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BV Solutions of a State-Dependent Prox-Regular Sweeping Process with Nonconvex Perturbations

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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In a separable Hilbert space, we consider a sweeping process with perturbation. The values of the moving set are prox-regular sets dependent on time and state. The variation of the moving set is controlled in the state in a Lipschitz way and in time by a positive Radon measure. The perturbation is the sum of two multivalued mappings with different semicontinuity properties in state variable. The first perturbation with closed, possibly, nonconvex values is lower semicontinuous. The second perturbation with closed convex values has weakly sequentially closed graph. We prove the existence of a right continuous solution of bounded variation. Various a priori estimates for the solution are given. The proof is based on a new method that does not use any version of the catching-up algorithm. We use classical approaches based on a priori estimates, the compactness of sets in the space of right continuous regular functions of bounded variation, and a fixed point theorem for multivalued mappings. Our theorem is new and it comprises known results for this class of solutions of sweeping processes.

Keywords: BV-solution, prox-regular sets, bounded variation, Radon measures.

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1. Introduction

Let H be a separable Hilbert space with the norm $\|\cdot\|$, the inner product $\langle\cdot,\cdot\rangle$, the null element Θ and the unit closed ball $\overline{B}$ centered at Θ.

Let us introduce the following notation: $T = [0,a], a > 0$ is an interval of the real half-line $R^+ = [0,+\infty)$ with the Lebesgue measure λ and with σ-algebra Σ of Lebesgue measurable sets from T, ω-H is the space H endowed with the weak topology, $C : T \times H \rightrightarrows H$ is a multivalued mapping with r-prox regular values [25] and the range $\mathcal{R}(C)$

$$\mathcal{R}(C) = \{\cup C(t,x);\ t \in T,\ x \in H\}. \tag{1.1}$$

Let $\overline{\mathcal{R}(C)}$ be the closure of the range, $\mathcal{B}(\overline{\mathcal{R}(C)})$ be the σ-algebra of Borel sets from $\overline{\mathcal{R}(C)}$ and $\Sigma \otimes \mathcal{B}(\overline{\mathcal{R}(C)})$ be the σ-algebra on $T \times \overline{\mathcal{R}(C)}$ generated by sets $\Lambda \times A, \Lambda \in \Sigma, A \in \mathcal{B}(\overline{\mathcal{R}(C)}$

Let $d(x,A)$ be the distance from a point x to a set $A \subset H$, and

$$\|A\| = \sup\{\|x\|;\ x \in A\}.$$

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If $A \subset H$ is a closed convex set, then there exists a unique element $A^0 \in A$ of minimal norm, i.e.

$$\|A^0\| = d(\Theta, A).$$

In what follows, we always assume that the following Hypothesis $H(C)$ is valid:

(1) $C : T \times H \rightrightarrows H$ is a multivalued mapping with r-prox regular values [25];

(2) there exists a constant $L \in \,]0,1[$ and a positive Radon measure μ on T such that

$$\sup_{s\in]0,a]} \mu(\{s\}) < (1-L)\cdot\frac{r}{2}, \tag{1.2}$$

$$|d(z, C(t,x)) - d(z, C(s,y))| \le \mu(]s,t]) + L\|x-y\| \text{ a.e.}, \tag{1.3}$$

where $s \le t, \;\; s,t \in T, \;\; x,y \in H$. Consider the measurable sweeping process

$$\begin{aligned} -dx \in \mathcal{N}_{C(t,x(t))}(x(t)) + U(t,x(t)) + V(t,x(t)), \\ x(0) = x_0 \in C(0,x_0), \end{aligned} \tag{1.4}$$

where $\mathcal{N}_{C(t,y)}(z)$ is the proximal normal cone (see [1]) of the set $C(t,y)$ at a point $z \in C(t,y)$, $U, V : T \times \overline{\mathcal{R}(C)} \rightrightarrows H$ are multivalued mappings.

Definition 1.1. By a *solution* of the inclusion (1.4) we mean a pair $(x(f)(\cdot), f(\cdot))$ such that:

(1) $x(f)(\cdot)$ is a right continuous function of bounded variation from T to H,

$$\begin{aligned} x(f)(0) = x_0, \;\; x(f)(t) \in C(t, x(f)(t)), \;\; t \in T, \\ f(\cdot) = f_U(\cdot) + f_V(\cdot), \;\; f_U(\cdot), f_V(\cdot) \in L^1(T,H); \end{aligned} \tag{1.5}$$

(2) there exists a positive Radon measure ν on T absolutely continuously equivalent to the measure $\mu+\lambda$ such that the differential measure (Stieltjes measure) $dx(f)$ generated by the function $x(f)(\cdot)$ is absolutely continuous with respect to the measure ν and the density $\frac{dx(f)}{d\nu}(\cdot)$ of the measure $dx(f)$ with respect to the measure ν and the function $f_V(\cdot), f_U(\cdot)$ satisfy the inclusion

$$\frac{dx}{d\nu}(f)(\cdot) \in L^1_\nu(T,H),$$

$$-\frac{dx(f)}{d\nu}(t) - f(t)\frac{d\lambda}{d\nu}(t) \in \mathcal{N}_{C(t,x(f)(t))}(x(f)(t)) \;\; \nu \text{ a.e.}, \tag{1.6}$$

$$f_U(t) \in U(t, x(f)(t)) \text{ a.e.}, \tag{1.7}$$

$$f_V(t) \in V(t, x(f)(t)) \text{ a.e.} \tag{1.8}$$

Two positive Radon measures are absolutely continuously equivalent, if each of them is absolutely continuous with respect to the other. In the definition of a solution of the measurable sweeping process we follow [12]. Note that the definition of a solution does not depend of the measure ν in the sense that a function is a solution of inclusion (1.4), if and only if, the inclusion (1.6) takes place for any Radon measure ν which is absolutely continuously equivalent to the measure $\mu + \lambda$.

We make the following assumptions.

Hypothesis $H(U)$**:**

(1) *the multivalued mapping* $U : T\times\overline{\mathcal{R}(C)} \rightrightarrows H$ *with closed, not necessarily, convex values is weakly* $\Sigma \otimes \mathcal{B}(\overline{\mathcal{R}(C)})$ *measurable;*

(2) *the mapping* $x \mapsto U(t,x)$ *is lower semicontinuous for almost every* $t \in T$*;*

(3) *there exists* $l_U(\cdot) \in L^1(T,R^+)$*,* $l_U(t) > 0, t \in T$ *such that*

$$d(\Theta, U(t,x)) < l_U(t)(1+\|x\|), \quad t \in T. \tag{1.9}$$

Hypothesis $H(V)$**:**

(1) *the values of multivalued mapping* $V : T \times \overline{\mathcal{R}(C)} \rightrightarrows H$ *are closed convex sets and the mapping* $x \mapsto V(t,x)$ *has weakly sequentially closed graph;*

(2) *the mapping* $t \mapsto V^0(t,x)$ *is measurable;*

(3) *for some* $l_V(\cdot) \in L^1(T,R^+)$ *we have*

$$d(\Theta, V(t,x)) \le l_V(t)(1+\|x\|) \ \ a.e. \tag{1.10}$$

Consider the differential equation

$$\dot\rho(t) = \frac{2}{1-L} l(t)(1+\rho(t)), \tag{1.11}$$

$$\rho(0) = \|x_0\| + \frac{1}{1-L}\mu(]0,a]), \tag{1.12}$$

where

$$l(t) = l_U(t) + l_V(t), \tag{1.13}$$

that, as it is known, has the unique solution

$$\rho(t) = \rho(0)\exp\gamma(t) + \frac{2}{1-L}\int_0^t l(s)(\exp(\gamma(t)-\gamma(s))ds, t \in T,$$

where

$$\gamma(t) = \frac{2}{1-L}\int_0^t l(s)ds.$$

Hypothesis $H(C_S)$: For any $t\in T$ and any bounded set $\mathcal{D}\subset H$, $\|\mathcal{D}\| \le \rho(t)$, the set

$$C(t,\mathcal{D}) \cap \mathcal{D} \ \text{ is relatively compact.} \tag{1.14}$$

