

The latter implication is an equivalence whenever A is maximal monotone.

(b) *If* (CTA) *is valid and* $\{\operatorname{gph} \partial f_k\}_{k\in\mathbb{N}}$ *converges to* $\operatorname{gph} A$ *in a bounded mixed way* (*i.e.* (5) *holds true for* $S_k := \operatorname{gph} \partial f_k$ *and* $S := \operatorname{gph} A$), *then there exists an lsc convex function* $f : E \to \mathbb{R} \cup \{+\infty\}$ *such that* $A = \operatorname{gph} \partial f$.

For the Banach space $(E, \|\cdot\|)$ and an lsc proper convex function $f : E \to \mathbb{R} \cup \{+\infty\}$, we recall that the *slope of f at $x \in E$ with respect to the norm* $\|\cdot\|$ can be defined by means of the dual norm $\|\cdot\|_*$ by

$$s_f(x) := \inf\{\|x^*\|_* \mid x^* \in \partial f(x)\}, \tag{11}$$

with the usual convention that the latter infimum is $+\infty$ when $\partial f(x) = \emptyset$ (it is also known as a specific type of metric slope for f, see [39, page 266], see also [40] for the lower semicontinuity properties of slopes).

In [14, Proposition 6.7] it is stated that the epi-limit inferior and superior are monotone. As a simple consequence of those properties we can establish inequality connections between epi-limits superior of slopes related to equivalent norms.

Lemma 2.7. *Let E be a Banach space with two equivalent norms $\|\cdot\|$ and $\|\!|\cdot|\!\|$, so there are reals $\eta_1 > 0$ and $\eta_2 > 0$ such that for all $x^* \in E^*$*

$$\|x\|_* \leq \eta_1 \|\!|x|\!\|_* \leq \eta_2 \|x\|_*, \tag{12}$$

where $\|\cdot\|_*$ and $\|\!|\cdot|\!\|_*$ stand for the canonical dual norms, respectively.
Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions from E into $\mathbb{R}\cup\{+\infty\}$ and consider their slopes

$$s_{f_k}(x) := \inf\{\|x^*\|_* \mid x^* \in \partial f_k(x)\} \tag{13}$$

and

$$t_{f_k}(x) := \inf\{\|\!|x^*|\!\|_* \mid x^* \in \partial f_k(x)\} \tag{14}$$

with respect to the norms $\|\cdot\|$ and $\|\!|\cdot|\!\|$ respectively. Then for every $x \in E$ one has

$$\frac{\eta_2}{\eta_1}(\operatorname{ls} s_{f_k})(x) \geq (\operatorname{ls} t_{f_k})(x) \geq \frac{1}{\eta_1}(\operatorname{ls} s_{f_k})(x).$$

Moreover, for every sequence $\{c_k\}_{k\in\mathbb{N}}$ we have the following equivalence:

$$\lim_{k\to\infty} s_{f_k}(c_k) = 0 \text{ and } +\infty > \sum_{k=1}^{\infty}(s_{f_k}(c_k) + s_{f_k}(c_{k+1}))\|c_k - c_{k+1}\|$$

if and only if

$$\lim_{k\to\infty} t_{f_k}(c_k) = 0 \text{ and } +\infty > \sum_{k=1}^{\infty}(t_{f_k}(c_k) + t_{f_k}(c_{k+1}))\|c_k - c_{k+1}\|.$$

Imposing restrictions on the slope of f, a property of boundedness from below for f has been established in [31] with the help of [38, Lemma 4.2]. We state it as follows.

Proposition 2.8. [31] *Let $(E, \|\cdot\|)$ be a Banach space and $f : E \to \mathbb{R}\cup\{+\infty\}$ be an lsc proper convex function. For any sequence $\{x_k\}_{k\in\mathbb{N}}$ in E one has the implication*

$$\Big(\lim_{k\to\infty} s_f(x_k) = 0 \text{ and } +\infty > \sum_{k=1}^{\infty}(s_f(x_k) + s_f(x_{k+1}))\|x_k - x_{k+1}\|\Big)$$
$$\Longrightarrow \liminf_{k\to\infty} f(x_k) = \inf_E f > -\infty. \tag{15}$$

Moreover, if $\inf_E f > -\infty$, then there is a sequence $\{x_k\}_{k\in\mathbb{N}}$ such that the sequence $\{s_f(x_k)\}_{k\in\mathbb{N}}$ is nonincreasing and

$$\lim_{k\to\infty} s_f(x_k) = 0 \text{ and } +\infty > \sum_{k=1}^{\infty} s_f(x_k)\|x_k - x_{k+1}\|. \tag{16}$$

We now recall two results from [38] concerning slope inequalities and the determination of lower semicontinuous convex functions on Banach spaces via slopes (see [38, Proposition 5.1 and Theorem 5.1]). For related works, we refer the reader to [8, 15, 16, 17, 27, 30, 41], and the references therein.

Theorem 2.9. *Let $(E, \|\cdot\|)$ be a Banach space and let $f, g : E \to \mathbb{R}\cup\{+\infty\}$ be two lsc proper convex functions.*

(a) *If g is bounded from below and*

$$d(0, \partial f(x)) \leq d(0, \partial g(x)) \quad \text{for all } x \in E,$$

then f is also bounded from below and

$$f(x) - \inf_E f \leq g(x) - \inf_E g \quad \text{for all } x \in E.$$

(b) *Assume that either f or g is bounded from below. Then*

$$d(0, \partial f(x)) = d(0, \partial g(x)) \quad \text{for all } x \in E$$

if and only if there exists a real constant c such that

$$f(x) = g(x) + c \quad \text{for all } x \in E.$$

3. Convergence of slopes in separable Banach spaces

Under (CTA) we know by the theorem below from [31] that the epigraphical convergence of slopes is implied by the bounded mixed convergence of subdifferentials. Below we restate this property with any equivalent norm, noticing for a Banach space $(E, \|\cdot\|)$ that a norm $|||\cdot|||$ on E is equivalent to $\|\cdot\|$ if and and only if the respective canonical dual norms $\|\cdot\|_*$ and $|||\cdot|||_*$ on E^* are equivalent according to the equalities

$$|||x^*|||_* = \sup_{|||x|||\leq 1} \langle x^*, x\rangle \quad \text{and} \quad |||x||| = \sup_{|||x^*|||_*\leq 1} \langle x^*, x\rangle.$$

Theorem 3.1. [31] *Let $(E, \|\cdot\|)$ be a Banach space and $|||\cdot|||$ be any norm on E equivalent to $\|\cdot\|$. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions from E into $\mathbb{R}\cup\{+\infty\}$ and f be an lsc convex function from E into $\mathbb{R}\cup\{+\infty\}$. Let s_{f_k}, s_f and t_{f_k}, t_f be the slopes of f_k, f with respect to the norms $\|\cdot\|$ and $|||\cdot|||$ respectively (see* (13) *and* (14)*).*

(a) *If f is proper, one has the implication*

$$\partial f_k \xrightarrow{PK} \partial f \Longrightarrow t_{f_k} \xrightarrow{e} t_f.$$

(b) *If (CTA) is valid for the sequence $\{f_k\}_{k\in\mathbb{N}}$ with respect to the norm $\|\cdot\|$, then the other implication*

$$\partial f_k \xrightarrow{b\mu} \partial f \Longrightarrow t_{f_k} \xrightarrow{e} t_f$$

also holds true (without imposing the properness of f).

