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On the Infimum of the Upper Envelope of Certain Families of Functions

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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Given a topological space X, an interval $I \subseteq \mathbf{R}$ and five continuous functions $\varphi, \psi, \omega : X \to \mathbf{R}$, $\alpha, \beta : I \to \mathbf{R}$, we are interested in the infimum of the function $\Phi : X \to]-\infty, +\infty]$ defined by

$$\Phi(x) = \sup_{\lambda \in I}(\alpha(\lambda)\varphi(x) + \beta(\lambda)\psi(x)) + \omega(x)\,.$$

Using a recent minimax theorem of the author [see *Minimax theorems in a fully non-convex setting*, J. Nonlinear Var. Analysis 3 (2019) 45-52], we build a general scheme which provides the exact value of $\inf_X \Phi$ for a large class of functions Φ. When additional compactness conditions are satisfied, our scheme provides also the existence of (explicitly detected) functions $\gamma, \eta : X \to \mathbf{R}$ such that, for some $\tilde{x} \in X$, one has

$$\gamma(\tilde{x})\varphi(\tilde{x}) + \eta(\tilde{x})\psi(\tilde{x}) + \omega(\tilde{x}) = \inf_{x \in X}(\gamma(\tilde{x})\varphi(x) + \eta(\tilde{x})\psi(x) + \omega(x))\,.$$

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1. Introduction

Throughout the sequel, X is a topological space, $I \subseteq \mathbf{R}$ is an interval and

$$\varphi, \psi, \omega : X \to \mathbf{R}, \ \ \alpha, \beta : I \to \mathbf{R}$$

are five continuous functions.

According to a standard terminology, the upper envelope of the family of functions $\{\alpha(\lambda)\varphi(\cdot) + \beta(\lambda)\psi(\cdot) + \omega(\cdot)\}_{\lambda \in I}$ is the function $\Phi : X \to]-\infty, +\infty]$ defined by

$$\Phi(x) := \sup_{\lambda \in I}(\alpha(\lambda)\varphi(x) + \beta(\lambda)\psi(x) + \omega(x))$$

for all $x \in X$. The function Φ can have a quite complicated structure, depending on the specific choice of α and β. We are interested in the infimum of Φ. In this connection, we have the obvious estimate

$$\sup_{\lambda \in I} \inf_{x \in X}(\alpha(\lambda)\varphi(x) + \beta(\lambda)\psi(x) + \omega(x)) \leq \inf_X \Phi\,. \tag{1}$$

In general, the inequality in (1.1) is strict. A particularly simple example is provided by taking: $X = [0, 2]$, $I = [0, 1]$, $\varphi(x) = 1 - x^2$, $\psi(x) = x$, $\omega = 0$, $\alpha(\lambda) = \lambda$, $\beta(\lambda) = 1$.

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Actually, for each $x \in [0,1]$, we have

$$\sup_{\lambda\in[0,1]} (\lambda(1-x^2)+x) = \begin{cases} 1-x^2+x & \text{if } x \in [0,1], \\ x & \text{if } x \in]1,2]. \end{cases}$$

Consequently $\inf_{[0,2]} \Phi = 1$. Now, fix $\lambda \in [0,1]$. We have

$$\inf_{x\in[0,2]} (\lambda(1-x^2)+x) = \min\{\lambda, -3\lambda+2\}\,.$$

Therefore

$$\sup_{\lambda\in[0,1]} \inf_{x\in[0,2]} (\lambda(1-x^2)+x) = \frac{1}{2}\,,$$

and so the inequality in (1) is strict.

Of course, it is interesting to know when equality occurs in (1).

Before stating our contributions to this question, let us recall some definitions. Let Y be a topological space. A function $\eta : Y \to \mathbf{R}$ is said to be inf-connected (resp. inf-compact) if, for each $r \in \mathbf{R}$, the set $\eta^{-1}(]-\infty, r[)$ (resp. $\eta^{-1}(]-\infty, r])$ is connected (resp. compact). If $-\eta$ is inf-compact, then η is said to be sup-compact.

The aim of this paper is to estabilish the following results whose proofs are based on the use of a quite recent minimax theorem, that is Theorem 1.2 of [5].

Theorem 1.1. *Let $I := [a,b]$ be compact. Assume that:*

(i_1) *there exists a set $D \subseteq I$, dense in I, such that, for each $\lambda \in D$, the function $\alpha(\lambda)\varphi(\cdot)+\beta(\lambda)\psi(\cdot)+\omega(\cdot)$ is inf-connected;*

(i_2) *the functions α, β are C^1 in $]a,b[$, $\alpha'(\lambda) \neq 0$ for all $\lambda \in]a,b[$ and the function $g := \frac{\beta'}{\alpha'}$ is strictly monotone in $]a,b[$.*

Finally, assume that one of the following conditions holds:

(i_3) *g is increasing in $]a,b[$ and $\alpha'(\lambda)\psi(x) < 0$ for all $(\lambda, x) \in]a,b[\times(X \setminus \psi^{-1}(0))$;*

(i_4) *g is decreasing in $]a,b[$ and $\alpha'(\lambda)\psi(x) > 0$ for all $(\lambda, x) \in]a,b[\times(X \setminus \psi^{-1}(0))$.*

Set

$$\gamma := \inf_{]a,b[} g\,, \quad \textit{and} \quad \delta := \sup_{]a,b[} g$$

and let $h : \mathbf{R} \to [a,b]$ be the function defined by

$$h(\mu) = \begin{cases} g^{-1}(\mu) & \textit{if } \mu \in g(]a,b[) \\ a & \textit{if } \mu \leq \gamma \textit{ and } (i_3) \textit{ holds or } \mu \geq \delta \textit{ and } (i_4) \textit{ holds} \\ b & \textit{if } \mu \leq \gamma \textit{ and } (i_4) \textit{ holds or } \mu \geq \delta \textit{ and } (i_3) \textit{ holds}. \end{cases}$$

