Journal of Convex Analysis
Volume 32 (2025), No. 3, 767–788

Transversality and Strong Tangential Transversality of a Finite Number of Sets

Nadezhda Ribarska
(1) Faculty of Mathematics and Informatics, Sofia University, Sofia, Bulgaria;
(2) Inst. of Mathematics and Informatics, Bulgarian Acad. of Sciences, Sofia, Bulgaria
ribarska@fmi.uni-sofia.bg

Mariya Tasheva
Faculty of Mathematics and Informatics, Sofia University, Sofia, Bulgaria
mttasheva@uni-sofia.bg

Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

Received: July 20, 2024
Accepted: October 15, 2024

A definition of strong tangential transversality for a finite number of sets is proposed such that strong tangential transversality implies transversality. The main tool in the proofs are infinitesimal characterizations of transversality and subtransversality of a finite number of sets in the prime space which are of independent interest. A normal intersection property for Clarke normal cones, a sum rule for the Clarke subdifferential of a finite number of lower semicontinuous functions, and an abstract Lagrange multiplier rule are obtained as applications.

Keywords: Sub(transversality), primal space characterizations, strong tangential transversality, normal intersection properties, sum rule, Lagrange multiplier rule.

2020 Mathematics Subject Classification: 49K27, 49J53, 46N10.

1. Introduction

A classical concept of mathematical analysis, transversality has been developed by different authors in various directions. Its has been applied to problems in nonsmooth optimization, subdifferential calculus, to providing sufficient conditions for linear convergence of the alternating projections algorithm (cf. [8]), and in other parts of variational analysis. Transversality and its many existing variant properties such as subtransversality, intrinsic transversality, semitransversality have been studied in a considerable body of literature. Its close ties with metric regularity (cf., e.g., [1]) are one of the reasons for the central role transversality holds among the other transversality-type properties, and as such it is a fundamental concept with which newer properties in the field ought to be compared.

Strong tangential transversality of two sets appeared first as notion in a study investigating Pontryagin's type maximum principle for optimal control problems with terminal constraints in infinite dimensional state space (cf. [11]). It was introduced

The study is financed by the European Union-NextGenerationEU, through the National Recovery and Resilience Plan of the Republic of Bulgaria, project SUMMIT BG-RRP-2.004-0008-C01, and the Sofia University "St. Kliment Ohridski" fund "Research & Development" under contract 80-10-8 / 29.3.2024.

ISSN 0944-6532 / \$ 2.50

in [4] as a straightforward to verify in many classical settings property, which guarantees subtransversality of two closed sets at a common point as well as transversality of the respective Clarke tangent cones of the two closed set at the common point.

In the same article it is shown that, if $f_1 : X \to \mathbb{R} \cup \{+\infty\}$ and $f_2 : X \to \mathbb{R} \cup \{+\infty\}$ are lower semicontinuous and proper functions, $x_0 \in \mathrm{dom} f_1 \cap \mathrm{dom} f_2$, and

$$C_1 := \{(x, r_1, r_2) \in X \times \mathbb{R} \times \mathbb{R} \mid r_1 \geq f_1(x)\}$$
$$C_2 := \{(x, r_1, r_2) \in X \times \mathbb{R} \times \mathbb{R} \mid r_2 \geq f_2(x)\}$$

strong tangential transversality of C_1 and C_2 at $(x_0, f_1(x_0), f_2(x_0))$ yields

$$\partial_C(f_1 + f_2)(x_0) \subset \partial_C f_1(x_0) + \partial_C f_2(x_0)\,,$$

where ∂_C is the classical Clarke subdifferential. The result is based on a normal intersection property for Clarke normal cones. A Lagrange multiplier rule has also been proven using strong tangential transversality in [5]. Therefore, we see that strong tangential transversality appears to be useful in the study of some nontrivial problems in the field of nonsmooth analisys and optimal control.

In [12] the relation between transversality and strong tangential transversality of two sets has been studied. It has been shown that transversality follows from strong tangential transversality, but the reverse is not true. In the case when the sets are convex, it has been shown that the two properties coincide.

It is natural to try proposing a definition for strong tangential transversality of a finite number of sets. Firstly, because transversality of a finite number of sets has been a subject of wide studies for more than 20 years, and if the same relation between strong tangential transversality and transversality for a finite number of sets can be established, that would make strong tangential transversality a sufficient condition for transversality. More importantly, it would be very useful if the results obtained under the assumption of strong tangential transversality of two sets: normal intersection property for the Clarke normal cone, sum rule for the Clarke subdifferential, and a Lagrange multiplier rule, can be extended to finite number of sets. We propose a definition of strong tangential transversality of a finite number of sets and verify that it is consistent with the existing definition for two sets. From the definition immediately follows that in the case of convex sets, from transversality follows strong tangential transversality.

The main result of the paper is that from strong tangential transversality of a finite number of sets follows the transversality of the sets. The main tool used in the proof is an infinitesimal characterization of transversality of a finite number of sets in the prime space. This result is of independent interest. The proposed characterization of transversality tracks the distances between an auxiliary point and one point from each of the sets. Thus one has fewer points to work with than when trying to use the sufficient condition for transversality of Cuong and Kruger in [7]. On the other hand, fewer distances between points have to be tracked than Apostolov, Bivas and Ribarska's prime space charaterization if it were extented to the case of a finite number of sets. For that reason we believe that this characterization of transversality would be easier to apply. We use it to prove our main result. A similarly phrased infinitesimal characterization of subtransversality of a finite number of sets in the prime space is also obtained.

The paper is organized as follows: definitions of transversality of a finite number of sets and strong tangential transversality of two sets are given in the second section, as well as some of their known properties and their established relations to be used throughout the paper. In the third section infinitesimal characterizations of transversality and subtransversality of a finite number of sets in the prime space are studied. In the fourth section a definition of strong tangential transversality for a finite number of sets is proposed and its consistency with the existing definition for two sets is established. The characterization of transversality obtained in the previous section is used to prove that if a finite number of sets are strongly tangentially transversal at a point, they are also transversal at that point. A normal intersection property for Clarke normal cones is obtained in the last section, and it is then used to prove a sum rule for the Clarke subdifferential of a finite number of lower semi-continuous functions, and an abstract Lagrange multiplier rule.

2. Preliminaries

Throughout the paper if Y is a normed space, we will denote by $\mathbf{B}_Y$ $[\bar{\mathbf{B}}_Y]$ its open [closed] unit ball, centered at the origin. The index could be omitted if there is no ambiguity about the space.

The notion of transversality is central for our considerations. Here is its definition (cf., for example, [6]):

Definition 2.1. Let $\Omega_1, \dots, \Omega_n$ be subsets of the normed space X and $x_0 \in \bigcap_{i=1}^n \Omega_i$. We say that $\Omega_1, \dots, \Omega_n$ are *transversal* at the point x_0 if there exist constants $\alpha > 0$, $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\bigcap_{i=1}^{n} (\Omega_i - \omega_i - x_i) \cap (\rho \bar{\mathbf{B}}) \neq \emptyset$$

for every $\rho \in (0, \delta_1)$ and $\omega_i \in \Omega_i \cap (x_0 + \delta_2 \bar{\mathbf{B}})$, $x_i \in \rho\alpha\bar{\mathbf{B}}$, for $i = 1, 2, \dots, n$. $\square$

Remark 2.2. If the sets $\Omega_1, \dots, \Omega_n$ are transversal at $x_0 \in \bigcap_{i=1}^n \Omega_i$ then the sets $\Omega_1 \cap \Omega_2$, $\Omega_3, \dots, \Omega_n$ are also transversal at x_0.

Indeed, taking $\omega_1 = \omega_2 \in (\Omega_1 \cap \Omega_2) \cap (x_0 + \delta_2 \bar{\mathbf{B}})$ and $x_1 = x_2 \in \rho\alpha\bar{\mathbf{B}}$ with the rest of the variables selected as in the definition, it follows that

$$(\Omega_1 - \omega_1 - x_1) \cap (\Omega_2 - \omega_2 - x_2) = (\Omega_1 \cap \Omega_2) - \omega_1 - x_1$$

and we obtain

$$((\Omega_1 \cap \Omega_2) - \omega_1 - x_1) \cap \Big(\bigcap_{i=3}^{n} (\Omega_i - \omega_i - x_i) \cap (\rho\bar{\mathbf{B}})\Big) = \bigcap_{i=1}^{n} (\Omega_i - \omega_i - x_i) \cap (\rho\bar{\mathbf{B}}) \neq \emptyset$$

or $\Omega_1 \cap \Omega_2$, $\Omega_3, \dots, \Omega_n$ are transversal at x_0 with constants $\alpha > 0$, $\delta_1 > 0$ and $\delta_2 > 0$. $\square$

The following characterization of transversality (cf., for example, [9]), is very useful and it is often referred to as a definition:

Proposition 2.3. *Let $\Omega_1, \dots, \Omega_n$ be subsets of a normed space X and $x_0 \in \bigcap_{i=1}^n \Omega_i$. Then $\Omega_1, \dots, \Omega_n$ are transversal at the point x_0, if and only if there exist $K > 0$ and $\delta > 0$ such that*

$$d\Big(x, \bigcap_{i=1}^{n} (\Omega_i - x_i)\Big) \leq K \max_i d(x, \Omega_i - x_i)$$

for all $x \in x_0 + \delta \bar{\boldsymbol{B}}$ and $x_i \in \delta \bar{\boldsymbol{B}}$, $i = 1, 2, ..., n$.

