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The Minimal Time Problem of a Sweeping Process with a Discontinuous Perturbation

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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We investigate the minimal time problem where the dynamic data involves an autonomous sweeping process perturbed by a discontinuous *dissipative Lipschitz* multifunction. We show that under mild assumptions the minimal time function can be characterized in terms of Hamilton-Jacobi inequalities, one of which incorporates a limiting component that captures the discontinuous nature of the perturbation. Continuity of the minimal time function is proven under a Petrov-type condition.

Keywords: Minimal time function, sweeping process, Hamilton-Jacobi inequalities, dissipative Lipschitz multifunction.

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Introduction

The classical *sweeping process* is a state-constrained differential inclusion problem of the form

$$\begin{aligned} &\dot{x}(t) \in -N_{C(t)}(x(t)),\ \text{a.e. } t \geq 0, \\ &x(t) \in C(t),\ t \geq 0, \\ &x(0) = x_0 \in C(0), \end{aligned} \tag{$\mathcal{SP}$}$$

which was introduced by Moreau in the 70's [16, 17, 18, 20] as a mathematical framework for elastoplastic mechanics. In this dynamical system, the trajectory $x(t)$ either moves in the direction opposite to the normal cone when on the boundary of $C(t)$ or remains stationary when in the interior, but in both cases, it remains confined within $C(t)$ for all t. Moreau originally assumed the convexity of each $C(t)$, a condition later relaxed by Colombo and Goncharov [8] to what is now referred to as prox-regularity. See [10] for a thorough survey of this property whose main feature is that the projection onto the set $C(t)$ by nearby points is unique.

A major motivation for studying sweeping processes lies in the flexible structure of the differential inclusion defining ($\mathcal{SP}$), which allows for a variety of assumptions on the elements defining its data. For instance, significant advances in the theory have come from exploring various classes of *moving set* multifunctions $C(\cdot)$ and perturbations F added to the term $-N_{C(t)}(x(t))$. In the absence of such perturbations, and when $C(t)$ is convex, we obtain a dissipative differential inclusion with a

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unique absolutely continuous solution (see, e.g., [2],[10],[19] and references therein). An approach to optimal control of the sweeping process then involves adding functions or parameters that perturb the dynamic data. A commonly adopted method [9, 27] involves incorporating a parametrized Lipschitz perturbation of the form $F(x) := f(x, U)$, where f is single-valued and U is the set of control parameters $u \in U$, which permits optimal control considerations.

The wide range of assumptions applied to the moving set $C(\cdot)$ and the perturbation F highlight the versatility of the sweeping process models in describing phenomena such as elastoplastic behavior [19], traffic equilibria [4], electric circuits [1], hysteresis [15], and reflecting boundary problems [27], among others. The simplicity and relevance of the latter application have been the primary motivations for selecting the context of the perturbed sweeping processes investigated in this work. More specifically, the central theme of this article revolves around the so-called *minimal time function*

$$T_S(x_0) := \inf\{T > 0 : \exists\, x(\cdot) \text{ a trajectory of } (\mathcal{P}) \text{ with } x(T) \in S\}, \tag{$\mathcal{MT}$}$$

associated with the following perturbed (autonomous) sweeping process

$$\begin{aligned} &\dot{x}(t) \in -N_C(x(t)) + F(x(t)), \text{ a.e. } t \in I,\\ &x(t) \in C,\ t \in I,\\ &x(0) = x_0 \in C. \end{aligned} \tag{$\mathcal{P}$}$$

The data above consist of the interval $I := [0, \infty)$, a prescribed nonempty and closed subset S of $\mathbb{R}^n$ (*the target set*), the cone of normal directions $N_C(x)$ to C at x, and the perturbing multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Our main goal is to show that under a set of reasonable mild assumptions on the data of $(\mathcal{P})$, the function T_S can be characterized as the unique bounded below continuous function that is the solution of a pair of limiting Hamilton-Jacobi inequalities satisfying certain analytic boundary condition. The arguments of the current paper are implemented roughly following the ideas of Wolenski and Zhuang [30]. The investigation of the invariance properties of the epigraph of T_S was carried out in [30] under local Lipschitz hypotheses giving rise to a characterization of T_S as the unique lower semicontinuous function satisfying an exact Hamilton-Jacobi equation; see also [13] where these ideas appeared first for the Mayer problem. In the present work, we extend such a characterization to the more general framework provided in $(\mathcal{P})$ when the multifunction F is in general discontinuous and satisfies a quadratic perturbation of dissipative type.

The main result in [30] relies on a local version of a well-known strong invariance criterion for Lipschitz continuous dynamics (e.g. [6], [30]), which is not applicable here, in light of the discontinuity of the multifunction $-N_C + F$ defining $(\mathcal{P})$. This limitation was addressed in [9] in the case of a multivalued perturbation G that is Lipschitz continuous. To further leverage the instrumental role of invariance, we establish a new local Hamiltonian characterization of strong invariance within the fully discontinuous context expressed in $(\mathcal{P})$. The new criterion is formulated in terms of two Hamiltonian summands, aligning with the structure of the multifunctions governing the dynamics of $(\mathcal{P})$. Additionally, the replacement condition for strong invariance not only mitigates the discontinuity of the perturbation F in $(\mathcal{P})$ but also addresses the local hypo-monotonicity of the normal cone multifunction $N_C(\cdot)$, a property that holds only in certain enlargement of C. This obstacle prevents the dynamics in $(\mathcal{P})$ from satisfying a Dissipativity condition on the entire space, thus disqualifying from applying the characterization of strong invariance established earlier in [12] for similar dynamics (see also [23]). With this new invariant result, we incorporate pertinent modifications to the approach initiated in [30] to ultimately characterize T_S in the most general context presented by $(\mathcal{P})$.

We have focused the writing of the paper on the case where $C(\cdot)$ and F are time-independent to convey our ideas clearly and to avoid the complexities associated with a non-autonomous minimal-time function. However, our arguments are naturally extendable to the non-autonomous setting using standard modifications (see [25] and [23] for the corresponding time-dependent versions of the invariance and Filippov-type prerequisites used in this work, respectively).

We have organized the paper as follows. In Section 1 we specify the assumptions on the elements defining the data of our problem. We also briefly discuss the equivalence between our problem and a particular unconstrained differential inclusion. Section 2 is devoted to the dissipative Lipschitz condition, which is the main structural condition imposed on the perturbation of the sweeping process given in $(\mathcal{P})$, which greatly justifies the novelty of our work. Section 3 recalls the notion of escape time in terms of submultifunctions, and its relation with the minimal time function. The invariance theory that is required to connect the minimal time function and its characterizing Hamilton-Jacobi inequalities is developed in Section 4. We include a new strong invariance characterization of the Hamiltonian type. Section 5 presents the main result of the paper, which is a characterization of the minimal time function in terms of Hamilton-Jacobi inequalities. We briefly discuss the scenario where the statement of the main result is simplified and we recover an exact Hamilton-Jacobi equation. Finally, in Section 6 we present a sufficient condition of Petrov-type for the continuity of the minimal time function.

1. Nonsmooth preamble

The analysis of sweeping processes requires several key concepts from nonsmooth analysis. In this section we briefly outline the elements most relevant in our work, while a comprehensive exposition is available at [3],[5],[6],[7], and [21]. Let us consider a closed set $K \subseteq \mathbb{R}^n$. *The proximal normal cone to K at $x \in K$* is the set $N_K^P(x)$ of all vectors $\eta \in \mathbb{R}^n$ for which there is some $\sigma := \sigma(x, \eta) > 0$ such that

$$\langle \eta, y - x \rangle \leq \sigma \|y - x\|^2, \quad \text{for all } y \in K.$$

If $x \in \mathbb{R}^n \setminus K$ we set $N_K^P(x) = \emptyset$, as is commonly done. Given a lower semicontinuous function $f : K \to \mathbb{R} \cup \{+\infty\}$, the *proximal subdifferential* of f at a point x of $\mathrm{dom}(f) := \{x \in K : f(x) < +\infty\}$ is the set

$$\partial_P f(x) := \{v : (v, -1) \in N^P_{\mathrm{epi}(f)}(x, f(x))\},$$

where $\mathrm{epi}(f) := \{(x, r) : x \in \mathrm{dom}(f),\ r \geq f(x)\}$ is the *epigraph of f*.

Complementarily, if f is upper semicontinuous we define *the hypograph of f* as $\mathrm{hypo}(f) := \{(x, r) : x \in \mathrm{dom}(f),\ r \leq f(x)\}$. Notice that f is upper semicontinuous if and only if $-f$ is lower semicontinuous. This allows to consider *the proximal superdifferential of f at $x \in \mathrm{dom}(f)$* through the relationship $\partial^P f(x) := -\partial_P(-f)(x)$.

The following proposition gathers some basic observations about $\partial^P f$. The proofs of (a)–(c) are well known and use arguments that are symmetrical to those of Theorem 2.2.5 in [6] for $\partial_P f$. Part (d) is the recast of the well-known Rockafellar's horizontality theorem in terms of $\partial^P f(x)$ and $\mathrm{hypo}(f)$ (see [26]).

Proposition 1.1. *Let $f : K \to \mathbb{R}$ be upper semicontinuous. Then $v \in \partial^P f(x)$ if and only if $(-v, 1) \in N^P_{\mathrm{hypo}(f)}(x, f(x))$. Furthermore, for any $(x, r) \in \mathrm{hypo}(f)$ and $(v, \nu) \in N^P_{\mathrm{hypo}(f)}(x, r)$ we have*

(a) $\nu \geq 0$.

(b) *If $\nu > 0$, then $r = f(x)$.*

(c) *If $r < f(x)$ then $\nu = 0$, and in this case $(v, 0) \in N^P_{\mathrm{hypo}(f)}(x, f(x))$.*

(d) *If $(v, 0) \in N^P_{\mathrm{hypo}(f)}(x, f(x))$, there are sequences*

$$x_i \in K \text{ and } (v_i, \nu_i) \in N^P_{\mathrm{hypo}(f)}(x_i, f(x_i)),$$

with $\nu_i > 0$, and such that $x_i \to x$, $f(x_i) \to f(x)$, and $\nu_i \downarrow 0$.

