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Approximation Approach to Holditch's Theorem

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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Due to recent results by the authors, Holditch's original proof of his eponymous theorem is effective when applied to a sufficiently smooth convex curve with positive curvature, since such curves were shown to possess a smooth envelope generated by a traveling chord. In the present work, we smoothly approximate a convex curve satisfying only a mild local obtuseness condition, and thereby derive Holditch's theorem for such curves without requiring them to possess an envelope.

Keywords: Holditch's theorem, convex analysis, Broman's condition, local obtuseness, smooth approximation.

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1. Introduction and background

Throughout this article, $\mathbb{C}$ will denote a convex curve in $\mathbb{R}^2$, meaning that $\mathbb{C}$ is planar curve enclosing a compact convex region $\widehat{\mathbb{C}}$. A function $C : \mathbb{R} \to \mathbb{R}^2$ provides a parametrization. We assume that

(i) $C(\cdot)$ is continuous.

(ii) $C(t)$ moves counterclockwise as t increases.

(iii) $C(t) = C(t + T)$ for all $t \in \mathbb{R}$. In particular, one complete traverse is made for $0 \le t < T$, and $C(0) = C(T)$.

(iv) If t_1 and t_2 are distinct values in $[0, T)$, then $C(t_1) \neq C(t_2)$.

Since $\mathbb{C}$ is rectifiable, one such parametrization is given by arclength, but we do not insist on this choice.

We will make reference to the following hypothesis which makes precise the notion of a chord of length $L > 0$ which travels "properly" around the curve $\mathbb{C}$.

Good Travel Hypothesis $\mathbb{GT}(L)$: There exists a function $B : \mathbb{R} \to \mathbb{R}^2$ with the following properties:

(a) $B(\cdot)$ satisfies properties (i)-(iv) as does $C(\cdot)$.

(b) The *chordlength* $\|C(t) - B(t)\| = L,\ \forall t \in \mathbb{R}$.

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Here $\|\cdot\|$ denotes the Euclidean norm. Notationally, $L = a + b$, and D denotes the dividing point of the traveling chord $\overline{CB}$ so that $\|\overline{DB}\| = a$ and $\|\overline{CD}\| = b$. We regard B as the head of the chord, and C as the tail.

Holditch's Theorem [5] in its original 1858 form lacked specific hypotheses and a cogent proof. The following modern version, due to A. Broman, follows from Theorem 1 in [3]. It is a consequence of a more general result proven with line integral methods, and not related to Holdith's original technique.

Theorem 1.1. *Suppose that $\mathbb{GT}(L)$ holds and that the locus of D is a Jordan curve. Then the area of the region between the curve $\mathbb{C}$ and locus of D is equal to πab.*

We call the locus of any dividing point D of the traveling chord $\overline{CB}$ a *Holditch curve.* Note that the Jordan property permits us to speak of the region between a Holditch curve and the curve $\mathbb{C}$.

We now recall two geometric conditions utilized by the authors in [6] when deriving smoothness and convexity properties of Holditch curves.

The first of these is *Broman's condition*[1]:

$\mathbb{P}(\widehat{L})$**:** For every $L < \widehat{L}$, every circle centered on $\mathbb{C}$ with radius L intersects $\mathbb{C}$ at precisely two points.

When there is such an $\widehat{L}$, we simply say that Broman's condition holds. In [3], Broman proved that this condition is sufficient for good chord travel for every $0 < L < \widehat{L}$, and in [6] it was proven that good chord travel for L^* implies $P(\widehat{L})$ holds for some $0 < \widehat{L} < L^*$ and that all Holditch curves are Jordan for sufficiently small L.

The other geometric condition is of a different type. For a given $C \in \mathbb{C}$, the *normal cone* to $\widehat{\mathbb{C}}$ is given by

$$N_{\widehat{\mathbb{C}}}(C) := \{\zeta \in \mathbb{R}^2 : \langle \zeta, W - C\rangle \le 0,\ \forall W \in \widehat{\mathbb{C}}\},$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product. We say that $N_{\widehat{\mathbb{C}}}(C)$ is *acute* provided that

$$\langle \zeta_1, \zeta_2\rangle > 0, \quad \forall\, \zeta_1, \zeta_2 \in N_{\widehat{\mathbb{C}}}(C)\backslash\{0\}. \tag{1}$$

In other words, the maximum angle subtended by the normal cone at C is less than $\frac{\pi}{2}$ radians. This can be viewed as a *local obtuseness* condition on the curve $\mathbb{C}$.

