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The Gauss Map of a Nonsmooth Convex Cone and the Antipodal Mate Property

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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We discuss some aspects concerning the angular structure of a closed convex cone K in a Euclidean vector space E. The cone under consideration is assumed to be pointed and solid, but not necessarily smooth. Its Gauss map G_K is therefore to be understood in a multivalued sense. By definition, G_K assigns to a boundary point u of K the set $G_K(u) := N_K(u) \cap S_E$, where S_E is the unit sphere of E and N_K is the normal cone map of K in the sense of convex analysis. By a positive homogeneity argument, there is no loss of generality in assuming that u has unit length. Among other issues, we elaborate on the connection between $G_K(u)$ and the set $M_K(u)$ of antipodal mates of u. That v is an antipodal mate of u means that $\{u, v\}$ is a pair of unit vectors in the boundary of K achieving the maximum angle of the cone.

Keywords: Convex cone, maximum angle, largest angle function, oriented distance function, Gauss map, antipodal mate, diametric completeness.

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1. Introduction

Let E be a Euclidean space with inner product $\langle \cdot, \cdot \rangle$, associated norm $\| \cdot \|$, and unit sphere S_E. For avoiding trivialities, we assume that E is of dimension at least three. The notation $\mathbb{P}(E)$ stands for the set of proper cones in E. Recall that a proper cone is a closed convex cone that is pointed and has nonempty interior. Throughout this work, K is a proper cone in E and K^* is its dual cone. The interior and boundary of K are denoted by $\mathrm{int}(K)$ and $\mathrm{bd}(K)$, respectively.

There are several interesting functions in the literature that serve to characterize a proper cone or to study a particular aspect of such a cone. By way of example, the Koszul-Vinberg characteristic function $\varrho_K : \mathrm{int}(K) \to \mathbb{R}$ given by

$$\varrho_K(u) := \int_{K^*} e^{-\langle u,v \rangle} dv$$

is a strictly log-convex function that serves to analyze volumetric properties of K, see Güler [12] and Seeger and Torki [23]. Since ϱ_K is infinitely often differentiable, such

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a function is ill-suited to analyze corner points or angular properties of K. Various issues concerning the angular structure of K can be handled with the help of the *largest angle function* f_K or the *smallest angle function* h_K. These are nonsmooth continuous functions on S_E given by

$$f_K(u) := \max_{v \in K \cap S_E} \arccos \langle u, v \rangle , \tag{1}$$

$$h_K(u) := \min_{v \in K \cap S_E} \arccos \langle u, v \rangle , \tag{2}$$

respectively. The largest angle function has many uses, among which we mention the definition

$$\theta_{\max}(K) := \max_{u \in K \cap S_E} f_K(u) \tag{3}$$

$$= \max_{u,\, v \in K \cap S_E} \arccos \langle u, v \rangle \tag{4}$$

of the *maximum angle* of K. The term $\theta_{\max}(K)$ arises in a great variety of contexts, see for instance Clarke et al. [5], Peña and Renegar [19], and Iusem and Seeger [15, 16, 17]. A pair $\{u, v\}$ of unit vectors in K that solves the maximization problem (4) is called an *antipodal pair* of K.

As we shall see in Proposition 2.5, largest and smallest angle functions are related to oriented distance functions. The *oriented distance function* of K is the continuous function $\Delta_K : E \to \mathbb{R}$ given by

$$\Delta_K(u) := \begin{cases} \operatorname{dist}(u, K) & \text{if } u \in K^c \\ -\operatorname{dist}(u, K^c) & \text{if } u \in K , \end{cases}$$

where $K^c := E \backslash K$ is the set-complement of K. Note that Δ_K vanishes on the boundary of K, takes positive values in the exterior of K, and negative values in the interior of K. Hence, $\Delta_K(u)$ measures up to a sign how far is u from the boundary of K. Oriented distance functions of general closed convex sets were introduced in Hiriart-Urruty [13] and used in a variety of applications, see for instance Allevi et al. [1], Delfour and Zolésio [7, 8], Gintchev et al. [11], and Zaffaroni [25]. See also the recent book of Thibault [24] for additional information on this subject.

Quite different in spirit is the Gauss map $G_K : \operatorname{bd}(K) \rightrightarrows E$ associated to K. The double-arrow notation underlines the multivalued character of G_K. This map assigns to $u \in \operatorname{bd}(K)$ the set

$$G_K(u) := N_K(u) \cap S_E ,$$

where N_K is the normal cone map of K in the sense of convex analysis, i.e.,

$$N_K(u) := \{\xi \in E : \langle \xi, x - u \rangle \le 0 \text{ for all } x \in K\}.$$

A smooth boundary point of K is a nonzero $u \in \operatorname{bd}(K)$ such that $G_K(u)$ is a singleton. One says that K is smooth if all nonzero boundary points of K are smooth. The Gauss map G_K not only serves to discriminate between smooth and nonsmooth boundary points, but it also provides valuable information on the angular structure of K. It turns out that $G_K(u)$ is related to the set

$$M_K(u) := \{v \in K \cap S_E : \{u, v\} \text{ is an antipodal pair of } K\}$$

of *antipodal mates* of u. To the best of our knowledge, the link between the maps G_K and M_K has not been explored yet in the literature.

The organization of the paper is as follows. Section 2 establishes various relations between f_K, h_K, and Δ_K. It also displays a number of properties of the solution set

$$A_K(u) := \{v \in K \cap S_E : \arccos\langle u, v\rangle = f_K(u)\}$$

to problem (1). Each element of $A_K(u)$ is called an *antipode* of u (within K). Beware that each antipode of u belongs to $K \cap S_E$, but u itself can be anywhere on S_E. Of special importance is the case in which the argument u belongs to

$$\Sigma_K := \mathrm{bd}(K) \cap S_E\,,$$

a set that we call the *core* of K. Antipodes and antipodal mates are not quite the same thing. The last part of Section 2 recalls some ingredients of the theory of critical pairs and critical angles in proper cones. Section 3 explores the link between G_K and M_K. This last section contains the most significant contributions of this work.