Then, we have

$$\Phi(x) = \begin{cases} \alpha\Big(h\Big(-\frac{\varphi(x)}{\psi(x)}\Big)\Big)\varphi(x) + \beta\Big(h\Big(-\frac{\varphi(x)}{\psi(x)}\Big)\Big)\psi(x)+\omega(x) & \textit{if } x \in X \setminus \psi^{-1}(0) \\ \max\{\alpha(a)\varphi(x), \alpha(b)\varphi(x)\}+\omega(x) & \textit{if } x \in \psi^{-1}(0) \end{cases}$$

and

$$\inf_X \Phi = \sup_{\lambda\in I} \inf_{x\in X} (\alpha(\lambda)\varphi(x)+\beta(\lambda)\psi(x)+\omega(x))\,.$$

Theorem 1.2. *Assume that:*

(i_0) *there exists a (possibly different) topology τ_1 on X such, for every $\lambda \in I$, the function $\alpha(\lambda)\varphi(\cdot) + \beta(\lambda)\psi(\cdot) + \omega(\cdot)$ is τ_1-lower semicontinuous and, for some $\lambda_0 \in I$, the function $\alpha(\lambda_0)\varphi(\cdot) + \beta(\lambda_0)\psi(\cdot) + \omega(\cdot)$ is τ_1-inf-compact;*

(i_1) *there exists a set $D \subseteq I$, dense in I, such that, for each $\lambda \in D$, the function $\alpha(\lambda)\varphi(\cdot) + \beta(\lambda)\psi(\cdot) + \omega(\cdot)$ is inf-connected;*

(i_2') *the functions α, β are derivable in A, where $\mathrm{int}(I) \subseteq A \subseteq I$, $\alpha'(\lambda) \neq 0$ for all $\lambda \in A$, the function $g := \frac{\beta'}{\alpha'}$ is strictly monotone in A and $-\frac{\varphi(x)}{\psi(x)} \in g(A)$ for all $x \in X \setminus \psi^{-1}(0)$.*

Furthermore, assume that one of the following conditions holds:

(i_3') *g is increasing in A and $\alpha'(\lambda)\psi(x) < 0$ for all $(\lambda, x) \in A \times (X \setminus \psi^{-1}(0))$;*

(i_4') *g is decreasing in A and $\alpha'(\lambda)\psi(x) > 0$ for all $(\lambda, x) \in A \times (X \setminus \psi^{-1}(0))$.*

Then, we have

$$\Phi(x) = \begin{cases} \alpha\Big(g^{-1}\Big(-\frac{\varphi(x)}{\psi(x)}\Big)\Big)\varphi(x) + \beta\Big(g^{-1}\Big(-\frac{\varphi(x)}{\psi(x)}\Big)\Big)\psi(x) + \omega(x) & \text{if } x \in X \setminus \psi^{-1}(0) \\ \sup_{\lambda \in I}(\alpha(\lambda)\varphi(x) + \omega(x)) & \text{if } x \in \psi^{-1}(0) \end{cases}$$

and

$$\inf_X \Phi = \sup_{\lambda \in I} \inf_{x \in X} (\alpha(\lambda)\varphi(x) + \beta(\lambda)\psi(x) + \omega(x)).$$

Theorem 1.3. *Let the assumptions of Theorem 1.2 be satisfied. Moreover, assume that $\psi^{-1}(0) = \emptyset$ and that, for some $x_0 \in X$, the function $\alpha(\cdot)\varphi(x_0) + \beta(\cdot)\psi(x_0)$ is sup-compact in I.*

Then, there exists $\tilde{x} \in X$ such that

$$\begin{aligned} &\alpha\Big(g^{-1}\Big(-\frac{\varphi(\tilde{x})}{\psi(\tilde{x})}\Big)\Big)\varphi(\tilde{x}) + \beta\Big(g^{-1}\Big(-\frac{\varphi(\tilde{x})}{\psi(\tilde{x})}\Big)\Big)\psi(\tilde{x}) + \omega(\tilde{x}) \\ &= \inf_{x \in X} \Big(\alpha\Big(g^{-1}\Big(-\frac{\varphi(\tilde{x})}{\psi(\tilde{x})}\Big)\Big)\varphi(x) + \beta\Big(g^{-1}\Big(-\frac{\varphi(\tilde{x})}{\psi(\tilde{x})}\Big)\Big)\psi(x) + \omega(x)\Big). \end{aligned}$$

An extremely important remark is that, even when, for each $\lambda \in I$, the function $\alpha(\lambda)\varphi(\cdot) + \beta(\lambda)\psi(\cdot) + \omega(\cdot)$ is convex, the above theorems cannot covered by the classical Sion's result ([6]), since, for some $x \in X$, the function $\alpha(\cdot)\varphi(x) + \beta(\cdot)\psi(x)$ can be not quasi-concave.