Proof. (a) The implication

$$\partial f_k \xrightarrow{PK} \partial f \Longrightarrow s_{f_k} \xrightarrow{e} s_f$$

was established in [31]. Since the PK limit is not depending on the choice of equivalent norm in E or E^*, so considering the Banach space $(E, |||\cdot|||)$ we get the implication

$$\partial f_k \xrightarrow{PK} \partial f \Longrightarrow t_{f_k} \xrightarrow{e} t_f.$$

(b) If (CTA) is valid for the sequence $\{f_k\}_{k\in\mathbb{N}}$ with respect to the norm $\|\cdot\|$, then by Proposition 2.6 we get

$$(\operatorname{gph}\partial f_k \xrightarrow{b\mu} \operatorname{gph}\partial f) \Longrightarrow (\operatorname{gph}\partial f_k \xrightarrow{PK} \operatorname{gph}\partial f).$$

Thus employing (a) we get (b). $\square$

Remark 3.2. Assume that $(E, \|\cdot\|)$ is a separable Banach space and keep notation t_{f_k} and t_f as in Theorem 3.1. Let us notice that

$$\partial f_k \xrightarrow{b\mu} \partial f \Longleftrightarrow t_{f_k} \xrightarrow{e} t_f,$$

when f is not proper, i.e. $f \equiv +\infty$, which is equivalent to $\operatorname{gph} \partial f = \emptyset$, see for example [7, Theorem 4.3.2 (Brondsted-Rockafellar)] or [39, Theorem 3.142]. Notice also that $\operatorname{gph} \partial f = \emptyset$ if and only if $t_f \equiv +\infty$.

If $t_f \equiv +\infty$ and $\partial f_k \xrightarrow{b\mu} \partial f$, then we have $\liminf_{k\to\infty} t_{f_k}(x_k) = +\infty$, whenever $x_k \longrightarrow x$, otherwise $\mathcal{A}_s(\{\partial f_k\}_{k\in\mathbb{N}}) \neq \emptyset$, so $\operatorname{gph} \partial f \neq \emptyset$, a contradiction. Thus the implication

$$\partial f_k \xrightarrow{b\mu} \partial f \Longrightarrow t_{f_k} \xrightarrow{e} t_f \equiv +\infty$$

is established in this case.

In order to establish the reverse implication, that is

$$t_{f_k} \xrightarrow{e} t_f \equiv +\infty \Longrightarrow \partial f_k \xrightarrow{b\mu} \partial f$$

let us assume that $t_{f_k} \xrightarrow{e} t_f$ and $\mathcal{A}_s(\{\partial f_k\}_{k\in\mathbb{N}}) \neq \emptyset$. It follows from Proposition 2.5 that there is an increasing function $\gamma : \mathbb{N} \longrightarrow \mathbb{N}$ and a subset S of $E \times E^*$ such that

$$\mathcal{A}_i(\{\partial f_{\gamma(k)}\}_{k\in\mathbb{N}}) = S = \mathcal{A}_s(\{\partial f_{\gamma(k)}\}_{k\in\mathbb{N}}) \neq \emptyset.$$

It is easy to notice that $\operatorname{Dom} S \subset \operatorname{dom} t_f$, hence we get a contradiction. It ensues that $\mathcal{A}_s(\{\partial f_k\}_{k\in\mathbb{N}}) = \emptyset$, which justifies the implication

$$t_{f_k} \xrightarrow{e} t_f \equiv +\infty \Longrightarrow \partial f_k \xrightarrow{b\mu} \partial f. \qquad \square$$

Let us also observe other features on the convergence of subdifferentials and slopes without requiring knowledge of the limiting set or function. Therefore, this indicates that, in some sense, the bounded mixed and PK convergences of subdifferentials, and the epigraphical convergence of slopes are close for convex functions.

Theorem 3.3. *Let $(E, \|\cdot\|)$ be a separable Banach space and $|\!|\!|\cdot|\!|\!|$ be any norm on E equivalent to $\|\cdot\|$. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions from E into $\mathbb{R}\cup\{+\infty\}$ such that that* (CTA) *is valid with respect to the norm $\|\cdot\|$. Consider the following properties with slopes t_{f_k} with respect to the norm $|\!|\!|\cdot|\!|\!|$ (see* (14)) :

(a) $\{\partial f_k\}_{k\in\mathbb{N}}$ *converges in a bounded mixed way.*

(b) $\{t_{f_k}\}_{k\in\mathbb{N}}$ *epigraphically converges to some function.*

(b') $\{t_{f_k}\}_{k\in\mathbb{N}}$ *epigraphically converges to some function ψ such that there is a sequence $\{c_k\}_{k\in\mathbb{N}}$ for which*

$$\lim_{k\to\infty} \psi(c_k) = 0 \quad \textit{and} \quad +\infty > \sum_{k=1}^{\infty} (\psi(c_k) + \psi(c_{k+1}))\|c_k - c_{k+1}\|.$$

Then one has (a)⇒(b) *and* (b')⇒(a).

Moreover, in case (a) *the limit is given by the graph of the subdifferential of an lsc convex function* $f : E \to \mathbb{R} \cup \{+\infty\}$ *as well as in case* (b) *(*f *must be the same in both cases, whenever* (a) *is assumed). If only case* (b') *is assumed, then there is an lsc proper convex function* f *bounded from below such that* $\psi \equiv t_f$*.*

Proof. First, let $j : \mathbb{N} \to \mathbb{N}$ be an increasing function and $A : E \rightrightarrows E^*$ be an operator such that $\partial f_{j(k)} \xrightarrow{b\mu} A$. We now claim that there is an lsc convex function $\phi : E \to \mathbb{R} \cup \{+\infty\}$ such that

$$\operatorname{gph} A = \operatorname{gph} \partial\phi \quad \text{and} \quad t_{f_{j(k)}} \xrightarrow{e} t_\phi. \tag{17}$$

Consider first the case $\operatorname{Dom} A \neq \emptyset$, so A is maximal cyclically monotone, as stated in Proposition 2.6(b). Therefore, $\operatorname{gph} A$ is nothing else than the graph, $\operatorname{gph} \partial\phi$, of the subdifferential of an lsc proper convex function $\phi : E \to \mathbb{R} \cup \{+\infty\}$, so Theorem 3.1(b) ensures that $t_{f_{j(k)}} \xrightarrow{e} t_\phi$. In the other case $\operatorname{Dom} A = \emptyset$, we have $\partial f_{j(k)} \xrightarrow{b\mu} \partial\phi$ for $\phi \equiv +\infty$, hence Remark 3.2 entails $t_{f_{j(k)}} \xrightarrow{e} t_\phi$. The claim is then proved.

Now let us justify the implications of the theorem.

(a)⇒(b) This implication directly follows from the claim (17).