2. The largest angle function of a proper cone

Let us give a closer look at the largest angle function f_K. Note that $f_K(u) = \Theta(\vec{u}, K)$, where $\vec{u} := \{tu : t \geq 0\}$ is the ray generated by $u \in S_E$ and

$$\Theta(P, Q) := \max_{\substack{y \in P \cap S_E \\ x \in Q \cap S_E}} \arccos\langle y, x\rangle \tag{5}$$

is the maximum angle between a pair of nonzero closed convex cones in E. Geometrically speaking, $f_K(u)$ is the maximum angle between the ray $\vec{u}$ and the proper cone K. Similarly, $h_K(u)$ can be interpreted as the minimum angle between $\vec{u}$ and K. The minimum angle between a pair of nonzero closed convex cones is defined as in (5) with "max" changed by "min". Most results concerning problem (5) obtained in Seeger and Sossa [21, 22] can be adapted to the case of problem (1). For instance, the next proposition is a consequence of [21, Theorem 5.2]. Let $\mathbb{P}(E)$ be equipped with the metric

$$\varrho(K_1, K_2) := \mathrm{haus}(K_1 \cap S_E, K_2 \cap S_E)\,,$$

where
$$\mathrm{haus}(\Omega_1, \Omega_2) := \max\left\{\max_{x \in \Omega_1} \mathrm{dist}(x, \Omega_2), \max_{x \in \Omega_2} \mathrm{dist}(x, \Omega_1)\right\}$$

is the Pompeiu-Hausdorff distance between a pair of nonempty compact subsets of E.

Proposition 2.1. *There exists a positive constant β such that*

$$|f_{K_1}(u_1) - f_{K_2}(u_2)| \leq \beta \max\{\|u_1 - u_2\|, \varrho(K_1, K_2)\}$$

for all $u_1, u_2 \in S_E$ and $K_1, K_2 \in \mathbb{P}(E)$.

Proposition 2.1 asserts that $f_K(u)$ behaves in a Lipschitz continuous manner with respect to simultaneous perturbations in $u \in S_E$ and $K \in \mathbb{P}(E)$. There are good reasons for studying the largest angle function associated to a proper cone K. One of such reasons is that the largest angle function determines the cone itself, cf. Proposition 2.2.

We could equally well work with the smallest angle function of K. Indeed, the identity

$$f_K(-u) + h_K(u) = \pi \qquad \text{for all } u \in S_E \tag{6}$$

allows to pass from h_K to f_K, and viceversa. The next result is obtained by combining (6) and

$$\cos[h_K(u)] \;=\; \sigma_K(u) \;:=\; \max_{v\in K\cap S_E} \langle v, u\,\rangle\,. \tag{7}$$

Although σ_K is defined on the whole space E, the first equality in (7) applies only to $u \in S_E$.

Proposition 2.2. *Let $K_1, K_2 \in \mathbb{P}(E)$ be such that $f_{K_1} = f_{K2}$. Then $K_1 = K_2$.*

Proof. Let $f_{K_1} = f_{K_2}$. Identity (6) yields $h_{K_1} = h_{K_2}$. By passing to cosine on each side of the later equality, we get $\sigma_{K_1} = \sigma_{K_2}$. Hence,

$$\mathrm{co}(K_1 \cap S_E) = \mathrm{co}(K_2 \cap S_E)\,, \tag{8}$$

where "co" stands for convex hull. But a proper cone K in E is determined by $K^\diamond := \mathrm{co}(K \cap S_E)$. Indeed, we have $K = \{tx : t \geq 0,\ x \in K^\diamond\}$. Hence, (8) yields $K_1 = K_2$. □

For each $u \in S_E$, the maximization problem (1) has the same solution set as the problem which consists in minimizing the linear form $\langle u, \cdot\,\rangle$ on $K \cap S_E$. In other words,

$$A_K(u) = \mathrm{argmin}_{v\in K\cap S_E}\langle u, v\rangle\,. \tag{9}$$

The advantage of working with (9) is that we do not need to bother with the arc-cosine function. Note that $A_K(u)$ is nonempty and compact for all $u \in S_E$. The next proposition recalls other properties of the map $A_K : S_E \rightrightarrows E$. The faces of a closed convex cone K are defined in the usual way, cf. Barker [2, Definition 1.3]. A one-dimensional face of K is called an extreme ray of K. The *set of generators* of K is defined as

$$\mathrm{gen}(K) := \{x \in K \cap S_E : \vec{x} \text{ is an extreme ray of } K\}.$$

The generators of K are of course on the boundary of K.

Proposition 2.3. *Let $K \in \mathbb{P}(E)$.*

(a) *If $u \in S_E\backslash(-K)$, then $A_K(u) \subseteq \Sigma_K$.*

(b) *If $u \in \mathrm{int}(K^*) \cap S_E$, then $A_K(u) \subseteq \mathrm{gen}(K)$.*

(c) *If $u \in S_E\backslash K^*$, then $A_K(u)$ is a singleton.*

Proof. Part (a) is easy and, therefore, we omit the proof. Part (b) can be shown as in [17, Lemma 1]. For proving (c), let $u \in S_E\backslash K^*$. Since $\cos[f_K(u)] < 0$, a positive homogeneity argument shows that

$$A_K(u) \;=\; \mathrm{argmin}_{v\in K\cap B_E}\langle u, v\rangle\,, \tag{10}$$

where B_E is the closed unit ball of E. Since the right-hand side of (10) is a convex set lying on S_E, the strict convexity of B_E implies that $A_K(u)$ is a singleton. □

Corollary 2.4. *Let $K \in \mathbb{P}(E)$ be acute in the sense that $\theta_{\max}(K) < \pi/2$. Then $A_K(u) \subseteq \mathrm{gen}(K)$ for all $u \in K \cap S_E$.*

The above corollary is a direct consequence of Proposition 2.3 (b). As shown in the next proposition, largest and smallest angular functions are related to oriented distance functions.