The needed geometric hypothesis is

Condition $\mathbb{O}_{\mathbb{C}}$: $N_{\widehat{\mathbb{C}}}(C)$ is acute for all $C \in \mathbb{C}$.

In Theorem 16 of [6] it was proven that when $\mathbb{O}_{\mathbb{C}}$ holds, so does Broman's condition $\mathbb{P}(\widehat{L})$ for a certain $\widehat{L} > 0$. The reverse implication is seen to be false, upon consideration of the unit square with $\widehat{L} \le 1$; then one has good chord travel, but $\mathbb{O}_{\mathbb{C}}$ fails to hold. Also, it is not difficult to show that good chord travel is impossible if, at some point of $\mathbb{C}$, the normal cone is obtuse.

Since Theorem 16 of [6] features in what follows, for convenience, its proof will be provided in the next section.

The above discussion shows that the hypotheses which Broman required in Theorem 1.1 hold when $\mathbb{O}_{\mathbb{C}}$ holds and the chordlength is sufficienctly small.

[1] This condition was incorrectly stated in [6].

In his original 1858 article, Reverend Holditch assumed the existence of an *envelope* $\mathbb{E}$, meaning a curve with the property that the traveling chord is tangent to $\mathbb{E}$ at a unique contact point. Holditch apparently required the envelope to be C^1-regular (where "regular" means nonvanishing derivative) and strictly convex (i.e. containing no line segments). When such an envelope exists, Holditch's proof is not difficult to follow; see Broman [3]. It utilizes the idea of "sweeping tangents" in Mamikon's theorem, as popularized by Apostol and Manatsakanian [1]; in this regard, Holditch appears to have been ahead of his time.

In [7], the authors proved that an envelope as described exists when $\mathbb{C}$ is C^2-regular with positive curvature, and the chordlength L is sufficiently small. So in this case, one can say that Holditch's original method of proof applies. For a more general convex curve $\mathbb{C}$, however, we do not have an envelope at our disposal.

In the present article, we apply results of Azagra and Stolyarov [2] in order to approximate a convex curve $\mathbb{C}$ satisfying the mild condition $\mathbb{O}_{\mathbb{C}}$ with C^∞-regular approximations having positive curvature. We then go on to use the approximations in order to show that Holditch's Theorem holds for $\mathbb{C}$, without concern about the existence of a suitable envelope for $\mathbb{C}$.

1.1. On Broman's condition (former Theorem 16 from [6])

We denote by $\widehat{N}_{\widehat{\mathbb{C}}}(c)$ the set of unit vectors in $N_{\widehat{\mathbb{C}}}(c)$.

Theorem 1.2. *Suppose that condition $\mathbb{O}_{\mathbb{C}}$ holds. Then Broman's condition holds.*

Proof. It is not difficult to see that for every $C \in \mathbb{C}$ and every $\zeta \in \widehat{N}_{\widehat{\mathbb{C}}}(C)$, the acuteness hypothesis on the normal cone implies that the open half-line

$$L_{C,\zeta} := \{C - \alpha\zeta : \alpha > 0\}$$

intersects $\mathbb{C}$ at precisely one point $P(C,\zeta) = C - \alpha(C,\zeta)\zeta$.

As well, the relative interior of the line segment $\overline{C, P(C,\zeta)}$, which is of length $\alpha(C,\zeta) > 0$, is contained in the interior of $\widehat{\mathbb{C}}$. Also, as is readily noted, the function $P(C,\cdot) : \widehat{N}_{\widehat{\mathbb{C}}}(C) \to \mathbb{C}$ is continuous, and therefore the function $\alpha(C,\cdot) : \widehat{N}_{\widehat{\mathbb{C}}}(C) \to \mathbb{R}$ is continuous as well. Hence for any $C \in \mathbb{C}$ we can define

$$m(C) := \min\{\alpha(C,\zeta) : \zeta \in \widehat{N}_{\widehat{\mathbb{C}}}(C)\} > 0.$$

We now claim that the function $m : \mathbb{C} \to \mathbb{R}$ is lower semicontinuous. To this end, let C_i be any sequence in $\mathbb{C}$ converging to C and assume that $m(C_i)$ is convergent.