The above results should properly be regarded as a general scheme from which one can derive, in a unitary way, boundless particular cases depending on the particular choices of the involved functions. Some samples will be highlighted in the third section. For instance, we will get the following proposition whose statement is particularly simple:

Proposition 1.4. *Let X be a real Banach space, let φ be linear and continuous and let ψ be non-negative and Lipschitzian, with Lipschitz constant equal to $\|\varphi\|_{X^*}$. Then, for each constant $c \geq 1 + \sqrt{2}$, one has*

$$\inf_{x \in X}(c\varphi(x) + (c - \sqrt{2})\psi(x) + \sqrt{|\varphi(x)|^2 + |\psi(x)|^2}) = \Big(c - \frac{1}{\sqrt{2}}\Big) \inf_{x \in X}(\varphi(x) + \psi(x)).$$

2. Proofs of the basic results

As we said, the main tool used to prove the above results is Theorem 1.2 of [5]. For the reader's convenience, here is its statement:

Theorem 2.1. *Let I be compact and let $f : X \times I \to \mathbf{R}$ be an upper semicontinuous function such that $f(\cdot, \lambda)$ is continuous for all $\lambda \in I$. Assume also that, for each λ in a dense subset of I, the function $f(\cdot, \lambda)$ is inf-connected and that, for each $x \in X$, the set of all global maxima of the function $f(x, \cdot)$ is connected. Then, one has*

$$\sup_I \inf_X f = \inf_X \sup_I f\,.$$

Proof of Theorem 1.1. First, notice that, since g is continuous and strictly monotone, $g(]a, b[)$ agrees with the open interval $]\gamma, \delta[$. Fix $x \in X \setminus \psi^{-1}(0)$. Let us show that the point $h(-\frac{\varphi(x)}{\psi(x)})$ is the unique global maximum in $[a, b]$ of the function $\alpha(\cdot)\varphi(x) + \beta(\cdot)\psi(x)$. We prove this only for the case (i_3). The proof in the case (i_4) is similar. So, assume that (i_3) holds.

Assume that $-\frac{\varphi(x)}{\psi(x)} \in]\gamma, \delta[$. Let $\tilde{\lambda} \in]a, b[$ be such that

$$\frac{\beta'(\tilde{\lambda})}{\alpha'(\tilde{\lambda})} = -\frac{\varphi(x)}{\psi(x)}\,. \tag{2}$$

Since the function $\frac{\beta'}{\alpha'}$ is increasing, we have

$$\frac{\beta'(\lambda)}{\alpha'(\lambda)} < -\frac{\varphi(x)}{\psi(x)} \ \text{ for all } \lambda \in]a, \tilde{\lambda}[\quad \text{and} \quad \frac{\beta'(\lambda)}{\alpha'(\lambda)} > -\frac{\varphi(x)}{\psi(x)} \ \text{ for all } \lambda \in]\tilde{\lambda}, b[\,.$$

Then, by (i_3), we have

$$\alpha'(\lambda)\varphi(x) + \beta'(\lambda)\psi(x) > 0 \ \text{ for all } \lambda \in]a, \tilde{\lambda}[$$

and

$$\alpha'(\lambda)\varphi(x) + \beta'(\lambda)\psi(x) < 0 \ \text{ for all } \lambda \in]\tilde{\lambda}, b[\,.$$

Hence $\tilde{\lambda}$ is a global maximum of the function $\alpha(\cdot)\varphi(x) + \beta(\cdot)\psi(x)$. Finally, since

$$\alpha'(\lambda)\varphi(x) + \beta'(\lambda)\psi(x) \neq 0$$

for all $\lambda \in]a, b[\setminus \{\tilde{\lambda}\}$, it follows that $\tilde{\lambda}$ is the unique global maximum of that function in $[a, b]$ thanks to Rolle's theorem. From (2) it follows that

$$\tilde{\lambda} = g^{-1}\left(-\frac{\varphi(x)}{\psi(x)}\right) = h\left(-\frac{\varphi(x)}{\psi(x)}\right)\,.$$

Now, suppose that $-\frac{\varphi(x)}{\psi(x)} \notin]\gamma, \delta[$. If $-\frac{\varphi(x)}{\psi(x)} \leq \gamma$, then, by (i_3), we have

$$\alpha'(\lambda)\varphi(x) + \beta'(\lambda)\psi(x) < 0$$

for all $\lambda \in]a, b[$. Hence, the point a is the unique global maximum of the function $\alpha(\cdot)\varphi(x) + \beta(\cdot)\psi(x)$ in $[a, b]$. If $-\frac{\varphi(x)}{\psi(x)} \geq \delta$, by (i_3) again, we have

$$\alpha'(\lambda)\varphi(x) + \beta'(\lambda)\psi(x) > 0 \ \text{ for all } \lambda \in]a, b[\,.$$

So, b is the unique global maximum of the function $\alpha(\cdot)\varphi(x)+\beta(\cdot)\psi(x)$ in $[a,b]$, and the claim is proved. Now suppose that $x \in \psi^{-1}(0)$. In this case, the set of all global maxima of the function $\alpha(\cdot)\varphi(x)$ is either the whole I (if $\varphi(x)=0$), or $\{a\}$, or $\{b\}$ in view of the strict monotonicity of α. Therefore, the function $f : X \times I \to \mathbf{R}$ defined by

$$f(x,\lambda) = \alpha(\lambda)\varphi(x) + \beta(\lambda)\psi(x) + \omega(x)$$

satisfies the hypotheses of Theorem 2.1, and so we have

$$\sup_{\lambda\in I}\inf_{x\in X}(\alpha(\lambda)\varphi(x) + \beta(\lambda)\psi(x) + \omega(x)) = \inf_{x\in X}\sup_{\lambda\in I}(\alpha(\lambda)\varphi(x) + \beta(\lambda)\psi(x) + \omega(x))\,.$$