Main result

Theorem 1.2. *Let the Hypotheses H(C), H(C_S), H(U), H(V) hold. Then, the sweeping process* (1.4) *has a solution* $(x(f)(\cdot), f(\cdot))$ *satisfying the inequalities*

$$\|x(f)(t)\| \le \rho(t), \ t \in T, \tag{1.15}$$

$$\|\frac{dx}{d\nu}(t)\| \le \frac{1}{1-L}\frac{d\mu}{d\nu}(t) + \dot\rho(t)\frac{d\lambda}{d\nu}(t) \ \ \nu \ \ a.e., \tag{1.16}$$

$$\|x(f)(t) - x(f)(t-0)\| \le \frac{1}{1-L}\mu(\{t\}), \ \ t \in]0,a], \tag{1.17}$$

$$\|f_U(t)\| \le l_U(t)(1+\rho(t)) \ \ a.e., \tag{1.18}$$

$$\|f_V(t)\| \le l_V(t)(1+\rho(t)) \ \ a.e. \tag{1.19}$$

Corollary 1.3. *Assume that the Hypotheses $H(U)$, $H(V)$ hold for a multivalued mapping $C : T \times H \rightrightarrows H$ with r-prox regular values, that Hypothesis $H(C_S)$ is valid, and*

$$|d(z, C(t,x)) - d(z, C(s,y))| \leq |\beta(t) - \beta(s)| + L\|x - y\|, \tag{1.20}$$

$s \leq t,\ x, y, z \in H,\ L \in]0,1]$, *where $\beta : T \to R$ is an absolutely continuous function. Then, there exists a pair $(x(f)(\cdot),\ f(\cdot))$ such that $x(f) : T \to H$, $x(f)(0) = x_0$ is an absolutely continuous function,*

$$x(f)(t) \in C(t, x(f)(t)),\ t \in T,\ f(\cdot) \in L^1(T,H),\ f(\cdot) = f_U(\cdot) + f_V(\cdot),$$

the inclusions (1.7), (1.8) *hold,*

$$-\frac{dx(f)(t)}{dt} \in \mathcal{N}_{C(t,x(f)(t))}(x(f)(t)) + f(t) \ \ a.e. \tag{1.21}$$

and the inequalities (1.15), (1.18), (1.19) *are valid. In addition,*

$$\|\frac{dx(f)(t)}{dt}\| \leq \frac{1}{1-L}\ |\dot{\beta}(t)| + \dot{\rho}(t) \ \ a.e. \tag{1.22}$$

In Corollary 1.3, $\rho(t)$ is the solution of equation (1.11) with $\rho(0) = \|x_0\| + \|\dot{\beta}\|_{L^1}$, where $\|\dot{\beta}\|_{L^1}$ is the norm of $\dot{\beta}(\cdot)$ in the space $L^1(T,R)$.

The existence of absolutely continuous solutions of sweeping processes with state-dependent r-prox regular sets were considered in the works [2], [7], [24]. The results of these works follow from Corollary 1.3, because in these works the values of perturbations are convex sets, and they have properties which are particular cases of our properties.

Theorem 1.2 implies Corollary 5.4 [23] in which one assumes that the multivalued perturbation $G : T \times H \rightrightarrows H$ is an analogue of our perturbation $V : T \times \overline{\mathcal{R}(C)}$. This mapping enjoys the same Hypotheses $H(V)$, (2), (3), and instead of Hypothesis $H(V)$ (1) it is supposed that the mapping $x \mapsto \mathcal{G}(t,x)$ is scalarly upper semicontinuous.

Note that from the scalar upper semicontinuity of the mapping $x \mapsto \mathcal{G}(t,x)$ follows the weak sequential closedness of its graph. Hence, the mapping $V : T \times \overline{\mathcal{R}(C)} \rightrightarrows H$ is more general than the mapping $\mathcal{G} : T \times H \rightrightarrows H$ in [23].

The main difference between the present work and those cited above is that we consider a new approach for studying state-dependent prox-regular sweeping process. In all the cited works, various versions of the catching-up algorithm originated from [22] are used. Our approach is based on the use of a priori estimates and the Ky Fan theorem [13] on fixed points of multivalued mappings.

At the end of introduction, we mention the works [3], [15], [18] in which approaches other than those mentioned above were used to study state-dependent prox-regular sweeping processes. In [3, 15] theorems of the existence of absolutely continuous solutions of state-dependent prox-regular sweeping processes were proved in a finite-dimensional space. The perturbations in these works possess the so-called mixed semicontinuity property and their values can be nonconvex sets. The approach is based on an important result of the coincidence between the solution sets of constrained and unconstrained first order differential inclusions and Kakutani fixed point theorem for multimappings.

The existence of absolutely continuous and right-continuous solutions of bounded variation was proved in [18] for the sweeping process without perturbation in a Hilbert space. In this work, the family $\{C(t,x);\ (t,x) \in T \times H\}$ is equi-uniformly subsmooth and the Moreau-Yosida regularization technique is used in the proof.

2. Main notation and definitions

Let X be a Banach space with the norm $\|\cdot\|$, the null element Θ and the unit open B and closed $\overline{B}$ balls centered at Θ. By $d(y,A)$ we denote the distance from a point $y \in X$ to a set $A \subset X$ and

$$\|A\| = \sup\{\|y\|;\ y \in A\}.$$

For a set $A \subset X$ the symbols $\operatorname{co} A$ and $\overline{\operatorname{co}}\, A$ denote the convex and closed convex hulls of the set A, respectively. By ω-X we denote the space X endowed with the weak topology and ω-A denotes the set $A \subset X$ endowed with the topology induced by that of the space ω-X. The Hausdorff distance between closed sets $A_i \subset X$, $i = 1,2$ is denoted by $\operatorname{haus}(A_1, A_2)$,

$$\operatorname{haus}(A_1,A_2) = \max\{\sup_{x \in A_2} d(x,A_1), \sup_{y \in A_1} d(y,A_2)\}.$$

It is known that

$$\operatorname{haus}(A_1,A_2) = \sup_{z \in X} |d(z,A_1) - d(z,A_2)|. \tag{2.1}$$

Let Y be a metric space. A multivalued mapping $F : Y \rightrightarrows X$ is a mapping whose values are nonempty sets from X. Denote by

$$F^{-1}(U) = \{y \in Y; F(y) \cap U \neq \phi\},\ \ U \subset X.$$

A multivalued mapping $F : T \rightrightarrows X$ is called *measurable (weakly measurable)*, if $F^{-1}(U) \in \Sigma$ for any closed (open) set $U \subset X$ [16].

If $F : Y \rightrightarrows X$, then in the definition of measurability (weak measurability) we take the σ-algebra $\mathcal{B}(Y)$ of Borel sets from Y. Similarly, if $F : T \times Y \rightrightarrows X$, then the definitions of measurability are understood with respect to the σ-algebra $\Sigma \otimes \mathcal{B}(Y)$ on $T \times Y$ generated by sets of the form $\Lambda \times A$, $\Lambda \in \Sigma$, $A \in \mathcal{B}(Y)$.

The following properties can be found in [16]. If X is separable, then the $\Sigma \otimes \mathcal{B}(Y)$ measurability of a multivalued mapping $F : T \times Y \rightrightarrows X$ with closed values implies its weak $\Sigma \otimes \mathcal{B}(Y)$ measurability.

If $F : T \rightrightarrows X$ is a mapping with closed values, then the measurability and weak measurability are equivalent.

A multivalued mapping $F : Y \rightrightarrows X$ is *lower semicontinuous* at a point $y_0 \in Y$, if for any open set $U \subset X$, $F(y_0) \cap U \neq \oslash$ there exists a neighborhood $V(y_0)$ of the point y_0 such that $F(y) \cap U \neq \oslash$ for any $y \in V(y_0)$.

It follows directly from the definition that a multivalued mapping $F : Y \rightrightarrows X$ is lower semicontinuous at a point $y_0 \in Y$, if and only if, for any $x_0 \in F(y_0)$ and any sequence $y_n \in Y$, $n \geq 1$, convergent to y_0, there exists a sequence $x_n \in F(y_n)$, $n \geq 1$, converging to x_0.

