Journal of Convex Analysis
Volume 32 (2025), No. 3, 757–766

Existence of a Fixed Point for Boyd-Wong Type Contractions on Metric Spaces with Graphs

Simeon Reich, Alexander J. Zaslavski
Department of Mathematics, The Technion – Israel Institute of Technology, Haifa, Israel
sreich@technion.ac.il, ajzasl@technion.ac.il

Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

Received: July 25, 2024
Accepted: October 11, 2024

We study Boyd-Wong type contractions on a complete metric space with a graph and show the existence of a fixed point for such mappings. We pay special attention to convex metric spaces.

Keywords: Complete metric space, contraction, convex metric space fixed point, graph.

2010 Mathematics Subject Classification: 47H09, 47H10, 54E50.

1. Introduction

For more than sixty years now, there has been a lot of research activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) and contractive mappings. See, for example, [7, 8, 9, 13, 14, 16, 22] and references cited therein. This activity stems from Banach's classical theorem [2] concerning the existence of a unique fixed point for a strict contraction. It also concerns the convergence of (inexact) orbits of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, studies of feasibility, common fixed points and optimization problems, which find important applications in engineering, medical and the natural sciences [6, 22]. In particular, the study of nonexpansive and contractive mappings on complete metric spaces with graphs has recently become a rapidly growing area of research. See, for instance, [1, 3, 4, 10, 12, 17, 18, 19, 20, 21]. In [5] D. W. Boyd and J. S. W. Wong introduced an interesting class of mappings which contains Banach contractions (that is, strict contractions) and showed the existence of fixed points for these mappings. Boyd-Wong type contractions were studied in [11, 15, 16]. In the present paper we establish the existence of fixed points for Boyd-Wong type contractions on complete metric spaces with graphs (see Section 3 below). We pay special attention to convex metric spaces (see Section 4 below).

More precisely, assume that (X, ρ) is a complete metric space, $T : X \to X$ is a mapping, and that a function $\phi : [0, \infty) \to [0, \infty)$ is upper semicontinuous from the right at any positive t and satisfies

$$\phi(t) < t \text{ for all } t \in (0, \infty).$$

ISSN 0944-6532 / \$ 2.50

Assume that for each $(x, y) \in E(G)$, where $E(G)$ is the set of edges of a graph G, we have

$$(T(x), T(y)) \in E(G)$$

and

$$\rho(T(x), T(y)) \leq \phi(\rho(x, y)).$$

Then T is called a Boyd-Wong contraction. In the present paper we consider Boyd-Wong type contractions on a complete metric space with a graph and show that the corresponding fixed point problem has a solution. We pay special attention to metrically convex spaces.

2. Convergence results

Assume that (X, ρ) is a complete metric space endowed with a graph G. We denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges. For each mapping $S : X \to X$, set $S^0(x) = x$ for each $x \in X$ and for each integer $i \geq 0$, define $S^{i+1} := S \circ S^i$. Assume that $\phi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous function,

$$\phi(t) < t \text{ for all } t \in (0, \infty), \tag{1}$$

$T : X \to X$ is a mapping and that for each $(x, y) \in E(G)$, we have

$$(T(x), T(y)) \in E(G) \tag{2}$$

and

$$\rho(T(x), T(y)) \leq \phi(\rho(x, y)). \tag{3}$$

Proposition 2.1. *Let $0 < \epsilon < M$ be given. Then there exists a natural number n_0 such that for each $(x, y) \in E(G)$ satisfying*

$$\rho(x, y) \leq M \tag{4}$$

and for each integer $n \geq n_0$, we have $\rho(T^n(x), T^n(y)) \leq \epsilon$.