Some properties that we shall employ in our analysis are given in terms of the proximal subdifferential of *the distance function to a closed set $K \subset \mathbb{R}^n$*

$$d_K(x) := \inf_{y \in K} \|x - y\|. \tag{1}$$

The infimum in (1) is well defined since for every $x \in \mathbb{R}^n$ the *set of projections of x onto K*,

$$\mathrm{proj}_K(x) := \{s \in K : d_K(x) = \|x - s\|\}$$

is nonempty. Let $\mathbb{S}$, $\mathbb{B}$ and $\mathrm{int}(\mathbb{B})$ be the unit sphere, and the closed and open unit balls in $\mathbb{R}^n$, respectively. Given $s > 0$, the *s-open and s-closed enlargements of K* are the sets defined as $K_s := K + s\,\mathrm{int}(\mathbb{B})$ and $\overline{K}_s := K + s\mathbb{B}$. We have the following well-known formula

$$\partial_P d_K(x) = \begin{cases} N^P_K(x) \cap \mathbb{B}, & x \in K \\ N^P_{\overline{K}_s} \cap \mathbb{S}, & x \notin K, \end{cases} \tag{2}$$

where $s = d_K(x) > 0$ (see [3] and [6]). When f is locally Lipschitz, for every $x \in \mathbb{R}^n$ and every vector $v \in \mathbb{R}^n$, we define *the generalized directional derivative of f at x in the direction of v* as

$$f^\circ(x; v) := \limsup_{y \to x,\ t \downarrow 0} \frac{f(y + tv) - f(y)}{t}.$$

The *Clarke generalized gradient at x* is given by

$$\partial f(x) := \{\zeta \in \mathbb{R}^n : f^\circ(x; v) \geq \langle \zeta, v \rangle \ \forall v \in \mathbb{R}^n\},$$

which defines a multifunction $\partial f : \mathbb{R}^n \to \mathbb{R}^n$ that is nonempty, compact, and convex valued. More even, if $k_f > 0$ is a Lipschitz rank for f near x, then it follows that $\sup\{\|\zeta\| : \zeta \in \partial f(x)\} \leq k_f$. By applying the closure operation to the nonnegative scalar multiples of $\partial d_K(x)$ we obtain the so-called *Clarke normal cone to K at x*, that is

$$N_K(s) := \mathrm{cl}\Big(\bigcup_{\lambda\geq 0}\lambda\partial d_K(x)\Big).$$

Using the closure operation again, we obtain the following alternative definition of $N_K(x)$ in terms of elements in the image of $N_K^P(\cdot)$ when $x \in K$:

$$N_K(x) = \mathrm{cl}\big(\mathrm{co}\big\{\lim_{i\to\infty}\zeta_i : \zeta_i \in N_K^P(x_i),\ x_i \xrightarrow{K} x\big\}\big),$$

where "co" stands for convex hull and the notation $x_i \xrightarrow{K} x$ signifies that $x_i \in K$ and x_i converges to x. It is clear that $N_K^P(x) \subseteq N_K(x)$ for all $x \in \mathbb{R}^n$. We say that K is *regular at x* if $N_K^P(x) = N_K(x)$. In the next section, we will recall a particular type of regularity satisfied by the set C featuring in $(\mathcal{P})$ and we will highlight some of its properties.

2. Assumptions

We proceed to organize a group of standard assumptions **(SA)** on the multifunction F and the set C defining $(\mathcal{P})$.

- Assumptions on F: they guarantee existence of solutions to $(\mathcal{P})$ (see Theorem 3.1 in [28]).

F1 $F(x)$ is a nonempty, convex, and closed set for every $x \in \mathbb{R}^n$.

F2 There exists $\alpha > 0$ such that

$$\sup\{\|v\| : v \in F(x)\} \leq \alpha, \quad \text{for all } x \in \mathbb{R}^n.$$

F3 $F(\cdot)$ is upper semicontinuous: under the previous assumptions this property is equivalent to the graph of F

$$\mathrm{gr}\, F(\cdot) := \{(x, v) : v \in F(x)\}$$

being closed in $\mathbb{R}^n \times \mathbb{R}^n$ (see [2], [6]).

- Hypotheses on the set $C \subseteq \mathbb{R}^n$: they assure nonemptiness, and hypomonotonicity of $N_C(\cdot)$ (see [10], [22]).

C1 C is nonempty and closed.

C2 The set C is ρ-prox-regular, which implies the existence of $\rho > 0$ such that for every $x, x' \in C$ and every $\eta \in N_C^P(x)$,

$$\langle \eta, x' - x\rangle \leq \frac{\|\eta\|}{2\rho}\|x' - x\|^2.$$

The following proposition summarizes some important facts about prox-regular sets that we shall employ. Recall that a multifunction $H : \mathbb{R}^n \to \mathbb{R}^n$ is *hypomonotone* if there exists $\beta \in \mathbb{R}$ satisfying

$$\langle v - u, y - x\rangle \geq \beta\|y - x\|^2, \quad \text{for all } (x, u), (y, v) \in \mathrm{gr}\, H(\cdot).$$

Proposition 2.1. *Assume C satisfies* C1–C2. *Then the following conditions hold.*

(a) $N_C(x) = N_C^P(x)$ *for all* $x \in C$.

(b) $\partial d_C(y) = \partial_P d_C(y)$ *for all* $y \in C_\rho$.

(c) *For each* $y \in C_\rho$ *there exists a unique* $\overline{x} = \mathrm{proj}_C(y) \in C$ *such that*

$$d_C(y)\partial d_C(y) = \{y - \overline{x}\}. \tag{3}$$

(d) $\partial d_C(\cdot)$ *is hypomonotone in* C_s *for each* $s \in (0, \rho)$.

Proof. Assertions (a)–(c) are well known (see for example Theorem 2.8(g) in [21]). To establish (d) note that according to (b) and Theorem 5(b) in [22] the set C is ρ-prox regular if and only if the following estimate holds for any $s \in (0, \rho)$: for all $x, y \in C_s$ and $\zeta \in \partial d_C(x)$ we have

$$\langle \zeta, y - x \rangle \leq \frac{\rho}{2(\rho - s)^2} \|y - x\|^2 + d_C(y) - d_C(x). \tag{4}$$

Exchanging the roles of x and y in (4), for $\xi \in \partial d_C(y)$ we have

$$\langle -\xi, y - x \rangle \leq \frac{\rho}{2(\rho - s)^2} \|y - x\|^2 + d_C(x) - d_C(y). \tag{5}$$

Adding (4) and (5) and then multiplying by -1 the resulting inequality, we obtain

$$\langle \xi - \zeta, y - x \rangle \geq -\frac{\rho}{(\rho - s)^2} \|y - x\|^2, \tag{6}$$

which implies the hypomonotonicity of $\partial d_C(\cdot)$ as stated. □

2.1. Trajectories

We will refer to a solution (or trajectory) to $(\mathcal{P})$ as an absolutely continuous function $x(\cdot)$ satisfying the differential inclusion, the state constraint, and the initial condition defining $(\mathcal{P})$ on some non-degenerate interval $[0, T]$. The existence of solutions to $(\mathcal{P})$ is well known. It has been established, for example, in Corollary 3.2 of [28] under a set of standard time-dependent assumptions that subsume F1–F3 and C1–C2, and involves both a moving set multifunction $C(\cdot)$ (rather than a constant set C) and a time-dependent perturbation multifunction $G(\cdot, \cdot)$. To be more specific and be able to access a version of this result that is appropriate in our setting, we will say that $G : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a *submultifunction of* F if it satisfies:

- $G(t, x) \subset F(x)$ for all $(t, x) \in I \times \mathbb{R}^n$.
- For each $(t, x) \in I \times \mathbb{R}^n$, $G(t, x)$ nonempty, convex, and closed.
- $G(\cdot, \cdot)$ is $\mathcal{L}(I) \times \mathcal{B}(\mathbb{R}^n)$-measurable.
- $G(t, \cdot)$ is upper semicontinuous for each $t \in I$: under the previous assumptions this property is equivalent to $G(\cdot, \cdot)$ being *almost upper semicontinuous*, that is, for every compact interval $\widetilde{I} \subset I$ and each $\varepsilon > 0$, there exists a closed set $\mathcal{N}_\varepsilon \subseteq \widetilde{I}$ with Lebesgue measure $\mu(\widetilde{I} \setminus \mathcal{N}_\varepsilon) < \varepsilon$, so that the graph of F

  $$\mathrm{gr}\, G := \{(t, x, v) : v \in G(t, x)\}$$

  is closed in $\mathcal{N}_\varepsilon \times \mathbb{R}^n \times \mathbb{R}^n$ (see Theoren 3.3 in [29]).

Theorem 2.2. (c.f. Corollary 3.2 of [28]) *Assume* C1–C2 *and let G be a submultifunction of F satisfying*

$$\sup\{\|v\| : v \in G(t,x)\} \leq m(t),\ t \in [0,T], \tag{7}$$

for some integrable function $m : [0,T] \to \mathbb{R}$ with $\rho > \int_0^T m(s)ds$. Then an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ is a solution to the state-constrained problem

$$\begin{aligned}
&\dot{x}(t) \in -N_C(x(t)) + G(t,x(t)),\ \text{a.e. } t \in [\tau, \tau+T],\\
&x(t) \in C,\ t \in [\tau, \tau+T],\\
&x(\tau) = x_0 \in C,
\end{aligned} \tag{$\mathcal{P}_G$}$$

if and only if it is a solution to the unconstrained initial value problem

$$\begin{aligned}
&\dot{x}(t) \in -m(t)\partial d_C(x(t)) + G(t,x(t)),\ \text{a.e. } t \in [0,T],\\
&x(0) = x_0 \in C.
\end{aligned} \tag{$\mathcal{UP}_G$}$$

Furthermore, the set of solutions to $(\mathcal{P}_G)$ is nonempty.

Note that in its role of ensuring the existence of solutions, Theorem 2.2 connects the state-constrained problem $(\mathcal{P})$ and the unconstrained problem $(\mathcal{UP})$ so that resources from the standard theory differential inclusions can be applied. We will benefit from this by-pass to employ the arguments used in [30] on the unconstrained differential inclusion associated with the minimal time problem $(\mathcal{MT})$. Note that the latter approach is legit since the map $x \longmapsto \partial d_C(x)$ is nonempty, convex, and compact valued, and it is also globally bounded and upper semicontinuous (e.g. Proposition 2.1.5 of [6]). Therefore, the multifunction $\mathcal{D}_F(x) := -\alpha\partial d_C(x) + F(x)$ satisfies F1–F3 and is in particular globally bounded by the constant 2α, which readily implies that each solution to $(\mathcal{P})$ can be extended on the whole interval I, according to a well-known argument involving Gronwall's inequality.

Given a closed target set $S \subset \mathbb{R}^n$, the purpose of the minimal time function T_S is to record the earliest moment at which an optimal trajectory associated with $(\mathcal{P})$ exits the open set $S^c = \mathbb{R}^n \setminus S$. This event has an intrinsic interpretation in terms of invariance properties involving S^c, $\operatorname{epi}(T_S)$, and $\operatorname{hypo}(T_S)$ that we will explore in the following sections, and which requires the notion of escape time from an open set. Since differential inclusions typically admit multiple solutions, the notion of escape time depends on the specific trajectory in question. However, it still represents the maximal interval during which the trajectory remains in the open set.

To ensure consistency with the forthcoming discussion, we define escape time within the broader context of submultifunctions.

Consider the set-valued maps

$$\mathcal{M}_G(t,x) := -N_C(x) + G(t,x), \quad \text{and} \quad \mathcal{D}_G(t,x) := -\alpha\partial d_C(x) + G(t,x), \tag{8}$$

and let $(\tau, x_0) \in I \times \mathbb{R}^n$ and $T > \tau$. Let $U \subset \mathbb{R}^n$ be open and $x(\cdot)$ be a trajectory of $(\mathcal{P}_G)$ with $x(\tau) = x_0 \in U$. We say that T is an *escape time from U for $x(\cdot)$*, in which case we write $T := \operatorname{Esc}(x(\cdot); U)$, provided at least one of the following conditions holds:

(a) $T = \infty$ and $x(t) \in U$ for all $t \geq \tau$,

(b) $x(t) \in U$ for all $t \in [\tau, \tau + T)$ and $\|x(t)\| \to \infty$ as $t \uparrow \tau + T$, or

(c) $T < \infty$, $x(t) \in U$ for all $t \in [\tau, \tau + T)$, and $d_{U^c}(x(t)) \to 0$ as $t \uparrow \tau + T$.