We are to show that $$\lim_{i\to\infty} m(C_i) \geq m(C).$$

For each i, choose $\zeta_i \in \widehat{N}_{\widehat{\mathbb{C}}}(C_i)$ such that $m(C_i) = \alpha(C_i,\zeta_i)$. Taking a subsequence, we may assume that the sequence of unit vectors ζ_i converges, say to a unit vector ζ. We have

$$m(C_i)\zeta_i = C_i - P(C_i,\zeta_i) \to [\lim_{i\to\infty} m(C_i)]\zeta.$$

Taking a further subsequence, we may assume that $P(C_i,\zeta_i) \to P \in \mathbb{C}$.

The multifunction $\widehat{N}_{\widehat{\mathbb{C}}}(\cdot) : \mathbb{C} \to \overline{B}$ possesses the closed graph property, see, for example, Clarke, Ledyaev Stern and Wolenski, [4].

Hence $\zeta \in \widehat{N}_{\widehat{\mathbb{C}}}(C)$, and therefore $\lim_{i\to\infty} m(C_i) = \alpha(C,P)$. But $\alpha(C,\zeta) \geq m(C)$. Hence the function $m(\cdot)$ is lower semicontinuous, as claimed. It follows that the minimum of $m(\cdot)$ on $\mathbb{C}$ is attained, and is positive. We denote it by $\Lambda_{\mathbb{C}}$.

Figure 1 illustrates the rest of the proof of the theorem. (The figure is compressed vertically. Also, we have drawn the parts of $\mathbb{C}$ shown to appear smooth, but this is not assumed.)

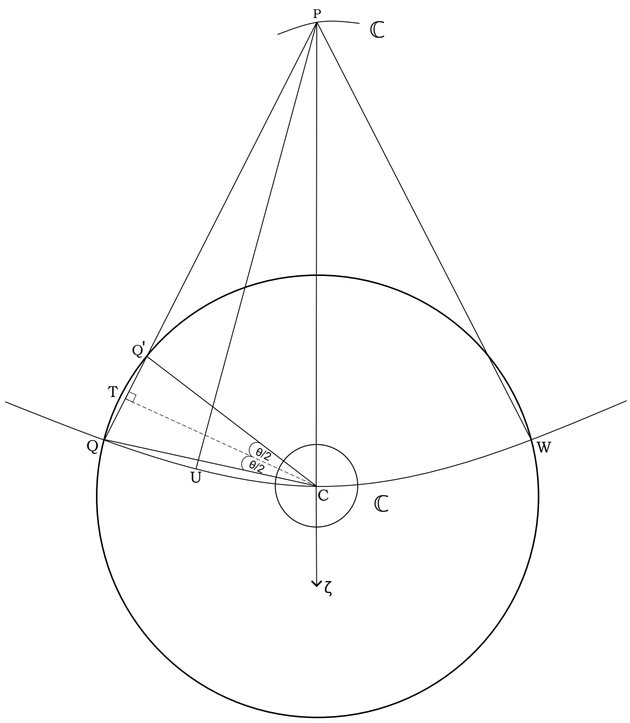

Figure 1: Part of proof of Theorem 1.2

Let $0 < r < \Lambda_{\mathbb{C}}$. For a given $C \in \mathbb{C}$, let ζ be a unit normal to $\mathbb{C}$ at C. (We have oriented the figure so that ζ points vertically downward.) Let $P = P(C,\zeta)$. Now consider a circle of radius r centered at C; this is the larger circle in the figure. Since the relative interior of the line segment $\overline{CP}$ is in the interior of $\widehat{\mathbb{C}}$, there are points Q and W on $\mathbb{C}$ such that the arc QCW contains no other intersection points of the circle with $\mathbb{C}$.