(b')⇒(a) Assume that $t_{f_k} \xrightarrow{e} \psi$ for some function ψ for which there is a sequence $\{c_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \psi(c_k) = 0$ and $+\infty > \sum_{k=1}^{\infty} (\psi(c_k) + \psi(c_{k+1}))\|c_k - c_{k+1}\|$. Take any $b\mu$-convergent subsequence $\operatorname{gph} \partial f_{\gamma(k)} \xrightarrow{b\mu} \operatorname{gph} A^\gamma$ (such subsequences exist by Proposition 2.5). By the claim (17) we must have

$$\operatorname{gph} A^\gamma = \operatorname{gph} \partial\phi^\gamma \quad \text{and} \quad t_{f_{\gamma(k)}} \xrightarrow{e} t_{\phi^\gamma}$$

for some lsc convex function ϕ^γ on E, which also gives $\psi = t_{\phi^\gamma}$ and therefore $\lim_{k\to\infty} t_{\phi^\gamma}(c_k) = 0$ and $+\infty > \sum_{k=1}^{\infty} (t_{\phi^\gamma}(c_k) + t_{\phi^\gamma}(t_{k+1}))\|c_k - c_{k+1}\|$. This lsc convex function ϕ^γ is then proper and bounded from below, see Proposition 2.8.

Consider any other $b\mu$-convergent subsequence $\operatorname{gph} \partial f_{\ell(k)} \to \operatorname{gph} A^\ell$. By what precedes there is an lsc proper convex function ϕ^ℓ on E bounded from below such that $\operatorname{gph} A^\ell = \operatorname{gph} \partial\phi^\ell$ and $t_{\phi^\ell} = \psi$, so $t_{\phi^\ell} = t_{\phi^\gamma}$. We deduce from Theorem 2.9 that $\operatorname{dom} \phi^\gamma = \operatorname{dom} \phi^\ell =: D$ (with D independent of γ) and that ϕ^ℓ and ϕ^γ are equal up to an additive constant. Then we may fix $a \in D$ (with a independent of γ), so the lsc proper convex function $f : E \to \mathbb{R} \cup \{+\infty\}$ with

$$f(x) := \phi^\gamma(x) - \phi^\gamma(a) = \phi^\ell(x) - \phi^\ell(a) \quad \text{for all } x \in E$$

is well defined and $t_{f_k} \xrightarrow{e} t_f$ since $t_f = \psi$ and $\operatorname{gph} A^\gamma = \partial f$.

Hence, by Proposition 2.6 we get $\partial f_{l(k)} \xrightarrow{PK} \partial f$.

It follows from Lemma 2.2 that $\partial f_k \xrightarrow{PK} \partial f$ and Proposition 2.6 assures us that $\partial f_k \xrightarrow{b\mu} \partial f$, which finishes the proof of the theorem. □

4. Convergence of subdifferentials and slopes in general Asplund spaces

In this section, we extend the previous results to general Banach spaces without assuming the separability. The key technique for this purpose is the use of *rich families* in the theory of separable reducibility/reduction.

Consider an arbitrary Banach space E and denote by $\mathcal{S}(E)$ the family of all (norm-) separable closed linear subspaces of E. Following [6] a set $\mathcal{R} \subset \mathcal{S}(E)$ is called a *rich family* on E (of closed separable linear subspaces) if it satisfies the following properties:

(P_1) For every $W \in \mathcal{S}(E)$, there exists $V \in \mathcal{R}$ such that $W \subset V$.

(P_2) If a sequence $\{V_n\}_{n\in\mathbb{N}}$ in $\mathcal{R}$ satisfies $V_n \subset V_{n+1}$, then $\mathrm{cl}(\bigcup_{n\in\mathbb{N}} V_n) \in \mathcal{R}$.

A fundamental property is that, for any countable collection $(\mathcal{R}_i)_{i\in I}$ of rich families on E, the family $\bigcap_{i\in I} \mathcal{R}_i$ is a rich family on E (see [6]).

The separable reduction technique has been successfully employed to characterize significant properties in analysis, subdifferential calculus, and metric regularity, among others, on the entire space E (which may not be separable). This is achieved by verifying the same property in separable subsets or linear spaces. For more details, we refer to [6, 12, 21, 22, 26] and the references therein.

Let us recall for convex functions the following result, which is a direct consequence of a more general statement in [21, Theorem 4.1].

Theorem 4.1. *Let $(E, \|\cdot\|)$ be an arbitrary Banach space and $f : E \to \cup\{+\infty\}$ be any proper convex function and $r \geq 0$. Then, there exists a rich family $\mathcal{R}(r) \subset \mathcal{S}(E)$ such that for every $V \in \mathcal{R}(r)$ and every $v \in V$, denoting $|\cdot|$ the norm induced on V by $\|\cdot\|$, the following holds: there exists $x^* \in \partial f(v)$ with $\|x^*\|_* \leq r$ if and only if there is $w^* \in \partial f_{|V}(v)$ with $|w^*|_* \leq r$, where $f_{|V}$ is the restriction of f to V.*

The following result can follow from [23, Theorem 4.2] and [38, Lemma 3.2]. We provide a more direct proof in the context of convex functions. In the statement and in the rest of the paper we denote $s_{(f)_{|V}}(\cdot)$ the slope of the function $f_{|V}$.

Theorem 4.2. *Let $(E, \|\cdot\|)$ be an arbitrary Banach space and $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be any proper convex function. Then there exists a rich family $\mathcal{R} \subset \mathcal{S}(E)$ on E such that for any $V \in \mathcal{R}$ (endowed with the induced norm) the following equality holds:*

$$s_{(f)_{|V}}(v) = s_f(v), \ \textit{for all } v \in V. \tag{18}$$

Proof. For f as in the statement, let us consider for each $r \geq 0$ with $r \in \mathbb{Q}$ the rich family $\mathcal{R}(r)$ associated to r ensured by Theorem 4.1, and define the family

$$\mathcal{R} := \bigcap_{r\geq 0; r\in\mathbb{Q}} \mathcal{R}(r). \tag{19}$$

It follows that $\mathcal{R}$ is also a rich family as being the intersection of countably many rich families. We will show that $\mathcal{R}$ satisfies the desired property. Indeed, let us consider any $V \in \mathcal{R}$ and $v \in V$. Since the inequality $s_{(f)_{|V}}(v) \leq s_f(v)$ is obvious, see for example [38, Definition 3.2 and Lemma 3.2], so it is enough to justify the reverse inequality. If $s_{(f)_{|V}}(v) = \infty$, then the desired inequality is fixed. Assume that $s_{(f)_{|V}}(v) < \infty$ and choose any decreasing sequence of rational numbers converging to $s_{(f)_{|V}}(v)$, say $\{r_n\}_{n\in\mathbb{N}}$. By the construction of each family $\mathcal{R}(r_n)$ (keep also in mind the inclusion $\mathcal{R} \subset \mathcal{R}(r_n)$), we infer that there are $x_n^* \in \partial f(v)$ such that $\|x_n^*\|_* \leq r_n$, for all $n \in \mathbb{N}$.

We conclude that

$$s_{(f)_{|V}}(v) = \lim_{n\to\infty} r_n \geq \limsup_{n\to\infty} \|x_n^*\|_* \geq s_f(v),$$

and this ends the proof. $\square$

Let us now state another result which follows directly from the arguments in [26, Lemma 1] and [22, Proposition 7].