A multivalued mapping $F : Y \rightrightarrows X$ is *weakly upper semicontinuous* at a point $y_0 \in Y$, if for any open set $U \subset \omega\text{-}X,\ F(y_0) \subset U$ there exists a neighborhood $V(y_0)$ of the point y_0 such that $F(y) \subset U$ for any $y \in V(y_0)$.

Denote by $\operatorname{gr} F$ the graph of a multivalued mapping $F : Y \rightrightarrows X$

$$\operatorname{gr} F = \{(y, x) \in Y \times X;\ \ x \in F(y)\}.$$

In the rest of the paper, $F : Y \rightrightarrows \omega\text{-}X$ is a multivalued mapping with convex closed values. It is well known that closed convex sets from X and $\omega\text{-}X$ coincide.

A multivalued mapping $F : Y \rightrightarrows \omega\text{-}X$ has a weakly sequentially closed graph at a point $y_0 \in Y$, if for any sequence $(y_n, x_n) \in \operatorname{gr} F,\ n \geq 1$ converging in the space $Y \times \omega\text{-}X$ to the point (y_0, x_0) the inclusion $x_0 \in F(y_0)$ holds.

In the sequel, for convenience, we speak about the weak sequential closedness of a mapping F at a point y_0 instead of the weak sequential closedness of the graph of the mapping at the point y_0.

A multivalued mapping $F : Y \rightrightarrows \omega\text{-}X$ is called *weakly locally compact* at a point $y_0 \in Y$, if there exists a neighborhood $V(y_0)$ of the point y_0 such that the set $F(V(y_0)) = \{\cup F(y); y \in V(y_0)\}$ is relatively compact in the space $\omega\text{-}X$.

If all the properties of a mapping F mentioned above are valid at every point $y_0 \in Y$, then we speak about the lower semicontinuity, weak upper semicontinuity, weak sequential closedness and weak local compactness of the mapping F on Y.

If $A \subset X$, then by $\omega\text{-}\operatorname{cl} A$ we denote the closure of the set A in the space $\omega\text{-}X$. Similarly, if a sequence $x_n \in X,\ \ n \geq 1$ converges to x in the space $\omega\text{-}X$, then we write $x = \omega\text{-}\lim_{n\to\infty} x_n$.

Take a sequence $x_n \in X$, $n \geq 1$. Considering the elements x_n, $n \geq 1$ as singletons $\{x_n\}$, $n \geq 1$, we can introduce the weak upper limit of the sequence of sets $\{x_n\}$, $n \geq 1$.

Definition 2.1. The *weak upper limit* of the sequence $x_n,\ n \geq 1$ is the set

$$\omega\text{-}\operatorname*{Ls}_{n\to\infty} \{x_n\} = \{x \in X;\ x = \omega\text{-}\lim_{k\to\infty} x_{n_k},\ n_1 < n_2 \ldots < n_k \ldots\}.$$

If a sequence $x_n \in X,\ n \geq 1$ is relatively weakly compact, then from the Eberlein-Šmulian theorem it follows that the set $\omega\text{-}\operatorname*{Ls}_{n\to\infty} \{x_n\}$ is not empty and

$$\omega\text{-}\operatorname*{Ls}_{n\to\infty} \{x_n\} = \bigcap_{n=1}^{\infty} \omega\text{-}\operatorname{cl}\Big(\bigcup_{k=n}^{\infty} x_k\Big) \tag{2.2}$$

A function $x : T \to X$ is called *regular* [10], if at the points $t \in (0, a)$ it has the left $x(t-0)$ and right limits $x(t+0)$, at the point $t = 0$ the right limit $x(0+)$, and at the point $t = a$ the left limit $x(a-0)$.

The space of all regular functions $x : T \to X$ is denoted by $\Pi(T, X)$, and by $\Pi_+(T, H)$ we denote the subset of the space $\Pi(T, H)$ consisting of right continuous functions. The space $\Pi(T, X)$ is endowed with the topology of uniform convergence on T, and $\Pi_+(T, X)$ with the topology induced from $\Pi(T, X)$. A function $x : T \to H$ from the space $\Pi_+(T, X)$ is called *step right continuous*, if there exists a partition $0 = c_0 < c_1 < \ldots < c_i < \ldots < c_n = a$ of the interval T such that the function x is constant on every interval $[c_{i-1}\ c_i[, 1 \leq i \leq n$.

Denote by $BV_+(T,X)$ the space of all right continuous functions $x : T \to H$ of bounded variation, and by $S_+(T,X)$ the space of all right continuous step functions.

Let $U \subset \Pi_+(T,X)$. The set U is called *right equicontinuous* at a point $s \in [0,a[$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x(s) - x(t)\| \le \varepsilon$ for all $x(\cdot) \in U$ and $t \in [s, s+\delta[$.

A set $U \subset \Pi_+(T,H)$ is called *left equicontinuous* at a point $s \in]0,a]$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x(s-0) - x(t)\| \le \varepsilon$ for all $x(\cdot) \in U$ and $t \in]s-\delta, s[$.

Definition 2.2. A set $U \subset \Pi_+(T,X)$ is called *unilaterally equicontinuous* on T, if it is simultaneously right and left equicontinuous at every point $s \in]0,a[$, right equicontinuous at the point 0 and left equicontinuous at the point a.

The *variation* of a function $x(\cdot) \in BV_+(T,H)$ is denoted by $\operatorname{var} x(\cdot)$.

A set $U \subset BV_+(T,H)$ is called *uniformly bounded in variation*, if there exists a constant $M > 0$ such that

$$\operatorname{var} x(\cdot) \le M \quad \forall x(\cdot) \in U.$$

Let H be a Hilbert space and $S \subset H$ be a nonempty subset and

$$\operatorname{Proj}_S(x) = \{y \in S;\ d(x,S) = \|x - y\|\}.$$

If $\operatorname{Proj}_S(x) = \{\overline{y}\}$ for some $\overline{y} \in S$, then in this case $\operatorname{Proj}_S(x) = \overline{y}$.

The *proximal normal cone* to a closed set $S \subset H$ at a point $x \in S$ is defined as follows [26]:

$$N^P(S;x) = \{v \in H; \exists r > 0, x \in \operatorname{Proj}_S(x + rv)\}.$$

It is a convex cone containing the point Θ. It is supposed that $N^P(S;x) = \varnothing$, if $x \in H \backslash S$.

Definition 2.3. [25] A nonempty closed set $S \subset H$ is called *r-prox regular*, if for any $x \in S$ and for all $v \in N^P(S;x) \cap \overline{B}$ and all $t \in]0,r[$ the inclusion $x \in \operatorname{Proj}_S(x+tv)$ holds.

From Theorem 2.2 in [1] it follows that if S is r-prox regular, then $N^P(S;x),\ x \in S$ is a closed convex cone.

In what follows, similarly to [1] the cone $N^P(S;x)$ is denoted by $N_S(x)$.

3. Radon measures

In this section, we introduce some notions from the measure theory.

Let $\mathcal{B}$ be the σ-algebra of all Borel sets from $T = [0,a]$. Scalar and vector measures on $\mathcal{B}$ are called Borel or Stieltjes measures [11]. Since every positive scalar Stieltjes measure is regular [11], it is called a Radon measure [20]. In what follows, a Radon measure is a scalar positive measure defined on the σ-algebra $\mathcal{B}$. For convenience, we say that the measure is defined on T. We denote by λ the Lebesgue measure on T.

For a Radon measure ν on T, we denote by $L^1_\nu(T,H)$ the space of equivalence classes with respect to ν of mappings from T to H which are ν Bochner integrable. Note that a function $x : T \to H$ is an element of the space $L^1_\nu(T,H)$, if it is ν Bochner measurable on T and $\int_T \|f(\tau)\| d\nu(\tau) < \infty$ (see, e.g., [9, Ch.2, Definition 1]).

For more details on the Bochner integral and the space $L^1_\nu(T,H)$ we refer to [11, 9] and references therein. Note in particular that the space $L^1_\nu(T,H)$ with the norm $\|f\|_{L^1} = \int_T \|t(\tau)\| d\nu(\tau)$ is Banach. The space of Bochner integrable functions is denoted by $L^1(T,H)$.