Consider the sector $PQCWP$, which is a convex subset of $\widehat{\mathbb{C}}$, whose boundary has the arc QCW in common with $\mathbb{C}$. Now consider any point $U \in \mathbb{C}$ strictly between Q and W on the arc QCW. One such point U is shown in the simple scenario depicted in the figure; C itself is another. Now note that the relative interior of the line segment $\overline{UP}$ is contained in the interior of $\widehat{\mathbb{C}}$. (If not, the line spanned by U and P would support $\widehat{\mathbb{C}}$ with Q and W on opposite sides of the line, which is impossible.) It follows that the relative interior of $\overline{UP}$ contains no points of $\mathbb{C}$. Hence the interior of the sector $PQCWP$ contains no points of $\mathbb{C}$.

We seek a bound independent of C for the distance from C to the line segment $\overline{QP}$, so that the portion of an open disk of radius less than this bound which intersects the open sector $PQCWP$ contains no points of $\mathbb{C}$. Then any circle of radius less than this bound and centered at C intersects $\mathbb{C}$ at precisely two points. (One such smaller circle is shown in the figure.)

Assume first that the segment $\overline{PQ}$ also intersects the circle at a point Q' between Q and P. It is readily noted that the chord QQ' subtends an angle $\theta \leq \frac{\pi}{2}$ at C. The perpendicular bisector of this chord also bisects θ. We let T denote its intersection with the chord. Then

$$\|\overline{TC}\| = r\cos(\frac{\theta}{2}) \geq r\cos(\frac{\pi}{4}) = \frac{r}{\sqrt{2}}$$

and any radius less than $\frac{\Lambda_{\mathbb{C}}}{\sqrt{2}}$ has the desired property.

If Q and Q' are reversed (that is, Q is closer to P than is Q') or if $Q = Q'$ (that is, $\overline{PQ}$ is tangent to the circle), then the simpler bound that the radius be less than $\Lambda_{\mathbb{C}}$ applies. Hence in all cases, the bound $\frac{\Lambda_{\mathbb{C}}}{\sqrt{2}}$ applies; that is, Broman's condition $\mathbb{P}(\frac{\Lambda_{\mathbb{C}}}{\sqrt{2}})$ holds. □

1.2. Approximation and parametrization

From Azagra and Stolyarov, [2], we have that for any given $\varepsilon > 0$ there exists a C^∞ function $g : \mathbb{R}^2 \to \mathbb{R}$ with positive definite Hessian everywhere (i.e. g is *strongly convex*) such that there exists $r > \inf g$ for which

$$K_\varepsilon := \{z : g(z) = r\} \subset \{\widehat{\mathbb{C}} + \varepsilon B\} \backslash \widehat{\mathbb{C}}$$

where B denotes the open unit ball.

Since the epigraph of g is *strictly convex* (i.e. contains no line segments), so is $\widehat{K}_\varepsilon$, the convex region enclosed by the curve K_ε.

Also, the gradient $\nabla g \neq 0$ on K_ε, since K_ε contains no minimizers of g. It follows that K_ε is a C^∞ manifold.

We also note that strong convexity of g implies that the graph of g has positive Gauss curvature. This in turn implies that the curve K_e has positive curvature.

Consider the Gauss Map $\quad \theta(z) := \dfrac{\nabla g(z)}{\|\nabla g(z)\|} : K_\varepsilon \to S^1$

given by $\quad \theta(z) = (\cos\theta, \sin\theta),$

where θ is the angle of $\nabla g(z)$ measured counterclockwise from the nonnegative x-axis. Here $\|\cdot\|$ is the Euclidean norm and S^1 is the unit circle. This map is clearly surjective, and strict convexity implies that it is injective as well. In fact, it is a C^∞ diffeomorphism between K_ε and S^1.

The Inverse Gauss Map $S^1 \to K_\varepsilon$ is specified by the unique point $z(\theta) \in K_\varepsilon$ where $\nabla g(z)$ subtends an angle θ with the nonnegative x-axis, measured counterclockwise. It follows that θ can be used to parametrize the curve K_ε; that is, $z = z(\theta)$ along the curve. This parametrization gives counterclockwise motion as θ increases as can be seen by locally viewing $K\varepsilon$ as the graph of a smooth convex function and applying monotonicity of its derivative.