Proposition 4.3. *Let $(E, \|\cdot\|)$ be a Banach space and let $(S_i)_{i\in I}$ be a countable collection of subsets of E. Then there exists a rich family $\mathcal{R} \subset \mathcal{S}(E)$ on E such that for every $i \in I$ and every $V \in \mathcal{R}$*

$$d(x, S_i) = d(x, S_i \cap V) \text{ for all } x \in V.$$

Moreover, if E is a Cartesian product of Banach spaces $E_1, \dots, E_k$, then $\mathcal{R}$ can be chosen as a rich rectangle-family, that is, formed by elements $V_1 \times \ldots \times V_k$ with $V_i \in \mathcal{S}(E_i)$ for $i = 1, \dots, k$.

Let us derive from Proposition 4.3 a similar feature for epigraphs of functions.

Proposition 4.4. *Let $(E, \|\cdot\|)$ be an arbitrary Banach space and $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions on E. Then, fixing on $E \times \mathbb{R}$ any product norm of the norm $\|\cdot\|$ on E and the absolute value on $\mathbb{R}$, there exists a rich family $\mathcal{R} \subset \mathcal{S}(E)$ on E such that for any $k \in \mathbb{N}$ and any $V \in \mathcal{R}$ (endowed with the induced norm)*

(a) $s_{(f_k)_{|V}}(v) = s_{f_k}(v)$ *for all* $v \in V$,

(b) $d((v,\alpha), \operatorname{epi} s_{(f_k)_{|V}}) = d((v,\alpha), \operatorname{epi} s_{f_k})$ *for all* $(v,\alpha) \in V \times \mathbb{R}$,

(c) *if* $\{v_1, v_2, \ldots\} \subset V$, *then* $\limsup_{k\to\infty} s_{(f_k)_{|V}}(v_k) = \limsup_{k\to\infty} s_{f_k}(v_k)$.

Proof. For every $k \in \mathbb{N}$ let $\mathcal{R}_{k,1}$ be the rich family on E (defined in (19)) associated to the function f_k such that the statement of Theorem 4.2 is valid, that is the equality in (18) holds true for every $V \in \mathcal{R}_{k,1}$ and $v \in V$, whenever f_k is taken instead of f in (18). Let us consider $\mathcal{R}_1 := \cap_{k\in\mathbb{N}} \mathcal{R}_{k,1}$. Let $\widetilde{\mathcal{R}}_2 \subset \mathcal{S}(E\times\mathbb{R})$ be a rich rectangle-family on $E \times \mathbb{R}$ given by Proposition 4.3 associated to the sequence of sets $\operatorname{epi} s_{f_k}$. Let us define $\mathcal{R}_2 := \{V \in \mathcal{S}(E) \mid V \times \mathbb{R} \in \widetilde{\mathcal{R}}_2\}$. It is easy to see that $\mathcal{R}_2$ is a rich family on E. Consider the rich family $\mathcal{R} := \mathcal{R}_1 \cap \mathcal{R}_2$ on E. For each fixed $k \in \mathbb{N}$ and each fixed $V \in \mathcal{R}$ we claim that the following holds

$$d\big((v,\alpha), \operatorname{epi} s_{(f_k)_{|V}}\big) = d\left((v,\alpha), \operatorname{epi} s_{f_k}\right) \text{ for all } (v,\alpha) \in V \times \mathbb{R}.$$

Indeed, the inclusion $V \times \mathbb{R} \in \widetilde{\mathcal{R}}_2$ implies that

$$d\big((v,\alpha), \operatorname{epi} s_{f_k}\big) = d\big((v,\alpha), \operatorname{epi} s_{f_k} \cap (V \times \mathbb{R})\big) \text{ for all } (v,\alpha) \in V \times \mathbb{R}.$$

Moreover, the other inclusion $V \in \mathcal{R}_{k,1}$ gives $(s_{f_k})_{|V} = s_{(f_k)_{|V}}$, hence we conclude that for all $(v,\alpha) \in V \times \mathbb{R}$

$$\begin{aligned} d\big((v,\alpha), \operatorname{epi} s_{f_k}\big) &= d\big((v,\alpha), \operatorname{epi} s_{f_k} \cap (V \times \mathbb{R})\big) = d\big((v,\alpha), \operatorname{epi} (s_{f_k})_{|V}\big) \\ &= d\big((v,\alpha), \operatorname{epi} s_{(f_k)_{|V}}\big). \end{aligned}$$

Therefore, the rich family $\mathcal{R}$ satisfies items (a) and (b) and concludes the proof, since (c) is a direct consequence of (a). $\square$

The next corollary provides a separable reduction characterization of the epigraphical convergence of slopes.

Corollary 4.5. *Let $(E, \|\cdot\|)$ be an arbitrary Banach space, $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions on E and $\psi : E \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a proper function. Then there exists a rich family $\mathcal{R} \subset \mathcal{S}(E)$ on E such that $s_{f_k} \xrightarrow{e} \psi$ if and only if $s_{(f_k)_{|V}} \xrightarrow{e} \psi_{|V}$ for every $V \in \mathcal{R}$ (endowed with the induced norm).*

Proof. Let $\mathcal{N}$ be any product norm on $E \times \mathbb{R}$ of the norm $\|\cdot\|$ on E and the absolute value on $\mathbb{R}$. Let us consider a rich family $\mathcal{R}_1$ on E given by Proposition 4.4 associated to the sequence $\{f_k\}_{k\in\mathbb{N}}$, and let $\mathcal{R}_2$ be a rich rectangle-family on $E\times\mathbb{R}$ given by Proposition 4.3 applied to the constant sequence built from the set $\operatorname{epi}\psi$. Define $\mathcal{R}_3 := \{V \in \mathcal{S}(E) \mid V \times \mathbb{R} \in \mathcal{R}_2\}$ and note that $\mathcal{R}_3$ is a rich family on E, as easily seen. Consider any V in the rich family $\mathcal{R} := \mathcal{R}_1 \cap \mathcal{R}_3$ on E, and observe that

$$s_{(f_k)_{|V}}(v) = s_{f_k}(v) \quad \text{for all } v \in V \text{ and all } k \in \mathbb{N}, \tag{20}$$

$$d((v,\alpha), \operatorname{epi} s_{(f_k)_{|V}}) = d((v,\alpha), \operatorname{epi} s_{f_k}) \quad \text{for all } (v,\alpha)\in V\times\mathbb{R} \text{ and all } k\in\mathbb{N} \tag{21}$$

and

$$d((v,\alpha), \operatorname{epi}\psi_{|V}) = d((v,\alpha), (\operatorname{epi}\psi)\cap(V\times\mathbb{R})) = d((v,\alpha), \operatorname{epi}\psi) \tag{22}$$

for all $(v,\alpha) \in V\times\mathbb{R}$. The equalities in (21) yield

$$\limsup_{k\to\infty} d((v,\alpha), \operatorname{epi} s_{(f_k)_{|V}}) = \limsup_{k\to\infty} d((v,\alpha), \operatorname{epi} s_{f_k}) \quad \text{for all } (v,\alpha)\in V\times\mathbb{R}. \tag{23}$$