In view of what seen above, the conclusion follows. □

Proof of Theorem 1.2. Let $\{I_n\}$ be a non-decreasing sequence of compact intervals such that $I = \cup_{n\in\mathbf{N}} I_n$. Fix $n \in \mathbf{N}$. Fix also $x \in X$. Denote by M_n the set of all global maxima of the restriction of the function $\alpha(\cdot)\varphi(x)+\beta(\cdot)\psi(x)$ to I_n. We claim that M_n is connected. First, suppose that $\psi(x) \neq 0$. If $-\frac{\varphi(x)}{\psi(x)} \in g(A \cap I_n)$, then reasoning as in the proof of Theorem 1.1, we have

$$M_n = \Big\{g^{-1}\Big(-\frac{\varphi(x)}{\psi(x)}\Big)\Big\}.$$

If $-\frac{\varphi(x)}{\psi(x)} \notin g(A \cap I_n)$, the function $\alpha(\cdot)\varphi(x) + \beta(\cdot)\psi(x)$ turns out to be strictly monotone in I_n (since its derivative does not vanish in $\mathrm{int}(I_n)$) and so we have either $M_n = \{\inf I_n\}$ or $M_n = \{\sup I_n\}$. Now, let $\psi(x) = 0$. In this case, we have $M_n = I_n$ if $\varphi(x) = 0$. If $\varphi(x) \neq 0$, we still have either $M_n = \{\inf I_n\}$ or $M_n = \{\sup I_n\}$, by strict monotonicity. So, after noticing that $D \cap I_n$ is dense in I_n, we realize that the function $f : X \times I_n \to \mathbf{R}$ defined by

$$f(x,\lambda) = \alpha(\lambda)\varphi(x) + \beta(\lambda)\psi(x) + \omega(x)$$

satisfies the hypotheses of Theorem 2.1. Consequently, we have

$$\sup_{\lambda\in I_n}\inf_{x\in X}(\alpha(\lambda)\varphi(x) + \beta(\lambda)\psi(x) + \omega(x)) = \inf_{x\in X}\sup_{\lambda\in I_n}(\alpha(\lambda)\varphi(x) + \beta(\lambda)\psi(x) + \omega(x))\,.$$

At this point, (i_0) allows us to use Proposition 2.1 of [4] to obtain

$$\sup_{\lambda\in I}\inf_{x\in X}(\alpha(\lambda)\varphi(x) + \beta(\lambda)\psi(x) + \omega(x)) = \inf_{x\in X}\sup_{\lambda\in I}(\alpha(\lambda)\varphi(x) + \beta(\lambda)\psi(x) + \omega(x))\,.$$

Finally, in view of (i_2'), reasoning as in the proof of Theorem 1.1, we have

$$\sup_{\lambda\in I}(\alpha(\lambda)\varphi(x) + \beta(\lambda)\psi(x)) = \alpha\Big(g^{-1}\Big(-\frac{\varphi(x)}{\psi(x)}\Big)\Big)\varphi(x) + \beta\Big(g^{-1}\Big(-\frac{\varphi(x)}{\psi(x)}\Big)\Big)\psi(x)$$

for all $x \in X \setminus \psi^{-1}(0)$, and we are done. □

Proof of Theorem 1.3. Notice that the function $\sup_{\lambda\in I}(\alpha(\lambda)\varphi(\cdot)+\beta(\lambda)\psi(\cdot)+\omega(\cdot))$ turns out to be τ_1-lower semicontinous and τ_1-inf-compact. Consequently, there exists $\tilde{x} \in X$ such that

$$\inf_{x\in X}\sup_{\lambda\in I}(\alpha(\lambda)\varphi(\cdot) + \beta(\lambda)\psi(\cdot) + \omega(\cdot)) = \sup_{\lambda\in I}(\alpha(\lambda)\varphi(\tilde{x}) + \beta(\lambda)\psi(\tilde{x}) + \omega(\tilde{x}))\,. \quad (3)$$

Also, the function $\inf_{x\in X}(\alpha(\cdot)\varphi(x)+\beta(\cdot)\psi(x)+\omega(x))$ turns out to be sup-compact in I, and hence there exists $\tilde{\lambda}\in I$ such that

$$\sup_{\lambda\in I}\inf_{x\in X}(\alpha(\lambda)\varphi(x)+\beta(\lambda)\psi(x)+\omega(x))=\inf_{x\in X}(\alpha(\tilde{\lambda})\varphi(x)+\beta(\tilde{\lambda})\psi(x)+\omega(x))\,. \quad (4)$$

Thanks to Theorem 1.2, the left-hand sides of (3) and (4) are equal. Therefore, we have

$$\begin{aligned}\alpha(\tilde{\lambda})\varphi(\tilde{x})+\beta(\tilde{\lambda})\psi(\tilde{x})+\omega(\tilde{x}) &= \inf_{x\in X}(\alpha(\tilde{\lambda})\varphi(x)+\beta(\tilde{\lambda})\psi(x)+\omega(x))\\ &= \sup_{\lambda\in I}(\alpha(\lambda)\varphi(\tilde{x})+\beta(\lambda)\psi(\tilde{x})+\omega(\tilde{x}))\,.\end{aligned}$$

So, $\tilde{\lambda}$ is a global maximum of the function $\alpha(\cdot)\varphi(\tilde{x})+\beta(\cdot)\psi(\tilde{x})$ and hence, in view of (i_2'), reasoning as in the proof of Theorem 1.1, we have $\tilde{\lambda}=g^{-1}\big(-\frac{\varphi(\tilde{x})}{\psi(\tilde{x})}\big)$, and we are done. □

3. Some applications

The following proposition will be used in the subsequent alternative theorem.