The set of all trajectories of $(\mathcal{P}_G)$ originating from $x_0 \in C \cap U$ that remain in U over a maximal interval is denoted by $\Upsilon_{(\mathcal{M}_G;U)}(\tau, x_0)$. That is, $\Upsilon_{(\mathcal{M}_G;U)}(\tau, x_0)$ consists of those trajectories $x(\cdot)$ of $(\mathcal{P}_G)$ with $x(\tau) = x_0 \in C \cap U$ and for which $T = \mathrm{Esc}(x(\cdot); U)$.

Remark 2.3. In light of Theorem 2.2 it is clear that $\Upsilon_{(\mathcal{M}_G;U)}(\tau, x_0) = \Upsilon_{(\mathcal{D}_G;U)}(\tau, x_0)$. In the time-independent case $G = F$ we can suppose without loss of generality that $\tau = 0$ for any trajectory $x(\cdot)$ of G with initial point $x(\tau) = x_0 \in U$, and then we can write $\Upsilon_{(\mathcal{M}_F,U)}(x_0)$ instead of $\Upsilon_{(\mathcal{M}_F,U)}(0, x_0)$. □

Notice that if G is a submultifunction of F and C1–C2 hold, then any trajectory of $(\mathcal{P}_G)$ has an escape time associated with U, that is, the set $\Upsilon_{(\mathcal{M}_G;U)}(\tau, x_0)$ is nonempty for each $(\tau, x_0) \in I \times U$. We record this in the next result.

Proposition 2.4. *Every trajectory $x(\cdot)$ of $(\mathcal{AP})$ with $x(\tau) \in U$ has an escape time from U.*

Proof. Let us first note that the proof proceeds identically to that of Proposition 2.5 in [30] in the case of the dynamics $(\mathcal{UP}_G)$ due to our assumptions on the submultifunctions G. The result then follows from the equality $\Upsilon_{(\mathcal{M}_G;U)}(\tau, x_0) = \Upsilon_{(\mathcal{D}_G;U)}(\tau, x_0)$ (see Remark 2.3). □

We close this section with the adaptation of Proposition 2.6 of [30] to the context of the problem $(\mathcal{P})$, which establishes, among other facts, the lower semicontinuity of T_S. Note again the dynamics in $(\mathcal{UP})$ operates under assumptions that are equivalent to those used in Proposition 2.6 of [30], and thanks to Theorem 2.2 we have

$$T_S(x_0) := \inf\{T > 0 : \exists\, x(\cdot) \text{ a trajectory of } (\mathcal{UP}) \text{ with } x(T) \in S\}. \tag{9}$$

Therefore, the proof of the result below is a replica of its predecessor in [30].

Proposition 2.5. *If $x_0 \in S^c \cap \mathrm{dom}\, T_S$, then there exists $x(\cdot) \in \Upsilon_{(\mathcal{M}_F,S^c)}(x_0)$ with $\mathrm{Esc}(x(\cdot), S^c) = T_S(x_0)$ and $x(T_S(x_0)) \in S$ (that is, the infimum in $(\mathcal{MT})$ is attained). Furthermore, $T_S(\cdot)$ is lower semicontinuous on C.*

3. The dissipative Lipschitz condition

In this paper, we aim to characterize T_S using Hamilton-Jacobi inequalities without imposing continuity assumptions on the multifunction F in $(\mathcal{P})$. This approach is naturally motivated by sweeping processes in which control mechanisms involve perturbations with discontinuous behavior. A significant example of such behavior is discussed below.

Definition 3.1. Let $\mathcal{D} : \Omega \to \mathbb{R}^n$ be a nonempty-valued multifunction satisfying $\emptyset \neq \Omega \subset \mathbb{R}^n$. We say that $\mathcal{D}$ is *Dissipative Lipschitz* (*DL*), if there is a constant $k_{\mathcal{D}}$ (not necessarily positive) such that for each pair $x, y \in \Omega$, and each $u \in \mathcal{D}(x)$, there is $v \in \mathcal{D}(y)$ that satisfies $\langle v - u, y - x\rangle \leq k_{\mathcal{D}}\|y - x\|^2$. When the above property is satisfied we say that $k_{\mathcal{D}}$ is the *dissipative Lipschitz rank* of $\mathcal{D}$.

The following result identifies the DL property satisfied by the multifunction governing the dynamics of $(\mathcal{UP})$. Notably, this property holds not throughout the entire space but only within a family of enlarged sets C_s, for certain values of $s > 0$.

Proposition 3.2. *Let us assume that the multifunction F is DL with rank k_F. Then for any $s \in (0, \rho)$ the multifunction $\mathcal{D}_F$ given in* (8) *is DL in the open enlargement C_s, and its rank is $\frac{\alpha\rho}{(\rho - s)^2} + k_F$.*

Proof. Let $s \in (0, \rho)$ and consider $x, y \in C_s$. Given $u \in \mathcal{D}_F(x)$, there are $u_1 \in \partial d_C(x)$ and $u_2 \in F(x)$ such that $u = -\alpha u_1 + u_2$. Since $F(\cdot)$ is DL with rank k_F, there is some $v_2 \in F(y)$ such that

$$\langle v_2 - u_2, y - x\rangle \leq k_F\|y - x\|^2. \tag{10}$$

Recall that $\alpha > 0$. If $v_1 \in \partial d_C(y)$ condition (6) implies

$$\langle -\alpha(v_1 - u_1), y - x\rangle \leq \frac{\alpha\rho}{(\rho - s)^2}\|y - x\|^2. \tag{11}$$

Let us define $v = -\alpha v_1 + v_2$. It is clear that $v \in \mathcal{D}_F(y)$, and from (10) and (11) we obtain

$$\begin{aligned}\langle v - u, y - x\rangle &= \langle -\alpha v_1 + v_2 - (-\alpha u_1 + u_2), y - x\rangle \\ &= \langle -\alpha(v_1 - u_1), y - x\rangle + \langle v_2 - u_2, y - x\rangle \\ &\leq \big(\frac{\alpha\rho}{(\rho - s)^2} + k_F\big)\|y - x\|^2,\end{aligned}$$

which confirms the assertion on $\mathcal{D}_F$. □

Our work relies heavily on the use of the following Hamiltonian function. Given a submultifunction $G : I \times \mathbb{R}^n \to \mathbb{R}^n$, its (minimized) *Hamiltonian* is defined as

$$h_G(t, x, \zeta) := \inf\{\langle v, \zeta\rangle : v \in G(t, x)\} \quad \forall\,(t, x, \zeta) \in I \times \mathbb{R}^n \times \mathbb{R}^n.$$

In the language of h_G, a convex-valued multifunction G is (globally) *Lipschitz* if there exists a constant $L_G > 0$ such that

$$|h_G(t, y, \zeta) - h_G(\tau, x, \zeta)| \leq L_G\|\zeta\|\,(|t - \tau| + \|y - x\|) \tag{12}$$

for all pairs $(t, x), (\tau, y) \in I \times \mathbb{R}^n$, and all $\zeta \in \mathbb{R}^n$. As mentioned before, one of the novelties of this article is the use of a perturbation F that satisfies the DL condition on $\mathbb{R}^n$, which in terms of autonomous h_F can be expressed as

$$h_F(y, y - x) - h_F(x, y - x) \leq k_F\|y - x\|^2 \tag{13}$$

for all $x, y \in \mathbb{R}^n$ and some constant k_F. It is clear that in the autonomous case condition (12) implies the DL property (13). However, the converse does not hold, as the following examples corroborate: let $F_1(x) := \{-\operatorname{sgn}(x)\sqrt{|x|}\}$, for all $x \in \mathbb{R}$. Then the multifunction F_1 satisfies (13) with $k_{F_2} = 0$. However, F_1 is not Lipschitz at the origin. Another simple example is given by $F_2(x) = -\partial\|x\| + \{x\}$ for all $x \in \mathbb{R}$. The multifunction F_2 satisfies (13) with $k_{F_1} = 1$, but not (12). Notice that F_1 is continuous, while F_2 is discontinuous.

4. Invariance

In this section we discuss the invariance requirements of the article. We start by adapting the corresponding classical definitions to the escape time setting.

Let $U \subset \mathbb{R}^n$ be open. Consider a multifunction $\mathcal{M} : I \times \mathbb{R}^n \to \mathbb{R}^n$ and a set $\Sigma \subset \mathbb{R}^n$ such that $\Sigma \cap U \neq \emptyset$. In the same spirit of the terminology used for escape time:

- We say that $(\Sigma, \mathcal{M})$ is *weakly invariant in* U provided that for every $\tau \in I$ and every $x_0 \in \Sigma \cap U$, there exists $x(\cdot) \in \Upsilon_{(\mathcal{M},U)}(\tau, x_0)$ that satisfies $x(t) \in \Sigma$ for all $t \in [\tau, \tau + \mathrm{Esc}(x(\cdot); U))$.
- We say that $(\Sigma, \mathcal{M})$ is *strongly invariant in* U provided that for all $\tau \in I$ and all $x_0 \in \Sigma \cap U$, every $x(\cdot) \in \Upsilon_{(\mathcal{M},U)}(\tau, x_0)$ satisfies $x(t) \in \Sigma$ for all times $t \in [\tau, \tau + \mathrm{Esc}(x(\cdot); U))$.

An immediate consequence of Theorem 2.2 and its remarks is the equivalence between the invariance properties of $(C \cap E, \mathcal{M}_F)$ and $(C \cap E, \mathcal{D}_F)$ for every closed set $E \subset \mathbb{R}^n$ with $C \cap E \neq \emptyset$. For example, it is clear that $(C \cap E, \mathcal{M}_F)$ is weakly invariant in U if and only if $(C \cap E, \mathcal{D}_F)$ is weakly invariant in U, etc. This type of relationship naturally extends to the context of a particular augmented system involving $\mathrm{epi}\, T_S$ and $\mathrm{hypo}\, T_S$ as the next two propositions assert. Recall that the augmented multifunction $\mathcal{M}_F \times \{-1\} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is defined as

$$(\mathcal{M}_F \times \{-1\})(x, r) := \{(v, -1) : v \in \mathcal{M}_F(x)\} \subset \mathbb{R}^{n+1}.$$

It is well known that $\mathrm{epi}\, T_S$ ($\mathrm{hypo}\, T_S$) is closed when T_S is lower semicontinuous (upper semicontinuous).

Proposition 4.1. *Suppose (SA) are satisfied.*

(a) *If $E := \mathrm{epi}\, T_S$, then $(E, \mathcal{M}_F \times \{-1\})$ is weakly invariant in $U := S^c \times \mathbb{R}$.*

(b) *Additionally, if we assume T_S upper semicontinuous and $E = \mathrm{hypo}\, T_S$, then $(E, \mathcal{M}_F \times \{-1\})$ is strongly invariant in $\mathbb{R}^{n+1}$.*

Proof. (a) From Proposition 2.5 the set $E = \mathrm{epi}\, T_S$ results closed. Recall that (9) holds and the multifunction $\mathcal{D}_F$ satisfies the hypotheses used in Proposition 3.1 from [30]. Following the lines of part (a) in the same result we obtain that $(E, \mathcal{D}_F \times \{-1\})$ is weakly invariant in $U := S^c \times \mathbb{R}$. By Theorem 2.2 the system associated with $\mathcal{D}_F \times \{-1\}$, namely

$$\begin{aligned} &\dot{\tau}(t) = -1, \quad \dot{x}(t) \in -\alpha \partial d_C(x(t)) + F(x(t)), \text{ a.e. } t \in I, \\ &(x(0), \tau(0)) = (x_0, t_0), \end{aligned} \tag{14}$$

has the same trajectories as

$$\begin{aligned} &\dot{\tau}(t) = -1, \quad \dot{x}(t) \in -N_C(x(t)) + F(x(t)), \text{ a.e. } t \in I, \\ &(x(t), \tau(t)) \in C \times \mathbb{R}, \ t \in I, \\ &(x(0), \tau(0)) = (x_0, t_0), \end{aligned} \tag{15}$$

for each $(x_0, t_0) \in C \times I$. This in turn implies the weak invariance of $(E, \mathcal{M}_F \times \{-1\})$ in U.