Suppose that $s_{f_k} \xrightarrow{e} \psi$. Let us fix $V \in \mathcal{R}$ and any pair $(x,\alpha) \in \operatorname{epi}\psi_{|V}$. It follows from Definition 2.3 and the inequality in (4) that

$$\limsup_{k\to\infty} d((x,\alpha), \operatorname{epi} s_{f_k}) \le d((x,\alpha), \operatorname{epi}\psi).$$

Now using (21), (23) and (22), we easily see that

$$\begin{aligned}\limsup_{k\to\infty} d((x,\alpha), \operatorname{epi} s_{(f_k)_{|V}}) &= \limsup_{k\to\infty} d((x,\alpha), \operatorname{epi} s_{f_k})\\ &\le d((x,\alpha), \operatorname{epi}\psi) = d((x,\alpha), \operatorname{epi}\psi_{|V}) = 0,\end{aligned}$$

which means that $(x,\alpha) \in \operatorname{Li}_{\mathcal{N}} \operatorname{epi} s_{(f_k)_{|V}}$ and, since (x,α) is arbitrary in $\operatorname{epi}\psi_{|V}$, this inclusion yields $\operatorname{epi}\psi_{|V} \subset \operatorname{Li}_{\mathcal{N}} \operatorname{epi} s_{(f_k)_{|V}}$. Let $(y,\beta) \in \operatorname{Ls}_{\mathcal{N}} \operatorname{epi} s_{(f_k)_{|V}}$. Choose an increasing function $j : \mathbb{N} \longrightarrow \mathbb{N}$ and a sequence $\{(y_k,\beta_k)\}_{k\in\mathbb{N}}$ in $V\times\mathbb{R}$ such that $(y_k,\beta_k) \in \operatorname{epi} s_{(f_{j(k)})_{|V}}$ for all $k\in\mathbb{N}$ and $\lim_{k\to\infty}(y_k,\beta_k) = (y,\beta)$. Notice that the epi-convergence $s_{f_k} \xrightarrow{e} \psi$ ensures $s_{f_{j(k)}} \xrightarrow{e} \psi$ (see (2)) and that by (20) we have $s_{(f_{j(k)})_{|V}}(y_k) = s_{f_{j(k)}}(y_k)$ for all k, so we can write

$$\beta = \lim_{k\to\infty} \beta_k \ge \liminf_{k\to\infty} s_{(f_{j(k)})_{|V}}(y_k) = \liminf_{k\to\infty} s_{f_{j(k)}}(y_k) \ge \psi(y) = \psi_{|V}(y),$$

where the equality in the middle of the chain is due to (20) again. Thus we get $(y,\beta) \in \operatorname{epi}\psi_{|V}$, and this gives us the inclusion $\operatorname{Ls}_{\mathcal{N}} \operatorname{epi} s_{(f_k)_{|V}} \subset \operatorname{epi}\psi_{|V}$.

Since both inclusions in (1) are satisfied, we have $\operatorname{epi} s_{(f_k)_{|V}} \xrightarrow{PK} \operatorname{epi} \psi_{|V}$, which by Definition 2.3 yields the implication

$$s_{f_k} \xrightarrow{e} \psi \Longrightarrow s_{(f_k)_{|V}} \xrightarrow{e} \psi_{|V} \text{ for all } V \in \mathcal{R}.$$

Conversely, suppose that $s_{(f_k)_{|V}} \xrightarrow{e} \psi_{|V}$ for all $V \in \mathcal{R}$. Consider any $x \in E$ and any sequence $\{x_k\}_{k\in\mathbb{N}}$ in E with $x_k \to x$ as $k \to \infty$, and by (P_1) choose $V \in \mathcal{R}$ such that $x_k \in V$ for all $k \in \mathbb{N}$ (which implies that $x \in V$). Hence, by (20)

$$s_{(f_k)_{|V}}(x_k) = s_{f_k}(x_k) \text{ for all } k \in \mathbb{N},$$

which implies that $\liminf_{k\to\infty} s_{f_k}(x_k) \geq \psi_{|V}(x) = \psi(x)$. On the other hand, take any $x \in E$ and any $V \in \mathcal{R}$ with $x \in V$. Since $s_{(f_k)_{|V}} \xrightarrow{e} \psi_{|V}$, there exits $x_k \in V$ such that $x_k \to x$ and $\limsup_{k\to\infty} s_{(f_k)_{|V}}(x_k) \leq \psi_{|V}(x)$. Since by (20) we have the equality $s_{(f_k)_{|V}}(x_k) = s_{f_k}(x_k)$ for all $k \in \mathbb{N}$ and $\psi_{|V}(x) = \psi(x)$, we deduce that $\limsup_{k\to\infty} s_{f_k}(x_k) \leq \psi(x)$, which ends the proof. $\square$

The next step requires some features related to rich families in the setting of Asplund spaces. First, for our development we have to recall the result in Theorem 3.1 in [12]. According to its statement and its proof, this Theorem 3.1 in [10, Theorem 3.1], see also [10, Theorem 2.3 (iii)], can be reformulated in our convex setting as follows.

Theorem 4.6. *Let $(E, \|\cdot\|)$ be an Asplund space, let $\|\cdot\|_*$ be the associated dual norm on E^*, and let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Then there exists a rich-rectangle-family $\mathcal{R} \subset \mathcal{S}(E \times E^*)$ such that:*

(i) *$Y_1 \subset Y_2$ whenever $V_1 \times Y_1 \in \mathcal{R}$, $V_2 \times Y_2 \in \mathcal{R}$ and $V_1 \subset V_2$,*

(ii) *for every $V \times Y \in \mathcal{R}$, endowing V with the norm $|\cdot|$ as restriction of $\|\cdot\|$ to V and V^* with the associated dual norm $|\cdot|_*$, the assignment $Y \ni x^* \mapsto x^*|_V \in V^*$ is a surjective linear isometry from $(Y, \|\cdot\|_*)$ onto $(V^*, |\cdot|_*)$,*

(iii) *for every $V \times Y \in \mathcal{R}$ one has*

$$((\partial f(v)) \cap Y)|_V = (\partial f(v))|_V = \partial(f|_V)(v),$$

in fact, if $v^* \in \partial(f|_V)(v)$ there exists a unique $x^* \in \partial f(v) \cap Y$ such that $x^*|_V = v^*$ and $\|x^*\|_* = |v^*|_*$.

Second, we need the following result, which is a direct consequence of a more general result stated in [13, Theorem 13].

Theorem 4.7. *Let $(E, \|\cdot\|)$ be an Asplund space. Then there exists a rich-rectangle-family $\mathcal{R} \subset S(E \times E^*)$ such that the family*

$$\mathcal{R}^E := \{V \in \mathcal{S}(E) \mid V \times Y \in \mathcal{R} \textit{ for some } Y \in \mathcal{S}(E^*)\}$$

on E is rich.

From the above theorem we can deduce the following useful proposition concerning the (CTA) property of sequences of suitable restrictions of convex functions.