We will need a similar in spirit characterization of subtransversality (cf., for example, [9]), which is often referred to as a definition as well:

Proposition 2.4. *Let $\Omega_1, \dots, \Omega_n$ be subsets of a normed space X and $x_0 \in \bigcap_{i=1}^n \Omega_i$. Then $\Omega_1, \dots, \Omega_n$ are subtransversal at the point x_0, if and only if there exist $K > 0$ and $\delta > 0$ such that, for all $x \in (x_0 + \delta \bar{\boldsymbol{B}})$,*

$$d\Big(x, \bigcap_{i=1}^{n} \Omega_i\Big) \leq K \max_i d(x, \Omega_i). \tag{1}$$

A comparison between the above characterizations of transversality and subtransversality shows that, roughly speaking, transversality is equivalent to subtransversality with one and the same constant K of all "small" displacements of the sets.

The other notion we are interested in, the one of strong tangential transversality, is based on the notion of uniform tangent set, which first appears in [11]. It has been proven in [5] that the following definition is equivalent to the original one.

Definition 2.5. (Uniform tangent set, [11], [5]) Let S be a closed subset of the Banach space X and $x_0 \in S$. We say that the bounded set $D_S(x_0)$ is a *uniform tangent set* to S at the point x_0 if for each $\varepsilon > 0$ there exist $\delta > 0$ and $\lambda > 0$ such that for each $\nu \in D_S(x_0)$ and for each point $x \in S \cap (x_0 + \delta \bar{\mathbf{B}})$ the set $S \cap (x + t(\nu + \varepsilon \bar{\mathbf{B}}))$ is non empty for each $t \in [0, \lambda]$. □

It is clear from the above definition that any uniform tangent set to S at $x_0 \in S$ is contained in the Clarke tangent cone to S at x_0. Here is the definition we are interested in.

Definition 2.6. (Strong tangential transversality, [4]) Let A and B be closed subsets of the Banach space X and $x_0 \in A \cap B$. We say that A and B are *strongly tangentially transversal* at x_0 if there exist a uniform tangent set $D_A(x_0)$ to A at the point x_0 and a uniform tangent set $D_B(x_0)$ to B at the point x_0, and $\rho > 0$ such that

$$\rho \bar{\mathbf{B}} \subset \overline{co}(D_A(x_0) - D_B(x_0))$$

where $\bar{\mathbf{B}}$ is the closed unit ball of X. □

Some basic properties of uniform tangent sets are gathered in the next proposition.

Theorem 2.7. ([5], [11], [4]) *Let S be a closed subset of the Banach space X and $x_0 \in S$. Let $D_S(x_0)$ be a uniform tangent set to S at the point x_0. Then the following hold true:*

(a) *if $c > 0$ is fixed, $cD_S(x_0)$ is a uniform tangent set to S at x_0,*

(b) *if $D'_S(x_0) \subseteq D_S(x_0)$, $D'_S(x_0)$ is a uniform tangent set to S at x_0,*

(c) *if $D'_S(x_0)$ is another uniform tangent set to S at x_0, then $D_S(x_0) \cup D'_S(x_0)$ is a uniform tangent set to S at x_0,*

(d) *$\overline{co}(D_S(x_0))$ is a uniform tangent set to S at x_0,*

(e) *if S is convex, then $(S - x_0) \cap M\bar{\boldsymbol{B}}$ is a uniform tangent set to S at x_0 for any $M > 0$.*

The following statement and its corollary are taken from [12], Proposition 3.1 and Corollary 3.4.

Theorem 2.8. ([12]) *Let A and B be closed subsets of the Banach space X and $x_0 \in A \cap B$. If A and B are strongly tangentially transversal at x_0, then A and B are transversal at x_0.*

Corollary 2.9. ([12]) *Let A and B be closed convex bounded subsets of the Banach space X, and let the algebraic difference $A - B$ be dense in a neighbourhood of the origin. Then $A - B$ contains a neighbourhood of the origin.*

The characterization of transversality for convex sets formulated below has been obtained by Bui, Cuong, and Kruger in [6].

Theorem 2.10. (Transversality of convex sets, [6]) *Let $\Omega_1, \dots, \Omega_n$ be convex subsets of the normed space X and $x_0 \in \bigcap_{i=1}^n \Omega_i$. The sets $\Omega_1, \dots, \Omega_n$ are transversal at x_0 if and only if there exist constants $\alpha > 0$ and $\delta > 0$ such that*

$$\bigcap_{i=1}^{n}(\Omega_i - x_i) \cap (x_0 + \delta\bar{\boldsymbol{B}}) \neq \emptyset$$

for every $x_i \in \alpha\delta\bar{\boldsymbol{B}}$, $i = 1, 2, \dots, n$.

Indeed, Proposition 2 from [6] gives that from transversality follows the weaker property semi-transversality, and in Proposition 8 from the same paper it is shown that in the presence of convexity from semi-transversality follows transversality. Proposition 6(i) gives a characterization, in the above form, of semi-transversality for convex sets, which consequently applies to transversality as well.

We will make use of another result from [6]:

Theorem 2.11. ([6]) *Let $\Omega_1, \dots, \Omega_n$ be subsets of the normed space X and let $x_0 \in \bigcap_{i=1}^n \Omega_i$. If $\Omega_1, \dots, \Omega_n$ are transversal at x_0, then $\overline{conv}(\Omega_1)$,..., $\overline{conv}(\Omega_n)$ are also transversal at x_0.*

Again, it is used that from transversality follows semi-transversality (Proposition 2). Then from the definition of semi-transversality, because $\overline{conv}(\Omega_i) \supset \Omega_i$, it is obtained that $\overline{conv}(\Omega_1)$,..., $\overline{conv}(\Omega_n)$ are semi-transversal. Due to the convexity of the sets, they are also transversal from Proposition 8.

3. Characterizations of transversality and subtransversality

The following result by Cuong and Kruger was one of our starting points:

Theorem 3.1. (Sufficient condition for transversality, [7]) *Let $\Omega_1,\dots,\Omega_n$ be closed subsets of the Banach space X and let $x_0 \in \bigcap_{i=1}^n \Omega_i$. The sets $\Omega_1,\dots,\Omega_n$ are transversal at x_0 with constants $\alpha > 0$, $\delta_1 > 0$ and $\delta_2 > 0$ if for some $\gamma > 0$ and any $\omega_i' \in \Omega_i \cap (x_0 + \delta_2 \bar{\boldsymbol{B}})$ $(i = 1,2,\dots,n)$ and $\xi \in (0, \alpha\delta_1)$, there exists $\lambda \in (\alpha^{-1}\xi, \delta_1)$ such that the inequality*

$$\sup_{\substack{u_i \in \Omega_i, u \in X \\ (u_1,\dots,u_n,u) \neq (\omega_1,\dots,\omega_n,x)}} \frac{\max_i \|\omega_i - x_i - x\| - \max_i \|u_i - x_i - u\|}{\|(u_1,\dots,u_n,u) - (\omega_1,\dots,\omega_n,x)\|_\gamma} \geq \alpha \tag{2}$$

holds true for all x and $x_i \in X$, $\omega_i \in \Omega_i$ $(i = 1,2,\dots,n)$ satisfying

$$\|x - x_0\| < \lambda, \ \max_i \|\omega_i - \omega_i'\| < \frac{\lambda}{\gamma} \tag{3}$$

and
$$0 < \max_i \|\omega_i - x_i - x\| \leq \max_i \|\omega_i' - x_i - x_0\| = \xi \tag{4}$$

Remark 3.2. With $\gamma > 0$, the norm $\|\cdot\|_\gamma$ on X^{n+1} is defined as

$$\|(x_1, x_2, \dots, x_n, x)\|_\gamma := \max\{\|x\|, \gamma\|x_1\|, \dots\gamma\|x_n\|\}, \text{ for } x, x_1, \dots, x_n \in X. \qquad \square$$

We were looking to reduce the number of points being tracked while keeping the main idea: to find for any given points $\omega_1 \in \Omega_1, \dots, \omega_n \in \Omega_n$ and $x \in X$, some points belonging to the same sets and such that the distances between the initial points ω_i and x, for all i, are reduced for the corresponding new points.

Remark 3.3. The characterization of transversality of two sets given in [2] can be extended with minimal changes of the proof to the case of a finite number of sets, obtaining the following statement:

Proposition 3.4. (Characterization of transversality of finite number of sets) *Let $\Omega_1,\dots,\Omega_n$ be closed subsets of the Banach space X and let $x_0 \in \bigcap_{i=1}^n \Omega_i$. Then $\Omega_1,\dots,\Omega_n$ are transversal at x_0 if and only if there exist $\delta > 0$ and $M > 0$ such that for any $a_i \in \delta\bar{\boldsymbol{B}}$, $x_i \in \Omega_i \cap (x_0 + a_i + \delta\bar{\boldsymbol{B}})$ for $i = 1,2,\dots,n$, with*

$$\max_{i \neq j} \|x_i - a_i - (x_j - a_j)\| > 0,$$

there exist $\theta > 0$, $\hat{x}_i \in \Omega_i$, $i = 1,2,\dots,n$, such that

$$\max_i \|x_i - \hat{x}_i\| \leq \theta M$$

and
$$\max_{i \neq j} \|\hat{x}_i - a_i - (\hat{x}_j - a_j)\| \leq \max_{i \neq j} \|x_i - a_i - (x_j - a_j)\| - \theta.$$

In order to use the above characterization of transversality one has to verify that the maximum of all distances between the displaced new points $\hat{x}_i - a_i$ (which count at $n(n-1)/2$) is strictly less than the maximum of all distances between the displaced initial points $x_i - a_i$. $\square$

We want to simplify Theorem 3.1 and find a condition on $\Omega_1,\dots,\Omega_n$ which is close in spirit, but already characterizes transversality of $\Omega_1,\dots,\Omega_n$. The main technical tool is the following lemma:

Lemma 3.5. *Let $\Omega_1, \dots, \Omega_n$ be closed subsets of the Banach space X and let $x_0 \in X$. Let there exist $\delta > 0$ and $M > 0$ such that for any $x_i \in \Omega_i \cap (x_0 + \delta \bar{\boldsymbol{B}})$, $i = 1, 2, \dots, n$, and any $x \in (x_0 + \delta \bar{\boldsymbol{B}})$ with $\max_i \|x_i - x\| > 0$, there exist $\theta > 0$, $\hat{x}_i \in \Omega_i$ for $i = 1, 2, \dots, n$, and $\hat{x} \in X$, such that*

$$\max_i \|x_i - \hat{x}_i\| \leq \theta M \quad \textit{and} \quad \|x - \hat{x}\| \leq \theta M$$

and

$$\max_i \|\hat{x}_i - \hat{x}\| \leq \max_i \|x_i - x\| - \theta.$$

Let $x_i \in \Omega_i \cap (x_0 + \dfrac{\delta}{2M+1} \bar{\boldsymbol{B}})$ be such that $\max_{i \neq j} \|x_i - x_j\| > 0$. Then there exists $x' \in \bigcap_{i=1}^{n} \Omega_i$ satisfying

$$\max_i \|x' - x_i\| \leq M \max_{i \neq j} \|x_i - x_j\|.$$

Proof. Put $\bar{\delta} := \frac{\delta}{2M+1}$ and choose an arbitrary $x \in conv\{x_1, x_2, ..., x_n\} \subset (x_0 + \bar{\delta}\bar{\mathbf{B}})$. Since $\max_{i \neq j} \|x_i - x_j\| > 0$ it follows that $\max_i \|x_i - x\| > 0$.