(b) It is well known that when T_S is upper semicontinuous the set $E = \mathrm{hypo}\, T_S$ is closed. Let $(\overline{x}, r) \in C \times \mathbb{R}$ be such that we have $r \leq T_S(\overline{x})$. Now we consider $(x(\cdot), \tau(\cdot)) \in \Upsilon_{(\mathcal{M}_F \times \{-1\}, \mathbb{R}^{n+1})}(\overline{x}, r)$. Then we have necessarily $x(\cdot) \in \Upsilon_{(\mathcal{M}_F, \mathbb{R}^n)}(\overline{x})$ and $\tau(t) = r - t$ for all $t \in I$. By Remark 2.3 $x(\cdot) \in \Upsilon_{(\mathcal{D}_F, \mathbb{R}^n)}(\overline{x})$ and F2 implies $x(t) \in \mathbb{R}^n$ for all $t \in I$.

For every $t \in I$ the principle of optimality and (9) yield

$$r \le T_S(\overline{x}) = T_S(x(0)) \le T_S(x(t)) + t,$$

which implies $\tau(t) = r - t \le T_S(x(t))$, that is $(x(t), \tau(t)) \in E$ for all $t \in I$. Therefore, $(E, \mathcal{M}_F \times \{-1\})$ is strongly invariant in $\mathbb{R}^{n+1}$. □

Proposition 4.2. *Assume* (SA) *are satisfied and let* $\theta : C \to (-\infty, \infty]$ *be a function such that* $\theta(s) = 0$ *holds for all* $s \in S$. *Let* $U := S^c \times \mathbb{R}$.

(a) *If* $\theta(\cdot)$ *is bounded below and lower semicontinuous on* C, *and* $(E, \mathcal{M}_F \times \{-1\})$ *is weakly invariant in* U *with* $E := \operatorname{epi} \theta$, *then* $\theta(x) \ge T_S(x)$ *for all* $x \in C$.

(b) *If* $\theta(\cdot)$ *is upper semicontinuous on* C *and* $(E, \mathcal{M}_F \times \{-1\})$ *is strongly invariant in* $\mathbb{R}^{n+1}$, *where* $E = \operatorname{hypo} \theta$, *then* $\theta(x) \le T_S(x)$ *for all* $x \in C$.

Proof. Since that the pairs

$$(E, \mathcal{M}_F \times \{-1\}) \text{ and } (E, \mathcal{D}_F \times \{-1\})$$

have equivalent invariance properties and (9) holds, part (a) follows by applying Proposition 3.1(a) from [30] on $(E, \mathcal{D}_F \times \{-1\})$. To show (b), consider $\overline{x} \in C$. If $T_S(\overline{x}) = \infty$ or $\overline{x} \in S$, then in either case $\theta(\overline{x}) \le T_S(\overline{x})$. Assume that $\overline{x} \in S^c \cap \operatorname{dom} T_S$. Let $\varepsilon > 0$ be given. By definition of $T_S(\overline{x})$ and Proposition 2.5 there exists some $x(\cdot) \in \Upsilon_{(\mathcal{M}_F, S^c)}(\overline{x})$ with $\operatorname{Esc}(x(\cdot); S^c) =: T$, $x(T) \in S$, and such that $T < T_S(\overline{x}) + \varepsilon$. Since $\Upsilon_{(\mathcal{M}_F, S^c)}(\overline{x}) = \Upsilon_{(\mathcal{D}_F, S^c)}(\overline{x})$, by $F2$ we can extend $x(\cdot)$ on the whole interval I. Define $z(s) := (x(s), \theta(\overline{x}) - s)$, for $s \in I$. It is clear that $z(\cdot)$ is a trajectory of (15), with $z(0) = (\overline{x}, \theta(\overline{x})) \in E$. Since $(E, \mathcal{M}_F \times \{-1\})$ is strongly invariant in $\mathbb{R}^{n+1}$, we have $\theta(\overline{x}) - s \le \theta(x(s))$ for all $s \in I$. In particular, for $s = T$ we obtain $\theta(\overline{x}) - T \le \theta(x(T)) = 0$, that is $\theta(\overline{x}) \le T < T_S(\overline{x}) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary we obtain $\theta(\overline{x}) \le T_S(\overline{x})$ and the proof is complete. □

4.1. Hamiltonian inequalities

In this part, we provide Hamiltonian interpretations of the previous invariance properties. We start with a weak invariant result that is a time-dependent local version of Theorem 4.2.10 of [6] adapted to the multifunction $\mathcal{D}_G := -\alpha \partial d_C + G$ (Theorem 4.3). The reader interested in the argument used in its proof may consult Theorem 5.8 of [23], which is a Hamiltonian criterion involving only the perturbation G. This result easily extends to the case of the dynamics defined by $\mathcal{D}_G$ due to additivity of the Hamiltonian with respect to its associated multifunction and once again the equivalence between the invariance properties of $(\mathcal{P}_G)$ and $(\mathcal{UP}_G)$. More specifically, given a submultifunction G of F, it is clear from the definition of $h_{\mathcal{D}_G}$ that for all $(t, x, \zeta) \in I \times \mathbb{R}^n \times \mathbb{R}^n$ we have

$$h_{\mathcal{D}_G}(t, x, \zeta) = h_{-\alpha \partial d_C}(x, \zeta) + h_G(t, x, \zeta). \tag{16}$$

Using Proposition 2.1(b) the previous equality becomes

$$h_{\mathcal{D}_G}(t, x, \zeta) = h_{-\alpha \partial_P d_C}(x, \zeta) + h_G(t, x, \zeta), \quad \forall (t, x, \zeta) \in I \times C_\rho \times \mathbb{R}^n, \tag{17}$$

and formula (2) implies

$$h_{\mathcal{D}_G}(t, x, \zeta) = h_{-N_C^P \cap \alpha \mathbb{B}}(x, \zeta) + h_G(t, x, \zeta), \quad \forall (t, x, \zeta) \in I \times C \times \mathbb{R}^n. \tag{18}$$

Theorem 4.3. *Let $U \subset \mathbb{R}^n$ be open and $E \subset \mathbb{R}^n$ be closed such that $(C \cap E) \cap U \neq \emptyset$. If G is a submultifunction of F and C satisfies* C1–C2, *then $(C \cap E, \mathcal{M}_G)$ is weakly invariant in U if and only if for each subinterval $\widetilde{I} \subseteq I$ there exists a set with null Lebesgue measure $A \subset \widetilde{I}$ such that*

$$h_{-N_C^P \cap \alpha\mathbb{B}}(x, \zeta) + h_G(t, x, \zeta) \leq 0, \tag{19}$$

for all $t \in \widetilde{I} \setminus A$, $x \in (C \cap E) \cap U$, and $\zeta \in N_{C \cap E}^P(x)$. In this case, (19) holds at all points of density of a certain countable family of pairwise disjoint closed sets $I_j \subset \widetilde{I}$ $(j = 1, 2, \ldots,)$ for which the restriction of $G(\cdot, \cdot)$ to every $I_j \times \mathbb{R}^n$ is upper semicontinuous.

Remark 4.4. Recall that a point $t \in \mathbb{R}$ is called a point of density of a Lebesgue measurable set $\mathcal{J}$ of $\mathbb{R}$ if

$$\lim_{h \downarrow 0} \frac{\mu(\mathcal{J} \cap [t - h, t + h])}{2h} = 1.$$

The set of all points of density $\widetilde{\mathcal{J}}$ of $\mathcal{J}$ has full measure in $\mathcal{J}$, that is $\mu(\mathcal{J} \setminus \widetilde{\mathcal{J}}) = 0$. Moreover, the equality $\mathrm{cl}(\widetilde{\mathcal{J}}) = \mathrm{cl}(\widetilde{\mathcal{J}} \setminus \mathcal{V})$ holds for any set of null Lebesgue measure $\mathcal{V} \subset \mathbb{R}$, and when $\mathcal{J}$ is closed, it follows that $\widetilde{\mathcal{J}} \subset \mathcal{J}$ (see [14] for more aspects on density points). □

The following theorem establishes the local-autonomous version of the strong invariance characterization contained in [25]. We include its proof for completeness, as it is simpler than the original. Given the presence of the multifunction $-N_C$ in the dynamics defining $(\mathcal{P})$ and the liminf term in the second summand on the left-hand side of (20), the result is novel and subsumes Theorem 4 of [12] (in the autonomous case) by setting $C = \mathbb{R}^n$. Additionally, it is more general than Theorem 4.4 of [9] regarding the type of perturbation F on the right-hand side of $(\mathcal{P})$, as in the theorem below we employ a discontinuous F, whereas in [9] such a perturbation is assumed to be Lipschitz.

Theorem 4.5. *Let us assume (SA) and that F is DL. Let $E \subseteq \mathbb{R}^n$ be closed and $U \subseteq \mathbb{R}^n$ be open such that $(C \cap E) \cap U \neq \emptyset$. Then the pair $(C \cap E, \mathcal{M}_F)$ is strongly invariant in U if and only if*

$$-h_{-N_C^P \cap \alpha\mathbb{B}}(x, \zeta) + \liminf_{y \to_\zeta x} h_F(y, -\zeta) \geq 0, \tag{20}$$

for all $x \in (C \cap E) \cap U$, and all $\zeta \in N_{C \cap E}^P(x)$.

Remark 4.6. Given a nonzero vector $\zeta \in \mathbb{R}^n$, the notation $y \to_\zeta x$ in (20) signifies the limit of y approaching x along the vector ζ; in other words, $y \to_\zeta x$ if and only if $y \to x$ and $\dfrac{y - x}{\|y - x\|} \to \dfrac{\zeta}{\|\zeta\|}$. □

Proof. We first establish that (20) is sufficient for strong invariance. As we discussed previously, we may focus on the strong invariance property of $(C \cap E, \mathcal{D}_F)$ in U. Let $x(\cdot)$ be a trajectory of $(\mathcal{UP})$ satisfying $x(0) \in (C \cap E) \cap U$ and such that $x(\cdot) \in \Upsilon_{(\mathcal{D}_F, U)}(x(0))$. Let $T = \mathrm{Esc}(x(\cdot), U)$. Recall that due to (SA) the trajectory

$x(\cdot)$ is defined on the whole I. The multifunctions $\{\dot{x}(t)\} + \alpha\partial d_C(x(t))$ and $F(x(t))$ are both measurable on I, and their intersection is nonempty for almost all $t \in I$ since $x(t)$ is a solution of the differential inclusion in $(\mathcal{UP})$. Therefore, Proposition 3.4(a) of [11] implies the multifunction

$$H(t) := \Big(\{\dot{x}(t)\} + \alpha\partial d_C(x(t))\Big) \bigcap F(x(t)),\ t \in I,$$

is measurable. Consequently, we can invoke Proposition 3.2 in [11] to guarantee the existence of a measurable selection $h(t) \in H(t)$, a.e., $t \in I$. Let us define the multifunction

$$G(t,x) := \big\{v \in F(x) : \langle h(t) - v, x(t) - x\rangle \leq k_F\|x(t) - x\|^2\big\},\ (t,x) \in I \times \mathbb{R}^n.$$

It is a known routine to check that G is a submultifunction of F. Let us fix a subinterval $\widetilde{I} \subseteq I$. The almost upper semicontinuity of G implies that of $\mathcal{D}_G$, hence for any $\varepsilon > 0$ there is a closed set $I_\varepsilon \subset \widetilde{I}$, with Lebesgue measure $\mu(\widetilde{I} \setminus I_\varepsilon) < \varepsilon$, such that the restriction of $\mathcal{D}_G$ to $I_\varepsilon \times \mathbb{R}^n$ is upper semicontinuous in (t,x).