Proof. Clearly the function $t - \phi(t)$ is lower semicontinuous. Therefore there exists a number $\delta > 0$ such that

$$t - \phi(t) > \delta \text{ for each } t \in [\epsilon, M]. \tag{5}$$

Choose a natural number n_0 such that

$$\delta n_0 > M. \tag{6}$$

Assume that $(x, y) \in E(G)$ satisfies (4). We claim that for each integer $n \geq n_0$,

$$\rho(T^n(x), T^n(y)) \leq \epsilon.$$

In view of (1) and (3), it is sufficient to show that

$$\rho(T^{n_0}(x), T^{n_0}(y)) \leq \epsilon.$$

Suppose to the contrary that this does not hold. Then by (1) and (3),

$$\rho(T^n(x), T^n(y)) > \epsilon,\ n = 0, \dots, n_0. \tag{7}$$

Inequalities (1), (3) and (4) imply that

$$\rho(T^i(x), T^i(y)) \leq M, \; i = 0, 1, \dots . \tag{8}$$

It follows from (3), (5), (7) and (8) that for each integer $n \in \{0, \dots, n_0 - 1\}$,

$$\rho(T^n(x), T^n(y)) - \rho(T^{n+1}(x), T^{n+1}(y))$$
$$\geq \rho(T^n(x), T^n(y)) - \phi(\rho(T^n(x), T^n(y))) > \delta.$$

By the above relation,

$$M > \rho(x, y) - \rho(T^{n_0}(x), T^{n_0}(y))$$
$$= \sum_{i=0}^{n_0-1} (\rho(T^i(x), T^i(y)) - \rho(T^{i+1}(x), T^{i+1}(y))) > \delta n_0$$

and

$$n_0 < M\delta^{-1}.$$

This contradicts (6). The contradiction we have reached proves our claim and Proposition 2.1 itself. □

Proposition 2.1 implies the following result.

Proposition 2.2. *Let $\epsilon \in (0, 1)$, $M > 1$ and let q be a natural number. Then there exists a natural number n_0 such that for each $(x, z) \in E(G)$ for which there exist*

$$\tilde{q} \in \{1, \dots, q\}, \; y_i \in X, \quad i = 0, \dots, \tilde{q},$$

such that

$$y_0 = x, \; y_{\tilde{q}} = z, \; (y_i, y_{i+1}) \in E(G), \; \rho(y_i, y_{i+1}) \leq M, \; i = 0, \dots, \tilde{q} - 1,$$

and for each integer $n \geq n_0$, $\rho(T^n(x), T^n(y)) \leq \epsilon$.

Proposition 2.2 in its turn implies the following theorem.

Theorem 2.3. *Assume that $x \in X$, q is a natural number, $y_i \in X$, $i = 0, \dots, q$,*

$$y_0 = x, \; y_q = T(x_q),$$

and for each $i \in \{0, \dots, q - 1\}$, $(y_i, y_{i+1}) \in E(G)$. Then

$$\lim_{i \to \infty} \rho(T^i(x), T^{i+1}(x)) = 0.$$

3. The main result

Assume that (X, ρ) is a complete metric space endowed with a graph G. We denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges. Assume that $\phi : [0, \infty) \to [0, \infty)$ is a function which is upper semicontinuous from the right at any positive number t,

$$\phi(t) < t \text{ for all } t \in (0, \infty), \tag{9}$$

$T : X \to X$ is a mapping and that for each $(x, y) \in E(G)$, we have

$$(T(x), T(y)) \in E(G) \tag{10}$$

and

$$\rho(T(x), T(y)) \leq \phi(\rho(x, y)). \tag{11}$$

Fix

$$\Delta \in (0, 1).$$

In the sequel we use the following property:

(P) For each $x, y, z \in X$ satisfying $(x, y), (y, z) \in E(G)$ and $\rho(x, y), \rho(y, z) \leq \Delta$, we have $(x, z) \in E(G)$.

Theorem 3.1. *Assume that property (P) holds, $x \in X$, q is a natural number, $y_i \in X$, $i = 0, \dots, q$, for each $i \in \{0, \dots, q-1\}$,*

$$(y_i, y_{i+1}) \in E(G) \tag{12}$$

and that

$$x = y_0,\ T(x) = y_q. \tag{13}$$

Then the sequence $\{T^i(x)\}_{i=0}^{\infty}$ converges to a point $x_* \in X$. If the graph of T is closed, then $T(x_*) = x_*$ and if $E(G)$ is closed, then $(x_*, x_*) \in E(G)$.