We are going to inductively construct $n+2$ transfinite sequences indexed by ordinal numbers. More precisely, we will prove that there exist an ordinal number α_0 and transfinite sequences $\{x_\alpha^i\}_{\alpha \leq \alpha_0} \subset \Omega_i$ for $i = 1, 2, \dots, n$, $\{x_\alpha\}_{\alpha \leq \alpha_0} \subset X$ and $\{\theta_\alpha\}_{\alpha \leq \alpha_0} \subset \mathbb{R}$ such that $\max_i \|x_{\alpha_0}^i - x_{\alpha_0}\| = 0$ (that is $x_{\alpha_0}^1 = \dots = x_{\alpha_0}^n = x_{\alpha_0}$), and for each $\alpha \in [1, \alpha_0]$ the following properties hold true

(a) $x_\alpha^i \in \Omega_i \cap (x_0 + \delta \bar{\mathbf{B}})$ for $i = 1, 2, \dots, n$ and $x_\alpha \in (x_0 + \delta \bar{\mathbf{B}})$

(b) $\max_i \|x_\alpha^i - x_\alpha\| \leq \max_i \|x_i - x\| - \theta_\alpha$

(c) $\|x_\alpha^i - x_0\| \leq \|x_i - x_0\| + M\theta_\alpha$ for $i = 1, 2, \dots, n$ and
$\|x_\alpha - x_0\| \leq \|x - x_0\| + M\theta_\alpha$

(d) $\|x_\alpha^i - x_\gamma^i\| \leq M(\theta_\alpha - \theta_\gamma)$ for $i = 1, 2, \dots, n$ and $\|x_\alpha - x_\gamma\| \leq M(\theta_\alpha - \theta_\gamma)$,
for each $\gamma \leq \alpha$

The induction process will stop when for some α it is true that $\max_i \|x_\alpha^i - x_\alpha\| = 0$, and this α is named α_0. We start with $x_0^i := x_i$, $x_0 := x$ and $\theta_0 := 0$. The properties are trivially true for $\alpha = 0$.

Assume that for $\alpha > 0$ the sequences $\{x_\beta^i\}_{\beta < \alpha}$ for $i = 1, 2, \dots, n$, $\{x_\beta\}_{\beta < \alpha}$ and $\{\theta_\beta\}_{\beta < \alpha}$ are constructed and for any $\beta < \alpha$ the properties (a)–(d) hold true. Let us also assume that the process has not been terminated.

First, let us consider the case when α is a successor ordinal, i.e. there exists β such that $\alpha = \beta + 1$. As the process has not been terminated it follows $\max_i \|x_\beta^i - x_\beta\| > 0$. Then (a) yields that we can apply the assumption of the theorem to the points x_β^i, $i = 1, 2, \dots, n$ and x_β, obtaining that there exist $\theta > 0$, $\hat{x}_i \in \Omega_i$, for $i = 1, 2, \dots, n$, and $\hat{x} \in X$ such that

$$\max_i \|x_\beta^i - \hat{x}_i\| \leq \theta M \text{ and } \|x_\beta - \hat{x}\| \leq \theta M \tag{5}$$

and it holds true that

$$\max_i \|\hat{x}_i - \hat{x}\| \leq \max_i \|x_\beta^i - x_\beta\| - \theta. \tag{6}$$

We set $x^i_\alpha := \hat{x}_i$ for $i = 1, 2, \ldots, n$, $x_\alpha := \hat{x}$ and $\theta_\alpha := \theta_\beta + \theta$. We will verify the properties (a)–(d).

(b) By using (b) for β and (6) we obtain

$$\max_i \|x^i_\alpha - x_\alpha\| \le \max_i \|x^i_\beta - x_\beta\| - \theta \le \max_i \|x_i - x\| - \theta_\beta - \theta = \\ = \max_i \|x_i - x\| - \theta_\alpha$$

(c) From (c) for β and (5) it follows

$$\|x^i_\alpha - x_0\| \le \|x^i_\beta - x_0\| + \|x^i_\beta - x^i_\alpha\| \le \|x_i - x_0\| + M\theta_\beta + M\theta = \\ = \|x_i - x_0\| + M\theta_\alpha$$

$$\|x_\alpha - x_0\| \le \|x_\beta - x_0\| + \|x_\beta - x_\alpha\| \le \|x - x_0\| + M\theta_\beta + M\theta = \\ = \|x - x_0\| + M\theta_\alpha$$

(d) Let $\gamma < \alpha$. Then from (d) for γ and β, and (5) we have

$$\|x^i_\alpha - x^i_\gamma\| \le \|x^i_\beta - x^i_\gamma\| + \|x^i_\beta - x^i_\alpha\| \le M(\theta_\beta - \theta_\gamma) + M\theta = M(\theta_\alpha - \theta_\gamma),$$
$$\|x_\alpha - x_\gamma\| \le \|x_\beta - x_\gamma\| + \|x_\beta - x_\alpha\| \le M(\theta_\beta - \theta_\gamma) + M\theta = M(\theta_\alpha - \theta_\gamma).$$

(a) From $x_i \in (x_0 + \overline{\delta}\bar{\mathbf{B}})$ and $x \in (x_0 + \overline{\delta}\bar{\mathbf{B}})$ it follows

$$\max_i \|x_i - x\| \le \max_i \|x_i - x_0\| + \|x - x_0\| \le \overline{\delta} + \overline{\delta} = 2\overline{\delta}.$$

So from (b) for α we obtain $\theta_\alpha \le \max_i \|x_i - x\| \le 2\overline{\delta}$.

Therefore from (c) for α it follows

$$\|x^i_\alpha - x_0\| \le \|x_i - x_0\| + M\theta_\alpha \le \overline{\delta} + M2\overline{\delta} = (2M+1)\overline{\delta} = \delta$$

and $$\|x_\alpha - x_0\| \le \|x - x_0\| + M\theta_\alpha \le \overline{\delta} + M2\overline{\delta} = (2M+1)\overline{\delta} = \delta.$$

Thus we have obtained that $x^i_\alpha \in \Omega_i \cap (x_0 + \delta\bar{\mathbf{B}})$ for $i = 1, 2, \ldots, n$ and $x_\alpha \in (x_0 + \delta\bar{\mathbf{B}})$, that is (a) holds true.

Now let α be a limit ordinal – i.e. for any $\beta < \alpha$ it holds $\beta + 1 < \alpha$. As the process has not been terminated at $\beta + 1$, it follows that $\max_i \|x^i_\beta - x_\beta\| > 0$, so the properties (a)–(d) hold for β. From (b) for β we have that $\theta_\beta < \max_i \|x_i - x\|$, so the real-valued transfinite sequence $\{\theta_\beta\}_{\beta<\alpha}$ is bounded and strictly increasing and therefore convergent. Let $\theta_\alpha := \lim_{\beta\to\alpha} \theta_\beta$. Therefore for any $i \in \{1, 2, \ldots, n\}$ the transfinite sequence $\{x^i_\beta\}_{\beta<\alpha} \in \Omega_i$ is fundamental, because of (d), that is

$$\|x^i_\beta - x^i_\gamma\| \le M(\theta_\beta - \theta_\gamma).$$

We set $x^i_\alpha := \lim_{\beta\to\alpha} x^i_\beta \in \Omega_i$ (the limit exists, because X is complete). Moreover, $x^i_\alpha \in \Omega_i$, as Ω_i are closed. Analogically $\{x_\beta\}_{\beta<\alpha}$ is fundamental and we put $x_\alpha := \lim_{\beta\to\alpha} x_\beta$. The properties (a)–(d) immediately follow, because the norm is continuous.

We have thus inductively constructed the transfinite sequences $\{x^i_\alpha\} \subset \Omega_i$ for all $i = 1, 2, \ldots, n$, $\{x_\alpha\} \subset X$ and $\{\theta_\alpha\} \subset \mathbb{R}$. The process will terminate when we get

$\max_i \|x^i_{\alpha_0} - x_{\alpha_0}\| = 0$ for some α_0. Since $\max_i \|x^i_\alpha - x_\alpha\| + \theta_\alpha \le \max_i \|x_i - x\|$ and the transfinite sequence $\{\theta_\alpha\}_\alpha$ is strictly increasing, the equality $\max_i \|x^i_\alpha - x_\alpha\| = 0$ will be satisfied for some $\alpha = \alpha_0$ strictly preceding the first uncountable ordinal number. Indeed, the successor ordinals indexing the so constructed transfinite sequences, form a countable set (because to every successor ordinal $\alpha + 1$ corresponds the open interval $(\theta_\alpha, \theta_{\alpha+1}) \subset \mathbb{R}$, these intervals are disjoint, and the rational numbers are countably many and dense in $\mathbb{R}$). Therefore, α_0 is countably accessible. On the other hand, assuming the Axiom of countable choice, ω_1 is not countably accessible. Hence our inductive process terminates before ω_1.