Proposition 4.8. *Let $(E, \|\cdot\|)$ be an Asplund space and $\|\cdot\|_*$ be the associated dual norm on E^*. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lower semicontinuous proper convex functions on E for which* (CTA) *holds. Then, there exists a rich family $\mathcal{R}_{CTA}$ on E such that for every $V \in \mathcal{R}_{CTA}$ the* (CTA) *condition is valid on V (endowed with the induced norm) for the sequence $\{(f_k)_{|V}\}_{k\in\mathbb{N}}$.*

Proof. For each $k \in \mathbb{N}$ let us consider the rich family $\mathcal{R}_k$ on $E \times E^*$ given by Theorem 4.6 for the function f_k. Moreover, let us denote $\mathcal{R}_0$ the rich family given by Theorem 4.7. Put

$$\mathcal{R} := \bigcap_{k\in\mathbb{N}\cup\{0\}} \mathcal{R}_k$$

and notice that any element in $\mathcal{R}$ is a rectangle. If $V \times Y \in \mathcal{R}$ and $|\cdot|$ is the norm induced on V by $\|\cdot\|$, then, as a consequence of Theorem 4.6(ii), we have that for every $v^* \in V^*$ and all $\xi \in Y$ such that $v^* = \xi_{|V}$ the following equality $|v^*|_* = \|\xi\|_*$ holds true and at least one $\zeta \in Y$ such that $v^* = \zeta_{|V}$ exists. Consider the class of closed separable subspaces of E defined by

$$\mathcal{R}_{CTA} := \{V \in \mathcal{S}(E) \mid V \times Y \in \mathcal{R} \text{ for some } Y \in \mathcal{S}(E^*)\}.$$

Let us verify that $\mathcal{R}_{CTA}$ is a rich family on E. For any $W \in \mathcal{S}(E)$ there exists $V_0 \in \mathcal{R}_0^E$ such that $W \subset V_0$. Then, by definition of $\mathcal{R}_0^E$ there exists $Y_0 \in \mathcal{S}(E^*)$ such that $V_0 \times Y_0 \in \mathcal{R}_0$. Now, since $\mathcal{R}$ is a rich family, we have that there exists $V \times Y \in \mathcal{R}$ such that $V_0 \times Y_0 \subset V \times Y$, which implies that $W \subset V$ and $V \in \mathcal{R}_{CTA}$, so (P1) is justified.

Regarding (P2), let $V_n \in \mathcal{R}_{CTA}$ for all $n \in \mathbb{N}$ with $V_n \subset V_{n+1}$. By definition of $\mathcal{R}_{CTA}$, for each $n \in \mathbb{N}$ there is $Y_n \in \mathcal{S}(E^*)$ such that $V_n \times Y_n \in \mathcal{R}$. Choosing $k_0 \in \mathbb{N}$ we have $V_n \times Y_n$ and $V_{n+1} \times Y_{n+1}$ in $\mathcal{R}_{k_0}$, hence by Theorem 4.6 we see that $Y_n \subset Y_{n+1}$. It follows with $V := \mathrm{cl}\big(\bigcup_{n\in\mathbb{N}} V_n\big)$ and $Y := \mathrm{cl}\big(\bigcup_{n\in\mathbb{N}} Y_n\big)$ that

$$\bigcup_{n\in\mathbb{N}} (V_n \times Y_n) = \Big(\bigcup_{n\in\mathbb{N}} V_n\Big) \times \Big(\bigcup_{n\in\mathbb{N}} Y_n\Big), \text{ so } \mathrm{cl}\Big(\bigcup_{n\in\mathbb{N}} (V_n \times Y_n)\Big) = V \times Y,$$

thus $V \in \mathcal{R}_{CTA}$ since $\mathrm{cl}\big(\bigcup_{n\in\mathbb{N}}(V_n \times Y_n)\big)$ is in the rich family $\mathcal{R}$. The latter feature translates $(P2)$, confirming with $(P1)$ above that $\mathcal{R}_{CTA}$ is a rich family on E.

Now take any $V \in \mathcal{R}_{CTA}$. We endow V with the norm $|\cdot|$, given by the restriction of $\|\cdot\|$ to V, and denote V^* its topological dual with the associated dual norm $|\cdot|_*$. Fix any $a \in V$, any $a^* \in V^*$, any increasing function $j : \mathbb{N} \to \mathbb{N}$, and any bounded sequence $\{(b_k, b_k^*)\}_{k\in\mathbb{N}}$ in $V \times V^*$ such that $b_k^* \in \partial((f_{j(k)})_{|V})(b_k)$ for all $k \in \mathbb{N}$ and such that

$$\lim_{k\to\infty} \big(|b_k^* - a^*|_*^2 + |b_k - a|^2 + 2\langle b_k^* - a^*, b_k - a\rangle\big) = 0.$$

There exist $Y \in \mathcal{S}(E^*)$ such that $V \times Y \in \mathcal{R}$ and an $\alpha^* \in Y$ for which $(\alpha^*)_{|V} = a^*$. We recall that $V \times Y \in \mathcal{R}_k$, for all $k \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$ there exists a $\beta_k^* \in \partial f_{j(k)}(b_k) \cap Y$ such that $(\beta_k^*)_{|V} = b_k^*$ and $\beta_k^* - \alpha^* \in Y$. Moreover, according to the surjective linear isometry $Y \ni x^* \to x^*_{|V}$ from $(Y, \|\cdot\|_*)$ onto $(V^*, |\cdot|_*)$ we obtain

$$|b_k^* - a^*|_* = |(\beta_k^* - \alpha^*)_{|V}|_* = \|\beta_k^* - \alpha^*\|_*,$$
$$|b_k - a| = \|b_k - a\|,$$
$$\langle b_k^* - a^*, b_k - a\rangle = \langle \beta_k^* - \alpha^*, b_k - a\rangle,$$

which implies that

$$\lim_{k\to\infty}\left(\|\beta_k^*-\alpha^*\|_*^2+\|b_k-a\|^2+2\langle\beta_k^*-\alpha^*,b_k-a\rangle\right)=0.$$

Since the (CTA) condition holds on E for the sequence $\{f_k\}_{k\in\mathbb{N}}$, we deduce that $\{b_k\}_{k\in\mathbb{N}}$ has a subsequence converging in E, hence in V according to the closedness of V. This concludes the proof. □

We are now in a position to establish a first theorem of equivalence between convergences of subdifferentials and slopes in any Asplund space, whenever (CTA) condition is assumed. The theorem does not require a priori properties of the limits involved in assertions (a) and (b).

Theorem 4.9. *Let* $(E,\|\cdot\|)$ *be an Asplund space, let* $\{f_k\}_{k\in\mathbb{N}}$ *be a sequence of lsc proper convex functions on E such that that* (CTA) *with respect to* $\|\cdot\|$ *is valid. Let* $|\!|\!|\cdot|\!|\!|$ *be any norm on E equivalent to* $\|\cdot\|$ *and* t_{f_k} *be the slope of* f_k *with respect to this norm* $|\!|\!|\cdot|\!|\!|$*. Consider the following properties.*

(a) $\{\partial f_k\}_{k\in\mathbb{N}}$ *converges in a bounded mixed way.*

(b) $\{t_{f_k}\}_{k\in\mathbb{N}}$ *epigraphically converges to some function.*

(b') $\{t_{f_k}\}_{k\in\mathbb{N}}$ *epigraphically converges to some function* ψ *such that there is a sequence* $\{c_k\}_{k\in\mathbb{N}}$ *for which*

$$\lim_{k\to\infty}\psi(c_k)=0 \quad \textit{and} \quad +\infty>\sum_{k=1}^{\infty}(\psi(c_k)+\psi(c_{k+1}))\|c_k-c_{k+1}\|.$$

Then one has (a)⇒(b) *and* (b')⇒(a).