Proof. Let $i \in \{0, \dots, q-1\}$. In view of (12), for each integer $n \geq 0$,

$$(T^n(y_i), T^n(y_{i+1})) \in E(G). \tag{14}$$

We claim that

$$\lim_{n\to\infty} \rho((T^n(y_i), T^n(y_{i+1})) = 0. \tag{15}$$

Suppose to the contrary that this is not true. Then in view of (9) and (11), the sequence

$$\{(T^n(y_i), T^n(y_{i+1}))\}_{n=0}^{\infty}$$

is decreasing and there exists

$$\gamma = \lim_{n\to\infty} \rho((T^n(y_i), T^n(y_{i+1})) > 0. \tag{16}$$

It follows from (9), (11) and (16) that the sequence $\{\rho(T^n(y_i), T^n(y_{i+1}))\}_{n=0}^{\infty}$ is strictly decreasing. By our assumptions and (16), the function $t - \phi(t)$ is lower semicontinuous from the right at γ and there exists an integer $n_0 \geq 1$ such that for each integer $n \geq n_0$,

$$\rho(T^n(y_i), T^n(y_{i+1})) - \phi(\rho((T^n(y_i), T^n(y_{i+1}))) > 2^{-1}(\gamma - \phi(\gamma)). \tag{17}$$

Inequalities (11) and (17) imply that for each integer $n \geq n_0$,

$$\begin{aligned}&\rho(T^n(y_i), T^n(y_{i+1})) - \rho(T^{n+1}(y_i), T^{n+1}(y_{i+1}))\\ &\geq \rho(T^n(y_i), T^n(y_{i+1})) - \phi(\rho(T^n(y_i), T^n(y_{i+1})) \geq 2^{-1}(\gamma - \phi(\gamma)).\end{aligned}$$

It follows from the above relation that for each integer $m > n_0$, we have

$$\begin{aligned}\rho(T^{n_0}(y_i), T^{n_0}(y_{i+1})) &\geq \rho(T^{n_0}(y_i), T^{n_0}(y_{i+1})) - \rho(T^m(y_i), T^m(y_{i+1}))\\ &= \sum_{j=n_0}^{m-1} (\rho(T^j(y_i), T^j(y_{i+1})) - \rho(T^{j+1}(y_i), T^{j+1}(y_{i+1}))\\ &\geq 2^{-1}(\gamma - \phi(\gamma))(m - n_0) \to \infty \text{ as } m \to \infty.\end{aligned}$$

The contradiction we have reached proves that (15) does hold after all for each $i \in \{0, \dots, q-1\}$. In view of (12) and (15),

$$\lim_{n\to\infty} \rho(T^n(x), T^{n+1}(x)) = 0. \tag{18}$$

Property (P), (14) and (15) imply that there exists a natural number n_1 such that for each integer $n \geq n_1$,

$$(T^n(x), T^{n+1}(x)) \in E(G). \tag{19}$$

Let $\epsilon \in (0, \Delta)$. Since the function $t - \phi(t)$ is lower semicontinuous from the right at ϵ, there exists

$$\delta \in (0, 4^{-1}(\epsilon - \phi(\epsilon)) \tag{20}$$

such that for each $t \in (\epsilon, \epsilon + \delta]$,

$$t - \phi(t) > 2^{-1}(\epsilon - \phi(\epsilon)). \tag{21}$$

By (18), there exists an integer $n_2 > n_1$ such that for each integer $n \geq n_2$,

$$\rho(T^n(x), T^{n+1}(x)) \leq \delta. \tag{22}$$

Let $j > i \geq n_2$ be integers. We claim that

$$\rho(T^j(x), T^i(x)) \leq \epsilon. \tag{23}$$

Suppose to the contrary that this is not true. Then

$$\rho(T^j(x), T^i(x)) > \epsilon. \tag{24}$$

In view of (22) and (24),

$$j > i + 1. \tag{25}$$

By (22), (24) and (25), we may assume without any loss of generality that

$$\rho(T^s(x), T^i(x)) \leq \epsilon,\ s = i+1, \dots, j-1, \tag{26}$$

and, in particular,

$$\rho(T^i(x), T^{j-1}(x)) \leq \epsilon. \tag{27}$$

Property (P), (19), (22) and (26) imply that

$$(T^i(x), T^s(x)) \in E(G),\ s = i+1, \dots, j. \tag{28}$$

In view of (22), (24) and (27), we find that

$$\epsilon < \rho(T^i(x), T^j(x)) \leq \rho(T^i(x), T^{j-1}(x)) + \rho(T^{j-1}(x), T^j(x)) \leq \epsilon + \delta. \tag{29}$$