1.2.1. Relationship of the parametrization with the support function

Denote the *support function* of K_ε by

$$H(p,q) := \max\{\langle (p,q),(x,y)\rangle : (x,y) \in K_\varepsilon\}.$$

Then, as a function on S^1,

$$h(\theta) := H(\cos\theta, \sin(\theta) = x(\theta)\cos\theta + y(\theta)\sin\theta,$$

where $z(\theta) = (x(\theta), y(\theta))$. In view of the fact that

$$x'(\theta)\cos\theta + y'(\theta)\sin\theta = 0,$$

we obtain
$$h'(\theta) = -x(\theta)\sin\theta + y(\theta)\cos\theta.$$

From this[2] we see that h is a C^∞ function, and it is not hard to show that the above parametrization and its derivatives can be completely expressed in terms of the support function of K_ε and its derivatives. In particular, we have

$$z(\theta) = h(\theta)(\cos\theta, \sin\theta)) + h'(\theta)(-\sin\theta, \cos\theta)$$

and
$$z'(\theta) = (h(\theta) + h''(\theta))(-\sin\theta, \cos\theta)).$$

One can also show that the quantity $h(\theta) + h''(\theta)$ is the radius of curvature of K_ε at $z(\theta)$.

2. Main results

Let $z_i \in K_{\varepsilon_i}$, where $\varepsilon_i \downarrow 0$, and let $\zeta_i \in \widehat{N}_{\widehat{K}_{\varepsilon_i}}(z_i)$. By compactness, for a subsequence (not relabeled), we have $z_i \to z \in \mathbb{C}$ and $\zeta_i \to \zeta$.

Lemma 2.1. *For a limiting ζ as above, we have $\zeta \in \widehat{N}_{\widehat{\mathbb{C}}}(z)$.*

Proof. If $\zeta \notin \widehat{N}_{\widehat{\mathbb{C}}}(z)$, then there exists $w \in \widehat{\mathbb{C}}$ such that $\langle \zeta, w - z\rangle > 0$. But then $\langle \zeta_i, w - z_i\rangle > 0$ for large i. Since $w \in \widehat{K}_{\varepsilon_i}$, this contradicts $\zeta_i \in \widehat{N}_{\widehat{K}_{\varepsilon_i}}(z_i)$. □

Now, we can prove the following:

Theorem 2.2. *Suppose that condition $\mathbb{O}_{\mathbb{C}}$ holds. Then there exists $M > 0$ such that $\Lambda_{K_{\varepsilon_i}} \geq M$ for i sufficiently large.*

Proof. Suppose by way of contradiction that there are sequences $z_i^* \in K_{\varepsilon_i}$ and $\zeta_i^* \in \widehat{N}_{K_{\varepsilon_i}}(z_i^*)$ such that $z_i^* \to z \in \mathbb{C}$ and $\zeta_i^* \to \zeta \in \widehat{N}_{\mathbb{C}}(z)$, while

$$\Lambda_{K_{\varepsilon_i}} = \|z_i^* - P_{z_i^*,\zeta_i^*}\| \to 0. \tag{2}$$

Now refer to Figure 2[3]. In view of Theorem 1.2, we can inscribe a ball centered on the line segment from z to $P_{z,\zeta}$ and contained in interior of $\widehat{\mathbb{C}}$. For large i, the line segment from z_i^* to $P_{z_i^*,\zeta_i^*}$ intersects this ball. But, then, (2) is violated. □

[2] Cognoscenti will note that this also follows from results in Chapter 25 of Rockafellar [8] (see also Schneider [9]) since the strict convexity of K_ε implies that ∇H exists on $\mathbb{R}^2\backslash\{(0,0)\}$ and in particular $\nabla H(\cos\theta, \sin\theta) = z(\theta)$.

[3] We would like to thank MSc. student Fadia Ounissi for drawing the figures.