Proposition 2.5. *Let $K \in \mathbb{P}(E)$. Then, for all $u \in S_E$, one has*

$$h_K(u) \;=\; \arccos[\,\Delta_{-K^*}(u)] \tag{11}$$

$$f_K(u) \;=\; \arccos[-\Delta_{K^*}(u)]. \tag{12}$$

Proof. If $\Omega \subseteq E$ is a nonempty closed convex set with nonempty set-complement, then its oriented distance function admits the representation

$$\Delta_\Omega(u) \;=\; \sup_{x \in S_E} \inf_{v \in \Omega} \langle v - u, x\rangle$$

for all $u \in E$. Ginchev et al. [11, p. 17] observe that such a sup-inf becomes $\sigma_K(-u)$ when $\Omega = K^*$. Hence,

$$\cos[h_K(u)] \;=\; \sigma_K(u) \;=\; \Delta_{K^*}(-u) \;=\; \Delta_{-K^*}(u)$$

for all $u \in S_E$. This takes care of (11). Formula (12) follows afterwards by using (6). □

Obtaining an easily computable formula for evaluating $f_K(u)$ is difficult in general, but this can be done if K has a simple structure. Two such examples are given below.

Example 2.6. An *orthocone* in E is a proper cone which is usually written as $K_W := \mathrm{cone}\{w_1, \ldots, w_n\}$, where $\{w_1, \ldots, w_n\}$ is an orthonormal basis of E. We identify such an orthonormal basis with the linear isometry $W : \mathbb{R}^n \to E$ given by $Wz := \sum_{i=1}^n z_i w_i$. Orthocones are self-dual cones. Hence, Proposition 2.5 yields the explicit formula

$$f_{K_W}(u) = \arccos(-\Delta_{K_W}(u))$$

with $$\Delta_{K_W}(u) = \begin{cases} \left[\sum_{i=1}^n (\min\{0, \langle w_i, u\rangle\})^2\right]^{1/2} & \text{if } u \notin K_W \\ -\min_{1\le i\le n}\langle w_i, u\rangle & \text{if } u \in K_W\,. \end{cases}$$

Example 2.7. A *revolution cone* in E is a proper cone of the form

$$\mathrm{Rev}(a, \phi) := \{x \in E : \langle a, x\rangle \ge \|x\| \cos\phi\}\,, \tag{13}$$

where $a \in S_E$ generates the central axis and $0 < \phi < \pi/2$ is the half-aperture angle of the cone. The dual cone of (13) is yet another revolution cone, namely, $[\mathrm{Rev}(a, \phi)]^* = \mathrm{Rev}(a, (\pi/2) - \phi)$. For K as in (13), Proposition 2.5 yields

$$h_K(u) \;=\; \max\{0, \arccos\langle a, u\rangle - \phi\}$$

$$f_K(u) \;=\; \min\{\pi, \arccos\langle a, u\rangle + \phi\}$$

for all $u \in S_E$. These formulas are consistent with those obtained in [22, Corollary 2.2].

2.1. Critical angles in proper cones

A necessary condition for a pair $\{u, v\}$ of unit vectors in K to be a solution to the nonconvex optimization problem (4) is that

$$\begin{cases} -(v - \langle u, v\rangle u) \in N_K(u) \\ -(u - \langle u, v\rangle v) \in N_K(v). \end{cases} \tag{14}$$

These optimality conditions can be worked out a bit further and used to introduce the concept of critical angle.

Definition 2.8. Let $K \in \mathbb{P}(E)$. A pair $\{u, v\}$ of distinct vectors in E is a *critical pair* of K if

$$u, v \in K \cap S_E\,, \quad v - \langle u, v\rangle u \in K^*\,, \quad u - \langle u, v\rangle v \in K^*.$$

The angle formed by a critical pair is called a critical angle of K. The set of critical angles, denoted by $\mathrm{AS}(K)$, is called the angular spectrum of K.

We deviate from [16, Definition 4.2] in just one detail: we do not consider a critical pair as a couple (u, v), but as an unordered pair $\{u, v\}$. Consistently, the case $u = v$ is ruled out and 0 is not viewed as a critical angle. Both components of a critical pair of K are on the boundary of K, cf. [16, Corollary 4.4]. Note also that any antipodal pair of K is a critical pair of K, but not conversely. By using (14) one can easily check that $\{u, v\}$ is a critical pair of K if and only if u and v are distinct unit vectors in K satisfying

$$-\frac{v - \langle u, v\rangle u}{\sqrt{1 - \langle u, v\rangle^2}} \in G_K(u) \tag{15}$$

$$-\frac{u - \langle u, v\rangle v}{\sqrt{1 - \langle u, v\rangle^2}} \in G_K(v). \tag{16}$$

Hence, the Gauss map G_K not only serve to discriminate between smooth and nonsmooth boundary points of K, but it also plays a role in the characterization of criticality. We comment in passing that if $\{u, v\}$ is a pair of noncollinear unit vectors in E, then

$$a := \frac{v - \langle u, v\rangle u}{\sqrt{1 - \langle u, v\rangle^2}}, \quad b := \frac{u - \langle u, v\rangle v}{\sqrt{1 - \langle u, v\rangle^2}} \tag{17}$$

is a new pair of noncollinear unit vectors in E. We say that $\{a, b\}$ is the *conjugate pair* of $\{u, v\}$. The nonlinear transformation (17) is an involution.

3. Link between M_K and the Gauss map G_K

This section explores the connection between M_K and G_K. We work with the restrictions of these maps to the core of the proper cone K. Note that $G_K(u)$ is a nonempty compact set for all $u \in \Sigma_K$. By contrast, depending on the choice of $u \in \Sigma_K$, the set $M_K(u)$ can be empty, a singleton, or contain more than one element. For subsequent use, we introduce the notation

$$c_K := \cos[\theta_{\max}(K)]\,, \quad s_K := \sin[\theta_{\max}(K)]\,.$$

The term s_K is always positive, whereas c_K could be of any sign.