By (11), (28), (29) and the choice of δ (see (21)), we have

$$\begin{aligned} &\rho(T^i(x), T^j(x)) - \rho(T^{i+1}(x), T^{j+1}(x)) \\ &\geq \rho(T^i(x), T^j(x)) - \phi(\rho(T^i(x), T^j(x))) > 2^{-1}(\epsilon - \phi(\epsilon)). \end{aligned} \tag{30}$$

It follows from (20), (22) and (30) that

$$\begin{aligned}&\rho(T^i(x), T^j(x))\\&\le \rho(T^i(x), T^{i+1}(x)) + \rho(T^{i+1}(x), T^{j+1}(x)) + \rho(T^{j+1}(x), T^j(x))\\&\le 2\delta + \rho(T^i(x), T^j(x)) - 2^{-1}(\epsilon - \phi(\epsilon)) < \rho(T^i(x), T^j(x)).\end{aligned}$$

The contradiction we have reached proves that (23) does hold for each pair of integers $j > i \ge n_2$, as claimed. Since ϵ is any sufficiently small positive number, we conclude that $\{T^n(x)\}_{n=0}^{\infty}$ is a Cauchy sequence and there exists

$$x_* = \lim_{n\to\infty} T^n(x).$$

Property (P), (19) and (23) imply that for each pair of integers $j > i > n_2$,

$$(T^i(x), T^j(x)) \in E(G).$$

Clearly, if the graph of T is closed, then $x_* \in T(x_*)$. It is also not difficult to see that if the set $E(G)$ is closed, then

$$(T^i(x), x_*) \in E(G)$$

for each integer $i > n_2$ and this implies that $(x_*, x_*) \in E(G)$. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Assume that $x, x_* \in X$, $x_* \in T(x_*)$, q is a natural number, $y_i \in X$, $i = 0, \dots, q$,*

$$y_0 = x,\ y_q = x_*$$

and that for each $i \in \{0, \dots, q-1\}$, $(y_i, y_{i+1}) \in E(G)$. Then

$$\lim_{i\to\infty} \rho(T^n(x), x_*) = 0.$$

In order to prove this theorem, it is sufficient to note that arguing as in the proof of Theorem 3.1, we find that for each $i \in \{0, \dots, q-1\}$,

$$\lim_{n\to\infty} \rho(T^n(y_i), T^n(y_{i+1})) = 0.$$

Proposition 2.2 implies the following result.

Theorem 3.3. *Assume that the function ϕ is upper semicontinuous at any positive $t > 0$, $x_* \in X$, $x_* \in T(x_*)$, $M > 1$, and q is a natural number, and let*

$$X_{M,q} = \left\{ x \in X : \begin{array}{l} \text{there exists } p \in \{0, \dots, q\},\ y_i \in X,\ i = 0, \dots, p, \text{ satis-} \\ \text{fying } y_0 = x,\ y_p = x_*, \text{ and for each } i \in \{0, \dots, p-1\}, \\ \rho(y_i, y_{i+1}) \le M,\ (y_i, y_{i+1}) \in E(G) \end{array} \right\}.$$

Let $\epsilon \in (0,1)$ be given. Then there exists an integer $n_0 \ge 1$ such that for each integer $n \ge n_0$ and each point $x \in X_{M,q}$,

$$\rho(T^n(x), x_*) \le \epsilon.$$

4. Convex metric spaces

Assume that (X, ρ) is a complete metric space endowed with a graph G. We denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges. Assume that

$$(x, x) \in E(G),\ x \in X,$$

and that for each $x, y \in X$, there exists $z \notin \{x, y\}$ for which

$$\rho(x, y) = \rho(x, z) + \rho(z, y).$$

In other words, (X, ρ) is metrically convex [5].

We now recall the following auxiliary result [5].