Proposition 3.1. *Let X be a Hausdorff and let $f: X\to \mathbf{R}$ be a lower semicontinuous function such that, for some $r>\inf_X f$, the set $f^{-1}(]-\infty,r])$ is compact and disconnected. Then, f has at least two local minima.*

Proof. Let C_1, C_2 be two non-empty, closed and disjoint subsets of X such that

$$f^{-1}(]-\infty,r])=C_1\cup C_2\,.$$

Notice that C_1, C_2 are compact. So, since X is Hausdorff, there are two open and disjoint sets $\Omega_1,\Omega_2\subseteq X$ such that $C_1\subseteq\Omega_1$, $C_2\subseteq\Omega_2$. Since f is lower semicontinuous, there are $x_1^*\in C_1$, $x_2^*\in C_2$ such that

$$f(x_1^*)=\inf_{C_1} f \quad\text{and}\quad f(x_2^*)=\inf_{C_2} f\,.$$

Without loss of generality, we assume that $f(x_1^*)\le f(x_2^*)$. Fix $x\in\Omega_2$. We claim that $f(x_2^*)\le f(x)$. Indeed, when $f(x)>r$, the above inequality holds since $f(x_2^*)\le r$.

So, assume that $f(x)\le r$. This implies that $x\in C_1\cup C_2$. Hence, since $\Omega_2\cap C_1=\emptyset$, we have $x\in C_2$ and so the inequality is satisfied. Therefore, x_2^* is a local minimum for f. But, x_1^* is even a global minimum for f, and the proof is complete. □

The following remarkable alternative theorem is a consequence of Theorem 1.3.

Theorem 3.2. *Let (i_2') be satisfied jointly with either (i_3') or (i_4'). Moreover, assume that X is Hausdorff and that, for every $\lambda\in I$, the function $\alpha(\lambda)\varphi(\cdot)+\beta(\lambda)\psi(\cdot)+\omega(\cdot)$ is inf-compact. Finally, suppose that $\psi^{-1}(0)=\emptyset$ and that, for some $x_0\in X$, the function $\alpha(\cdot)\varphi(x_0)+\beta(\cdot)\psi(x_0)$ is sup-compact in I. Then, at least one of the following assertions holds:*

(a) *there exists $\tilde{x} \in X$ such that*

$$\alpha\Big(g^{-1}\Big(-\tfrac{\varphi(\tilde{x})}{\psi(\tilde{x})}\Big)\Big)\varphi(\tilde{x}) + \beta\Big(g^{-1}\Big(-\tfrac{\varphi(\tilde{x})}{\psi(\tilde{x})}\Big)\Big)\psi(\tilde{x}) + \omega(\tilde{x})$$
$$= \inf_{x\in X}\Big(\alpha\Big(g^{-1}\Big(-\tfrac{\varphi(\tilde{x})}{\psi(\tilde{x})}\Big)\Big)\varphi(x) + \beta\Big(g^{-1}\Big(-\tfrac{\varphi(\tilde{x})}{\psi(\tilde{x})}\Big)\Big)\psi(x) + \omega(x)\Big);$$

(b) *there exists an open interval $J \subseteq I$ such that, for each $\lambda \in J$, the function $\alpha(\lambda)\varphi(\cdot) + \beta(\lambda)\psi(\cdot) + \omega(\cdot)$ has at least two local minima.*

Proof. Assume that (a) does not hold. Notice that (i_0) is satisfied if τ_1 is the topology of X. Therefore, in view of Theorem 1.3, (i_1) does not hold. That is, there exists an open interval $J \subseteq I$ such that, for each $\lambda \in J$, some sublevel set of the function $\alpha(\lambda)\varphi(\cdot) + \beta(\lambda)\psi(\cdot) + \omega(\cdot)$ is not connected. But, by assumption, such a sublevel set is also compact and hence the conclusion follows directly from Proposition 3.1. □

Remark 3.3. Theorem 3.2 can be usefully applied to get innovative multiplicity results for the solutions of classical nonlinear difference equations.

Here is a consequence of Theorem 1.1.

Theorem 3.4. *Let $\inf_X \psi \geq 0$ and let $c, d \in \mathbf{R}$ be such that the set*

$$\Big\{\lambda \in \big[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\big] : (\sin\lambda + c)\varphi(\cdot) + (\cos\lambda + d)\psi(\cdot) + \omega(\cdot) \quad \textit{is inf-connected}\Big\}$$

is dense in $\big[-\frac{\pi}{2}, \frac{\pi}{2}\big]$. Then, one has

$$\inf_{x\in X}(c\varphi(x) + d\psi(x) + \sqrt{|\varphi(x)|^2 + |\psi(x)|^2} + \omega(x))$$
$$= \sup_{\lambda\in[-\frac{\pi}{2},\frac{\pi}{2}]}\inf_{x\in X}((\sin\lambda + c)\varphi(x) + (\cos\lambda + d)\psi(x) + \omega(x)).$$

Proof. We apply Theorem 1.1 taking

$$I = \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big], \quad \alpha(\lambda) = \sin\lambda + c, \quad \text{and} \quad \beta(\lambda) = \cos\lambda + d.$$

Therefore $g(\lambda) = -\tan\lambda$, (i_4) holds and $h(\mu) = -\arctan\mu$ for all $\mu \in \mathbf{R}$.