Let J_ε denote the points of density of I_ε, and define $A := \widetilde{I} \setminus (\cup_{\varepsilon>0} J_\varepsilon)$, which has null measure in $\widetilde{I}$ (see page 274 in [14]). Let $x_0 \in (C \cap E) \cap U$, and $\zeta \in N^P_{C\cap E}(x_0)$. Given $t \in \widetilde{I} \setminus A$, there is some $\varepsilon > 0$ for which $t \in J_\varepsilon$. By taking the lim inf all over the sequences $(s,z) \to (t,x_0)$ in $I_\varepsilon \times \mathbb{R}^n$, property (18) and the lower semicontinuity of $h_G(\cdot,\cdot,\zeta)$ yield

$$\begin{aligned} h_{\mathcal{D}_G}(t,x_0,\zeta) &\leq h_{-N^P_C \cap \alpha\mathbb{B}}(t,x_0,\zeta) + \liminf_{s\to t,\ z\to x_0} h_G(s,z,\zeta) \\ &\leq h_{-N^P_C \cap \alpha\mathbb{B}}(t,x_0,\zeta) + \liminf_{s\to t,\ y\to_\zeta x_0} h_G(s,y,\zeta) \\ &\leq h_{-N^P_C \cap \alpha\mathbb{B}}(t,x_0,\zeta) + \limsup_{s\to t,\ y\to_\zeta x_0} h_G(s,y,\zeta). \end{aligned} \tag{21}$$

Recall that $\limsup_{y\to_\zeta x_0}[-h_F(y,-\zeta)] = -\liminf_{y\to_\zeta x_0} h_F(y,-\zeta)$. Since $h_G(\cdot,\cdot,\zeta) \leq -h_F(\cdot,-\zeta)$, inequalities (20) and (21) now imply $h_{\mathcal{D}_G}(t,x_0,\zeta) \leq 0$, and this inequality holds for almost all $t \in \widetilde{I}$. This translates into the weak invariance of $(C\cap E, \mathcal{M}_G)$ in U according to Theorem 4.3, and thus that of $(C \cap E, \mathcal{D}_G)$. Let $y(\cdot)$ be an invariant trajectory of $(\mathcal{UP}_G)$ in U with $y(0) = x_0$ and let $\widetilde{T} = \mathrm{Esc}(y(\cdot), U)$. Proceeding as above, we find measurable functions $u(t) \in \partial d_C(x(t))$, $v(t) \in \partial d_C(y(t))$, and $w(t)$ such that $h(t) = \dot{x}(t) + \alpha u(t) \in F(x(t))$ and $w(t) = \dot{y}(t) + \alpha v(t) \in G(t, y(t))$ a.e., $t \in I$. Let $s = \rho/2$. Recall that $x(t), y(t) \in C \subset C_{\rho/2}$ for all $t \in I$. The definition of G and the hypomonotonicity of ∂d_C in $C_{\rho/2}$ (see (6)) yield

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|x(t) - y(t)\|^2 &= \langle \dot{x}(t) - \dot{y}(t), x(t) - y(t)\rangle \\ &= \langle h(t) - \alpha u(t) - (w(t) - \alpha v(t)), x(t) - y(t)\rangle \\ &= \langle h(t) - w(t), x(t) - y(t)\rangle - \alpha\langle u(t) - v(t), x(t) - y(t)\rangle \\ &\leq \Big(\frac{4}{\rho}\alpha + k_F\Big)\|x(t) - y(t)\|^2, \end{aligned}$$

which implies $x(t) = y(t)$, for all $t \in I$ via Gronwall's lemma.

This assures $x(t) \in E$ for all $t \in [0, \widetilde{T})$. Notice that $\widetilde{T} < T$ is impossible, since otherwise $y(t) = x(t) \in U$ for $\widetilde{T} \le t < T$, which violates the definition of $\widetilde{T}$. Therefore, $x(t) \in E$ for all $t \in [0, T)$. This establishes the strong invariance property of $(E, \mathcal{M}_F)$ in U.

Conversely, assume the system $(C \cap E, \mathcal{M}_F)$ is strongly invariant in U. Let $x \in (C \cap E) \cap U$, $\zeta \in N_{C\cap E}(x)$, and a sequence $y_i \to_\zeta x$ be given. For each i, consider $u_i \in \partial d_C(y_i)$ and $v_i \in F(y_i)$ such that $h_{\mathcal{D}_F}(y_i, -\zeta) = \langle -\alpha u_i, -\zeta\rangle + \langle v_i, -\zeta\rangle$, and define $G_i : \mathbb{R}^n \to \mathbb{R}^n$ by

$$G_i(y) := \{w \in F(y) : \ \langle v_i - w, y_i - y\rangle \le k_F\|y_i - y\|^2\}.$$

It is easy to check that G_i satisfies F1–F3, which means that G_i is an autonomous submultifunction of F. The strong invariance of $(C\cap E, \mathcal{M}_F)$ in U yields, in particular, the weak invariance of $(C\cap E, \mathcal{M}_{G_i})$ in U, for $i \ge 1$. If $d \in \partial d_C(x)$ and $w_i \in G_i(x)$ satisfy $\langle -\alpha d + w_i, \zeta\rangle = h_{\mathcal{D}_{G_i}}(x, \zeta)$, then Theorem 4.3 implies $\langle -\alpha d + w_i, \zeta\rangle \le 0$ (see also Theorem 4.2.10 of [6]). Furthermore, due to F1–F2 the sequences u_i, v_i, and w_i are bounded. Let $\varepsilon > 0$ be given and fix $s \in (0, \rho)$. For sufficiently large i we have $y_i \in C_s$ and the lower semicontinuity of $h_{-\alpha\partial d_C}(\cdot, \zeta)$ guarantees

$$h_{-\alpha\partial d_C}(x_0, \zeta) \le h_{-\alpha\partial d_C}(y_i, \zeta) + \varepsilon \le -h_{-\alpha\partial d_C}(y_i, -\zeta) + \varepsilon.$$

Therefore,

$$\begin{aligned} -h_{-\alpha\partial d_C}(x_0, \zeta) + h_F(y_i, -\zeta) &\ge h_{-\alpha\partial d_C}(y_i, -\zeta) + h_F(y_i, -\zeta) - \varepsilon \\ &= h_{\mathcal{D}_F}(y_i, -\zeta) - \varepsilon, \end{aligned} \tag{22}$$

for i sufficiently large. On the other hand, rearranging terms from the definition of $G_i(x)$, using the hypomonotonicity of $\partial d_C(\cdot)$ in C_s, and the properties of the above sequences we have

$$\begin{aligned} &\liminf_{i\to\infty} h_{\mathcal{D}_F}(y_i, -\zeta) = \liminf_{i\to\infty}\ \langle -\alpha u_i + v_i, -\zeta\rangle \\ &= \liminf_{i\to\infty}\ \|\zeta\| \Big\langle -\alpha u_i + v_i, \frac{x - y_i}{\|x - y_i\|}\Big\rangle \\ &\ge \liminf_{i\to\infty}\ \|\zeta\| \Big(-k_F\|y_i - x\| + \Big\langle w_i - \alpha d, \frac{x - y_i}{\|y_i - x\|}\Big\rangle + \alpha\Big\langle d - u_i, \frac{x - y_i}{\|y_i - x\|}\Big\rangle\Big) \\ &\ge \liminf_{i\to\infty}\ \|\zeta\| \Big(-(\alpha/(\rho - s) + k_F)\|y_i - x\| + \Big\langle w_i - \alpha d, \frac{x - y_i}{\|y_i - x\|}\Big\rangle\Big) \\ &= \liminf_{i\to\infty}\ \langle \alpha d - w_i, \zeta\rangle \ge 0. \end{aligned}$$

Taking the liminf on (22) and using Proposition 2.1(b) in combination with formula (2) and the preceding inequality, yields

$$-h_{-N_C^P\cap\alpha\mathbb{B}}(x_0, \zeta) + \liminf_{i\to\infty} h_F(y_i, -\zeta) \ge \liminf_{i\to\infty} h_{\mathcal{D}_F}(y_i, -\zeta) - \varepsilon \ge -\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, condition (20) is satisfied with the liminf taken all over the sequences $y_i \to_\zeta x$. Therefore, (20) holds as stated. □

Remark 4.7. Taking the liminf over $y \to_\zeta x$ in the previous result can be replaced by the a priori weaker condition of taking this limit over $\delta \to 0^+$ and with $y = x + \delta\zeta$ without changing the equivalence with strong invariance. □

5. Hamilton-Jacobi theory for T_S

In this section, we apply the previously established invariance results to provide Hamiltonian interpretations of the invariance properties of the auxiliary augmented system $(E, \mathcal{M}_F \times \{-1\})$, first considering $E = \operatorname{epi} T_S$ and subsequently $E = \operatorname{hypo} T_S$. As a result, under the additional assumption of continuity for $T_S(\cdot)$, we derive a characterization of the minimal time function as a proximal solution to a pair of Hamilton-Jacobi inequalities. We begin by establishing the first of these interpretations, which parallels Proposition 3.3 in [30].

Proposition 5.1. *Suppose* (SA) *are satisfied and let* $\theta : C \to (-\infty, \infty]$ *be a given function.*

(a) *If* θ *is lower semicontinuous and* $E = \operatorname{epi}\theta$*, then* $(E, \mathcal{M}_F \times \{-1\})$ *is weakly invariant in* $S^c \times \mathbb{R}$ *if and only if*

$$1 + h_{-N_C^P \cap \alpha\mathbb{B}}(x, \xi) + h_F(x, \xi) \le 0, \tag{23}$$

for all $x \in C \cap S^c$*, and all* $\xi \in \partial_P \theta(x)$.

(b) *If additionally* F *is DL,* θ *is upper semicontinuous on* C*, and* $E = \operatorname{hypo}\theta$*, then* $(E, \mathcal{M}_F \times \{-1\})$ *is strongly invariant in* $\mathbb{R}^{n+1}$ *if and only if*

$$1 - h_{-N_C^P \cap \alpha\mathbb{B}}(x, \xi) + \liminf_{y \to_\xi x} h_F(y, -\xi) \ge 0, \tag{24}$$

for all $x \in C$*, and all* $\xi \in -\partial^P \theta(x)$.