Now let $\overline{x}_i := x^i_{\alpha_0} \in \Omega_i$ and $\overline{x} := x_{\alpha_0}$. Since the process terminates at α_0 it follows that $\max_i \|x^i_{\alpha_0} - x_{\alpha_0}\| = \max_i \|\overline{x}_i - \overline{x}\| = 0$, and so for $i = 1, 2, \ldots, n$ we have

$$\overline{x}_i = \overline{x} \in \bigcap_{i=1}^{n} \Omega_i. \tag{*}$$

Since $x \in conv\{x_1, x_2, ..., x_n\}$ we have $x = \sum_{j=1}^n \lambda_j x_j$ for $\lambda_j \ge 0$, $\sum_{j=1}^n \lambda_j = 1$, and so for any i

$$\|x_i - x\| = \Big\|x_i - \sum_{j=1}^n \lambda_j x_j\Big\| = \Big\|\sum_{j=1}^n \lambda_j (x_i - x_j)\Big\| \le \sum_{j=1}^n \lambda_j \|x_i - x_j\| \le$$
$$\le \sum_{j=1}^n \lambda_j \max_{k \ne l} \|x_k - x_l\| = \max_{k \ne l} \|x_k - x_l\|.$$

It follows that
$$\max_i \|x_i - x\| \le \max_{i \ne j} \|x_i - x_j\| \tag{**}$$

From (*), (d), (b) and (**) we obtain

$$\max_i \|\overline{x} - x_i\| = \max_i \|\overline{x}_i - x_i\| = \max_i \|x^i_{\alpha_0} - x_i\| \le M(\theta_{\alpha_0} - \theta_0) =$$
$$= M\theta_{\alpha_0} \le M \max_i \|x_i - x\| \le M \max_{i \ne j} \|x_i - x_j\|.$$

By taking $x' := \overline{x}$, the proof is complete. $\square$

The above lemma yields a characterization of subtransversality:

Theorem 3.6. *Let $\Omega_1, \ldots, \Omega_n$ be closed subsets of the Banach space X and let $x_0 \in \bigcap_{i=1}^n \Omega_i$. The sets $\Omega_1, ..., \Omega_n$ are subtraversal at x_0* ***if and only if*** *there exist $\delta > 0$ and $M > 0$ such that for any $x_i \in \Omega_i \cap (x_0 + \delta \bar{\boldsymbol{B}})$, $i = 1, 2, \ldots, n$, and any $x \in (x_0 + \delta \bar{\boldsymbol{B}})$ with $\max_i \|x_i - x\| > 0$, there exist $\theta > 0$, $\hat{x}_i \in \Omega_i$ for $i = 1, 2, \ldots, n$, and $\hat{x} \in X$, such that*

$$\max_i \|x_i - \hat{x}_i\| \le \theta M \quad and \quad \|x - \hat{x}\| \le \theta M$$

and
$$\max_i \|\hat{x}_i - \hat{x}\| \le \max_i \|x_i - x\| - \theta.$$

Moreover, if the sets $\Omega_1, ..., \Omega_n$ are subtransversal at x_0 with constants $K > 0$ and $\hat{\delta} > 0$, then the above statement holds for $\delta = \hat{\delta}$ and $M = 2K + 3$. Conversely, if the above statement holds for some $\delta > 0$ and $M > 0$, then the sets $\Omega_1, ..., \Omega_n$ are subtransversal at x_0 with constants $K = 2M + 1$ and $\hat{\delta} = \frac{\delta}{4(2M+1)}$.

Proof. First, let $\Omega_1, \dots, \Omega_n$ be subtransversal at x_0. Then there exist $K > 0$ and $\hat{\delta} > 0$ such that

$$d\Big(x, \bigcap_{i=1}^{n} \Omega_i\Big) \le K \max_i d(x, \Omega_i) \text{ for all } x \in (x_0 + \hat{\delta}\bar{\mathbf{B}}). \tag{7}$$

We set $\delta := \hat{\delta}$. Let us fix arbitrary $x_i \in \Omega_i \cap (x_0 + \delta\bar{\mathbf{B}})$, $i = 1, 2, \dots, n$, and let $x \in x_0 + \delta\bar{\mathbf{B}}$, where additionally $\max_i \|x_i - x\| > 0$. We set $\theta := \max_i \|x_i - x\|$. Let $\hat{x} \in \bigcap_{i=1}^n \Omega_i$ be such that $\|x_1 - \hat{x}\| \le d\left(x_1, \bigcap_{i=1}^n \Omega_i\right) + \theta$. Then we have

$$\begin{aligned}\|x_1 - \hat{x}\| &\le d\Big(x_1, \bigcap_{i=1}^{n} \Omega_i\Big) + \theta \le K \max_i d(x_1, \Omega_i) + \theta \le \\ &\le K \max_i(\|x_1 - x\| + d(x, \Omega_i)) + \theta \le \\ &\le K \max_i(\|x_1 - x\| + \|x - x_i\|) + \theta \le \\ &\le K \max_i(\theta + \theta) + \theta = (2K + 1)\theta.\end{aligned}$$

Then for any $i \in 2, 3, \dots, n$ it holds true that

$$\|x_i - \hat{x}\| \le \|x_i - x\| + \|x - x_1\| + \|x_1 - \hat{x}\| \le 2\theta + (2K + 1)\theta = (2K + 3)\theta.$$

We set $\hat{x}_i := \hat{x}$ for $i = 1, 2, \dots, n$. Then we have the estimates

$$\|x_i - \hat{x}_i\| = \|x_i - \hat{x}\| \le (2K + 3)\theta,$$

$$\|x - \hat{x}\| \le \|x - x_1\| + \|x_1 - \hat{x}\| \le \theta + (2K + 1)\theta \le (2K + 3)\theta.$$

Since
$$\max_i \|\hat{x}_i - \hat{x}\| = 0 = \max_i \|x_i - x\| - \theta,$$

we have obtained the characterization in the theorem with the constants δ and $M = 2K + 3$.

Now, let the characterization in the theorem holds for some $\delta > 0$ and $M > 0$. Let us put $\hat{\delta} := \dfrac{\delta}{4(2M + 1)}$.

Let $x \in (x_0 + \hat{\delta}\bar{\mathbf{B}})$ be arbitrary and $\varepsilon \in (0, \hat{\delta})$. Choose $x_i \in \Omega_i$ such that

$$\|x - x_i\| \le d(x, \Omega_i) + \varepsilon \quad \text{for} \quad i = 1, 2, \dots, n. \tag{8}$$

Since $x_0 \in \bigcap_{i=1}^n \Omega_i$, it follows that $d(x, \Omega_i) \le d(x, x_0) \le \hat{\delta}$, and so we have that for any $i \in 1, 2, ..., n$

$$\begin{aligned}\|x_i - x_0\| &\le \|x_i - x\| + \|x - x_0\| \le d(x, \Omega_i) + \varepsilon + \|x - x_0\| \\ &\le \hat{\delta} + \varepsilon + \hat{\delta} \le 3\hat{\delta} < \frac{\delta}{2M + 1}.\end{aligned}$$

Thus $x_i \in \Omega_i \cap \left(x_0 + \frac{\delta}{2M+1}\bar{\mathbf{B}}\right)$ for $i = 1, 2, \dots, n$. Therefore we can apply Lemma 3.5 to $x_1, \dots, x_n$ and obtain $x' \in \bigcap_{i=1}^n \Omega_i$ such that

$$\max_i \|x' - x_i\| \le M \max_{i \ne j} \|x_i - x_j\|.$$

For any $i \in 1, 2, ..., n$, from (8) and the above inequality we have

$$\|x' - x_i\| \le M \max_{k\neq j} \|x_k - x_j\| \le M \max_{k\neq j}(\|x_k - x\| + \|x - x_j\|) \le$$
$$\le M \max_{k\neq j}[d(x, \Omega_k) + 2\varepsilon + d(x, \Omega_j)] \le 2M \max_j d(x, \Omega_j) + 2M\varepsilon$$

Making use of this inequality and of (8) we obtain

$$d\Big(x, \bigcap_{i=1}^{n} \Omega_i\Big) \le \|x - x'\| \le \|x - x_i\| + \|x_i - x'\| \le$$
$$\le d(x, \Omega_i) + \varepsilon + 2M \max_j d(x, \Omega_j) + 2M\varepsilon \le$$
$$\le (2M + 1) \max_i d(x, \Omega_i) + (2M + 1)\varepsilon$$

As $\varepsilon \to 0$ we have

$$d\Big(x, \bigcap_{i=1}^{n} \Omega_i\Big) \le (2M + 1) \max_i d(x, \Omega_i)$$

which is (7) for $\hat{\delta} = \frac{\delta}{4(2M+1)}$ and $K = 2M + 1$ since $x \in (x_0 + \hat{\delta}\bar{\mathbf{B}})$ was arbitrary. $\square$

From the obtained characterization of subtransversality, as in [2], we arrive at

Theorem 3.7. (Characterization of transversality) *Let $\Omega_1, \dots, \Omega_n$ be closed subsets of the Banach space X and let $x_0 \in \bigcap_{i=1}^n \Omega_i$. The sets $\Omega_1, \dots, \Omega_n$ are transversal at x_0 if and only if there exist $\delta > 0$ and $M > 0$ such that for any $a_i \in \delta\bar{\boldsymbol{B}}$, $x_i \in \Omega_i \cap (x_0 + a_i + \delta\bar{\boldsymbol{B}})$ for $i = 1, 2, ..., n$, and $x \in (x_0 + \delta\bar{\boldsymbol{B}})$ with $\max_i \|x_i - a_i - x\| > 0$, there exist $\theta > 0$, $\hat{x} \in X$ and $\hat{x}_i \in \Omega_i$, $i = 1, 2, \dots, n$, such that*

$$\max_i \|x_i - \hat{x}_i\| \le \theta M, \quad \|x - \hat{x}\| \le \theta M,$$

and
$$\max_i \|\hat{x}_i - a_i - \hat{x}\| \le \max_i \|x_i - a_i - x\| - \theta.$$