Proposition 3.1. *Let $K \in \mathbb{P}(E)$ be obtuse in the sense that $\theta_{\max}(K) > \pi/2$ and let $u \in \Sigma_K$. Then $M_K(u)$ is either empty or a singleton.*

Proof. Suppose that $M_K(u)$ has at least one element, say $\tilde{v}$. Since $M_K(u) \subseteq A_K(u)$, it suffices to check that $A_K(u)$ is a singleton. Note that $\tilde{v} \in K \cap S_E$ and

$$\langle u, \tilde{v} \rangle = \min_{v \in K \cap S_E} \langle u, v \rangle = \cos[f_K(u)] \, = \, c_K.$$

But $c_K < 0$, because K is obtuse. So, $u \in S_E \backslash K^*$ and we are in the context of Proposition 2.3 (c). □

The next proposition gives an upper estimate of $M_K(u)$ in terms of $G_K(u)$.

Proposition 3.2. *Let $K \in \mathbb{P}(E)$. Then, for all $u \in \Sigma_K$, one has*

$$M_K(u) \, \subseteq \, c_K u - s_K G_K(u) \,. \tag{18}$$

In particular, if $u \in \Sigma_K$ is smooth and ξ_u is the unique element of $G_K(u)$, then $M_K(u)$ is either empty or it contains $v_u := c_K u - s_K \xi_u$ as unique element. In the later case, the antipodal mate of u belongs to the linear span of u and ξ_u.

Proof. Let $u \in \Sigma_K$. If $M_K(u)$ is empty, then we are done. Let $v \in M_K(u)$. Then $\langle u, v \rangle = c_K$ and (15) yields $(1/s_K)(c_K u - v) \in G_K(u)$. Hence, $v = c_K u - s_K \xi$ with $\xi \in G_K(u)$. This proves the inclusion (18). The second part of the proposition follows from such an inclusion. □

If K is neither obtuse nor smooth, then a nonsmooth point $u \in \Sigma_K$ may have more than one antipodal mate. For instance, a generator of an orthocone has infinitely many antipodal mates. The next corollary is a consequence of Proposition 3.2.

Corollary 3.3. *Let $\{u, v\}$ be an antipodal pair of $K \in \mathbb{P}(E)$. Then*

$$\begin{cases} v \in c_K u - s_K G_K(u) \\ u \in c_K v - s_K G_K(v) \,. \end{cases}$$

If K is smooth, then the above system becomes

$$\begin{cases} v = c_K u - s_K \xi_u \\ u = c_K v - s_K \xi_v \,. \end{cases}$$

In particular, the pair $\{\xi_u, \xi_v\}$ is conjugate to $\{u, v\}$ and the unit vectors $\{u, v, \xi_u, \xi_v\}$ are all on a same plane.

3.1. The Antipodal Mate Property

The next definition introduces a new class of proper cones. We do not know if such a definition has been considered already in the literature, at least in the way we are formulating it. In any case, Definition 3.4 is quite natural in the context of our work.

Definition 3.4. $K \in \mathbb{P}(E)$ enjoys the *Antipodal Mate Property* (in short, K is AMP) if every unit vector in $\mathrm{bd}(K)$ has an antipodal mate.

The Antipodal Mate Property can be rephrased in many equivalent ways. Below we display just a short sample of alternative formulations.

Proposition 3.5. *Let $K \in \mathbb{P}(E)$. Then the following statements are equivalent:*

(a) *K is AMP.*

(b) *f_K is constant on Σ_K.*

(c) *$A_K(u) = M_K(u)$ for all $u \in \Sigma_K$.*

Proof. (a) ⇒(b). Let K be AMP. Pick any $u \in \Sigma_K$. Then there exists $v \in K \cap S_E$ such that

$$\theta_{\max}(K) = \arccos\langle u, v\rangle \leq f_K(u) \leq \theta_{\max}(K).$$

Hence, $f_K(u) = \theta_{\max}(K)$ for all $u \in \Sigma_K$.

(b) ⇒(c). Let f_K be constant on Σ_K. Pick any $u \in \Sigma_K$. Clearly, any $v \in A_K(u)$ is an antipodal mate of u. Indeed, $\arccos\langle u, v\rangle = f_K(u) = \theta_{\max}(K)$, where the first equality is due to the definition of $A_K(u)$ and the second equality is because f_K is constant on Σ_K. Hence, $A_K(u) \subseteq M_K(u)$. The reverse implication is always true.

(c) ⇒(a). This implication is obvious because A_K is nonempty-valued on Σ_K. □

The Antipodal Mate Property has many uses. Recall that a proper cone K in E is said to be *rotund* if each unit vector in $\mathrm{bd}(K)$ is a generator of the cone. In other words, a rotund cone is a proper cone such that every nonzero boundary point belongs to an extreme ray of the cone. Rotundity and smoothness are dual concepts in the sense that rotundity in a proper cone is equivalent to smoothness in its dual cone.

Proposition 3.6. *If $K \in \mathbb{P}(E)$ is AMP and acute, then K is rotund.*

Proof. Let $v \in \Sigma_K$. We must check that v is a generator of K. Since K is AMP, there exists a point $u \in \Sigma_K$ such that $\{u, v\}$ is an antipodal pair of K. In particular, v is an antipode of u. That K is acute is equivalent to saying that c_K is positive. Hence, $\langle u, x\rangle \geq \cos[f_K(u)] = c_K$ is positive for all $x \in K \cap S_E$. In particular, $u \in \mathrm{int}(K^*)$. That v is a generator of K is therefore a consequence of Proposition 2.3 (b). □

The next result is in the same vein: acuteness is changed by obtuseness and, simultaneously, rotundity is changed by smoothness.