A set $K \subset L^1_\nu(T,H)$ is called *uniformly integrable* [9], if

$$\lim_{\nu(E)\to 0} \int_E \|f(\tau)\| d\nu(\tau) = 0, \quad E \in \mathcal{B}$$

uniformly in $f(\cdot) \in K$.

The *variation* of a measure $m : \mathcal{B} \to H$ on a set $A \in \mathcal{B}$ is denoted by $|m|(A)$ [11]. If $|m|(T) < \infty$, then we say that the measure m has bounded variation. In this case, the variation $|m|$ of measure m is a Radon measure.

A Radon measure ν is *absolutely continuous* with respect to a Radon measure μ, if from the equality $\mu(A) = 0$, $A \in \mathcal{B}$ it follows that $\nu(A) = 0$. If the inverse is also true, then the Radon measures ν and μ are called *absolutely continuously equivalent.*

A Stieltjes measure m is absolutely continuous with respect to a Radon measure μ, if its variation $|m|$ is absolutely continuous with respect to the measure μ. If a Stieltjes measure $m : \mathcal{B} \to H$ of bounded variation is absolutely continuous with respect to a measure μ, then according to the Radon-Nikodym theorem [9] there exists a function $\hat{m}(\cdot) \in L^1_\mu(T,H)$ such that

$$m(A) = \int_A \hat{m}(\tau) d\mu(\tau), \quad A \in \mathcal{B} \tag{3.1}$$

The function $\hat{m}(\cdot)$ is called the *density of measure* m with respect to the Radon measure μ and is denoted by $\frac{dm}{d\mu}(t) = \hat{m}(t)$, $t \in T$.

Let $T(t,\tau) = T \cap [t-\tau, t+\tau]$, $\tau > 0$, $t \in T$.

Lemma 3.1. [20] *For any two Radon measures ν and μ the following statements are true:*

(1) *the limit*
$$\frac{dm}{d\mu}(t) = \lim_{\tau \downarrow 0} \frac{\nu(T(t,\tau))}{\mu(T(t,\tau))} \tag{3.2}$$
(with the convention that $\frac{0}{0} = 0$) exists for every $t \in T$ and is finite for μ almost every $t \in T$. The function $t \mapsto \frac{dm}{d\mu}(t)$ is Borel from T to $\overline{R}^+ = [0, +\infty]$ and it is called the derivative of measure ν with respect to the measure μ;

(2) *the measure ν is absolutely continuous with respect to the measure μ, if and only if, $\frac{d\nu}{d\mu}(\cdot)$ is the density of measure ν with respect to the measure μ, i.e., if and only if, we have*
$$\nu(A) = \int_A \frac{d\nu}{d\mu}(\tau) d\mu(\tau), \quad A \in \mathcal{B}. \tag{3.3}$$

Theorem 3.2. [11] *Let $u : T \to H$ be a right continuous function of bounded variation. Then, there exists a unique Stieltjes measure $m : \mathcal{B} \to H$ of bounded variation such that for any $0 \le c \le d \le a$ it holds*

$$m(]c,d]) = u(d) - u(c), \quad m(]c,d[) = u(d-0) - u(c), \quad m\{0\} = \Theta. \tag{3.4}$$

In an alternative terminology [see, e.g., [12, 21] and others], the measure $m(\cdot)$ is called the differential measure generated by the function $u(\cdot)$ and is denoted by du. From (3.4) it follows that

$$u(t) = u(s) + \int_{]s,t]} du, \tag{3.5}$$

$$u(t-0) = u(s) + \int_{]s,t[} du, \quad s \leq t, \quad s,t \in T. \tag{3.6}$$

If $\hat{u}(\cdot) \in L^1_\nu(T,H)$ and

$$u(t) = u(0) + \int_{]0,t]} \hat{u}(\tau) d\nu(\tau), \quad t \in T, \tag{3.7}$$

then the function $t \to u(t)$ is a right continuous function of bounded variation, the differential measure du generated by $u(\cdot)$ is absolutely continuous with respect to the measure ν and $\hat{u}(\cdot)$ is the density of measure du with respect to the measure ν, i.e.

$$\frac{du}{d\nu}(t) = \hat{u}(t) \ \ \nu \text{ a.e.} \tag{3.8}$$

and for the variation $|du|$ of measure du we have

$$|du|(]s,t]) = \int_{]s,t]} \|\hat{u}(\tau)\| d\nu(\tau), \quad s \leq t, \quad s,t \in T, \tag{3.9}$$

$$|du|(]s,t[) = \int_{]s,t[} \|\hat{u}(\tau)\| d\nu(\tau), \quad s \leq t, \quad s,t \in T. \tag{3.10}$$

Let the Lebesgue measure λ is absolutely continuous with respect to a Radon measure ν and $\frac{d\lambda}{d\nu}(\cdot)$ is the density of measure λ with respect to the measure ν. Then, a function $\hat{u}(\cdot) \in L^1(T,H)$, if and only if, the function $t \mapsto \hat{u}(t)\frac{d\lambda}{d\nu}(t)$ is ν integrable [20]. In this case, we have

$$\int_T \hat{u}(t) d\lambda(t) = \int_T \hat{u}(t) \frac{d\lambda}{d\nu}(t) d\nu(t). \tag{3.11}$$

Denote by $\{t\}$ the singleton with the element t. From (3.2) it follows that

$$\frac{d\lambda}{d\nu}(t) = \frac{\lambda(\{t\})}{\nu(\{t\})} = 0 \ \text{ for all } \ t \in T \ \text{ with } \ \nu(\{t\}) > 0.$$

From this equality and (3.11) it follows that

$$\int_{]0,t]} \hat{u}(\tau) d\lambda(\tau) = \int_{]0,t]} \hat{u}(\tau) \frac{d\lambda}{d\nu}(\tau) d\nu(\tau) \ \text{ for all } \ t \in T, \ \ t > 0. \tag{3.12}$$

Lemma 3.3. *Let* $\hat{u}(\cdot) \in L^1(T, H)$,

$$u(t) = u(0) + \int_{]0,t]} \hat{u}(s) d\lambda(s), \quad t \in T \tag{3.13}$$

and the Lebesgue measure λ *is absolutely continuous with respect to a Radon measure* ν. *Then, the function* $t \mapsto u(t)$ *is a right continuous function of bounded variation, the differential measure* du *generated by* $u(\cdot)$ *is absolutely continuous with respect to the measure* ν *and we have*

$$\frac{du}{d\nu}(t) = \hat{u}(t) \frac{d\lambda}{d\nu}(t) \quad \nu \text{ a.e.} \tag{3.14}$$

The lemma follows from the equalities (3.13), (3.12), (3.7), and (3.8).

4. Preliminaries

In this section, Y is a metric space, X is a Banach space, $F : Y \rightrightarrows X$ is a multivalued mapping with closed convex values.

Proposition 4.1. *If a mapping* $F : Y \rightrightarrows \omega\text{-}X$ *is weakly upper semicontinuous at a point* y_0, *then it is weakly sequentially closed at this point.*

Proposition 4.2. *Let a mapping* $F : Y \rightrightarrows \omega\text{-}X$ *be weakly sequentially closed and locally compact at a point* $y_0 \in Y$. *If* X *is separable or* X *is a reflexive Banach space, then the mapping* F *is weakly upper semicontinuous at the point* y_0.

Propositions 4.1 and 4.2 can be found in [17].

Theorem 4.3. *Assume that the space* $\Pi_+(T, X)$ *is Banach. The norm of an element* $x(\cdot) \in \Pi_+(T, X)$ *is defined by*

$$\|x(\cdot)\|_{\Pi_+} = \sup\{\|x(t)\|;\ t \in T\}.$$

The sets $S_+(T, H)$ *and* $BV_+(T, H)$ *satisfy the inclusions*

$$S_+(T, X) \subset BV_+(T, X) \subset \Pi_+(T, X) \tag{4.1}$$

and are dense in the space $\Pi_+(T, X)$.