Proof. Let $\Omega_1, \dots, \Omega_n$ be transversal at x_0 with constants $K > 0$ and $\delta > 0$. Then for all $a_i \in \frac{\delta}{4K+4}\bar{\mathbf{B}}$ we have that

$$\Big(\bigcap_{i=1}^{n} (\Omega_i - a_i)\Big) \cap \Big(x_0 + \frac{\delta}{4}\bar{\mathbf{B}}\Big) \neq \emptyset$$

and the sets $\Omega_1 - a_1, \dots, \Omega_n - a_n$ are subtransversal at any point belonging to the above intersection with constants $K > 0$ and $\delta/2 > 0$. Therefore, according to Theorem 3.6, the condition in Theorem 3.6 holds with $\delta/2$ and $M := 2K + 3$. Since

$$\tilde{x} + \frac{\delta}{2}\bar{\mathbf{B}} \supset x_0 + \frac{\delta}{4}\bar{\mathbf{B}} \quad \text{for every} \quad \tilde{x} \in \Big(\bigcap_{i=1}^{n} (\Omega_i - a_i)\Big) \cap \Big(x_0 + \frac{\delta}{4}\bar{\mathbf{B}}\Big),$$

we obtain that the condition in Theorem 3.7 is satisfied with $\dfrac{\delta}{4K + 4}$ and M.

Now let $\Omega_1, \dots, \Omega_n$ satisfy the condition in Theorem 3.7 with constants $\delta > 0$ and $M > 0$. We set $\hat{\delta} := \delta/(2M+1)(8M+9)$. Then for all $a_i \in \hat{\delta}\,\bar{\mathbf{B}}$, $i = 1, 2, \dots, n$, the condition of Lemma 3.5 for the sets $\Omega_1 - a_1, \dots, \Omega_n - a_n$ is satisfied with δ and M. Since $\hat{\delta} \le \delta/(2M+1)$, we have $(\Omega_i - a_i) \cap (x_0 + \hat{\delta}\,\bar{\mathbf{B}}) \neq \emptyset$ and Lemma 3.5 yields the existence of

$$\tilde{x} \in \Big(\bigcap_{i=1}^{n} (\Omega_i - a_i)\Big) \cap \Big(x_0 + (2M+1)\,\hat{\delta}\,\bar{\mathbf{B}}\Big).$$

Therefore the condition of Theorem 3.6 holds for $\Omega_1 - a_1, \dots, \Omega_n - a_n$ around $\tilde{x}$ with constants $\delta - (2M+1)\,\hat{\delta}$ and M. Hence $\Omega_1 - a_1, \dots, \Omega_n - a_n$ are subtransversal around $\tilde{x}$ with constants $[\delta - (2M+1)\hat{\delta}]/4(2M+1)$ and $2M+1$. Now a straightforward computation shows that

$$\tilde{x} + \frac{\delta - (2M+1)\hat{\delta}}{4(2M+1)}\,\bar{\mathbf{B}} \supset x_0 + \hat{\delta}\,\bar{\mathbf{B}}.$$

Therefore the sets $\Omega_1, \dots, \Omega_n$ are transversal at x_0 with $K := 2M+1$ and $\hat{\delta}$. □

4. Transversality and strong tangential transversality of a finite number of sets

At the time of writing of this paper strong tangential transversality has been defined and studied as a property of two sets. Transversality of a finite number of sets has been a subject of wide studies for more than 20 years, and the question naturally arises whether a definition for strong tangential transversality can be given for a finite number of sets, and such that as in the case of two sets, the transversality property for the given sets would follow.

Definition 4.1. (Strong tangential transversality, n) Let $\Omega_1, \dots, \Omega_n$ be closed subsets of the Banach space X with $x_0 \in \bigcap_{i=1}^{n} \Omega_i$. We say that $\Omega_1, \dots, \Omega_n$ are strongly tangentially transversal at the point x_0 if there exist uniform tangent sets $D_i(x_0)$ to Ω_i at x_0 for $i = 1, 2, \dots, n$ such that $D_1(x_0), \dots, D_n(x_0)$ are transversal at the origin. □

Remark 4.2. Without loss of generality, due to Theorem 2.11 and the properties of uniform tangent sets (see Theorem 2.7), we may assume that the uniform tangent sets $D_i(x_0)$ to Ω_i at x_0 for $i = 1, 2, \dots, n$ are closed and convex. □

Remark 4.3. We will verify that the proposed definition is consistent with the existing one in the case of two sets. Let A and B be closed subsets of the Banach space X with $x_0 \in A \cap B$. We will make use of the following assertion, namely Lemma 3.3 from [12]:

Lemma 4.4. *Let C and D be closed subsets of the normed vector space X and $x_0 \in C \cap D$. Then the following are equivalent:*

(a) *there exist $\alpha > 0$ and $\delta > 0$ such that, for all $x_C, x_D \in \alpha\delta\bar{\boldsymbol{B}}$,*

$$(C - x_C) \cap (D - x_D) \cap (x_0 + \delta\bar{\boldsymbol{B}}) \neq \emptyset;$$

(b) *there exist $\rho > 0$ and $\delta' > 0$ such that*

$$\rho\bar{\boldsymbol{B}} \subset [C \cap (x_0 + \delta'\bar{\boldsymbol{B}}) - D \cap (x_0 + \delta'\bar{\boldsymbol{B}})].$$

Firstly, let the sets A and B be strongly tangentially transversal in the sense of Definition 2.6 at x_0: there exist $\rho > 0$ and $D_A(x_0)$ and $D_B(x_0)$, uniform tangent sets to A and B at x_0, respectively (we assume them to be closed and convex) such that

$$\rho\bar{\mathbf{B}} \subset \overline{co}(D_A(x_0) - D_B(x_0))$$

or equivalently, because of convexity,

$$\rho\bar{\mathbf{B}} \subset \overline{D_A(x_0) - D_B(x_0)}.$$

Since $D_A(x_0)$ and $D_B(x_0)$ are bounded in consequence of Corollary 2.9 there exists $\rho' > 0$ such that

$$\rho'\bar{\mathbf{B}} \subset D_A(x_0) - D_B(x_0).$$

Since $D_A(x_0)$ and $D_B(x_0)$ are convex, we can apply Lemma 4.4 followed by Theorem 2.10 to obtain that $D_A(x_0)$ and $D_B(x_0)$ are transversal at x_0, which is strong tangential transversality of A and B at x_0 in the sense of Definition 4.1.

Now let us have strong tangential transversality of A and B at x_0 in the sense of Definition 4.1: there exist uniform tangent sets $D_A(x_0)$ and $D_B(x_0)$ (we assume them to be closed and convex), which are transversal at x_0. Again because of the convexity of $D_A(x_0)$ and $D_B(x_0)$ we can apply Theorem 2.10 followed by Lemma 4.4 to obtain that there exists $\rho' > 0$ such that

$$\rho'\bar{\mathbf{B}} \subset D_A(x_0) - D_B(x_0).$$

By taking closure we obtain strong tangential transversality of A and B at x_0 in the sense of Definition 2.6. □

Immediately from the proposed definition for strong tangential transversality, for convex sets we have the following:

Proposition 4.5. *Let $\Omega_1, \ldots, \Omega_n$ be closed convex subsets of the Banach space X with $x_0 \in \bigcap_{i=1}^n \Omega_i$. If $\Omega_1, \ldots, \Omega_n$ are transversal at x_0, then they are strongly tangentially transversal at x_0.*

Proof. From the properties of uniform tangent sets (Theorem 2.7 (e)) we see that

$$D_i(x_0) := (\Omega_i - x_0) \cap M\bar{\mathbf{B}}$$

is a uniform tangent set to Ω_i at x_0 for any $M > 0$. Now let the transversality related constants of $\Omega_1, \ldots, \Omega_n$ be $\delta_1 > 0$, $\delta_2 > 0$ and $\alpha > 0$. Then if we take $M \geq \delta_1(1+\alpha) + \delta_2$, from the fact that $\Omega_1, \ldots, \Omega_n$ are transversal at x_0 follows that $D_1(x_0), \ldots, D_n(x_0)$ are transversal at the origin. □

We aim to prove that for a finite number of sets, from strong tangential transversality follows transversality at a given point, by applying the characterization for transversality obtained in the previous section.

Theorem 4.6. *Let $\Omega_1, \ldots, \Omega_n$ be closed subsets of the Banach space X satisfying $x_0 \in \bigcap_{i=1}^n \Omega_i$. If $\Omega_1, \ldots, \Omega_n$ are strongly tangentially transversal at x_0, then they are transversal at x_0.*

Proof. We are going to use Theorem 3.7 (characterization of transversality). Since $\Omega_1, \ldots, \Omega_n$ are strongly tangentially transversal at x_0, there exist respective uniform tangent sets $\tilde{D}_1(x_0), \ldots, \tilde{D}_n(x_0)$ tranversal at 0. Without loss of generality we may (and do) assume them to be closed, convex and containing the origin (see Remark 4.2). Because $\tilde{D}_1(x_0), \ldots, \tilde{D}_n(x_0)$ are assumed to be convex, their transversality at the origin is equivalent (from Theorem 2.10) to the existence of $\alpha > 0$ and $\eta > 0$ such that

$$\bigcap_{i=1}^{n} (\tilde{D}_i(x_0) - y_i) \cap \eta\bar{\mathbf{B}} \neq \emptyset$$

for all $y_i \in X$ such that $\|y_i\| \leq \alpha\eta$, $i = 1, 2, \ldots, n$.