Lemma 4.1. *For each $\alpha \in (0, 1)$ and each $x, y \in X$, there exists a point $z \in X$ such that*

$$\rho(x, z) = \alpha\rho(x, y),\ \rho(z, y) = (1 - \alpha)\rho(x, y).$$

In the sequel we assume the following assumption regarding the connection between the graph G and the convexity structure:

(A) If $(x, y) \in E(G)$ and a point $z \in X$ satisfies $\rho(x, z) + \rho(z, y) = \rho(x, y)$, then $(x, z) \in E(G)$ and $(z, y) \in E(G)$.

We note here that (A) holds if T is a locally uniformly nonexpansive operator and $(x, y) \in E(G)$ if and only if $\rho(x, y) \leq \Delta$, where Δ is the positive constant which appears in the definition of T. Assumption (A) also holds if X is an ordered normed space and $(x, y) \in E(G)$ if and only if $x \leq y$. Set

$$Q := \{\rho(x, y) :\ (x, y) \in E(G)\}.$$

In view of assumption (A), Q is a convex set. Clearly, Q is either $[0, b)$ or $[0, b]$, where $0 \leq b \leq \infty$.

Theorem 4.2. *Assume that $\phi : [0, \infty) \to [0, \infty)$, for each $t > 0$,*

$$\phi(t) < t \tag{31}$$

and that for each $(z, y) \in E(G)$,

$$(T(z), T(y)) \in E(G) \tag{32}$$

and

$$\rho(T(z), T(y)) \leq \phi(\rho(z, y)). \tag{33}$$

Then the followings assertions hold.

1. Assume that $x \in X$, q is a natural number, $y_i \in X$, $i = 0, \dots, q$,

$$(y_i, y_{i+1}) \in E(G),\ i = 0, \dots, q - 1, \tag{34}$$

and

$$y_0 = x,\ y_q = T(x). \tag{35}$$

Then there exists

$$x_* = \lim_{n\to\infty} T^n(x).$$

If the graph of T is closed, then $x_* = T(x_*)$.

2. Assume that $x, x_* \in X$, $x_* \in T(x_*)$, q is a natural number, $y_i \in X$, $i = 0, \dots, q$,

$$y_0 = x, \ y_q = x_*,$$

and that for each $i \in \{0, \dots, q-1\}$, $(y_i, y_{i+1}) \in E(G)$. Then

$$\lim_{i \to \infty} \rho(T^n(x), x_*) = 0.$$

Proof. Define a function $\psi : [0, \infty) \to [0, \infty)$ as follows. For each $t \in Q$, set

$$\psi(t) := \sup\{\rho(T(x), T(y)) : \ (x, y) \in E(G), \ \rho(x, y) = t\}, \tag{36}$$

if $b \in Q$, then for each $t > b$, set

$$\psi(t) := \psi(b)$$

and if $b \notin Q$, set $\psi(t) := t/2$ for each $t \geq b$.

By (33) and (36), for each $(x, y) \in E(G)$,

$$\rho(T(x), T(y)) \leq \psi(\rho(x, y)) \leq \phi(\rho(x, y)). \tag{37}$$

It is clear that for each $t \in Q$,

$$\psi(t) \leq \phi(t) < t \tag{38}$$

and for each $t \in R^1 \setminus \{0\}$, $\psi(t) < t$.

Now we show that ψ is upper semicontinuous from the right at any positive t. Clearly, this is true if $t \geq b$. Therefore we only need to consider the case where $t < b$. Assume that $0 < s, t < b$ and $s + t < b$, and that

$$(x, y) \in E(G), \ \rho(x, y) = s + t.$$

Since (X, ρ) is metrically convex, there exists a point $z \in X$ such that

$$\rho(x, z) = s, \ \rho(z, y) = t.$$

By assumption (A), we have

$$(x, z), \ (z, y) \in E(G),$$

$$\rho(T(x), T(y)) \leq \rho(T(x), T(z)) + \rho(T(z), T(y)) \leq \psi(s) + \psi(t).$$

Thus $\psi(t + s) \leq \psi(s) + \psi(t)$

for each $s, t > 0$ satisfying $s + t < b$.