The space $\Pi(T, Z)$ of regular functions from T to Z, where $(Z, \mathcal{U})$ is a Hausdorff uniformly complete space with the gage of pseudometrics of the uniformity $\mathcal{U}$ [19], with the topology of uniform convergence on T was studied in [27]. Since $\Pi_+(T, Z)$ is a closed subspace of $\Pi(T, Z)$, Theorem 4.3 follows from Corollary 1.1 in [27].

In the work [27], regular functions are called proper functions.

Theorem 4.4. *Let* $U \subset \Pi_+(T, X)$. *Then, the following conditions are equivalent:*

(1) *the set* U *is relatively compact;*

(2) *the set* U *is unilaterally equicontinuous and for every* $t \in T$ *the set*

$$U(t) = \{\cup x(t) : x(\cdot) \in U\} \tag{4.2}$$

is relatively compact.

A criterion of relative compactness of sets in the space $\Pi(T, Z)$ was given in Theorem 1.2 [27]. Since the space $\Pi_+(T, X)$ is a closed subspace of the space $\Pi(T, X)$, Theorem 4.4 follows from Theorem 1.2 [27]. A criterion of relative compactness of sets in the space $\Pi(T, R^n)$, where R^n is the n-dimensional space was obtained in the work [14]. In this work, unilaterally equicontinuous sets are called equiregulated. Since the criterion of relative compactness in the space $\Pi(T, Z)$ was obtained earlier than in the work [14], we follow the terminology of the work [27].

Proposition 4.5. *If a set $U \subset \Pi_+(T, X)$ is relatively compact, then the set*

$$U(T) = \{\cup x(t); t \in T, x(\cdot) \in U\} \tag{4.3}$$

is relatively compact.

The proposition follows from Corollary 1.2 in [27]. From Theorem 4.3 it follows that the space $\Pi_+(T, X)$ is the completion of the space $BV_+(T, X)$. Hence, following [5, Chapter 11, §4, Definition 2] a set $U \subset BV_+(T, X)$ is called precompact, if its closure $\overline{U}$ in $\Pi_+(T, X)$ is compact.

Proposition 4.6. *If a set $U \subset BV_+(T, X)$ is uniformly bounded in variation, then its closed in space $\Pi_+(T, X)$ convex hull is a uniformly bounded in variation set and, consequently, $\overline{\mathrm{co}}\, U \subset BV_+(T, X)$.*

Proof. Let $\text{var } x(\cdot) \leq M$ for all $x(\cdot) \in U$ for some $M > 0$. From this inequality it follows that

$$\text{var } x(\cdot) \leq M, \quad \forall x(\cdot) \in \mathrm{co}\, U. \tag{4.4}$$

Let $y(\cdot) \in \overline{\mathrm{co}}\, U$. Then, for any $\varepsilon > 0$ and any partition $\mathcal{T}$ of the interval $T = [0, a]$ by points $0 \leq t_0 < t_1 < \ldots < t_i < \ldots < t_n = a$ there exists an element $x(\cdot) \in \mathrm{co}\, U$ such that

$$\sum_{i=1}^{n} \|y(t_{i-1}) - y(t_i)\| \leq \sum_{i=1}^{n} \|x(t_{i-1}) - x(t_i))\| + \varepsilon \leq \text{var } x(\cdot) + \varepsilon.$$

From this inequality, the arbitrariness of partition $\mathcal{T}$ of the interval T and (4.4) we infer that

$$\| \text{var } y(\cdot)\| \leq \varepsilon + M.$$

Since $\varepsilon > 0$ is arbitrary, the proposition is proven. □

Theorem 4.7. *Let $U \subset BV_+(T, X)$ be a uniformly bounded in variation precompact set. Then, the closed in $\Pi_+(T, X)$ convex hull $\overline{\mathrm{co}}\, U$ is a compact subset of the space $BV_+(T, H)$.*

Since the set $U \subset BV_+(T, X)$ is precompact, the set $\overline{co}\, U$ is a compact subset of the space $\Pi_+(T, X)$. Now the theorem follows from Proposition 4.6.

In the sequel, H is a Hilbert space.

Theorem 4.8. *Let M be a metric space, $C : M \rightrightarrows H$ a continuous in the Hausdorff metric mapping whose values are r-prox regular sets, $r \in]0, \infty]$. If sequences $z_n \in M$, $x_n \in C(z_n)$, $n \geq 1$ converge to z, x, respectively,and a sequence $\xi_n \in N_{C(z_n)}(x_n)$, $n \geq 1$ weakly converges to ξ, then $x \in C(z)$ and $\xi \in N_{C(z)}(x)$.*

The theorem directly follows from Theorem 2.2 in [3] and Proposition 2.3 in [1].

Theorem 4.9. (Dunford theorem [9]) *Let μ be a Radon measure on T. A set $V \subset L^1_\mu(T,H)$ is relatively weakly compact, if:*

(i) *the set V is bounded in $L^1_\mu(T,H)$,*

(ii) *the set U is uniformly integrable.*

Proposition 4.10. *Let μ be a Radon measure, $f_n(\cdot) \in L^1_\mu(T,H)$, $n \geq 1$ and $f_n(\cdot)$, $n \geq 1$ weakly converge to $f(\cdot)$. If for μ almost all $t \in T$ the set $\{f_n(t);\ n \geq 1\} \subset H$ is bounded, then*

$$f(t) \in \overline{co}\,\omega\text{-}\operatorname*{Ls}_{n\to\infty}\{f_n(t)\} \quad \mu \ \ a.e.,$$

where ω-$\mathrm{Ls}\{f_n(t)\}$ is defined in (2.2).

The proposition follows from Proposition 4.7.44 in [8].

5. Selectors of the multivalued Nemytskii operator

In what follows, H is a separable Hilbert space. Let $\mathcal{K} \subset \Pi_+(T,H)$ be a compact set and

$$\mathcal{R}(\mathcal{K}) = \{x(t); t \in T, x(\cdot) \in \mathcal{K}\}. \tag{5.1}$$

According to Proposition 4.5, the set $\overline{\mathcal{R}(\mathcal{K})}$, where the bar stands for the closure in H, is compact. Assume that

$$\overline{\mathcal{R}(\mathcal{K})} \subset \overline{R(C)}, \tag{5.2}$$

where $\mathcal{R}(C)$ is the set defined in (1.1), $\overline{\mathcal{R}(C)}$ is its closure in H.

Let Hypothesis $H(U)$ hold. From inequality (1.9) it follows that the set

$$\Gamma(t,x) = U(t,x) \cap l_U(t)(1+\|x\|)B \tag{5.3}$$

is not empty for any $t \in T, x \in \overline{\mathcal{R}(\mathcal{K})}$. Denote by $\overline{\Gamma}(t,x)$ the closure of the set $\Gamma(t,x)$. Then, we can define the multivalued mapping $\overline{\Gamma} : T \times \overline{\mathcal{R}(\mathcal{K})} \rightrightarrows H$ with closed values.

Lemma 5.1. *Let Hypothesis $H(U)$ hold. Then, there exists a Borel set $\mathcal{T} \subset T$, $\mu(\mathcal{T}) = 0$ such that the multivalued mapping $\hat{\Gamma} : T \times \overline{\mathcal{R}(\mathcal{K})} \rightrightarrows H$,*

$$\hat{\Gamma}(t,x) = \overline{\Gamma}(t,x), \ \ t \in T \backslash \mathcal{T}, \ \ \hat{\Gamma}(t,x) = \Theta, \ \ t \in \mathcal{T}, \ \ x \in \overline{\mathcal{R}(\mathcal{K})} \tag{5.4}$$

with closed values is weakly $\Sigma \otimes \mathcal{B}(\overline{\mathcal{R}(\mathcal{K})})$ measurable and lower semicontinuous at x for all $t \in T$.

Since the set $\overline{\mathcal{R}(\mathcal{K})}$ is compact, it is a Suslin space. Therefore, the proof of lemma repeats verbatim the proof of Lemma 5.2 in [29] replacing the domain $T \times H$ of the mapping U with $T \times \overline{\mathcal{R}(\mathcal{K})}$.