Moreover, in the implication (a)⇒(b) (*respectively* (b')⇒(a)) *the limit in* (b) (*respectively in* (a)) *is given by* s_f (*respectively* ∂f) *for some lower semicontinuous convex function* $f:E\to\mathbb{R}\cup\{+\infty\}$*. If only the condition* (b') *is assumed, then the above lower semicontinuous convex function* f *is bounded from below, and also* $\psi\equiv t_f$*.*

Proof. First, assume that (a) of the theorem holds. We infer from Proposition 2.6(b) the existence of a lower semicontinuous convex function $f:E\longrightarrow\mathbb{R}\cup\{+\infty\}$ such that $\partial f_k \xrightarrow{b\mu} \partial f$. Thus Theorem 3.1(b) yields the statement in (b) of the present theorem.

Now, let us assume (b'), so in particular $t_{f_k}\xrightarrow{e}\psi$. Let us consider a rich family $\mathcal{R}_1$ satisfying the properties of Proposition 4.4 and Corollary 4.5 simultaneously with slopes with respect to the norm $|\!|\!|\cdot|\!|\!|$, see (14). By Proposition 4.8 let $\mathcal{R}_{CTA}$ be a rich family on E such that for every $V\in\mathcal{R}_{CTA}$ the (CTA) condition is valid on V for the sequence $\{(f_k)_{|V}\}_{k\in\mathbb{N}}$.

Consider $V_0\in\mathcal{R}_1$ such that $\{c_1,c_2,\ldots\}\subset V_0$ for the c_k given in (b'), so we can define the rich family

$$\mathcal{R}=\{V\in\mathcal{R}_1\cap\mathcal{R}_{CTA}\mid V_0\subset V\}. \tag{24}$$

Take any $V\in\mathcal{R}$. Since $V\supset V_0$ we have $c_k\in V$ for all $k\in\mathbb{N}$ and $\lim_{k\to\infty}\psi_{|V}(c_k)=0$ and $+\infty>\sum_{k=1}^{\infty}(\psi_{|V}(c_k)+\psi_{|V}(c_{k+1}))\|c_k-c_{k+1}\|$. Further, Corollary 4.5 ensures that $t_{(f_k)_{|V}}\xrightarrow{e}\psi_{|V}$.

Then, by Theorem 3.3 there exists a lower semicontinuous convex function bounded from below $\phi_V : V \to \mathbb{R} \cup \{+\infty\}$ for which $t_{(f_k)_{|V}} \xrightarrow{e} s_{\phi_V}$, and hence (see Theorem 3.3) for which the three (equivalent) conditions hold:

(i) $\partial (f_k)_{|V} \xrightarrow{b\mu} \partial \phi_V$;

(ii) $\partial (f_k)_{|V} \xrightarrow{PK} \partial \phi_V$;

(iii) $t_{(f_k)_{|V}} \xrightarrow{e} t_{\phi_V}$.

Claim. We have that $\partial f_k \xrightarrow{PK} A$ and $\partial f_k \xrightarrow{b\mu} A$, for some operator $A : E \rightrightarrows E^*$.

Indeed, it is enough for us to show that $\mathcal{A}_s(\{\partial f_k\}_{k\in\mathbb{N}}) \subset \mathrm{Li}(\{\partial f_k\}_{k\in\mathbb{N}})$. Fix any pair $(x, x^*) \in \mathcal{A}_s(\{\partial f_k\}_{k\in\mathbb{N}})$. It follows from the definition that there exists an increasing function $j : \mathbb{N} \to \mathbb{N}$ and $x_k^* \in \partial f_{j(k)}(x_k)$ such that $x_k \to x$ and

$$x^* \in \bigcap_{k\in\mathbb{N}} \mathrm{cl}^{w^*} \{x_k^*, x_{k+1}^*, \ldots\}. \tag{25}$$

Now, let us consider the functions $\tilde{f}_k := f_k - x^*$ and $\tilde{\phi}_V := \phi_V - x^*_{|V}$. The inclusion in (25) yields

$$0 \in \bigcap_{k\in\mathbb{N}} \mathrm{cl}^{w^*} \{z_k^*, z_{k+1}^*, \ldots\}, \tag{26}$$

where $z_k^* := x_k^* - x^* \in \partial f_{j(k)}(x_k) - x^*$. On the other hand, it follows from the equalities

$$\partial((\tilde{f}_k)_{|V}) = \partial((f_k)_{|V}) - x^*_{|V} \text{ for all } k \in \mathbb{N}$$
$$\partial\tilde{\phi}_V = \partial\phi_V - x^*_{|V} \text{ for all } V \in \mathcal{R}$$

and the items (i) and (ii) above, that for every $V \in \mathcal{R}$

(i') $\partial (\tilde{f}_k)_{|V} \xrightarrow{b\mu} \partial \tilde{\phi}_V$;

(ii') $\partial (\tilde{f}_k)_{|V} \xrightarrow{PK} \partial \tilde{\phi}_V$.

Now, for any $r \geq 0$ and $k \in \mathbb{N}$ consider a rich family $\mathcal{R}_{2,k}(r)$ on E given by Theorem 4.1 employed with the function $\tilde{f}_k$. Then, let us shrink the rich family $\mathcal{R}$, defined in (24), by considering the new rich family

$$\widetilde{\mathcal{R}} := \mathcal{R} \cap \bigcap_{r\geq 0,\ r\in\mathbb{Q},\ k\in\mathbb{N}} \mathcal{R}_{2,k}(r).$$

Let us fix $V \in \widetilde{\mathcal{R}}$ such that $\{x_k \,|\, k \in \mathbb{N}\} \subset V$. It follows from (26) and (i') that $0 \in \partial\tilde{\phi}_{|V}(x)$. Then, using item (ii') there exists w_k and $w_k^* \in \partial((\tilde{f}_k)_{|V})(w_k)$ such that $w_k \to x$ and $|w_k^*|_* \to 0$, where $|\cdot|_*$ is the dual norm of the norm $|\cdot|$ induced on V by the norm $\|\cdot\|$.

For each $k \in \mathbb{N}$ choose $r_k \in \mathbb{Q}$ with $|w_k^*|_* \leq r_k < |w_k^*|_* + 1/k$. Since $V \in \mathcal{R}_{2,k}(r_k)$, it follows that there exists $v_k^* \in \partial\tilde{f}_k(w_k) = \partial f_k(w_k) - x^*$ such that $\|v_k^*\|_* \leq r_k$.