Multiplying the above intersection by $\dfrac{1}{\alpha\eta}$, we obtain

$$\bigcap_{i=1}^{n} (D_i(x_0) - x_i) \cap \bar{\delta}\bar{\mathbf{B}} \neq \emptyset$$

for all $x_i \in X$ such that $\|x_i\| \leq 1$ where $D_i(x_0) := \frac{1}{\alpha\eta}\tilde{D}_i(x_0)$, $i = 1, 2, \ldots, n$ and $\bar{\delta} := \frac{1}{\alpha}$. Let

$$M_0 := \sup\big\{\|\nu\| : \text{there exists } i \in \{1, 2, \ldots, n\}, \text{ such that } \nu \in D_i(x_0)\big\}.$$

Let us fix a positive real $\varepsilon \in (0, \frac{1}{n})$.

Then the fact that $D_i(x_0)$ is a uniform tangent set to Ω_i at x_0 implies that there exist $\delta_i > 0$ and $\mu_i > 0$ such that for any $t \in (0, \mu_i]$, $x \in \Omega_i \cap (x_0 + \delta_i\bar{\mathbf{B}})$, and $\nu \in D_i(x_0)$ it is true that

$$\Omega_i \ \cap (x + t(\nu + \varepsilon\bar{\mathbf{B}})) \neq \emptyset.$$

We put $\tilde{\delta} := \min\{\bar{\delta}, \delta_1, ..., \delta_n\}$, $\mu := \min\{\mu_1, ..., \mu_n\}$ and $\delta := \frac{1}{2}\tilde{\delta} > 0$.

For the so defined $\delta > 0$ let us fix arbitrary $a_i \in \delta\bar{\mathbf{B}}$, $x_i \in \Omega_i \cap (x_0 + a_i + \delta\bar{\mathbf{B}})$ for $i \in \{1, 2, \ldots, n\}$, and $x \in (x_0 + \delta\bar{\mathbf{B}})$, such that $\max_i \|x_i - a_i - x\| > 0$.

Observe that $x_i \in x_0 + a_i + \delta\bar{\mathbf{B}} \subset x_0 + \delta\bar{\mathbf{B}} + \delta\bar{\mathbf{B}} = x_0 + 2\delta\bar{\mathbf{B}} = x_0 + \tilde{\delta}\bar{\mathbf{B}}$.

Let $\omega_i := x_i - a_i - x$. We have that

$$\max_i \|x_i - a_i - x\| = \max_i \|\omega_i\| > 0$$

by assumption for the points x_i, a_i and x. We set the vector $\tilde{\omega}_i \in X$ to be $-\frac{\omega_i}{\|\omega_i\|}$ in the case $\omega_i \neq \mathbf{0}$, and to $\mathbf{0}$ when $\omega_i = \mathbf{0}$. Either way $\tilde{\omega}_i \in \bar{\mathbf{B}}$.

Let us fix $i \in \{1, 2, ..., n\}$. Since $\tilde{\omega}_i \in \bar{\mathbf{B}}$, by transversality of $D_1(x_0), \ldots, D_n(x_0)$ at the origin we have

$$(D_i(x_0) - \tilde{\omega}_i) \cap \Big(\bigcap_{j \neq i} D_j(x_0)\Big) \neq \emptyset$$

or equivalently: there exist $b_i \in D_i(x_0)$ and $c_i \in \bigcap_{j\neq i} D_j(x_0)$ such that

$$c_i = b_i - \tilde{\omega}_i \in (D_i(x_0) - \tilde{\omega}_i) \cap \Big(\bigcap_{j \neq i} D_j(x_0)\Big).$$

Thus $\tilde{\omega}_i = b_i - c_i$. Since $c_j \in D_i(x_0)$ for any $j \neq i$ and $b_i \in D_i(x_0)$, by the convexity of $D_i(x_0)$ we have

$$\nu_i := \frac{1}{n} b_i + \sum_{j \neq i} \frac{1}{n} c_j \ \in D_i(x_0).$$

Let us fix $t > 0$ such that

$$t := \min \left\{ \mu, \frac{1}{2\varepsilon} \max_i \|\omega_i\|, n \min_i \{ \|\omega_i\| : \|\omega_i\| > 0 \} \right\}.$$

For any $\omega_i \neq \mathbf{0}$ it will then follow that $1 - \frac{t}{n\|\omega_i\|} \geq 0$.

Since $\|x_i - x_0\| \leq \tilde{\delta} \leq \delta_i$, $\nu_i \in D_i(x_0)$ and $t \leq \mu_i$, we have

$$\Omega_i \cap (x_i + t(\nu_i + \varepsilon \bar{\mathbf{B}})) \neq \emptyset.$$

Equivalently: there exists $\tilde{\nu}_i \in X$ such that $\|\nu_i - \tilde{\nu}_i\| \leq \varepsilon$ and $x_i + t\tilde{\nu}_i \in \Omega_i$. We set $\hat{x}_i := x_i + t\tilde{\nu}_i$ and $\hat{x} := x + t \sum_{j=1}^n \frac{1}{n} c_j$. Then we have the estimate:

$$\begin{aligned}
\|\hat{x}_i - a_i - \hat{x}\| &= \Big\| x_i + t\tilde{\nu}_i - a_i - \Big(x + t \sum_{j=1}^n \frac{1}{n} c_j \Big) \Big\| = \\
&= \Big\| x_i - a_i - x + t\Big(\nu_i + \tilde{\nu}_i - \nu_i - \sum_{j=1}^n \frac{1}{n} c_j \Big) \Big\| = \\
&= \Big\| \omega_i + t\Big(\frac{1}{n} b_i + \sum_{j \neq i} \frac{1}{n} c_j - \sum_{j=1}^n \frac{1}{n} c_j \Big) + t(\tilde{\nu}_i - \nu_i) \Big\| = \\
&= \Big\| \omega_i + t\Big(\frac{1}{n} b_i - \frac{1}{n} c_i \Big) + t(\tilde{\nu}_i - \nu_i) \Big\| = \\
&= \Big\| \omega_i + \frac{t}{n} \tilde{\omega}_i + t(\tilde{\nu}_i - \nu_i) \Big\|.
\end{aligned}$$

In the case when $\omega_i \neq \mathbf{0}$ we have that $\tilde{\omega}_i = -\frac{\omega_i}{\|\omega_i\|}$, and so

$$\begin{aligned}
\|\hat{x}_i - a_i - \hat{x}\| &= \Big\| \omega_i + \frac{t}{n} \tilde{\omega}_i + t(\tilde{\nu}_i - \nu_i) \Big\| = \\
&= \Big\| \omega_i + \frac{t}{n} \Big(-\frac{\omega_i}{\|\omega_i\|} \Big) + t(\tilde{\nu}_i - \nu_i) \Big\| \leq \\
&\leq \Big\| \omega_i + \frac{t}{n} \Big(-\frac{\omega_i}{\|\omega_i\|} \Big) \Big\| + t\|\tilde{\nu}_i - \nu_i\| \leq \\
&\leq \|\omega_i\| \Big(1 - \frac{t}{n\|\omega_i\|} \Big) + t\varepsilon = \\
&= \|\omega_i\| - \frac{t}{n} + t\varepsilon = \\
&= \|\omega_i\| - t\Big(\frac{1}{n} - \varepsilon \Big) = \|x_i - a_i - x\| - t\Big(\frac{1}{n} - \varepsilon \Big).
\end{aligned}$$

In the case when $\omega_i = \mathbf{0}$ we have that $\tilde{\omega_i} = \mathbf{0}$, and so

$$\begin{aligned}\|\hat{x}_i - a_i - \hat{x}\| &= \Big\|\omega_i + \frac{t}{n}\tilde{\omega_i} + t(\tilde{\nu}_i - \nu_i)\Big\| = \\ &= \Big\|\mathbf{0} + \frac{t}{n}\mathbf{0} + t(\tilde{\nu}_i - \nu_i)\Big\| = \\ &= t\big\|\tilde{\nu}_i - \nu_i\big\| \le t\varepsilon \le \frac{1}{2\varepsilon}\max_i \|\omega_i\|\varepsilon = \\ &= \frac{1}{2}\max_i \|x_i - a_i - x\| = \\ &= \max_i \|x_i - a_i - x\| - \frac{1}{2}\max_i \|x_i - a_i - x\|.\end{aligned}$$

With these two estimates in mind, we can now set

$$\theta := \min\Big\{t\Big(\frac{1}{n} - \varepsilon\Big), \frac{1}{2}\max_i \|x_i - a_i - x\|\Big\},$$

and it follows that for any i

$$\|\hat{x}_i - a_i - \hat{x}\| \le \max_i \|x_i - a_i - x\| - \theta.$$

Having in mind that from $\nu_i \in D_i(x_0)$ follows $\|\nu_i\| \le M_0$, and that $\varepsilon < \frac{1}{n}$ we can see

$$\begin{aligned}\|\hat{x}_i - x_i\| = \|x_i + t\tilde{\nu}_i - x_i\| = t\|\tilde{\nu}_i\| \le t(\|\tilde{\nu}_i - \nu_i\| + \|\nu_i\|) \le \\ \le t(\varepsilon + M_0) = t\left(\frac{1}{n} - \varepsilon\right)\frac{M_0 + \varepsilon}{(\frac{1}{n} - \varepsilon)} \le \theta M\end{aligned}$$

where we have set $M := \dfrac{M_0 + \varepsilon}{(\frac{1}{n} - \varepsilon)}$.