Proposition 3.7. *If $K \in \mathbb{P}(E)$ is AMP and obtuse, then K is smooth.*

Proof. Let $u \in \Sigma_K$. Since K is obtuse and AMP, Proposition 3.1 implies that u has exactly one antipodal mate, say v. By Corollary 3.3,

$$\xi := -(1/s_K)(v - c_K u) \in G_K(u).$$

Suppose to the contrary that u is nonsmooth. In such a case, the linear span of $N_K(u)$ is of dimension at least 2. Note also that $N_K(u)$ is a pointed closed convex cone. Hence, $N_K(u)$ has a generator, say $\tilde{\xi}$, that is different from the vector ξ (which could be generator of $N_K(u)$ or not). From the general theory of normal cones to convex sets, there exist a sequence $\{u_k\}_{k\in\mathbb{N}}$ of smooth boundary points of K converging to u and a sequence $\{\xi_k\}_{k\in\mathbb{N}}$ converging to $\tilde{\xi}$ such that $\xi_k \in N_K(u_k)$ for all $k \in \mathbb{N}$.

In fact, we may take ξ_k as the unique element of $N_K(u_k) \cap S_E$ and we may assume that $u_k \in S_E$. Next, consider the sequence $\{v_k\}_{k\in\mathbb{N}}$ defined by $v_k := c_K u_k - s_K \xi_k$. Since K is AMP and u_k is smooth, Proposition 3.2 implies that $v_k \in \Sigma_K$ is the unique antipodal mate of u_k. The limit

$$\tilde{v} := \lim_{k\to\infty} v_k \ = c_K u - s_K \tilde{\xi}$$

remains in Σ_K. Since $\langle u_k, v_k \rangle = c_K$ for all $k \in \mathbb{N}$, we also get $\langle u, \tilde{v} \rangle = c_K$. Hence, $\tilde{v}$ is an antipodal mate of u. Finally, since $M_K(u)$ is a singleton, we obtain

$$c_K u - s_K \tilde{\xi} = \tilde{v} = v = c_K u - s_K \xi$$

and then $\tilde{\xi} = \xi$, which is a contradiction. □

Remark 3.8. By Proposition 3.6 and Proposition 3.7, if $K \in \mathbb{P}(E)$ is AMP and polyhedral, then K is neither acute nor obtuse. The only possibility left is to be right-angled, i.e., $\theta_{\max}(K) = \pi/2$.

3.2. Diametric completeness in the angular sense

We now enter into the final and most important part of our work. Note that inclusion (18) could be strict, even if the point $u \in \Sigma_K$ under consideration is such that $M_K(u)$ is nonempty. To see this, it suffices to consider the case of a nonsmooth point $u \in \Sigma_K$ such that $M_K(u)$ is a singleton.

Example 3.9. In the low dimensional space $\mathbb{R}^3$, let K be the polyhedral proper cone generated by the unit vectors

$$g_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix},\ g_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix},\ g_3 = \frac{1}{\sqrt{1+\varepsilon^2}}\begin{bmatrix}\varepsilon\\0\\1\end{bmatrix},\ g_4 = \frac{1}{\sqrt{1+\varepsilon^2}}\begin{bmatrix}-\varepsilon\\0\\1\end{bmatrix},$$

where ε is a small positive parameter. The unique antipodal pair of K is $\{g_1, g_2\}$. In particular, $M_K(g_1) = \{g_2\}$ is a singleton. However, $G_K(g_1)$ is not a singleton because g_1 is a nonsmooth boundary point of K.

Examples as above can be constructed also in a non-polyhedral setting. Our endeavor in the remaining part of this work is to settle the next question.

Question 3.10. Identify the proper cones K in E for which the identity

$$M_K(u) = c_K u - s_K G_K(u) \tag{19}$$

holds true for all $u \in \Sigma_K$.

The identity (19) expresses the multivalued map M_K in terms of the Gauss map G_K, which is usually easier to evaluate. The coefficients c_K and s_K are linked by the trigonometric identity $c_K^2 + s_K^2 = 1$. Determining c_K is a matter of solving the maximization problem (4). Formula (19) can be rewritten as

$$G_K(u) = (1/s_K)(c_K u - M_K(u)), \tag{20}$$

expressing in this way G_K in terms of M_K. Note that (19) and (20) are equalities between compact subsets of E. For solving Question 3.10 we need to open a parenthesis and say some words on diametric completeness.

Let diam(Ω) denote the diameter of a nonempty bounded set Ω in E. One says that Ω is *diametrically complete* (DC) if

$$\operatorname{diam}(\Omega \cup \{x\}) > \operatorname{diam}(\Omega) \quad \text{for all} \quad x \in E \setminus \Omega\,. \tag{21}$$

DC sets have been widely discussed in the literature since the beginning of the twentieth century. Condition (21) says that Ω is not contained strictly in any set of the same diameter. DC sets are closed and convex. Furthermore, unless being a singleton, a DC set has interior points. Hence, there is no loss of generality in assuming from the very beginning that Ω is a convex body, i.e., a compact convex set with nonempty interior. A celebrated result of convex geometry asserts that a convex body Ω in E is DC if and only if Ω is of constant width, see the book of Eggleston [10] or the celebrated paper of Chakerian and Groemer [4]. Consider now the case of a proper cone. The next definition proposes an angular version of inequality (21).

Definition 3.11. $K \in \mathbb{P}(E)$ is *diametrically complete in the angular sense* (DCA) if

$$f_K(x) > \theta_{\max}(K) \quad \text{for all } x \in S_E \backslash K. \tag{22}$$

The above definition is essentially due to Lassak [18]. This author identifies a proper cone in E with a "convex" closed set on the sphere S_E. Convexity in Lassak's work corresponds to spherical convexity in a geodesic sense. We prefer to avoid the spherical convexity terminology and continue using the standard conic approach. Our concept of being DCA puts the emphasis on angles rather than on distances. Revolution cones are clearly DCA. As shown in the forthcoming Corollary 3.16, any self-dual cone is DCA. So, examples of DCA cones include the nonnegative orthant of $\mathbb{R}^n$, the n-dimensional Lorentz cone, the cone of m-by-n nonnegative matrices, and the Löwner cone of positive semidefinite symmetric matrices of order n. Besides self-dual cones and revolution cones, there are other DCA cones with a more involved geometric structure. Proposition 3.12 provides a useful theoretical characterization of DCA cones.