Proof. The equivalence between the invariance properties of $(E, \mathcal{M}_F \times \{-1\})$ and $(E, \mathcal{D}_F \times \{-1\})$ permits us to invoke Proposition 3.3(a) of [30] to obtain that $(E, \mathcal{M}_F \times \{-1\})$ is weakly invariant in $S^c \times \mathbb{R}$ if and only if $1 + h_{\mathcal{D}_F}(x, \xi) \le 0$, for all $x \in C \cap S^c$, and all $\xi \in \partial_P \theta(x)$, which turns into condition (23) after applying (18). To establish (b) observe that $F \times \{-1\}$ is dissipative Lipschitz and

$$\mathcal{D}_F \times \{-1\} = (-N_C^P \cap \alpha\mathbb{B}) \times \{0\} + F \times \{-1\}.$$

Since $(E, \mathcal{D}_F \times \{-1\})$ is strongly invariant in $\mathbb{R}^{n+1}$, we can apply (20) on the sets $\widetilde{C} = C \times \mathbb{R}$, $E = \operatorname{hypo}\theta$, and $\widetilde{U} = \mathbb{R}^{n+1}$ to obtain

$$-h_{(-N_C^P \cap \alpha\mathbb{B}) \times \{0\}}((x, r), (\xi, \eta)) + \liminf_{(y,s) \to_{(\xi,\eta)} (x,r)} h_{F \times \{-1\}}((y, s), -(\xi, \eta)) \ge 0 \tag{25}$$

for all $(x, r) \in (\widetilde{C} \cap E) \cap \widetilde{U}$ and all $(\xi, \eta) \in N^P_{\widetilde{C} \cap E}(x, r)$, where the liminf is taken all over the sequences of the form $(y, s) = (x, r) + \delta(\xi, \eta)$ with $\delta \downarrow 0$ (see Remark 4.7).

If $x \in C$ and $\xi \in -\partial^P \theta(x)$, then $(x, \theta(x)) \in \operatorname{hypo}(\theta) = (\widetilde{C} \cap E) \cap \widetilde{U}$ and according to Proposition (1.1) we have $(\xi, 1) \in N^P_{\operatorname{hypo}(\theta)}(x, \theta(x))$. If we elaborate on the left-hand side of (25) under these specific elements we have

$$\begin{aligned} &- h_{(-N_C^P \cap \alpha\mathbb{B}) \times \{0\}}((x, \theta(x)), (\xi, 1)) + \liminf_{(y,s) \to_{(\xi,1)} (x,\theta(x))} h_{F \times \{-1\}}((y, s), -(\xi, 1)) \\ &= - \inf_{v \in N_C^P(x) \cap \alpha\mathbb{B}} \langle (-v, 0), (\xi, 1) \rangle + \liminf_{(y,s) \to_{(\xi,1)} (x,\theta(x))} \inf_{w \in F(y)} \langle (w, -1), (-\xi, -1) \rangle \end{aligned}$$

$$= -\inf_{v\in N_C^P(x)\cap\alpha\mathbb{B}}\langle -v,\xi\rangle + \liminf_{y\to_\xi x}\Big(\inf_{w\in F(y)}\langle w,-\xi\rangle + 1\Big)$$
$$= 1 - h_{-N_C^P\cap\alpha\mathbb{B}}(x,\xi) + \liminf_{y\to_\xi x}\ h_F(y,-\xi),$$

which corroborates the necessity of (24) according to (25).

To prove sufficiency, we consider a trajectory $z(\cdot)$ of $\mathcal{D}_F\times\{-1\}$ with $z(0)\in\operatorname{hypo}\theta$, which is necessarily of the form $z(t)=(x(t),r-t)$, where $x(\cdot)$ satisfies the differential inclusion $\dot{x}(t)\in\mathcal{D}_F(x(t))$, a.e. $t\in I$, and $r\le\theta(x(0))$. We can proceed as in the proof of the sufficiency of Theorem 4.5 by considering again the submultifunction G of F defined by

$$G(t,y) := \{v\in F(y) : \langle h(t)-v, x(t)-y\rangle \le k_F\|x(t)-x\|^2\},\ (t,y)\in I\times\mathbb{R}^n,$$

where $h(\cdot)$ is a measurable selection of $H(t)=(\{\dot{x}(t)\}+\alpha\partial d_C(x(t)))\cap F(x(t))$ on I. It is clear that $G\times\{-1\}$ is a submultifunction of $F\times\{-1\}$. For a subinterval $\widetilde{I}\subset I$, let A be the set of null measure such that G is upper semicontinuous on $\widetilde{I}\setminus A$ and fix $t\in\widetilde{I}\setminus A$. For $(x,q)\in(C\times\mathbb{R})\cap E$ and any $(\xi,\eta)\in N^P_{(C\times\mathbb{R})\cap E}(x,q)$ Proposition 1.1 indicates that $\eta\ge 0$ and

$$\begin{aligned} &h_{\mathcal{D}_G\times\{-1\}}((t,x,q),(\xi,\eta))\\ &\quad= h_{(-N_C^P\cap\alpha\mathbb{B})\times\{0\}}((x,q),(\xi,\eta)) + h_{G\times\{-1\}}((t,x,q),(\xi,\eta))\\ &\quad= -\eta + h_{-N_C^P\cap\alpha\mathbb{B}}(x,\xi) + h_G((t,x),\xi). \end{aligned}\tag{26}$$

If $\eta>0$, Proposition 1.1(b) asserts $q=\theta(x)$ and since $N^P_{(C\times\mathbb{R})\cap E}(x,\theta(x))$ is a cone, we have $(\xi/\eta,1)\in N^P_{(C\times\mathbb{R})\cap E}(x,\theta(x))$, and then $\xi/\eta\in-\partial^P\theta(x)$ again by Proposition 1.1. Arguing as in (21) and combining (24) and (26) we have

$$\begin{aligned} &h_{\mathcal{D}_G\times\{-1\}}((t,x,\theta(x)),(\xi,\eta))\\ &\quad= h_{(-N_C^P\cap\alpha\mathbb{B})\times\{0\}}((x,\theta(x)),(\xi,\eta)) + h_{G\times\{-1\}}((t,x,\theta(x)),(\xi,\eta))\\ &\quad\le \eta\big(-1 + h_{-N_C^P\cap\alpha\mathbb{B}}(x,\xi/\eta) - \liminf_{y\to_{\xi/\eta}x} h_F(y,-\xi/\eta)\big)\le 0. \end{aligned}\tag{27}$$

If $\eta=0$, the combination of parts (c) and (d) from Proposition 1.1 guarantees the existence of sequences $x_i\to x$, $\xi_i\to\xi$, $\theta(x_i)\to\theta(x)$, and $\eta_i\to 0$, as $i\to\infty$, such that $(x_i,\theta(x_i))\in\operatorname{hypo}\theta$, $(\xi_i,\eta_i)\in N^P_{(C\times\mathbb{R})\cap E}(x_i,\theta(x_i))$, and $\eta_i>0$ for all i. Refreshing (27) in terms of the above sequences we get

$$h_{\mathcal{D}_G\times\{-1\}}((t,x_i,\theta(x_i)),(\xi_i,\eta_i))\le 0,\quad \text{for all } i\ge 1.\tag{28}$$

It is clear that

$$h_{\mathcal{D}_G\times\{-1\}}((t,x,r),(\xi,0)) = h_{\mathcal{D}_G\times\{-1\}}((t,x,\theta(x)),(\xi,0)).$$

The lower semicontinuity of $h_{\mathcal{D}_G\times\{-1\}}$ and (28) now yield

$$h_{\mathcal{D}_G\times\{-1\}}((t,x,r),(\xi,0)) \le \liminf_{i\to\infty} h_{\mathcal{D}_G\times\{-1\}}((t,x_i,\theta(x_i)),(\xi_i,\eta_i)\le 0.$$

Since (t,x,r) and (ξ,η) are arbitrary in $\widetilde{I}\times E$ and E, respectively, it follows from Theorem 4.3 that $(E,\mathcal{D}_G\times\{-1\})$ is weakly invariant in $\widetilde{U}$. If $w(\cdot)$ is an invariant trajectory of $(E,\mathcal{D}_G\times\{-1\})$ with $w(0)=z(0)$, then necessarily $w(t)=(y(t),r-t)$, where $y(\cdot)\in\Upsilon_{(\mathcal{D}_G,\mathbb{R}^n)}(x(0))$ satisfies $y(t)\in C$ and $r-t\le\theta(y(t))$ for all $t\in[0,+\infty)$.

The rest of the proof proceeds identically to the sufficiency of Theorem 4.5, where the hypomonotonicity of $\partial_P d_C(\cdot)$ and the DL property of $F(\cdot)$ are heavily used, obtaining at the end that $x(t)=y(t)$ for all $t\in[0,+\infty)$. Notice that this in particular implies $r-t\le\theta(x(t))$ for all $t\in[0,+\infty)$, which means $z(t)=(x(t),r-t)\in\widetilde{C}\cap E$ for all $t\in[0,+\infty)$. Therefore, $(E,\mathcal{M}_F\times\{-1\})$ is strongly invariant in $\mathbb{R}^{n+1}$. □

We now present the main result of this article. The following theorem is a characterization of the minimal time function $T_S(\cdot)$ in terms of Hamilton-Jacobi inequalities involving an analytical boundary condition on the target set S.

Theorem 5.2. *Assume that* (SA) *are satisfied, the DL condition holds on F, $S\subset\mathbb{R}^n$ is closed, and T_S is upper semicontinuous on C. Then T_S is the unique continuous and bounded below function on C that satisfies the following Hamilton-Jacobi inequalities*

$$\left.\begin{array}{ll} 1+h_{-N_C^P\cap\alpha\mathbb{B}}(x,\xi)+h_F(x,\xi)\le 0, & \forall\xi\in\partial_P T_S(x)\\ 1-h_{-N_C^P\cap\alpha\mathbb{B}}(x,\xi)+\liminf\limits_{y\to_\xi x} h_F(y,-\xi)\ge 0, & \forall\xi\in-\partial^P T_S(x)\end{array}\right\}\ \forall x\in C\cap S^c,\quad \text{(HJI)}$$

and the analytical boundary conditions

$$\left.\begin{array}{ll} 1-h_{-N_C^P\cap\alpha\mathbb{B}}(x,\xi)+\liminf\limits_{y\to_\xi x} h_F(y,-\xi)\ge 0, & \forall\xi\in-\partial^P T_S(x)\\ T_S(x)=0 & \end{array}\right\}\ \forall x\in C\cap S.\quad \text{(ABC)}$$

Proof. The continuity of $T_S(\cdot)$ results from the combination of the upper semicontinuity assumption and the lower semicontinuity established in Proposition 2.5. It is also clear that $T_S(\cdot)$ is bounded below by zero and that vanishes on $S\cap C$, which confirms the second condition in (ABC).

The first inequality in (HJI) follows from Proposition 4.1(a) in conjunction with Proposition 5.1(a). Similarly, the second inequality in (HJI) and the first inequality in (ABC) are derived by applying parts (b) of the same Propositions.

Let us consider a function $\theta(\cdot)$ that is bounded below and continuous on C. If $\theta(\cdot)$ satisfies the first inequality in (HJI) and the second condition in (ABC), Proposition 5.1(a) implies that $(\operatorname{epi}\theta,\mathcal{M}_F\times\{-1\})$ is weakly invariant in $S^c\times\mathbb{R}^n$. Using Proposition 4.1(a), we conclude that $\theta(x)\ge T_S(x)$ for all $x\in C$. If $\theta(\cdot)$ additionally satisfies the second inequality in (HJI) and the first inequality in (ABC), Proposition 5.1(b) guarantees $(\operatorname{hypo}\theta,\mathcal{M}_F\times\{-1\})$ is strongly invariant in $\mathbb{R}^{n+1}$. Applying Proposition 4.1(b), it follows that $\theta(x)\le T_S(x)$ for all $x\in C$. Thus, the functions $T_S(\cdot)$ and $\theta(\cdot)$ coincide on C, and the proof is complete. □

To close this section we reformulate the result above in terms of a continuous perturbation F.