Theorem 5.2. *Let Hypothesis $H(U)$ hold. Then, there exists a continuous mapping $g : \mathcal{K} \to L^1(T,H)$ such that*

$$g(x)(t) \in U(t,x(t)) \ a.e., \ x(\cdot) \in \mathcal{K}, \tag{5.5}$$

$$\|g(x)(t)\| \leq l_U(t)(1+\|x(t)\|) \ a.e., \ x(\cdot) \in \mathcal{K}. \tag{5.6}$$

Proof. Since functions $x(\cdot) \in \Pi_+(T, H)$ are measurable, from Lemma 5.1 we derive that for any $x(\cdot) \in \mathcal{K}$ the mapping $t \mapsto \hat{\Gamma}(t, x(t))$ is weakly measurable with closed values. Then, according to (5.4) such is also the multivalued mapping $t \mapsto \overline{\Gamma}(t, x(t))$.

Denote
$$\mathcal{G}_U(x) = \{u(\cdot) \in L^1(T, H); u(t) \in \overline{\Gamma}(t, x(t)) \text{ a.e.}\}. \tag{5.7}$$

From (5.3) it follows that the set $\mathcal{G}_U(x)$ is a nonempty closed subset of the space $L^1(T, H)$. Thus, we can define the multivalued mapping $\mathcal{G}_U : \mathcal{K} \rightrightarrows L^1(T, H)$ with closed decomposable values.

Let us prove that the mapping $x(\cdot) \mapsto \mathcal{G}_U(x)$, $x(\cdot) \in \mathcal{K}$ is lower semicontinuous. For this, it is enough to show that for any $x(\cdot) \in \mathcal{K}$, $u(\cdot) \in \mathcal{G}_U(x)$ and for any sequence $x_n(\cdot) \in \mathcal{K}$, $n \leq 1$ converging to $x(\cdot)$ there exists a sequence $u_n(\cdot) \in \mathcal{G}(x_n)$, $n \leq 1$, converging to $u(\cdot)$ in the space $L^1(T, H)$.

Let $x(\cdot) \in \mathcal{K}$ and $u(\cdot) \in \mathcal{G}(x)$ be fixed and a sequence $x_n(\cdot) \in \mathcal{K}$, $n \geq 1$ converge to $x(\cdot)$. Since the multivalued mapping $t \mapsto \overline{\Gamma}(t, x_n(t))$ with closed values is measurable, the function $t \mapsto d(u(t)), \overline{\Gamma}(t, x_n(t))$ is measurable. Therefore, the multivalued mapping $Q_n : T \rightrightarrows H$
$$Q_n(t) = u(t) + (d(u(t)), \overline{\Gamma}(t, x_n(t)) + \frac{1}{n})\overline{B}, \quad t \in T \tag{5.8}$$
with closed convex values is measurable. The same property has also the multivalued mapping $t \mapsto Q_n(t) \cap \overline{\Gamma}(t, x_n(t))$. From Theorem 5.1 [16] it follows that there exists a measurable selector $t \mapsto u_n(t)$ of this mapping, i.e.
$$u_n(t) \in Q_n(t) \cap \overline{\Gamma}(t, x_n(t)) \text{ a.e.}$$
From this inclusion, (5.7), (5.8) it follows that $u_n(\cdot) \in \mathcal{G}_U(x_n)$, $n \geq 1$ and
$$\|u(t) - u_n(t)\| \leq d(u(t),\ \overline{\Gamma}(t, x_n(t)) + \frac{1}{n} \text{ a.e.} \tag{5.9}$$
According to Lemma 5.1, the mapping $y \mapsto \overline{\Gamma}(t, y)$, $y \in \overline{\mathcal{R}(\mathcal{K})}$ is lower semicontinuous for almost all $t \in T$. Using Lemma 1.2 [24] we see that the function $y \mapsto d(z, \overline{\Gamma}(t, y))$, $z \in H$ is upper semicontinuous for almost all $t \in T$. Passing to the limit in (5.9) as $n \to \infty$ we obtain the inequality
$$\overline{\lim_{n\to\infty}} \|u(t) - u_n(t)\| \leq \overline{\lim_{n\to\infty}} d(u(t), \overline{\Gamma}(t, x_n(t)) \leq d(u(t),\ \overline{\Gamma}(t, x(t)) \text{ a.e.}$$
Since $d(u(t),\ \overline{\Gamma}(t, x(t)) = 0$ a.e., the sequence $u_n(t)$, $n \leq 1$ converges to $u(t)$ in H for almost all $t \in T$. From (5.3), (5.7) it follows that
$$\|u_n(t)\| \leq l_U(t)(1 + \|x_n(t)\|) \text{ a.e.}$$
From this inequality and Lebesgue's dominated convergence theorem we infer that the sequence $u_n(\cdot)$, $n \geq 1$ converges to $u(\cdot)$ in the space $L^1(T, H)$. Consequently, the multivalued mapping $\mathcal{G}_U : \mathcal{K} \rightrightarrows L^1(T, H)$ with closed decomposable values is lower semicontinuous. Using Theorem 3 from [6], we see that there exists a continuous selector $g : \mathcal{K} \to L^1(T, H)$ of the lower semicontinuous multivalued mapping $\mathcal{G}_u : \mathcal{K} \rightrightarrows L^1(T, H)$ with closed decomposable values, i.e.
$$g(x) \in \mathcal{G}_U(x), \quad x(\cdot) \in \mathcal{K}. \tag{5.10}$$
Now the statement of the theorem follows from this inclusion and (5.3), (5.7), and the theorem is proven. □

Let $$\mathcal{U}(x) = \{u(\cdot) \in L^1(T,H);\ u(t) \in U(t,x(t)) \text{ a.e.}\}, \quad x(\cdot) \in \mathcal{K}. \tag{5.11}$$

From Theorem 5.2 it follows that the set $U(x)$ is not empty. Hence, we can define the multivalued operator $\mathcal{U} : \mathcal{K} \rightrightarrows L^1(T,H)$ that is called the multivalued Nemytskii operator.

As follows from (5.11), (5.5), (5.6), the mapping $g : \mathcal{K} \to L^1(T,H)$ is a continuous selector of the Nemytskii operator $\mathcal{U}$.

Let Hypothesis $H(V)$ hold. Then, from the inequality (1.10) it follows that we can define the multivalued mapping $\widetilde{V} : \overline{\mathcal{R}(\mathcal{K})} \rightrightarrows H$

$$\tilde{V}(t,x) = V(t,x) \cap l_V(t)(1 + \|x\|)\overline{B} \tag{5.12}$$

with nonempty convex closed values.

Lemma 5.3. *Let Hypothesis $H(V)$ hold. Then, for any $x(\cdot) \in \mathcal{K}$ the mapping $t \to \tilde{V}(t,x(t))$ has an integrable selector.*

Proof. Let $x(\cdot) \in \mathcal{K}$. According to Proposition 4.5, the set $\overline{\mathcal{R}(\mathcal{K})} \subset H$ is compact. Using Theorem 1.5 [27] and considering $\overline{\mathcal{R}(\mathcal{K})}$ as an independent metric space, we conclude that there exists a sequence of right continuous step functions $x_k : T \to \overline{\mathcal{R}(\mathcal{K})}$, $k \geq 1$ converging uniformly on T to the function $x(\cdot)$.

From Hypothesis $H(V)$ (2) and (5.12) it follows that for any $k \geq 1$ there exists a measurable selector $t \to v_k(t)$ of the multivalued mapping $t \mapsto \tilde{V}(t,x_k(t))$,

$$v_k(t) = V^0(t,x_k(t)), \quad k \geq 1, \tag{5.13}$$

$$\|v_k(t)\| \leq l_V(t)(1 + \|x_k(t)\|). \tag{5.14}$$

Since the set $\overline{\mathcal{R}(\mathcal{K})} \subset H$ is compact, from (5.12) it follows that there exists $M > 0$ such that

$$\tilde{V}(t,x) \subset l_V(t)(1 + M)\overline{B} \quad \text{a.e.}, \quad x \in \overline{\mathcal{R}(\mathcal{K})}. \tag{5.15}$$

Since $v_k(t) \in \tilde{V}(t,x_k(t)), k \geq 1, x_k(\cdot) \in \overline{\mathcal{R}(\mathcal{K})}$, from (5.15) it follows that the sequence $v_k(\cdot), k \geq 1$ is relatively compact subset of the space ω-$L^1(T,H)$. Since any compact set of the space ω-$L^1(T,H)$ is metrizable, without loss of generality, we can assume that the sequence $v_k(\cdot), k \geq 1$ converges to some element $v(\cdot) \in L^1(T,H)$ in the space ω-$L^1(T,H)$.