Proposition 3.12. *Let $K \in \mathbb{P}(E)$. Then K is DCA if and only if*

$$K = \{x \in E : \varphi_K(x) \le 0\}\,, \tag{23}$$

where $\varphi_K : E \to \mathbb{R}$ is the continuous positively homogeneous function given by

$$\varphi_K(x) := c_K \|x\| - \min_{v \in K \cap S_E} \langle x, v\rangle\,.$$

Proof. That φ_K is positively homogeneous and continuous is clear. Beware that φ_K may not be convex, because we have no information on the sign of c_K. Let Q_K denote the sublevel set on the right-hand side of (23). For all $x \in E$, one has

$$\varphi_K(x) \le 0 \;\Leftrightarrow\; \langle x, v\rangle \ge c_K \,\|x\| \quad \text{for all } v \in K \cap S_E\,. \tag{24}$$

This shows that the inclusion $K \subseteq Q_K$ is true for any proper cone K. Hence, (23) is equivalent to $Q_K \subseteq K$. But

$$\begin{aligned} Q_K \subseteq K \quad &\Leftrightarrow \quad \{x \in S_E : \varphi_K(x) \le 0\} \subseteq K \cap S_E \\ &\Leftrightarrow \quad S_E \backslash K \subseteq \{x \in S_E : \varphi_K(x) > 0\} \\ &\Leftrightarrow \quad S_E \backslash K \subseteq \{x \in S_E : \mathfrak{f}_K(x) > \theta_{\max}(K)\} \end{aligned}$$

and the last inclusion expresses the fact that K is DCA. □

The equivalence (24) can be rephrased as an equality between sets, namely,

$$Q_K = \bigcap_{v \in K \cap S_E} \{x \in E : \langle x, v \rangle \geq c_K \|x\|\} . \tag{25}$$

The above formula applies to any proper cone K, not necessarily DCA. Note that if $0 < c_K < 1$, then each set forming the intersection in (25) is a revolution cone. Hence, any acute DCA cone is an intersection of revolution cones of equal half-aperture angle. This observation is reminiscent of a classical result of the theory of convex bodies according to which a DC convex body can be expressed as intersection of balls of equal radius, see Eggleston [9] for a more precise statement of such a result. Incidentally, we own to a referee the observation that a set which can be expressed as intersection of closed balls of equal radius R is sometimes called an R-strongly convex set.

The next proposition concerns the directional differentiability of φ_K. A function $g : E \to \mathbb{R}$ is called *tangentially convex* at a point $x \in E$ if, for all direction $d \in E$, the directional derivative

$$g'(x; d) := \lim_{t \to 0^+} \frac{g(x + td) - g(x)}{t}$$

exists, is finite, and is convex as a function of d. Tangential convexity is a property called quasi-differentiability in the book of Pshenichny [20]. Hörmander's theorem [14, Theorem 2.2.8] on positively homogeneous convex functions implies that $g'(x; \cdot)$ is the support function of a unique nonempty compact convex set in E, namely,

$$\partial g(x) := \{z \in E : \langle z, d \rangle \leq g'(x; d) \ \text{ for all } d \in E\}. \tag{26}$$

Such a set is called the *tangential subdifferential* of g at x. This name is used for instance in [1, Definition 2], but other names have also been considered in the literature.

Proposition 3.13. *Let $K \in \mathbb{P}(E)$. Then:*

(a) *φ_K is expressible as a difference of two finite-valued support functions. In particular, φ_K is directionally differentiable and Lipschitzian. More precisely,*

$$|\varphi_K(x) - \varphi_K(y)| \leq (1 + |c_K|)\|x - y\| \quad \textit{for all } x, y \in E.$$

(b) *φ_K is tangentially convex at each $u \in S_E$ and*

$$\varphi_K'(u; d) = \max_{v \in A_K(u)} \langle c_K u - v, d \rangle \quad \textit{for all } d \in E.$$

Furthermore, the tangential subdifferential of φ_K at u is given by

$$\partial \varphi_K(u) = \mathrm{co}[c_K u - A_K(u)]. \tag{27}$$

Proof. Part (a) is obvious. For proving (b) we observe that

$$x \in E \mapsto \varphi_K(x) = \max_{v \in K \cap S_E} \overbrace{c_K \|x\| - \langle x, v \rangle}^{F_v(x)}$$

is the upper envelope of a family $\{F_v : v \in K \cap S_E\}$ of functions that are continuously differentiable on $E \backslash \{0\}$.

Pick $d \in E$ and $u \in S_E$. By applying Danskin's directional differentiability theorem, cf. [6], we get

$$\varphi_K'(u,d) = \max_{v \in R(u)} \langle \nabla F_v(u), d \rangle \;=\; \max_{v \in R(u)} \langle c_K u - v, d \rangle \,,$$

where

$$\begin{aligned} R(u) &:= \{v \in K \cap S_E : F_v(u) = \varphi_K(u)\} \;=\; \{v \in K \cap S_E : c_K - \langle u, v\rangle = \varphi_K(u)\} \\ &= \{v \in K \cap S_E : \arccos\langle u, v\rangle = f_K(u)\} \;=\; A_K(u)\,. \end{aligned}$$

For obtaining the subdifferential of φ_K at u, we take of course the convex hull of the nonempty compact set $c_K u - A_K(u)$. □

Before continuing the discussion on DCA cones, we record the following easy corollary.

Corollary 3.14. *Let $K \in \mathbb{P}(E)$ and $u \in S_E$. Then*

$$A_K(u) \text{ is a singleton } \Leftrightarrow \varphi_K \text{ is differentiable at } u.$$

Under the above equivalent condition, one has $A_K(u) = c_K u - \nabla\varphi_K(u)$.

We underline the fact that Proposition 3.13 and Corollary 3.14 apply to any proper cone, not necessarily DCA. The next proposition compares DCA cones and AMP cones.