Since $c_j \in D_i(x_0)$ for any $j \ne i$ we have $\|c_j\| \le M_0$, and so

$$\begin{aligned}\|\hat{x} - x\| &= \Big\|x + t\sum_{j=1}^{n}\frac{1}{n}c_j - x\Big\| = t\Big\|\sum_{j=1}^{n}\frac{1}{n}c_j\Big\| \\ &\le \frac{t}{n}\sum_{j=1}^{n}\|c_j\| \le \frac{t}{n}\sum_{j=1}^{n}M_0 = tM_0 < \theta M.\end{aligned}$$

We have thus obtained that the inequalities of the characterization are satisfied:

$$\max_i \|x_i - \hat{x}_i\| \le \theta M \quad \text{and} \quad \|x - \hat{x}\| \le \theta M$$

and

$$\max_i \|\hat{x}_i - a_i - \hat{x}\| \le \max_i \|x_i - a_i - x\| - \theta.$$

With this result the assumptions of Theorem 3.7 are satisfied and the transversality of $\Omega_1, \dots, \Omega_n$ at x_0 follows. $\square$

5. Sum rule for the Clarke subdifferential and Lagrange multiplier rule

Throughout this section $\hat{T}_A(x_0)$ is going to denote the Clarke tangent cone to the closed set A at the point x_0, and $N_A(x_0)$ is going to denote the Clarke normal cone to A at x_0, i.e. $N_A(x_0) = (\hat{T}_A(x_0))^\circ$.

The main ingredient in most proofs of sum rules is the assertion that (under suitable assumptions) any normal to the intersection of some closed sets can be expressed as a sum of normals to the sets (we call it normal intersection property, see [3]). We know that two strongly tangentially transversal sets at some point possess normal intersection property for Clarke normals at the same point. More precisely, we have

Proposition 5.1. ([4], Corollary 3.11) *Let A and B be closed subsets of the Banach space X and let $x_0 \in A \cap B$. Let A and B be strongly tangentially transversal at x_0. Then the following hold true:*

(a) $\hat{T}_A(x_0) \cap \hat{T}_B(x_0) \subset \hat{T}_{A\cap B}(x_0)$

(b) $N_{A\cap B}(x_0) \subset N_A(x_0) + N_B(x_0)$

The generalization of the above to a finite number of sets is obtained by an induction argument and it makes use of the following facts:

Proposition 5.2. ([4], Proposition 3.4) *Let A and B be closed subsets of the Banach space X, and let A and B be tangentially transversal at $x_0 \in A \cap B$. If $D_A(x_0)$ is a uniform tangent set to A at the point x_0 and $D_B(x_0)$ is a uniform tangent set to B at the point x_0, then $D_A(x_0) \cap D_B(x_0)$ is a uniform tangent set to $A \cap B$ at the point x_0.*

Proposition 5.3. *Let $\Omega_1, ..., \Omega_n$ be closed subsets of the Banach space X, and let $\Omega_1, ..., \Omega_n$ be strongly tangentially transversal at $x_0 \in \bigcap_{i=1}^n \Omega_i$. Then $\Omega_1 \cap \Omega_2$, Ω_3,..., Ω_n are strongly tangentially transversal at x_0.*

Proof. $\Omega_1, \dots, \Omega_n$ are strongly tangentially transversal at x_0: there exist respective uniform tangent sets $D_1, \dots, D_n$ transversal at 0. Note that Theorem 4.6 implies that then $\Omega_1, \dots, \Omega_n$ are transversal at x_0. We apply Proposition 5.2 to Ω_1 and Ω_2 and their uniform tangent sets D_1 and D_2, obtaining that $D_1 \cap D_2$ is a uniform tangent set to $\Omega_1 \cap \Omega_2$ at x_0 (indeed, Ω_1 and Ω_2 are transversal directly from the definition of transversality of $\Omega_1, \dots, \Omega_n$, hence they are tangentially transversal). By Remark 2.2 we know that $D_1 \cap D_2$, $D_3, \dots, D_n$ are tranversal at 0, so it follows that $\Omega_1 \cap \Omega_2$, Ω_3,..., Ω_n are strongly tangentially transversal at x_0. □

Proposition 5.4. (Normal intersection property) *Let $\Omega_1, \dots, \Omega_n$ be closed subsets of the Banach space X, and let $\Omega_1, \dots, \Omega_n$ be strongly tangentially transversal at $x_0 \in \bigcap_{i=1}^n \Omega_i$. Then*

$$N_{\Omega_1 \cap \Omega_2 \dots \cap \Omega_n}(x_0) \subset \sum_{i=1}^n N_{\Omega_i}(x_0).$$

Proof. The base of the induction is $n = 2$, for which the assertion has already been verified. We assume that the assertion is true for some $n \in \mathbb{N}$ and we check that it remains true for $n + 1$. Indeed, for $\Omega_1, \dots, \Omega_n$, Ω_{n+1} by repeatedly applying Proposition 5.3 we have that $\Omega := \Omega_1 \cap \Omega_2 \cap \dots \cap \Omega_n$ and Ω_{n+1} are strongly tangentially transversal at x_0.

From Proposition 5.1 we have that

$$N_{\Omega\cap\Omega_{n+1}}(x_0) \subset N_\Omega(x_0) + N_{\Omega_{n+1}}(x_0).$$

Now we use the induction assumption to obtain

$$N_{\Omega_1\cap\Omega_2\ldots\cap\Omega_{n+1}}(x_0) = N_{\Omega\cap\Omega_{n+1}}(x_0) \subset N_\Omega(x_0) + N_{\Omega_{n+1}}(x_0) \subset$$
$$\subset \sum_{i=1}^{n} N_{\Omega_i}(x_0) + N_{\Omega_{n+1}}(x_0) = \sum_{i=1}^{n+1} N_{\Omega_i}(x_0). \qquad \square$$

Let us recall that

Definition 5.5. Let X be a Banach space, $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous and proper and $x \in X$ be in $\mathrm{dom} f$. It is said that $x^* \in X^*$ belongs to the *Clarke regular subdifferential* $\partial_C f(x)$ (to the *Clarke singular subdifferential* $\partial_C^\infty f(x)$) iff

$$(x^*, -1) \in N_{epif}(x, f(x))((x^*, 0) \in N_{epif}(x, f(x))). \qquad \square$$

Let $f_i : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, $i = 1, 2, \ldots, n$, be lower semicontinuous and proper functions and $x_0 \in X$ be in $\bigcap_{i=1}^n dom f_i$. We are going to apply the normal intersection property (Proposition 5.4) to the closed sets

$$C_i := \{(x, r_1, ..., r_n) \in X \times \mathbb{R}^n : \; r_i \geq f_i(x)\}, \quad i = 1, 2, \ldots, n$$

in order to obtain a sum rule for the Clarke subdifferential.

Proposition 5.6. (Sum rule) *Let the sets $C_1, ..., C_n$ be strongly tangentially transversal at $(x_0, f_1(x_0), \ldots, f_n(x_0))$. Then*

$$\partial_C\Big(\sum_{i=1}^n f_i\Big)(x_0) \subset \sum_{i=1}^n \partial_C f_i(x_0)$$

and

$$\partial_C^\infty\Big(\sum_{i=1}^n f_i\Big) \subset \sum_{i=1}^n \partial_C^\infty f_i(x_0).$$

Proof. It is straightforward to see that

$$C_1 \cap C_2 \cap \cdots \cap C_n \subset C := \Big\{(x, r_1, \ldots, r_n) \in X \times \mathbb{R}^n : \sum_{i=1}^n r_i \geq \sum_{i=1}^n f_i(x)\Big\}.$$

Using Lemma 2.4 from [10] (with the identity g, $C := C_1$, $D := C_2 \cap C_3 ... \cap C_n$ and $B := C$) we have that

$$\hat{T}_{C_1\cap C_2\cap\cdots\cap C_n}(x_0, f_1(x_0), ..., f_n(x_0)) \subset \hat{T}_C(x_0, f_1(x_0), ..., f_n(x_0)).$$

Hence by polarity and the normal intersection property (Proposition 5.4) we have that

$$N_C(x_0, f_1(x_0), \ldots, f_n(x_0)) \subset N_{C_1\cap C_2\cap\cdots\cap C_n}(x_0, f_1(x_0), \ldots, f_n(x_0)) \subset$$
$$\subset \sum_{i=1}^n N_{C_i}(x_0, f_1(x_0), \ldots, f_n(x_0)). \tag{9}$$

Apparently $\hat{T}_C(x_0, f_1(x_0), \ldots, f_n(x_0))$ coincides with the set

$$\Big\{(v, r_1, \ldots, r_n) \in X \times \mathbb{R}^n : \Big(v, \sum_{i=1}^n r_i\Big) \in \hat{T}_{epi\left(\sum\limits_{i=1}^n f_i\right)}\Big(x_0, \sum_{i=1}^n f_i(x_0)\Big)\Big\}.$$

By polarity we have

$$N_C(x_0, f_1(x_0), \ldots, f_n(x_0)) = \hat{T}_C^\circ(x_0, f_1(x_0), \ldots, f_n(x_0)) = \tag{10}$$

$$= \left\{\begin{array}{l}(x^*, s_1, \ldots, s_n) \in X^* \times \mathbb{R}^n : \langle x^* v\rangle + \sum\limits_{i=1}^n s_i r_i \leq 0 \\ \text{for all } \Big(v, \sum\limits_{i=1}^n r_i\Big) \in \hat{T}_{epi\left(\sum\limits_{i=1}^n f_i\right)}\Big(x_0, \sum\limits_{i=1}^n f_i(x_0)\Big).\end{array}\right\}. \tag{11}$$

Therefore we have that $N_C(x_0, f_1(x_0), ..., f_n(x_0))$ contains the set

$$\Big\{(x^*, s, ..., s) \in X^* \times \mathbb{R}^n : (x^*, s) \in N_{epi\left(\sum\limits_{i=1}^n f_i\right)}\Big(x_0, \sum_{i=1}^n f_i(x_0)\Big)\Big\},$$

hence

$$x^* \in \partial_C\Big(\sum_{i=1}^n f_i\Big) \implies (x^*, -1, ..., -1) \in N_C(x_0, f_1(x_0), ..., f_n(x_0))$$

and

$$x^* \in \partial_C^\infty\Big(\sum_{i=1}^n f_i\Big) \implies (x^*, 0, ..., 0) \in N_C(x_0, f_1(x_0), ..., f_n(x_0)).$$