Since the mapping $x \mapsto l_V(t)(1 + \|x\|)\overline{B}, x \in \overline{\mathcal{R}(\mathcal{K})}$ has weakly sequentially closed graph for almost every $t \in T$, from (5.12) and $H(V)1)$ it follows that the mapping $x \to \tilde{V}(t,x), x \in \overline{\mathcal{R}(\mathcal{K})}$ has weakly sequentially closed graph for almost every $t \in T$.

From (5.15) it follows that for almost every $t \in T$ the set $\omega\text{-}\operatorname*{Ls}_{k\to\infty} \{v_k(t)\}$ is not empty and

$$\omega\text{-}\operatorname*{Ls}_{k\to\infty} \{v_k(t)\} \subset \tilde{V}(t,x(t)) \ \text{ a.e.}$$

Since the set $\tilde{V}(t,x(t)), t \in T$ is convex and closed, from the last inclusion we see that

$$\overline{\text{co}}\, \omega\text{-}\operatorname*{Ls}_{k\to\infty} \{v_k(t)\} \subset \tilde{V}(t,x(t)) \ \text{ a.e.} \tag{5.16}$$

Using this inclusion and Proposition 4.10, we obtain the inclusion

$$v(t) \in \tilde{V}(t,x(t)) \ \text{ a.e.} \tag{5.17}$$

From this inclusion it follows that $v(\cdot)$ is an integrable selector of the mapping $t \mapsto \tilde{V}(t, x(t))$. The lemma is proven. □

Let
$$\mathcal{G}_{\tilde{V}}(x) = \{v(\cdot) \in L^1(T, H);\ v(t) \in \tilde{V}(t, x(t)) \text{ a.e.}\},\ x(\cdot) \in \mathcal{K}. \tag{5.18}$$

From Lemma 5.3 and (5.15), (5.18) it follows that $\mathcal{G}_{\tilde{V}}(x)$ is a nonempty convex compact subset of the space ω-$L^1(T, H)$.

Theorem 5.4. *Let Hypothesis $H(V)$ hold. Then $x \mapsto \mathcal{G}_{\tilde{V}}(x)$, $x(\cdot) \in \mathcal{K}$ is a weakly upper semicontinuous mapping from $\mathcal{K}$ to ω-$L^1(T, H)$ with convex weakly compact values and for any $v(\cdot) \in \mathcal{G}_{\tilde{V}}(x)$ we have*
$$v(x)(t) \in V(t, x(t)) \ \ a.e. \tag{5.19}$$

Proof. Let $x(\cdot) \in \mathcal{K}$ and $x_k(\cdot) \in \mathcal{K},\ v_k(\cdot) \in \mathcal{G}_{\tilde{V}}(x_k),\ k \geq 1$ be arbitrary sequences converging to $x(\cdot)$ in $\Pi_+(T, H)$ and to $v(\cdot)$ in ω-$L^1(T, H)$, respectively. Using (5.15), the weak sequential closedness of mapping $y \mapsto \tilde{V}(t, y)$ at the point $x(t)$ and reasoning as in the proof of Lemma 5.1, we see that the inclusion (5.16) holds. From this inclusion and Proposition 4.2 we infer the inclusion (5.17). Using this inclusion, the arbitrariness of sequence $x_k(\cdot),\ k \geq 1$ and (5.18), we conclude that the mapping $y(\cdot) \mapsto \mathcal{G}_{\tilde{V}}(y), y(\cdot) \in \mathcal{K}$ is weakly sequentially closed at the point $x(\cdot) \in \mathcal{K}$. Since $x(\cdot) \in \mathcal{K}$ is arbitrary, the mapping $x(\cdot) \to \mathcal{G}(x)$ is weakly sequentially closed in $\mathcal{K}$. Since the set
$$\{f(\cdot) \in L^1(T, H);\ \ \|f(t)\| \leq l_V(t)(1 + M) \ \text{ a.e.}\}$$
is a convex metrizable compact subset of the space ω-$L^1(T, H)$, from (5.15) it follows that the mapping $x(\cdot) \mapsto \mathcal{G}_{\tilde{V}}(x), x(\cdot) \in \mathcal{K}$ is weakly locally compact at every point $x(\cdot) \in \mathcal{K}$. Using Proposition 4.2, we see that the mapping $x(\cdot) \mapsto \mathcal{G}_{\tilde{V}}(x), x(\cdot) \in \mathcal{K}$ is weakly upper semicontinuous with convex weakly compact values. The inclusion (5.19) directly follows from (5.18), (5.12). The theorem is proven. □

Let $x(\cdot) \in \mathcal{K}$ and $\mathcal{V}(x) = \{v(\cdot) \in L^1(T, H);\ v(t) \in V(t, x(t)) \ \text{ a.e.}\}$.

From Theorem 5.2 and (5.12), (5.18) it follows that the set $\mathcal{V}(x)$ is not empty and the multivalued Nemytskii operator $\mathcal{V} : \mathcal{K} \rightrightarrows L^1(T, H)$ has weakly upper semicontinuous multivalued selector $\mathcal{G}_{\tilde{V}}(x) \subset \mathcal{V}(x), x(\cdot) \in \mathcal{K}$ with convex weakly compact values.

6. Solutions of the measurable sweeping process

Let $Q : T \rightrightarrows H$ be a multivalued mapping with r-prox regular values. Assume that there exists a Radon measure α on T such that
$$|d(y, Q(t)) - d(y, Q(s))| \leq \alpha(]s, t]),\ \ s \leq t,\ \ s, t \in T,\ \ y \in H \tag{6.1}$$
with
$$\sup_{s \in]0,a]} \alpha(\{s\}) < \frac{r}{2}. \tag{6.2}$$

Consider the measurable sweeping process
$$-dx \in \mathcal{N}_{Q(t)}(x(t)) + f(t),$$
$$x(0) = x_0 \in Q(0),\ \ f(\cdot) \in L^1(T, H). \tag{6.3}$$

Definition 6.1. By a solution to inclusion (6) we mean a right continuous function $x(f) : T \to H$ of bounded variation such that

(a) $x(f)(0) = x_0, \;\; x(f)(t) \in Q(t), \;\; t \in T;$ (6.4)

(b) there exists a Radon measure γ absolutely continuously equivalent to the measure $\alpha + \lambda$ with respect to which the differential measure $dx(f)$ generated by the function $x(f)(\cdot)$ is absolutely continuous and the density $\frac{dx(f)}{d\gamma}(\cdot)$ is an element of the space $L^1_\gamma(T, H)$;

(c) the following inclusion holds

$$-\frac{dx(f)}{d\gamma}(t) - f(t)\frac{d\lambda}{d\gamma}(t) \in \mathcal{N}_{Q(t)}(x(t)) \;\; \gamma \text{ a.e.} \tag{6.5}$$

In the definition of a solution of the measurable sweeping process we follow [1, 12] and others.

Theorem 6.2. *Let $r \in]0, +\infty]$ and $Q : T \rightrightarrows H$ be a multivalued mapping with r-prox regular values. Assume that the inequalities* (6.1), (6.2) *hold. Then, for any $x_0 \in Q(0)$ and any $f(\cdot) \in L^1(T, H)$ there exists a unique solution $x(f)(\cdot)$ of the measurable sweeping process* (6) *satisfying the inequalities*

$$\|x(f)(t) - x(f)(t-0)\| \le \alpha(\{t\}), \;\; t \in]0, a], \tag{6.6}$$

$$\|\frac{dx(f)}{d\gamma}(t) + f(t)\frac{d\lambda}{d\gamma}(t)\| \le \frac{d\mu}{d\gamma}(t) + \|f(t)\|\frac{d\lambda}{d\gamma}(t) \quad \nu \text{ a.e.} \tag{6.7}$$

for any Radon measure γ absolutely continuously equivalent to the Radon measure $\alpha + \lambda$.