Proposition 3.15. *Let $K \in \mathbb{P}(E)$.*

(a) *If K is DCA, then K is AMP.*

(b) *If K is AMP and nonobtuse, then K is DCA.*

Proof. (a) Let K be DCA. Using reductio ad absurdum, let $u \in \Sigma_K$ be such that $M_K(u)$ is empty. Hence, $f_K(u) < \theta_{\max}(K)$. The function $f_K : S_E \to \mathbb{R}$ being continuous, there exists a point $x \in S_E$ near u such that $\mathfrak{f}_K(x) \leq \theta_{\max}(K)$. Furthermore, given that u belongs to the boundary of K, we can take x in the exterior of K. The existence of such a point x contradicts (22).

(b) Let K be AMP and nonobtuse. That K is nonobtuse ensures the convexity of φ_K and, a posteriori, that Q_K is a closed convex cone. Pick any $\tilde{x} \in \operatorname{int}(K)$. Since such a point satisfies the Slater condition $\varphi_K(\tilde{x}) < 0$, we can write the formulas

$$\begin{aligned} \operatorname{int}(Q_K) &= \{x \in E : \varphi_K(x) < 0\} \\ \operatorname{bd}(Q_K) &= \{x \in E : \varphi_K(x) = 0\}. \end{aligned}$$

Since K is AMP, we have $f_K(x) = \theta_{\max}(K)$ for all $x \in \Sigma_K$, cf. Proposition 3.5. By passing to cosines, we get $\varphi_K(x) = 0$ for all $x \in \Sigma_K$. Hence, $\operatorname{bd}(K) \subseteq \operatorname{bd}(Q_K)$. Summarizing, K and Q_K are solid closed convex cones such that

$$K \subseteq Q_K \;,\; \operatorname{bd}(K) \cap \operatorname{int}(Q_K) = \emptyset. \tag{28}$$

We claim that $K = Q_K$. Suppose that there exists a point $w \in Q_K$ which is not in K. Hence, the line segment $[\tilde{x}, w]$ intersects $\operatorname{bd}(K)$ at some y. Since we have $\tilde{x} \in \operatorname{int}(K) \subseteq \operatorname{int}(Q)$ and $w \in Q_K$, we get $y \in \operatorname{int}(Q_K)$, contradicting the second condition in (28). In conclusion, the equality $K = Q_K$ is true and, therefore, K is DCA by Proposition 3.12. □

The next corollary shows that diametric completeness in the angular sense can be viewed as a relaxation of self-duality.

Corollary 3.16. *Let $K \in \mathbb{P}(E)$. Then K is self-dual if and only if K is DCA and right-angled.*

Proof. If K is DCA, then we can write

$$K = \bigcap_{v \in K \cap S_E} \{x \in E : \langle x, v\rangle \geq c_K \|x\|\}. \tag{29}$$

If, in addition, K is right-angled, then $c_K = 0$ and (29) becomes

$$K = \bigcap_{v \in K \cap S_E} \{x \in E : \langle x, v\rangle \geq 0\},$$

which is equivalent to saying that $K = K^*$. Conversely, suppose that K is self-dual. In such a case, K is right-angled and, on the other hand, Proposition 2.5 yields

$$f_K(u) = \arccos[-\Delta_{K^*}(u)] = \arccos[-\Delta_K(u)] = \arccos 0$$

for all $u \in \Sigma_K$. This constancy result and Proposition 3.5 show that K is AMP. A posteriori, K is DCA by Proposition 3.15 (b). □

Next, we state three lemmas that will be needed for handling Question 3.10. The first lemma is a well known rule for computing a normal cone to a sublevel set of a convex function. Such a rule pertains to the domain of subdifferential calculus and it is recorded for instance in Cabot and Thibault [3, p. 6591].

Lemma 3.17. *Let $g : E \to \mathbb{R}$ be a convex function such that $g(\tilde{x}) < 0$ for some $\tilde{x} \in E$. Then the normal cone map of $C := \{x \in E : g(x) \leq 0\}$ admits the representation*

$$N_C(x) = \{ty : y \in \partial g(x)\,, t \geq 0\} \tag{30}$$

for all $x \in \mathrm{bd}(C)$, where ∂ is the subdifferential operator in the sense of convex analysis.

The notation $\partial g(x)$ used in (30) is consistent with (26) because the tangential subdifferential of a convex function coincides with the subdifferential in the sense of convex analysis. The second lemma is part of the folklore on convex cones. Its proof is easy and, therefore, omitted.

Lemma 3.18. *Let Ω be a nonempty compact subset of E such that $0 \notin \mathrm{co}(\Omega)$. Then*

$$\mathrm{cone}(\Omega) := \{tw : t \geq 0, w \in \mathrm{co}(\Omega)\}$$

is a pointed closed convex cone and

$$\mathrm{gen}[\mathrm{cone}(\Omega)] \subseteq \left\{\|w\|^{-1} w : w \in \Omega\right\}.$$

The third lemma provides a formula for N_K when K is a nonobtuse DCA cone.

Lemma 3.19. *Let $K \in \mathbb{P}(E)$ be DCA and nonobtuse. Then, for all $u \in \Sigma_K$, one has*

$$N_K(u) = \mathrm{cone}\,[c_K u - M_K(u)] \tag{31}$$

$$\mathrm{gen}[N_K(u)] \subseteq (1/s_K)(c_K u - M_K(u)). \tag{32}$$

Proof. Let $u \in \Sigma_K$. Being DCA, the cone K is expressible as in (23). Since K is nonobtuse, the coefficient c_K is nonnegative and, therefore, the function φ_K is convex. If $\tilde{x} \in \mathrm{int}(K)$, then $\varphi_K(\tilde{x}) < 0$ and we are in the context of Lemma 3.17 with $g = \varphi_K$ and $C = K$. Hence,

$$N_K(u) = \{ty : y \in \partial\varphi_K(u)\,, t \geq 0\}. \tag{33}$$

We know from Proposition 3.13 (b) that $\partial\varphi_K(u)$ admits the representation (27). But K is AMP by Proposition 3.15. Hence, $A_K(u) = M_K(u)$ by Proposition 3.5. In conclusion,

$$\partial\varphi_K(u) \;=\; \mathrm{co}(\Omega) \quad \text{ with } \; \Omega := c_K u - M_K(u). \tag{34}$$