By using (11), (9) and

$$N_{C_i}(x_0, f_1(x_0), ..., f_n(x_0)) =$$
$$= \{(x^*, 0, .., 0, s_i, 0, .., 0) \in X^* \times \mathbb{R}^n : (x^*, s_i) \in N_{epif_i}(x_0, f_i(x_0))\}$$

we obtain respectively

$$\partial_C\Big(\sum_{i=1}^n f_i\Big)(x_0) \subset \sum_{i=1}^n \partial_C f_i(x_0)$$

and

$$\partial_C^\infty\Big(\sum_{i=1}^n f_i\Big)(x_0) \subset \sum_{i=1}^n \partial_C^\infty f_i(x_0).$$ □

Remark 5.7. An overview of the known results concerning sum rules for Clarke subdifferentials is presented in the introduction of [4]. As it is pointed out there, sufficient conditions for the normal intersection property for the Clarke normal cone (sum rule for the Clarke subdifferential) involve strong assumptions on one of the sets (epigraphs) – existence of a hypertangent, epi-Lipschitz-like (directionally Lipschitz, Lipschitz-like). Using strong tangential transversality, this should not be always the case. In Theorem 4.2 (see also Remark 4.7) of [12] we provided an application of the sum rule proved in [4] (whose generalization to more than two summands is

Proposition 5.6) for the case of two functions which are "Lipschitz in one variable" where neither nonempty interior nor finite codimensionality of the Clarke tangent cone to any of the epigraphs is needed. It is straightforward to extend it to n lower semicontinuous and proper functions, which are "Lipschitz in all but one variable". □

We prove another application of Proposition 5.4 (normal intresection property) of type Lagrange multiplier rule.

Definition 5.8. Let X be a Banach space and S be a subset of X. The set S is said to be *quasisolid* if its closed convex hull $\overline{co}S$ has nonempty interior in its closed affine hull, i.e. if there exists a point $x_0 \in \overline{co}S$ such that $\overline{co}(S - x_0)$ has nonempty interior in $\overline{span}(S - x_0)$ (the closed subspace spanned by $S - x_0$).

Theorem 5.9. (Lagrange multiplier rule) *Let X be Banach space and let us consider the optimization problem*

$$\varphi(x) \to min \quad subject\ to \quad x \in S := \bigcap_{i=1}^{n} S_i$$

where $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and proper function, and S_i are closed subsets of X, $i = 1, 2, ..., n$. Let $\overline{x}$ be a solution of the above problem. We set

$$\tilde{S} := S \times (-\infty, \varphi(\overline{x})] \quad and \quad \tilde{R} := epi\varphi.$$

Let D_{S_i} be a uniform tangent set to S_i at $\overline{x}$ for $i = 1, ..., n$, and let the set $D_S := \bigcap_{i=1}^n D_{S_i}$ be such that it generates $\hat{T}_S(\overline{x})$, $D_{\tilde{S}} := D_S \times [-1, 0]$. Additionally, let $S_1, \ldots, S_n$ be strongly tangentially transversal at $\overline{x}$. Let $D_{\tilde{R}}$ be a uniform tangent set to $\tilde{R}$ at $(\overline{x}, \varphi(\overline{x}))$ and such that it generates $\hat{T}_{\tilde{R}}(\overline{x}, \varphi(\overline{x}))$. We assume that the set $D_{\tilde{S}} - D_{\tilde{R}}$ is quasisolid.

Then there exists a pair $(\xi, \eta) \in X^* \times \mathbb{R}$ such that

(i) $(\xi, \eta) \neq (\boldsymbol{0}, 0)$;

(ii) $\eta \in \{0, 1\}$;

(iii) $\xi = \sum_{i=1}^n \xi_i$ *where* $\xi_i \in N_{S_i}(\overline{x})$, $i = 1, 2, ..., n$;

(iv) $-(\xi, \eta) \in N_{epi\varphi}(\overline{x}, \varphi(\overline{x}))$.

Proof. The proof is based on the following result:

Theorem 5.10. ([5], Theorem 3.4) *Let X be Banach space and let us consider the optimization problem*

$$\varphi(x) \to min \quad subject\ to \quad x \in S$$

where $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and proper function, S is a closed subset of X. Let $\overline{x}$ be a solution of the above problem. We set

$$\tilde{S} := S \times (-\infty, \varphi(\overline{x})] \quad and \quad \tilde{R} := epi\varphi.$$

Let D_S be a uniform tangent set to S at $\overline{x}$, $D_{\tilde{S}} := D_S \times [-1, 0]$ and let $D_{\tilde{R}}$ be a uniform tangent set to $\tilde{R}$ at $(\overline{x}, \varphi(\overline{x}))$.

We assume that the set $D_{\tilde{S}} - D_{\tilde{R}}$ is quasisolid. Then there exists a pair $(\xi, \eta) \in X^* \times \mathbb{R}$ such that

(i) $(\xi, \eta) \neq (\mathbf{0}, 0)$*;*

(ii) $\eta \in \{0, 1\}$*;*

(iii) $\langle \xi, v \rangle \leq 0$ *for every* $v \in D_S$*;*

(iv) $\langle \xi, w \rangle + \eta s \geq 0$ *for every* $(w, s) \in D_{\tilde{R}}$*.*

The sets $S_1, \dots, S_n$ are strongly tangentially transversal at $\overline{x}$, and $D_{S_1}, \dots, D_{S_n}$ are uniform tangent set to them, respectively, at $\overline{x}$. We do not know whether$D_{S_1}, \dots, D_{S_n}$ are transversal at the origin.

By Proposition 5.3 we have that $S_1 \cap S_2, S_3, \dots, S_n$ are strongly tangentially transversal at $\overline{x}$. By Proposition 5.2, because S_1 and S_2 are strongly tangentially transversal and therefore tangentially transversal, we have that $D_{S_1} \cap D_{S_2}$ is a uniform tangent set to $S_1 \cap S_2$ at $\overline{x}$. Using these observations recursively, we find that $D_S = \bigcap_{i=1}^n D_{S_i}$ is a uniform tangent set to $S = \bigcap_{i=1}^n S_i$ at $\overline{x}$.

Since the set $D_{\tilde{S}} - D_{\tilde{R}}$ is quasisolid, we apply Theorem 5.10 to obtain $(\xi, \eta) \in X^* \times \mathbb{R}$ such that

(i) $(\xi, \eta) \neq (\mathbf{0}, 0)$;

(ii) $\eta \in \{0, 1\}$;

(iii) $\langle \xi, v \rangle \leq 0$ for every $v \in D_S$;

(iv) $\langle \xi, w \rangle + \eta s \geq 0$ for every $(w, s) \in D_{\tilde{R}}$.

Since D_S generates $\hat{T}_S(\overline{x})$, (iii) is equivalent to $\xi \in N_S(\overline{x}) = N_{S_1 \cap \dots \cap S_n}(\overline{x})$. By applying Proposition 5.4 we get that there exist $\xi_i \in N_{S_i}(\overline{x})$, $i = 1, 2, \dots, n$, such that $\xi = \sum_{i=1}^n \xi_i$.

Since $D_{\tilde{R}}$ generates $\hat{T}_{\tilde{R}}(\overline{x}, \varphi(\overline{x}))$, (iv) is equivalent to

$$-(\xi, \eta) \in N_{\tilde{R}}(\overline{x}, \varphi(\overline{x})) = N_{epi\varphi}(\overline{x}, \varphi(\overline{x})). \qquad \square$$

References

[1] S. Apostolov, M. Bivas: *Characterizations of metric (sub)regularity via (sub)transversality*, J. Convex Analysis 32/4 (2025).

[2] S. Apostolov, M. Bivas, N. Ribarska: *Characterizations of some transversality-type properties*, Set-Valued Var. Analysis 30 (2022) 1041–1060.

[3] M. Bivas, M. Krastanov, N. Ribarska: *On tangential transversality*, J. Math. Analysis Appl. 481/1 (2020), art.no. 123445, 21p.

[4] M. Bivas, M. Krastanov, N. Ribarska: *On strong tangential transversality*, J. Math. Analysis Appl. 490/1 (2020), art.no. 124235, 16p.

[5] M. Bivas, N. Ribarska, M. Valkov: *Properties of uniform tangent sets and Lagrange multiplier rule*, C. R. Acad. Bulgare Sci. 71/7 (2018) 875–884.

[6] H. T. Bui, N. D. Cuong, A. Y. Kruger: *Geometric and metric characterizations of transversality properties*, Vietnam J. Math. 48/2 (2020) 277–297.

[7] N. D. Cuong, A. Y. Kruger: *Transversality properties: Primal sufficient conditions*, Set-Valued Var. Analysis 29 (2021) 221–256.

[8] D. Drusvyatskiy, A. D. Ioffe, A. S. Lewis: *Transversality and alternating projections for nonconvex sets*, Found. Comput. Math. 15/6 (2015) 1637–1651.

[9] A. Ioffe: *Variational Analysis of Regular Mappings. Theory and Applications*, Springer Monographs in Mathematics, Springer, Berlin (2017).

[10] A. Jourani, L. Thibault: *Extensions of subdifferential calculus rules in Banach spaces*, Canad. J. Math. 48/4 (1996) 834–848.

[11] M. I. Krastanov, N. K. Ribarska: *Nonseparation of sets and optimality conditions*, SIAM J. Control Optimization 55/3 (2017) 1598–1618.

[12] N. Ribarska, M. Tasheva: *Transversality versus strong tangential transversality*, J. Math. Analysis Appl. 524/2 (2023), art.no. 127215, 13p.