Hence, if $x \in X \setminus \psi^{-1}(0)$, recalling that $\psi(x) > 0$, we have

$$\Phi(x) = \Big(\sin\arctan\frac{\varphi(x)}{\psi(x)} + c\Big)\varphi(x) + \Big(\cos\arctan\frac{\varphi(x)}{\psi(x)} + d\Big)\psi(x) + \omega(x)$$
$$= \Big(\frac{\varphi(x)}{\sqrt{|\varphi(x)|^2 + |\psi(x)|^2}} + c\Big)\varphi(x) + \Big(\frac{\psi(x)}{\sqrt{|\varphi(x)|^2 + |\psi(x)|^2}} + d\Big)\psi(x) + \omega(x)$$
$$= c\varphi(x) + d\psi(x) + \sqrt{|\varphi(x)|^2 + |\psi(x)|^2} + \omega(x).$$

Now, the conclusion follows directly from Theorem 1.1 once we observe that

$$c\varphi(x) + d\psi(x) + \sqrt{|\varphi(x)|^2 + |\psi(x)|^2} + \omega(x) = \max\{(c+1)\varphi(x), (c-1)\varphi(x)\} + \omega(x)$$

for all $x \in \psi^{-1}(0)$. □

In the same spirit as that of Theorem 3.4, we also obtain:

Theorem 3.5. *Let $\psi(x) > 0$ for all $x \in X$ and let $J \subseteq \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ be a compact interval such that $\frac{\varphi(x)}{\psi(x)} \in \tan\left(J \cap \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right)$ for all $x \in X$. Let $c, d \in \mathbf{R}$. Assume that the set*

$$\{\lambda \in J : (\sin\lambda + c)\varphi(\cdot) + (\cos\lambda + d)\psi(\cdot) + \omega(\cdot) \quad \textit{is inf-connected}\ \}$$

is dense in J and that there exists (a possibly different) topology τ_1 on X such that, for every $\lambda \in J$, the function $(\sin\lambda + c)\varphi(\cdot) + (\cos\lambda + d)\psi(\cdot) + \omega(\cdot)$ is τ_1-lower semicontinuous and, for some $\lambda_0 \in J$, the function

$$(\sin\lambda_0 + c)\varphi(\cdot) + (\cos\lambda_0 + d)\psi(\cdot) + \omega(\cdot)$$

is τ_1-inf-compact. Then, there exists $\tilde{x} \in X$ such that

$$\begin{aligned}&\Big(\frac{\varphi(\tilde{x})}{\sqrt{|\varphi(\tilde{x})|^2 + |\psi(\tilde{x})|^2}} + c\Big)\varphi(\tilde{x}) + \Big(\frac{\psi(\tilde{x})}{\sqrt{|\varphi(\tilde{x})|^2 + |\psi(\tilde{x})|^2}} + d\Big)\psi(\tilde{x}) + \omega(\tilde{x})\\ &= \inf_{x\in X}\Big(\Big(\frac{\varphi(\tilde{x})}{\sqrt{|\varphi(\tilde{x})|^2 + |\psi(\tilde{x})|^2}} + c\Big)\varphi(x) + \Big(\frac{\psi(\tilde{x})}{\sqrt{|\varphi(\tilde{x})|^2 + |\psi(\tilde{x})|^2}} + d\Big)\psi(x) + \omega(x)\Big).\end{aligned}$$

Proof. We apply Theorem 1.3 taking $I = J$, $A = J \cap \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and proceed as in the proof of Theorem 3.4. □

Notice the following very particular corollary of Theorem 3.5:

Corollary 3.6. *Let X be a closed convex subset of a reflexive real Banach space and let φ, ψ be convex, with $\varphi(x) \geq 0, \psi(x) > 0$ for all $x \in X$. If X is unbounded, suppose also that $\lim_{\|x\|\to+\infty}\psi(x) = +\infty$. Then, there exists $\tilde{x} \in X$ such that*

$$|\varphi(\tilde{x})|^2 + |\psi(\tilde{x})|^2 = \inf_{x\in X}(\varphi(\tilde{x})\varphi(x) + \psi(\tilde{x})\psi(x)).$$

Proof. Apply Theorem 3.5 taking $J = \left[0, \frac{\pi}{2}\right]$ and $c = d = \omega = 0$. The assumptions are satisfisfied since, the function $(\sin\lambda)\varphi(\cdot) + (\cos\lambda)\psi(\cdot)$ is convex (and hence inf-connected) for all $\lambda \in J$ and also weakly inf-compact if $\lambda < \frac{\pi}{2}$. So, we can take the weak topology on X as τ_1. Then, by Theorem 3.5, there exists $\tilde{x} \in X$ such that

$$\begin{aligned}&\frac{|\varphi(\tilde{x})|^2}{\sqrt{|\varphi(\tilde{x})|^2 + |\psi(\tilde{x})|^2}} + \frac{|\psi(\tilde{x})|^2}{\sqrt{|\varphi(\tilde{x})|^2 + |\psi(\tilde{x})|^2}}\\ &= \inf_{x\in X}\Big(\frac{\varphi(\tilde{x})}{\sqrt{|\varphi(\tilde{x})|^2 + |\psi(\tilde{x})|^2}}\,\varphi(x) + \frac{\psi(\tilde{x})}{\sqrt{|\varphi(\tilde{x})|^2 + |\psi(\tilde{x})|^2}}\,\psi(x)\Big)\end{aligned}$$

and the conclusion follows multiplying both sides by $\sqrt{|\varphi(\tilde{x})|^2 + |\psi(\tilde{x})|^2}$. □

In turn, from Corollary 3.6 we directly get

Corollary 3.7. *Let X be a reflexive real Banach space and let φ, ψ be convex, bounded below and Gateaux differentiable, with $\lim_{\|x\|\to+\infty}\psi(x) = +\infty$.*
Then, for every $c \leq \inf_X \varphi$ and $d < \inf_X \psi$, there exists $\tilde{x} \in X$ such that

$$(\varphi(\tilde{x}) - c)\,\varphi'(\tilde{x}) + (\psi(\tilde{x}) - d)\,\psi'(\tilde{x}) = 0\,.$$

A further corollary of Theorem 3.5, where φ is not assumed to be convex, is as follows:

Corollary 3.8. *Let X be a real Hilbert space and let φ be C^1. Assume that φ' is Lipschitzian, with Lipschitz constant L. Moreover, let $\eta : X \to]0, +\infty[$ be a continuous, Gateaux differentiable, convex function such that*

$$|\varphi(0)| + \|\varphi'(0)\|\|x\| \leq \eta(x) \quad \textit{for all } x \in X. \tag{5}$$

Then, there exists $\tilde{x} \in X$ such that

$$\varphi(\tilde{x})\,\varphi'(\tilde{x}) + \Big(\frac{L}{2}\|\tilde{x}\|^2 + \eta(\tilde{x})\Big)(L\tilde{x} + \eta'(\tilde{x})) = 0\,.$$

Proof. Since φ is C^1, we have

$$\varphi(x) = \varphi(0) + \int_0^1 \langle \varphi'(tx), x\rangle dt$$

for all $x \in X$. From this, by Lipschtzianity, we get

$$\begin{aligned} |\varphi(x)| &\leq |\varphi(0)| + \int_0^1 |\langle \varphi'(tx), x\rangle| dt \leq |\varphi(0)| + \|x\| \int_0^1 \|\varphi'(tx)\| dt \\ &\leq \ |\varphi(0)| + \|x\| \int_0^1 (\|\varphi'(0)\| + L\|x\|t)dt = |\varphi(0)| + \|x\| \Big(\|\varphi'(0)\| + \frac{L}{2}\|x\|\Big) \end{aligned}$$

for all $x \in X$. Consequently, in view of (5), we have

$$\frac{|\varphi(x)|}{\frac{L}{2}\|x\|^2 + \eta(x)} \leq 1$$

for all $x \in X$. Now, noticing that

$$\sup_{\lambda \in [-\frac{\pi}{4}, \frac{\pi}{4}]} |\sin\lambda| = \inf_{\lambda \in [-\frac{\pi}{4}, \frac{\pi}{4}]} \cos\lambda\,,$$

for every $\lambda \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, in view of a known result ([7], Corollary 42.7) the function $(\sin\lambda)\varphi(\cdot) + \frac{L}{2}(\cos\lambda)\|\cdot\|^2$ is convex. At this point, we can apply Theorem 3.5 taking

$$J = \Big[-\frac{\pi}{4}, \frac{\pi}{4}\Big] \quad \text{and} \quad \psi(x) = \frac{L}{2}\,\|x\|^2 + \eta(x)\,.$$

The assumptions are satisfied with $c = d = \omega = 0$. Indeed, for what above, the function $(\sin\lambda)\varphi(\cdot) + (\cos\lambda)\psi(\cdot)$ is convex for all $\lambda \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ and

$$\frac{\varphi(x)}{\psi(x)} \in \tan\Big(\Big[-\frac{\pi}{4}, \frac{\pi}{4}\Big]\Big) = [-1, 1]$$

for all $x \in X$. We then take the weak topology as τ_1 and $\lambda_0 = 0$, and conclude as in the proof of Corollary 3.6. □

The next result is a further application of Theorem 1.1.

Theorem 3.9. *Let $I := [a, b]$ be compact and assume that X is a real Banach space, φ is continuous and linear, ψ is Lipschitzian, with Lipschitz constant L. Moreover, assume that (i_2) is satisfied jointly with either (i_3) or (i_4). Finally, assume that*

$$D := \{\lambda \in I : |\beta(\lambda)|L < |\alpha(\lambda)|\|\varphi\|_{X^*}\}$$

is dense in I. Then, if h is defined as in Theorem 1.1*, we have*

$$\Phi(x) = \begin{cases} \alpha\left(h\left(-\frac{\varphi(x)}{\psi(x)}\right)\right)\varphi(x) + \beta\left(h\left(-\frac{\varphi(x)}{\psi(x)}\right)\right)\psi(x) & \text{if } x \in X \setminus \psi^{-1}(0) \\ \max\{\alpha(a)\varphi(x), \alpha(b)\varphi(x)\} & \text{if } x \in \psi^{-1}(0) \end{cases}$$

and
$$\inf_X \Phi = \sup_{\lambda \in I \setminus D} \inf_{x \in X} (\alpha(\lambda)\varphi(x) + \beta(\lambda)\psi(x)) .$$

Proof. Under our assumptions, by Theorem 2 of [3], for each $\lambda \in D$, the function $\alpha(\lambda)\varphi(\cdot) + \beta(\lambda)\psi(\cdot)$ is inf-connected and unbounded below. So

$$\sup_{\lambda \in I} \inf_{x \in X} (\alpha(\lambda)\varphi(x) + \beta(\lambda)\psi(x)) = \sup_{\lambda \in I \setminus D} \inf_{x \in X} (\alpha(\lambda)\varphi(x) + \beta(\lambda)\psi(x))$$

and the conclusion follows directly from Theorem 1.1. □

Remark 3.10. Other results in the same spirit as that of Theorem 3.9 can be found in [1], [2] and [3].

Now, we are in a position to prove Proposition 1.4 stated in the Introduction.