The combination of (33) and (34) yields (31). Since $\varphi_K(\tilde{x}) < \varphi_K(u) = 0$, the point u is not a minimizer of φ_K and, therefore, $0 \notin \partial\varphi_K(u)$. The inclusion (32) is then obtained by applying Lemma 3.18. □

The proof of formula (31) relies not just on the sublevel set representation (23) of K, but also on the convexity of φ_K. This fact explains why we are asking K to be not just DCA, but also nonobtuse. Perhaps it is possible to derive a formula similar to (31) in a wider context, but this restrictive formulation suffices to our needs. We are now ready to address Question 3.10.

Theorem 3.20. *Let $K \in \mathbb{P}(E)$. That K is AMP is necessary and sufficient for having*

$$M_K(u) = c_K u - s_K G_K(u) \quad \textit{for all } \; u \in \Sigma_K. \tag{35}$$

Proof. *Necessity.* Let I_E be the identity map on E. The normal cone map N_K is nonempty-valued on the boundary of K. In particular, G_K is nonempty-valued on Σ_K. Hence, $M_K = c_K I_E - s_K G_K$ is nonempty-valued on Σ_K. In other words, K is AMP.

Sufficiency. Let K be AMP and $u \in \Sigma_K$. If K is obtuse, then K is smooth by Proposition 3.7 and (35) follows from the second part of Proposition 3.2. So, in what follows we assume that K is nonobtuse. In such a case, K is DCA and formulas (31) and (32) hold true. Let

$$D_K(u) := (1/s_K)[c_K u - M_K(u)]\,.$$

Since $D_K(u) \subseteq G_K(u)$ by Proposition 3.2, we just need to prove the reverse inclusion. Let $\xi \in N_K(u)$ with $\|\xi\| = 1$. We must check that

$$v := c_K u - s_K \xi \in M_K(u). \tag{36}$$

Since $\langle \xi, u\rangle = 0$, it is clear that $\|v\| = 1$ and $\langle v, u\rangle = c_K$. Hence, (36) amounts to saying that $v \in K$. But

$$\begin{aligned} v \in K \;\;\Leftrightarrow\;\; \varphi_K(v) \leq 0 \;\;&\Leftrightarrow\;\; \max_{z \in K\cap S_E} \{c_K - \langle c_K u - s_K\xi, z\rangle\} \;\leq 0 \\ &\Leftrightarrow\;\; c_K(1 - \langle u, z\rangle) + s_K\langle \xi, z\rangle \;\leq 0 \quad \text{for all } z \in K \cap S_E. \end{aligned}$$

So, we must prove that

$$\langle \xi, z\rangle \;\leq \gamma(z) \quad \text{for all } z \in K \cap S_E\,, \tag{37}$$

where

$$\gamma(z) := \overbrace{(c_k/s_K)}^{nonnegative}\; \overbrace{(\langle u, z\rangle - 1)}^{nonpositive} \leq 0\,.$$

For proving (37), we start by recalling that $N_K(u)$ is a pointed closed convex cone. Hence, it is possible to write

$$\xi = \Big\| \sum_{i=1}^m t_i \xi_i \Big\|^{-1} \sum_{i=1}^m t_i \xi_i$$

with $m \geq 1$, $t_1, \ldots, t_m > 0$, and $\xi_1, \ldots, \xi_m \in \mathrm{gen}[N_K(u)]$. We now define the coefficients $\beta_i := (t_1 + \ldots + t_m)^{-1} t_i$ for all $i \in \mathbb{N}_m$; then

$$\xi = \Big\| \sum_{i=1}^m \beta_i \xi_i \Big\|^{-1} \sum_{i=1}^m \beta_i \xi_i$$

with coefficients $\beta_1, \ldots, \beta_m > 0$ adding up to 1. By using (32) we get

$$v_i := c_K u - s_K \xi_i \in M_K(u)$$

for all $i \in \mathbb{N}_m$. Now, pick any $z \in K \cap S_E$. For all $i \in \mathbb{N}_m$, we have $v_i \in K \cap S_E$ and $\langle v_i, u \rangle = c_K$. Hence, $\langle \xi_i, z \rangle \leq \gamma(z)$. Multiplying on both sides by β_i and adding up, we get

$$\Big\langle \sum_{i=1}^m \beta_i \xi_i, z \Big\rangle \leq \gamma(z) \leq \gamma(z) \Big\| \sum_{i=1}^m \beta_i \xi_i \Big\|,$$

where the second inequality is because $\gamma(z) \leq 0$ and $\| \sum_{i=1}^m \beta_i \xi_i \| \leq 1$. This yields (37) and completes the proof of the theorem. $\square$

An important achievement of this work has been to bring into light a hidden connection between the multivalued maps G_K and M_K. Two auxiliary ideas emerged in the process of establishing the fundamental identity $M_K = c_K I_E - s_K G_K$, which is the main object of Theorem 3.20. The most important one is the introduction of the Antipodal Mate Property. Proper cones that satisfy this property (called AMP cones) have a number of striking geometric features. Proposition 3.5 mentions two alternative characterizations of AMP cones and Theorem 3.20 provides yet another one. This is however the tip of an iceberg. In a forthcoming paper of ours we shall present several additional characterizations of AMP cones, each one having its own interest. Such a material is highly technical and deserves a separate presentation. The second auxiliary idea consists in extending the concept of diametrically complete convex body to a conic setting. Proper cones that are diametrically complete in the angular sense (called DCA cones) also play an essential role in the overall exposition. We studied a number of properties of DCA cones that were needed for establishing Theorem 3.20. For reasons of space, we are leaving aside other results on DCA cones that we intend to present elsewhere.

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