Let $t_0 \in (0, b)$, $s > 0$, $t_0 + s < b$. Then

$$\psi(t_0 + s) \leq \psi(t_0) + \psi(s) < \psi(t_0) + s$$

and so,
$$\limsup_{t \to t_0^+} \psi(t) \leq \psi(t_0)$$

and ψ is upper semicontinuous from the right at $t_0 \in (a, b)$. Assertion 1 and our theorem now follow from Theorems 3.1 and 3.2, respectively. $\square$

Acknowledgments. Simeon Reich was supported in part by the Israel Science Foundation (Grant 820/17), by the Fund for the Promotion of Research at the Technion (Grant 2001893) and by the Technion General Research Fund (Grant 2016723). Both authors are grateful to an anonymous referee for several pertinent comments and useful suggestions.

References

[1] S. M. A. Aleomraninejad, Sh. Rezapour, N. Shahzad: *Some fixed point results on a metric space with a graph*, Topology Appl. 159 (2012) 659–663.

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. 3 (1922) 133–181.

[3] F. Bojor: *Fixed point of ϕ-contraction in metric spaces endowed with a graph*, An. Univ. Craiova Ser. Mat. Inform. 37 (2010) 85–92.

[4] F. Bojor: *Fixed point theorems for Reich type contractions on metric spaces with a graph*, Nonlinear Analysis 75 (2012) 3895–3901.

[5] D. W. Boyd, J. S. W. Wong: *On nonlinear contractions*, Proc. Amer. Math. Soc. 20 (1969) 458–464.

[6] Y. Censor, R. Davidi, G. T. Herman: *Perturbation resilience and superiorization of iterative algorithms*, Inverse Problems 26 (2010) 12 pp.

[7] S.-H. Cho: *Fixed point theorems for set-valued contractions in metric spaces*, Axioms 13/2 (2024), art. no. 86, 15 pp.

[8] K. Goebel, W. A. Kirk: *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge (1990).

[9] K. Goebel, S. Reich: *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York (1984).

[10] G. Gwóźdź-Łukawska, J. Jachymski: *IFS on a metric space with a graph structure and extensions of the Kelisky-Rivlin theorem*, J. Math. Analysis Appl. 356 (2009) 453–463.

[11] J. Jachymski: *Equivalence of some contractivity properties over metrical structures*, Proc. Amer. Math. Soc. 125 (1997) 2327–2335.

[12] J. Jachymski: *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc. 136 (2008) 1359–1373.

[13] W. A. Kirk: *Contraction mappings and extensions*, in: *Handbook of Metric Fixed Point Theory*, Kluwer, Dordrecht (2001) 1–34.

[14] E. Rakotch: *A note on contractive mappings*, Proc. Amer. Math. Soc. 13 (1962) 459–465.

[15] S. Reich, A. J. Zaslavski: *Convergence of iterates for a class of mappings of contractive type*, JP J. Fixed Point Theory Appl. 2 (2007) 69–78.

[16] S. Reich, A. J. Zaslavski: *Genericity in Nonlinear Analysis*, Developments in Mathematics Vol. 34, Springer, New York (2014).

[17] M. E. Samei: *Some fixed point results on intuitionistic fuzzy metric spaces with a graph*, Sahand Commun. Math. Analysis 13 (2019) 141–152.

[18] S. Suantai, M. Donganont, W. Cholamjiak: *Hybrid methods for a countable family of G-nonexpansive mappings in Hilbert spaces endowed with graphs*, Mathematics 7/10 (2019), art. no. 936.

[19] S. Suantai, K. Kankam, P. Cholamjiak, W. Cholamjiak: *A parallel monotone hybrid algorithm for a finite family of G-nonexpansive mappings in Hilbert spaces endowed with a graph applicable in signal recovery*, Comp. Appl. Math. 40/4 (2021), art. no. 145, 17 pp.

[20] R. Suparatulatorn, W. Cholamjiak, S. Suantai, *A modified S-iteration process for G-nonexpansive mappings in Banach spaces with graphs*, Numerical Algorithms 77 (2018) 479–490.

[21] R. Suparatulatorn, S. Suantai, W. Cholamjiak: *Hybrid methods for a finite family of G-nonexpansive mappings in Hilbert spaces endowed with graphs*, AKCE Int. J. Graphs Comb. 14 (2017) 101–111.

[22] A. J. Zaslavski: *Algorithms for Solving Common Fixed Point Problems*, Springer Optimization and its Applications, Springer, Cham (2018).