Corollary 5.3. *In addition to the hypotheses of Theorem* 5.2, *let us assume that F is continuous. Then T_S is the unique continuous function satisfying the following Hamilton-Jacobi inequalities*

$$\left.\begin{array}{ll} 1+h_{-N_C^P\cap\alpha\mathbb{B}}(x,\xi)+h_F(x,\xi)\le 0, & \forall\xi\in\partial_P T_S(x)\\ 1-h_{-N_C^P\cap\alpha\mathbb{B}}(x,\xi)+h_F(x,-\xi)\ge 0, & \forall\xi\in-\partial^P T_S(x)\end{array}\right\}\ \forall x\in C\cap S^c, \tag{HJI}$$

and the analytical boundary conditions

$$\left.\begin{array}{ll} 1-h_{-N_C^P\cap\alpha\mathbb{B}}(x,\xi)+h_F(x,-\xi)\ge 0, & \forall\xi\in-\partial^P T_S(x)\\ T_S(x)=0 & \end{array}\right\}\ \forall x\in C\cap S. \tag{ABC}$$

Remark 5.4. In the case where $\mathrm{int}(C)\neq\emptyset$, at every $x\in\mathrm{int}(C)$ the state constraint of the dynamics is inactive and the normal cone component becomes trivial, which discards $h_{-N_C^P\cap\alpha\mathbb{B}}$ from the above Hamiltonian inequalities. If additionally, $x\in\mathrm{int}(C)$ is such that $\partial_P T_S(x)\neq\emptyset$, and $\partial^P T_S(x)\neq\emptyset$, then the Fréchet derivative $\nabla T_S(x)$ exists and $\nabla T_S(x)=\partial_P T_S(x)=\partial^P T_S(x)$, and we recover the exact Hamilton-Jacobi equation

$$1+h_F(x,\nabla T_S(s))=0,$$

together with its analytical boundary conditions $1+h_F(x\nabla T_S(x))\ge 0$, $T_S(x)=0$.

6. Continuity of T_S

In this section, we provide a sufficient condition for the continuity of the minimal time function $T_S(\cdot)$ associated with $(\mathcal{P})$. This will require the compactness of the target set S and the following particular case of the Fillipov-type result about the dependence of the trajectories of $(\mathcal{P})$ on initial conditions that will appear on [24].

Proposition 6.1. *Assume (SA) are satisfied and F is DL with rank k_F. Let $y(\cdot)$ be an absolutely continuous arc that satisfies $y(t)\in C$ for all $t\in I$ and*

$$d(\dot y(t),-N_C(y(t))+F(y(t)))\le\lambda,\ \ a.e.\ t\in I,$$

for some $\lambda\ge 0$. Then for each $s\in(0,\rho)$ and each $x_0\in C$, there is a trajectory $x(\cdot)$ to $(\mathcal{P})$, with $x(0)=x_0$, that satisfies

$$\|y(t)-x(t)\|\le\Big(\|y(0)-x_0\|+\frac{2\lambda}{\delta}\Big)e^{\delta t}-\frac{2\lambda}{\delta},\ \text{ for all } t\in I,\ \delta:=\frac{\rho}{(\rho-s)^2}\alpha+k_F. \tag{29}$$

Proof. If $y(\cdot)$ is as in the statement and we define $P(x):=F(x)+\lambda\mathbb{B}$, then $y(\cdot)$ is an arc that satisfies the state-constrained differential inclusion

$$\begin{aligned}&\dot y(t)\in -N_C(y(t))+P(y(t)),\\ &y(t)\in C,\ t\in I.\end{aligned}\tag{30}$$

It is clear that the multifunction P satisfies F1 and F3 and the global bound condition

$$\sup\{\|v\| : v \in P(x)\} \leq \alpha + \lambda, \quad \text{for all } x \in \mathbb{R}^n.$$

It follows from Theorem 2.2 that $y(\cdot)$ satisfies the unconstrained differential inclusion

$$\dot{y}(t) \in -(\alpha + \lambda)\partial d_C(y(t)) + P(y(t)), \quad \text{a.e. } t \in I.$$

The latter inclusion implies

$$d(\dot{y}(t), -\alpha\partial d_C(y(t)) + F(y(t))) \leq 2\lambda, \quad \text{a.e. } t \in I. \tag{31}$$

Let $s \in (0, \rho)$ and consider the auxiliary multifunction

$$H(t,x) := (-\alpha\partial d_C(x) + F(x)) \bigcap D(t,x), \;\; (t,x) \in I \times \overline{C}_s,$$

where $D(t,x) := \Big\{ v \in \mathbb{R}^n : \langle \dot{y}(t) - v, y(t) - x\rangle \leq \delta\|y(t) - x\|^2 + 2\lambda\|y(t) - x\| \Big\}.$

We proceed to verify certain properties of the multifunction H.

(a) *$H(\cdot,\cdot)$ is nonempty, compact, and convex-valued.*

It is clear that the multifunction

$$t \longmapsto -\alpha\partial d_C(y(t)) + F(y(t))$$

is measurable. Therefore, we can apply Proposition 3.4(b) of [11] to find a measurable function $w(\cdot)$ satisfying $w(t) \in -\alpha\partial d_C(y(t)) + F(y(t))$ and

$$\|\dot{y}(t) - w(t)\| = d(\dot{y}(t), -\alpha\partial d_C(y(t)) + F(y(t))), \quad \text{a.e.}, t \in [0,T].$$

Let us fix t in the common subset of full measure of I where $w(t)$ is defined and $\dot{y}(t)$ satisfy (31), and let $x \in \overline{C}_s$. Since $y(t) \in C \subset \overline{C}_s$, by Proposition 3.2 there is some $v(t) \in -\alpha\partial d_C(x) + F(x)$ such that

$$\langle w(t) - v(t), y(t) - x\rangle \leq \left(\frac{\rho}{(\rho - s)^2}\alpha + k_F\right)\|y(t) - x\|^2 = \delta\|y(t) - x\|^2.$$

It follows that $v(\cdot)$ is also measurable and we have the estimate

$$\begin{aligned}\langle \dot{y}(t) - v(t), y(t) - x\rangle &= \langle w(t) - v(t), y(t) - x\rangle + \langle \dot{y}(t) - w(t), y(t) - x\rangle \\ &\leq \delta\|y(t) - x\|^2 + \|\dot{y}(t) - w(t)\|\|y(t) - x\| \\ &\leq \delta\|y(t) - x\|^2 + 2\lambda\|y(t) - x\|.\end{aligned}$$

The latter implies $v(t) \in (-\alpha\partial d_C(x) + F(x)) \bigcap D(t,x)$, which guarantees that $H(t,x) \neq \emptyset$. Notice that $-\alpha\partial d_C(x) + F(x)$ and $D(t,x)$ are compact and closed, respectively, and both sets are convex. Therefore, $H(t,x)$ is compact and convex.

(b) *$H(\cdot,\cdot)$ is globally bounded.*

It readily follows from F2 that

$$\sup\{\|v\| : v \in H(t,x)\} \leq 2\alpha.$$

(c) $H(\cdot,\cdot)$ *is almost upper semicontinuous.*

Let $\varepsilon > 0$. Invoking Lusin's theorem we can find a compact $I_\varepsilon \subseteq [0,T]$ with $\mathcal{L}([0,T] \setminus I_\varepsilon) < \varepsilon$, such that $\dot{y}(\cdot)$ is continuous on I_ε and, as noticed above, H is nonempty on $I_\varepsilon \times \overline{C}_s$. Let $J_\varepsilon \subseteq I_\varepsilon$ denote the set of points of density of I_ε and let $(t,x) \in J_\varepsilon \times \overline{C}_s$. If $(t_i, x_i, v_i) \in \operatorname{gr} D$ is such that $(t_i, x_i) \in I_\varepsilon \times \overline{C}_s$ and satisfies $(t_i, x_i, v_i) \mapsto (t, x, v)$, then $v_i \in D(t_i, x_i)$, which yields

$$\begin{aligned}\langle \dot{y}(t) - v, y(t) - x\rangle &= \lim_{i\to\infty} \langle \dot{y}(t_i) - v_i, y(t_i) - x_i\rangle \\ &\le \lim_{i\to\infty} (\delta \|y(t_i) - x_i\|^2 + 2\lambda \|y(t_i) - x_i\|) \\ &= \delta\|y(t) - x\|^2 + 2\lambda\|y(t) - x\|,\end{aligned}$$

thus $v \in D(t,x)$. Thus $\operatorname{gr} D(\cdot,\cdot)$ is almost closed. The t-independence of the term $-\alpha\partial d_C(\cdot) + F(\cdot)$ implies its upper semicontinuity on $[0,T] \times \mathbb{R}^n$, hence its graph is closed. Since the intersection of multifunctions with almost closed graphs results in a multifunction with the same property, from the above argument we get that

$$\operatorname{gr} H = \operatorname{gr}(-\alpha\partial d_C(\cdot) + F(\cdot)) \bigcap \operatorname{gr} D(\cdot,\cdot)$$

is almost closed in $[0,T] \times \overline{C}_s$ as well, which in the presence of properties (a) and (b) implies that $H(\cdot,\cdot)$ is almost upper semicontinuous on $[0,T] \times \overline{C}_s$.

Let us extend H over $[0,T] \times \mathbb{R}^n$ by defining

$$\widetilde{H}(t,x) := \operatorname{co} H(t, \operatorname{proj}_{\overline{C}_s}(x)), \quad \text{where} \quad H(t, \operatorname{proj}_{\overline{C}_s}(x)) = \bigcup_{y \in \operatorname{proj}_{\overline{C}_s}(x)} H(t,y).$$

It is standard to verify that the properties (a)–(b) remain valid for $\widetilde{H}$. We now proceed to verify that $\widetilde{H}(\cdot,\cdot)$ is also almost upper semicontinuous. For this purpose, we notice that for each $x \in \mathbb{R}^n$ the compactness of $\operatorname{proj}_{\overline{C}_s}(x)$ implies its separability. Therefore, there exists a sequence $y_i \in \operatorname{proj}_{\overline{C}_s}(x)$ such that

$$\operatorname{proj}_{\overline{C}_s}(x) = \overline{\{y_i\}} = \textstyle\bigcap_{k\ge1} \big(\{y_i\} + \frac{1}{k}\mathbb{B}\big).$$

Since $\{y_i\}$ is bounded, for almost all $t \in [0,T]$ the upper semicontinuity of $H(t,\cdot)$ implies $H\big(t, \bigcap_{k\ge1} \big(\{y_i\} + \frac{1}{k}\mathbb{B}\big)\big) = \bigcap_{k\ge1} H\big(t, \{y_i\} + \frac{1}{k}\mathbb{B}\big)$. Therefore,

$$\begin{aligned}H(t, \operatorname{proj}_{\overline{C}_s}(x)) &= \bigcap_{k\ge1} H\Big(t, \{y_i\} + \frac{1}{k}\mathbb{B}\Big) = \bigcap_{k\ge1}\Big(\bigcup_{i\ge1} H\Big(t, y_i + \frac{1}{k}\mathbb{B}\Big)\Big) \\ &= \bigcup_{i\ge1}\Big(\bigcap_{k\ge1} H\Big(t, y_i + \frac{1}{k}\mathbb{B}\Big)\Big) = \bigcup_{i\ge1} H(t, y_i).\end{aligned} \tag{32}$$

Let $\varepsilon > 0$. The almost upper semicontinuity of H on $[0,T] \times \overline{C}_s$ implies the existence of a compact $I_\varepsilon \subseteq [0,T]$ with $\mathcal{L}([0,T] \setminus I_\varepsilon) < \varepsilon$, and such that H is upper semicontinuous on $I_\varepsilon \times \overline{C}_s$. Let $J_\varepsilon \subseteq I_\varepsilon$ be the set of density points of I_ε and let $(t,x) \in J_\varepsilon \times \mathbb{R}^n$.