Proof of Proposition 1.4. Fix $c \geq 1 + \sqrt{2}$. Observe that the minimum of the function $\lambda \to \sin\lambda - \cos\lambda$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is equal to $-\sqrt{2}$ and is attained only at $-\frac{\pi}{4}$. This clearly implies that, if we take

$$I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad \alpha(\lambda) = \sin\lambda + c \quad \text{and} \quad \beta(\lambda) = \cos\lambda + c - \sqrt{2},$$

we have
$$\{\lambda \in I : |\beta(\lambda)| < |\alpha(\lambda)|\} = I \setminus \left\{-\frac{\pi}{4}\right\}.$$

Thus, since (i_2) and (i_4) hold, the assumptions of Theorem 3.9 are satisfied. So, proceedings as in the proof of Theorem 3.4, we get

$$\inf_{x \in X}(c\varphi(x)+(c-\sqrt{2})\psi(x)+\sqrt{|\varphi(x)|^2 + |\psi(x)|^2}) = \inf_{x \in X}\left(\alpha\left(-\frac{\pi}{4}\right)\varphi(x)+\beta\left(-\frac{\pi}{4}\right)\psi(x)\right)$$
$$= \inf_{x \in X}\left(\left(c - \frac{1}{\sqrt{2}}\right)\varphi(x) + \left(c - \sqrt{2} + \frac{1}{\sqrt{2}}\right)\psi(x)\right) = \left(c - \frac{1}{\sqrt{2}}\right)\inf_{x \in X}(\varphi(x) + \psi(x))$$

and the proof is complete. □

The final result is a further consequence of Theorem 1.3.

Theorem 3.11. *Let $\psi(x) > 0$ for all $x \in X$, and assume that $\inf_X \frac{\varphi}{\psi} > -\infty$. Set*

$$I := \left[\inf_X \frac{\varphi}{\psi}, \sup_X \frac{\varphi}{\psi}\right] \cap \mathbf{R}$$

and suppose that, for each λ in a dense subset of I, the function

$$e^{\lambda}(\varphi(\cdot) + (1-\lambda)\psi(\cdot)) + \omega(\cdot)$$

is inf-connected.

Finally, assume that there exists (a possibly different) topology τ_1 such that, for each $\lambda \in I$, the function $e^{\lambda}(\varphi(\cdot) + (1-\lambda)\psi(\cdot)) + \omega(\cdot)$ is τ_1-lower semicontinuous and, for some $\lambda_0 \in I$, the function $e^{\lambda_0}(\varphi(\cdot) + (1-\lambda_0)\psi(\cdot)) + \omega(\cdot)$ is τ_1-inf-compact.

Then, there exists $\tilde{x} \in X$ such that

$$\psi(\tilde{x}) + e^{-\frac{\varphi(\tilde{x})}{\psi(\tilde{x})}}\omega(\tilde{x}) = \inf_{x\in X}\left(\varphi(x) + \left(1 - \frac{\varphi(\tilde{x})}{\psi(\tilde{x})}\right)\psi(x) + e^{-\frac{\varphi(\tilde{x})}{\psi(\tilde{x})}}\omega(x)\right).$$

Proof. We are going to apply Theorem 1.3. In this connection, let $\alpha, \beta : I \to \mathbf{R}$ be defined by

$$\alpha(\lambda) = e^{\lambda} \quad \text{and} \quad \beta(\lambda) = (1-\lambda)e^{\lambda}$$

for all $\lambda \in I$. We conclude

$$\frac{\beta'(\lambda)}{\alpha'(\lambda)} = -\lambda\,.$$

Hence, (i_2') and (i_4') are satisfied, with $A = I$. Notice that, when I is not bounded, we have

$$\lim_{\lambda\to+\infty} e^{\lambda}(\varphi(x) + (\lambda-1)\psi(x)) = -\infty\,.$$

So, for each $x \in X$, the function $\lambda \to e^{\lambda}(\varphi(x) + (\lambda-1)\psi(x)) + \omega(x)$ is sup-compact in I. Therefore, all assumptions of Theorem 1.3 are satisfied. Consequently, there exists $\tilde{x} \in X$ such that

$$\begin{aligned} & e^{\frac{\varphi(\tilde{x})}{\psi(\tilde{x})}}\varphi(\tilde{x}) + \left(1 - \frac{\varphi(\tilde{x})}{\psi(\tilde{x})}\right) e^{\frac{\varphi(\tilde{x})}{\gamma(\tilde{x})}}\psi(\tilde{x}) + \omega(\tilde{x}) \\ &= \inf_{x\in X}\left(e^{\frac{\varphi(\tilde{x})}{\psi(\tilde{x})}}\varphi(x) + \left(1 - \frac{\varphi(\tilde{x})}{\psi(\tilde{x})}\right)e^{\frac{\varphi(\tilde{x})}{\psi(\tilde{x})}}\psi(x) + \omega(x)\right) \end{aligned}$$

and hence, multiplying by $e^{-\frac{\varphi(\tilde{x})}{\psi(\tilde{x})}}$, we get

$$\psi(\tilde{x}) + e^{-\frac{\varphi(\tilde{x})}{\psi(\tilde{x})}}\omega(\tilde{x}) = \inf_{x\in X}\left(\varphi(x) + \left(1 - \frac{\varphi(\tilde{x})}{\psi(\tilde{x})}\right)\psi(x) + e^{-\frac{\varphi(\tilde{x})}{\psi(\tilde{x})}}\omega(x)\right),$$

as claimed. □

Corollary 3.12. *Let the assumptions of Theorem* 3.11 *be satisfied. In addition, assume that X is an open set in a real normed space and that φ, ψ are Gateaux differentiable. Then, there exists $\tilde{x} \in X$ such that*

$$\varphi'(\tilde{x}) + \left(1 - \frac{\varphi(\tilde{x})}{\psi(\tilde{x})}\right)\psi'(\tilde{x}) + e^{-\frac{\varphi(\tilde{x})}{\psi(\tilde{x})}}\omega'(\tilde{x}) = 0\,.$$

Remark 3.13. Corollaries 3.7, 3.8 and 3.12 can be usefully applied to get existence theorems for different classes of nonlocal differential equations.

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