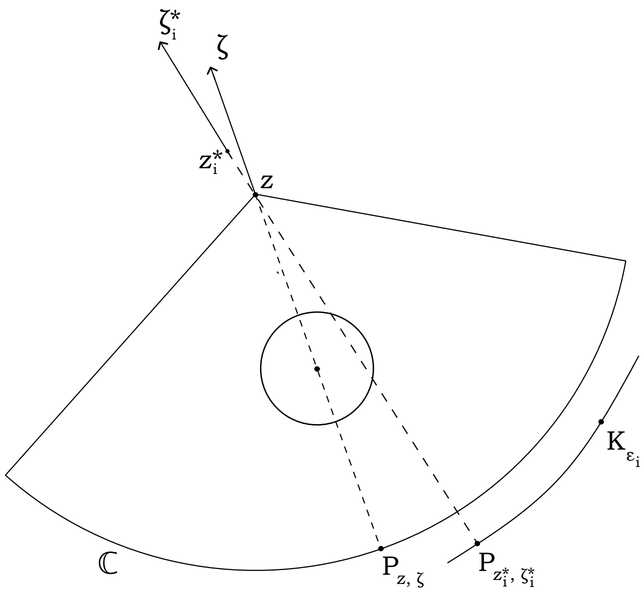

Figure 2: Proof of Theorem 2.2

It follows that we have good chord travel for a *common* chordlength $\widehat{L}$ for $\mathbb{C}$ and K_{ve_i} for all sufficiently large i.

Now we re-parametrize $\mathbb{C}$ as follows, using (for each such i) the parametrization of K_{ε_i} in terms of θ, as described in Subsection 2.2. We can write

$$K_{\varepsilon_i} = \{(z_i(\theta) : 0 \leq \theta < 2\pi\}.$$

Now, for each θ, let $\hat{z}_i(\theta)$ denote the unique point in $\widehat{\mathbb{C}}$ closest to $z_i(\theta)$.

We introduce the idea of *weak* good chord travel. This means that we allow *stalling* in the motion of $\hat{z}_i(\theta)$, in the sense that we may have $\hat{z}_i(\theta)$ constant on an interval of θ. This occurs when a point $\hat{z}_i(\theta)$ is closest to multiple points in K_{ε_i}, as when $\hat{z}_i(\theta)$ is a corner of $\widehat{\mathbb{C}}$.

It is clear that every point of $\mathbb{C}$ is accounted for in this parametrization, and while stalling is possible, we have that "backing up" or clockwise motion cannot occur. This is seen as follows, where $h > 0$ is small: For points $P = z_i(\theta)$, $Q = \hat{z}_i(\theta)$, $P' = z_i(\theta + h)$, $Q' = \hat{z}_i(\theta + h)$, Q' could not correspond to clockwise movement from Q, because this would imply, by convexity of $\mathbb{C}$, that $\|P'Q\| < \|P'Q'\|$, contradicting the fact that Q' is closest to P'.

Fix a chord $\overline{CB}$ of length $\widehat{L}$, and let $C_i \in K_{\varepsilon_i}$ converge to C. (Such a sequence can be constructed by intersecting the curves K_{ε_i} with a fixed normal direction to $\widehat{\mathbb{C}}$ at C.) By Broman's condition, there is a unique $B_i \in K_{\varepsilon_i}$ so that the chord $\overline{C_iB_i}$ has length $\widehat{L}$ with B_i is oriented counterclockwise with respect to C_i. We claim that $B_i \to B$. If not, a subsequence (not relabeled) could be extracted such that $B_i \to \tilde{B} \neq B$, with the chord $\overline{C\tilde{B}}$ having length $\widehat{L}$ with $\tilde{B}$ oriented counterclockwise with respect to C. But by Broman's condition, B is the only such point, verifying the claim.

At this point, one could formulate a reasonable approximate version of Holditch's theorem, by comparing the locus of the dividing point of the chord's motion around a fixed K_{ε_i} with motion around C. However, one can do better by noting that the estimates in [7] provide for a uniform chordlength guaranteeing Jordan dividing point loci and the existence of a smooth envelope as described above, for all the approximations K_{ε_i} where ε_i is small. Hence we have rescued Holditch's original approach for almost general convex curves. We summarize as follows:

Theorem 2.3. *Let $\mathbb{C}$ be a convex planar curve satisfying condition $\mathbb{O}_{\mathbb{C}}$. Then for sufficiently small chordlength, the locus of any dividing point D is a Jordan curve, and the area of the region between the curve $\mathbb{C}$ and locus of D is πab.*

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