Consider a sequence $(t_r, x_r, v_r) \in \operatorname{gr} H(\cdot, \operatorname{proj}_{\overline{C}_s}(\cdot))$ with $(t_r, x_r) \in I_\varepsilon \times \mathbb{R}^n$ and satisfying $(t_r, x_r, v_r) \mapsto (t, x, v)$. Since $v_r \in H(t_r, \operatorname{proj}_{\overline{C}_s}(x_r))$, due to (32) there exists

$y_{i_r} \in \overline{C}_s$, with $v_r \in H(t_r, y_{i_r})$, and such that $\|y_{i_r} - x_r\| = d_{\overline{C}_s}(x_r) \to d_{\overline{C}_s}(x)$, $r \to \infty$. The convergence of x_r implies y_{i_r} is bounded, and then passing to a convergent subsequence and relabelling, if necessary, we get $y_{i_r} \to y \in \overline{C}_s$. Therefore, $\|y - x\| = \lim_{r\to\infty} \|y_{i_r} - x_r\| = d_{\overline{C}_s}(x)$, that is $y \in \mathrm{proj}_{\overline{C}_s}(x)$. We have shown that $(t_r, y_{i_r}, v_r) \in \mathrm{gr}\, H(\cdot,\cdot)$, where $(t_r, y_{i_r}, v_r) \to (t, y, v)$ and $y \in \overline{C}_s$. Since H has closed graph on $I_\varepsilon \times \overline{C}_s$, we obtain $v \in H(t, y) \subset H(t, \mathrm{proj}_C(x))$, which implies $H(\cdot, \mathrm{proj}_C(\cdot))$ has almost closed graph. Therefore, $H(\cdot, \mathrm{proj}_C(\cdot))$ is almost upper semicontinuous on $[0, T] \times \mathbb{R}^n$. It finally follows from Proposition 2.2.11(b) in [29] that $\widetilde{H}(\cdot, \cdot)$ is almost upper semicontinuous on $[0, T] \times \mathbb{R}^n$.

We now invoke Theorem 5.3 in [11] to guarantee that for every $x_0 \in \mathbb{R}^n$ there is always a solution $x(\cdot)$ to the initial value problem $\dot{x}(t) \in \widetilde{H}(t, x(t))$, a.e., $t \in [0, T]$, $x(0) = x_0$. Furthermore, the very dynamics of a sweeping process implies that all of the trajectories of $(\mathcal{P})$ originated in C are completely contained in C, which implies that $(C, -\alpha\partial d_C + F)$ is strongly invariant in $\mathbb{R}^n$. Note also that

$$\widetilde{H}(t, x) = H(t, x) \subseteq -\alpha\partial d_C(x) + F(x) = \mathcal{D}_F$$

for almost all $t \in [0, T]$ and all $x \in C$, and any $x \in C$ is an interior point of $\overline{C}_s$. Proceeding as in (21) and applying Theorem 4.5, we obtain the following estimate for almost all $t \in [0, T]$, all $x \in C$, and all $\zeta \in N_C^P(x)$,

$$\begin{aligned} h_{\widetilde{H}}(t, x, \zeta) = h_H(t, x, \zeta) &\le \liminf_{\tau\to t,\, z\to x} h_H(\tau, z, \zeta) \le \limsup_{\tau\to t,\, y\to_\zeta x} h_H(\tau, y, \zeta) \\ &\le \limsup_{y\to_\zeta x} (-h_{\mathcal{D}_F}(y, -\zeta)) \le h_{-N_C^P\cap\alpha\mathbb{B}}(x, \zeta) - \liminf_{y\to_\zeta x} h_F(y, -\zeta) \le 0, \end{aligned}$$

which implies that $(C, \widetilde{H})$ is weakly invariant in $\mathbb{R}^n$, according to Theorem 1 of [12]. Therefore, given $x_0 \in C$ there is a trajectory $x(\cdot)$ to $\widetilde{H}$ such that $x(0) = x_0$ and $x(t) \in C$ for all $t \in C$. It is then clear that $x(\cdot)$ is also a trajectory of the differential inclusions problems associated with $-\alpha\partial d_C + F$ (and thus with $-N_C^P + F$) and D. The definition of D implies

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|y(t) - x(t)\|^2 &= \langle \dot{y}(t) - \dot{x}(t), y(t) - x(t)\rangle \\ &\le \delta\|y(t) - x(t)\|^2 + 2\lambda\|y(t) - x(t)\|, \quad \text{a.e., } t \in [0, T], \end{aligned}$$

from which we obtain

$$\frac{d}{dt}\|y(t) - x(t)\| \le \delta\|y(t) - x(t)\| + 2\lambda, \quad \text{a.e., } t \in [0, T].$$

Applying Gronwall's lemma on the last inequality ensues the estimate in (29). Therefore, $x(\cdot)$ is the sought-after trajectory of $(\mathcal{P})$ and the proof is complete. □

We now investigate the continuity of $T_S(\cdot)$. We start by recalling that a *modulus function* $m(\cdot) : [0, \infty) \to [0, \infty)$ is a nondecreasing continuous function satisfying $m(0) = 0$. We say that two modulus functions $m_1(\cdot)$ and $m_2(\cdot)$ *are equivalent* if there exists a constant $c > 0$ such that

$$0 < \liminf_{r\downarrow 0} \frac{m_1(cr)}{m_2(r)} \le \limsup_{r\downarrow 0} \frac{m_1(cr)}{m_2(r)} < \infty.$$

It is well known that the above condition defines an equivalence relation among modulus functions. We will call *modulus class* to any equivalence class of such modulus functions.

Let $\mathcal{M}$ be a modulus class and consider a function $f : U \to \mathbb{R}$, with $U \subset \mathbb{R}^n$. We will say that f is $\mathcal{M}$-*continuous on* U if there is some $m(\cdot) \in \mathcal{M}$ such that

$$|f(x) - f(y)| \leq m(\|x - y\|) \quad \text{for all } x, y \in U.$$

Proposition 6.2. *Assume* (SA) *are satisfied and* F *is DL with rank* k_F. *Let* $\mathcal{M}$ *be a modulus class and* $s \in (0, \rho)$ *be such that* $\delta > 0$, *with* δ *defined as in Proposition 6.1. Then the following assertions are equivalent.*

(a) *There exists* $\eta > 0$ *such that* $T_S(\cdot)$ *is* $\mathcal{M}$*-continuous on* $\overline{S}_\eta \cap C$.

(b) *There exists* $\eta > 0$ *and* $m(\cdot) \in \mathcal{M}$ *such that* $T_S(x) \leq m(d_S(x))$ *for all* $x \in \overline{S}_\eta \cap C$.

Proof. It is clear that assertion (a) implies (b). To show that (a) follows from (b), let $\eta > 0$ and $m(\cdot)$ satisfy condition (b). If we apply Proposition 2.2(a) from [30] with $\varepsilon = \eta/2$ and $K = \overline{S}_{\eta/2}$, we can find some $\tau > 0$ such that

$$\mathcal{R}^{(\leq \tau)}_{\mathcal{D}_F}(\overline{S}_{\eta/2} \cap C) \subset \mathcal{R}^{(\leq \tau)}_{\mathcal{D}_F}(\overline{S}_{\eta/2}) \subset \overline{S}_\eta,$$

where for any $\Omega \subset \mathbb{R}^n$ and any $t \geq 0$, the notation $\mathcal{R}^{(\leq t)}_{\mathcal{D}_F}(\Omega)$ represents the set of all points in $\mathbb{R}^n$ that can be reached from points $x \in \Omega$ within a time t or less using the trajectories associated with $\mathcal{D}_F$. Since $m(\cdot)$ is a modulus function, we can choose $\eta' > 0$ such that $m(\eta') \leq \min\{\tau, \eta/2\}$. The trajectories of $(\mathcal{UP})$ that emanate from C reside completely in C. This observation and the previous inclusions yield

$$\mathcal{R}^{(\leq m(\eta'))}_{\mathcal{D}_F}(\overline{S}_{\eta'} \cap C) \subset \overline{S}_\eta \cap C. \tag{33}$$

Let us define $m_1(t) := m(e^{\delta m(\eta')} t)$. It is clear that $m_1(\cdot)$ is a modulus function with $m_1(\cdot) \in \mathcal{M}$. Let $x, y \in \overline{S}_{\eta'} \cap C$. From condition (b) we obtain

$$T_S(x) \leq m(d_S(x)) \leq m(\eta') < \infty.$$

According to Proposition 2.5 there exists some $x(\cdot) \in \Upsilon(\mathcal{M}_F, S^c)(x)$ such that $T = T_S(x)$ and $x(T) \in S \cap C$. We invoke now Proposition 6.1 with $\lambda = 0$ to guarantee the existence of a trajectory $y(\cdot)$ for $(\mathcal{P})$ originating from y that satisfies

$$\|x(T) - y(T)\| \leq e^{\delta T} \|x - y\|. \tag{34}$$

Note that $y(T) \in \overline{S}_\eta \cap C$ according to (33). The application of the optimality principle in combination with condition (b), the monotonicity of $m(\cdot)$, and $x(T) \in S$ lead to

$$T_S(y) - T_S(x) \leq T_S(y(T)) \leq m(d_S(y(T))) \leq m(\|x(T) - y(T)\|).$$

Using again the monotonicity of $m(\cdot)$ in conjunction with (34) and the definition of $m_1(\cdot)$, from the last inequality we obtain

$$T_S(y) - T_S(x) \leq m_1(\|x - y\|).$$

If we switch the roles of x and y in the previous argument we finally obtain

$$|T_S(y) - T_S(x)| \leq m_1(\|x - y\|),$$

which corroborates the $\mathcal{M}$-continuity of $T_S(\cdot)$ in $\overline{S}_{\eta'} \cap C$. $\square$

We conclude our article with the following result, which establishes a sufficient condition for the continuity of $T_S(\cdot)$.

Theorem 6.3. *Assume the conditions of Proposition* 6.2 *are satisfied. Let $\mathcal{M}$ be a module class containing a C^2 function $m(\cdot)$ with $m'(r) > 0$ for all $r > 0$, and for which there is some $\eta > 0$ such that the Petrov-type condition*

$$h_{-N_C \cap \alpha \mathbb{B}}(x,\zeta) + h_F(x,\zeta) \leq -\frac{1}{m'(d_S(x))}, \ \forall \zeta \in \partial_P T_S(x) \tag{35}$$

is satisfied for all $x \in \overline{S}_\eta \cap C$. Then $T_S(\cdot)$ is $\mathcal{M}$-continuous on $\overline{S}_\eta \cap C$.

Proof. The proof is identical to the one contained in [30], with the sole distinction being the use of the localized domains $\overline{S}_\eta \cap C$ for $T_S(\cdot)$, as well as Proposition 6.2 above, and the Hamiltonian $h_{-N_C \cap \alpha \mathbb{B}} + h_F$, in place of $\overline{S}_\eta$, Proposition 6.1 of [30], and h_F, respectively. □

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