We derive the existence of some $x_k^* \in \partial f_k(w_k)$ such that $\|x_k^* - x^*\|_* \to 0$, and this ensures the existence of the sequence $(w_k, x_k^*) \in \operatorname{gph}\partial f_k$ such that $(w_k, x_k^*) \to (x, x^*)$, so that $\mathcal{A}_s(\{\partial f_k\}_{k\in\mathbb{N}}) \subset \operatorname{Li}_{d\times d^*} \operatorname{gph}\partial f_k$. This finishes the arguments confirming the claim, which itself translates the implication (b')⇒(a).

Further, considering the operator A given by the claim it follows from Proposition 2.6(b) that there exists a lower semicontinuous convex function $f : E \to \mathbb{R}\cup\{+\infty\}$ such that $A = \operatorname{gph}\partial f$. Using Theorem 3.1(b), we have that $t_{f_k} \to t_f$, so by uniqueness of the epigraphical limit, we get that $t_f = \psi$, which implies that f must be proper as well as bounded from below, see Lemma 2.7 and (15). That ends the proof of the theorem. □

Under the same condition (CTA) we also have the following second theorem on the equivalence between convergences of subdifferentials and slopes in any Asplund space.

Theorem 4.10. *Let $(E, \|\cdot\|)$ be an Asplund and let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of lsc proper convex functions on E for which the* (CTA) *condition with respect to $\|\cdot\|$ is fulfilled. Let $|||\cdot|||$ be any norm on E equivalent to $\|\cdot\|$ and let f be an lsc proper convex function on E bounded from below. Let t_{f_k} and t_f be the slopes of f_k and f respectively with respect to the norm $|||\cdot|||$ (see* (14)*). Then, the following statements are equivalent:* (a) $\partial f_k \xrightarrow{PK} \partial f$; (b) $\partial f_k \xrightarrow{b\mu} \partial f$; (c) $t_{f_k} \xrightarrow{e} t_f$.

Proof. The equivalence (a)⇔(b) is provided by Proposition 2.6 and the implication (b)⇒(c) is a consequence of Theorem 3.1.

To prove the final implication (c)⇒(b), assume that (c) holds. This assumption (c) and Proposition 2.8 ensure that (b') in Theorem 4.9 is satisfied with $\psi := t_f$. As a direct consequence the implication (b')⇒(a) in Theorem 4.9 along with the additional features therein entail that there is a lower semicontinuous convex function $\phi : E \to \mathbb{R}\cup\{+\infty\}$ such that $t_{f_k} \xrightarrow{e} s_\phi$ and $\partial f_k \xrightarrow{b\mu} \partial\phi$. The epigraphical convergence of $\{t_{f_k}\}_{k\in\mathbb{N}}$ to both t_ϕ and t_f assures us that $t_\phi = t_f$. Using this and the boundedness from below of the lower semicontinuous convex function f, Theorem 2.9 says that ϕ and f are equal up to an additive constant, so $\partial\phi = \partial f$, hence $\partial f_k \xrightarrow{b\mu} \partial f$. The proof is finished. □

The proofs of Theorem 4.9 and Theorem 4.10 clearly reveal that they still hold true for any sequence $\{f_k\}_{k\in\mathbb{N}}$ of lsc convex functions on a Banach space E possessing therein the (CTA) condition and for which there is a rich family $\mathcal{R}_{CTA}$ on E such that for every $V \in \mathcal{R}_{CTA}$ the (CTA) is valid on V for the sequence of restrictions $\{(f_k)_{|V}\}_{k\in\mathbb{N}}$. This leads to consider in addition to (CTA) the following stronger compactness type condition (SCTA) enjoying the particularity that every Banach space has a rich family $\mathcal{R}_{SCTA}$ such that every subspace from $\mathcal{R}_{SCTA}$ inherits the (SCTA) property, see Proposition 4.4(c).

We say that a sequence of lsc proper convex functions $\{f_k\}_{k\in\mathbb{N}}$ on a Banach space E possesses the *stronger compactness type assumption* (SCTA) provided that:

(SCTA) For every increasing function $j : \mathbb{N} \to \mathbb{N}$ and every bounded sequence $\{(x_k, x_k^*)\}_{k\in\mathbb{N}}$, such that $(x_k, x_k^*) \in \operatorname{gph}\partial f_{j(k)}$ for all $k \in \mathbb{N}$, the sequence $\{x_k\}_{k\in\mathbb{N}}$ admits a subsequence converging strongly to some $x \in E$.

Clearly, given a sequence $\{f_k\}_{k\in\mathbb{N}}$ of lsc proper convex functions on a Banach space E, the property (CTA) holds true whenever (SCTA) is satisfied. According to what said above, repeating reasoning from the proof of Theorem 4.9 or Theorem 4.10 with (SCTA) instead of (CTA) we get the following one.

Theorem 4.11. *Let* $(E, \|\cdot\|)$ *be a Banach space and let* $\{f_k\}_{k\in\mathbb{N}}$ *be a sequence of lsc proper convex functions on* E *such that that* (SCTA) *is valid. Let* $|\!|\!|\cdot|\!|\!|$ *be any norm on* E *equivalent to* $\|\cdot\|$ *and let* t_{f_k} *be the slope of* f_k *with respect to this norm* $|\!|\!|\cdot|\!|\!|$ *(see* (12)*).*

(I) *The implications* (a)⇒(b) *and* (b')⇒(a) *hold for the following assertions:*

(a) $\{\partial f_k\}_{k\in\mathbb{N}}$ *converges in a bounded mixed way;*

(b) $\{t_{f_k}\}_{k\in\mathbb{N}}$ *epigraphically converges to some function;*

(b') $\{t_{f_k}\}_{k\in\mathbb{N}}$ *epigraphically converges to some function* ψ *such that there is a sequence* $\{c_k\}_{k\in\mathbb{N}}$ *for which*

$$\lim_{k\to\infty}\psi(c_k) = 0 \quad \textit{and} \quad +\infty > \sum_{k=1}^{\infty}(\psi(c_k)+\psi(c_{k+1}))\|c_k - c_{k+1}\|.$$

(II) *Let* f *be an lsc proper convex function on* E *bounded from below. Then, the following assertions are equivalent:*

(a) $\partial f_k \xrightarrow{PK} \partial f$*;* (b) $\partial f_k \xrightarrow{b\mu} \partial f$*;* (c) $t_{f_k} \xrightarrow{e} t_f$*.*

Remark 4.12. It is worth mentioning that under the assumptions of either Theorem 4.9 or Theorem 4.10, we can easily derive a characterization of other convergences of functions, for instance, the Mosco convergence of a sequence of lsc proper convex functions $\{f_k\}_{k\in\mathbb{N}}$ to f (see, e.g., [3, Definition 3.17] or [3, Definition 7.4.6]). This is achieved in reflexive Banach spaces through Theorem 4.9 or 4.10 and Attouch's Theorem (see, e.g., [2, Theorem 3.66]), by means of the epigraphical convergence of slopes $\{s_{f_k}\}_{k\in\mathbb{N}}$ to s_f, with the addition of a normalization condition. Furthermore, since the results of Theorems 4.9, 4.10 and 4.11 are stated for diverse general Banach spaces, they can also be connected to the concept of *slice convergence*, using the result stated in [10, Theorem 2.5 and Corollary 2.7]. Similarly, the convergence of slopes can be related to other results through this type of connection; for more details, see [11, 42] and the references therein.

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