The theorem directly follows from Theorem 5.1 [1].

Let $\rho(t)$, $\rho(0) = \|x_0\| + \frac{1}{1-L}\,\mu(]0, a])$, $t \in T$ be the solution of the differential equation (1.11). Then, $\rho : T \to R^+$ is a continuous function of bounded variation, its differential measure $d\rho$ is a Radon measure and

$$d\rho(A) = \int_A \dot\rho(t)dt, \;\; A \in \mathcal{B}. \tag{6.8}$$

From this equality and Lemma 3.1 it follows that

$$\frac{d\rho}{d\lambda}(t) = \dot\rho(t) \;\text{ a.e.} \tag{6.9}$$

Lemma 6.3. *Let a Radon measure ν is absolutely continuously equivalent to the measure $\mu + \lambda$. Then, the measure $d\rho$ is absolutely continuous with respect to the measure ν and we have*

$$\frac{d\rho}{d\nu}(t) = \dot\rho(t)\frac{d\lambda}{d\nu}(t) \quad \nu \text{ a.e.} \tag{6.10}$$

Since the measure $d\rho$ is absolutely continuous with respect to the measure $\lambda + \mu$, it is absolutely continuous with respect to the measure ν. Using the equality (6.9), we obtain

$$\rho(t) = \rho(0) + \int_{]0,t]} \dot\rho(s)d\lambda(s), \;\; t \in T.$$

From this equality and Lemma 3.3 we obtain the equality (6.10).

In the sequel, ν is an arbitrary Radon measure absolutely continuously equivalent to the measure $\mu + \lambda$. Then, $\frac{d\mu}{d\nu}(\cdot), \frac{d\rho}{d\nu}(\cdot) \in L^1_\nu(T, R^+)$. Let

$$S(L^1_\nu) = \{f(\cdot) \in L^1_\nu(T, H);\ \|f(t)\| \leq \frac{1}{1-L}\frac{d\mu}{d\nu}(t) + \frac{d\rho}{d\nu}(t) \quad \nu \text{ a.e.}\}. \tag{6.11}$$

Lemma 6.4. *The set $S(L^1_\nu)$ is a convex weakly compact subset of the space $L^1_\nu(T, H)$.*

Proof. The convexity of the set $S(L^1_\nu)$ is evident. Show that the set $S(L^1_\nu)$ is relatively weakly compact. From (6.11) we have

$$\int_A \|f(t)\| d\nu \leq \frac{1}{1-L}\mu(A) + d\rho(A), \tag{6.12}$$

$A \in \mathcal{B}$, $f(\cdot) \in S(L^1_\nu)$. Since the measure $\frac{1}{1-L}\,\mu + d\rho$ is absolutely continuous with respect to the measure ν, according to Proposition 13 [11, §10, p. 177] for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any $A \in \mathcal{B}$ with $\nu(A) < \delta(\varepsilon)$ the inequality $\frac{1}{1-L}\,\mu(A) + d\rho(A) < \varepsilon$ holds. Now, from (6.12) it follows that the set $S(L^1_\nu)$ is uniformly integrable and bounded in the space $L^1_\nu(T, H)$. Using Theorem 4.9, we see that the set $S(L^1_\nu)$ is convex and relatively weakly compact in the space $L^1_\nu(T, H)$. Let us prove that $S(L^1_\nu)$ is closed in the space $L^1_\nu(T, H)$. Let a sequence $f_n(\cdot) \in S(L^1_\nu)$, $n \geq 1$ converge to $f(\cdot)$ in $L^1_\nu(T, H)$. Then, there exists a subsequence $f_{n_k}(\cdot), k \geq 1$ such that

$$f_{n_k}(t) \to f(t) \quad \nu \text{ a.e.}$$

Then, according to (6.11) $f(\cdot) \in S(L^1_\nu)$. Consequently, the set $S(L^1_\nu)$ is a convex closed subset of the space $L^1_\nu(T, H)$ and, consequently, a closed subset of the space ω-$L^1_\nu(T, H)$. Hence, the set $S(L^1_\nu)$ is a convex weakly compact subset of the space $L^1_\nu(T, H)$. T he lemma is proven. □

Let $AC_\nu(T, H)$ be a subspace of the space $BV_+(T, H)$ which consists of the functions $u : T \to H$ whose differential measures du are absolutely continuous with respect to the measure ν.

Define
$$S = \{u(\cdot) \in AC_\nu(T, H);\ u(0) = x_0,\ \frac{du}{d\nu}(\cdot) \in S(L^1_\nu)\}. \tag{6.13}$$

Lemma 6.5. *The set S is convex, unilaterally equicontinuous, uniformly bounded in variation and closed in the space $\Pi_+(T, H)$.*

For any $u(\cdot) \in S$ we have
$$\|u(t)\| \leq \rho(t), \ \ t \in T. \tag{6.14}$$

Proof. The convexity of the set S is evident. From (6.11), (6.13) we derive the inequalities

$$\|u(t) - u(s)\| \leq \frac{1}{1-L}\mu(]s, t]) + d\rho(]s, t]), \quad s \leq t,\ s, t \in T, \tag{6.15}$$

and
$$\|u(t-0) - u(s)\| \leq \frac{1}{1-L}\mu(]s, t[) + d\rho(]s, t[).$$

which is valid for any $u(\cdot) \in S$.

From these inequalities we infer that the set S is unilaterally equicontinuous. The uniform boundedness in variation of the set S follows from (6.15).

Denote by $\Lambda : L_\nu(T,H) \to \Pi_+(T,H)$ the operator defined by the rule

$$\Lambda(f)(t) = \int\limits_{]0,t]} f(s)d\nu(s), \quad t \in T. \tag{6.16}$$

The operator Λ is linear and continuous from the Banach space $L_\nu(T,H)$ to the Banach space $\Pi_+(T,H)$. Hence, it is continuous from ω-$L^1_\nu(T,H)$ to ω-$\Pi_+(T,H)$.

Then, according to Lemma 6.4 the set

$$\Big\{\Lambda\Big(\frac{du}{d\nu}\Big); \frac{du}{d\nu}(\cdot) \in S(L^1_\nu)\Big\} \tag{6.17}$$

is convex and compact in the space ω-$\Pi_+(T,H)$ and, thus, it is a closed subset of the space $\Pi_+(T,H)$. From (6.16), (6.17) we see that

$$u(t) = u_0 + \Lambda\Big(\frac{du}{d\nu}\Big)(t), \quad t \in T, \quad u(\cdot) \in S.$$

Consequently, the set S is a convex closed subset of the space $\Pi_+(T,H)$.

From (6.15) we infer that

$$\|u(t) - u(0)\| \leq \frac{1}{1-L}\, \mu(]0,t]) + d\rho(]0,t]). \tag{6.18}$$

Since
$$\rho(t) - \rho(0) = d\rho]0,t]), \quad t \in T,$$

the inequality (6.14) follows from (6.18), (1.12). The lemma is proven. □

Let
$$S(L^1) = \{f(\cdot) \in L^1(T,H); \;\; \|f(t)\| \leq l(t)(1+\rho(t)) \quad \text{a.e.}\}, \tag{6.19}$$
where $l(t)$ is defined by (1.13). If $f(\cdot) \in L^1(T,H)$, then the function

$$g(t) = \int\limits_{]0,t]} f(s)ds, \quad t \in T \tag{6.20}$$

is a function of bounded variation, its differential measure dg is absolutely continuous with respect to the measure ν and according to Lemma 3.3 we have

$$\frac{dg}{d\nu} = f(t)\frac{d\lambda}{d\nu}(t) \quad \nu \text{ a.e.} \tag{6.21}$$

Lemma 6.6. *Let a sequence $f_n(\cdot) \in S(L^1)$, $n \geq 1$ converge to $f(\cdot)$ in the space ω-$L^1(T,H)$. Then, the sequence $\frac{dg_n}{d\nu}(\cdot)$, $n \geq 1$ converges to $\frac{dg}{d\nu}(\cdot)$ in the space ω-$L^1_\nu(T,H)$.*

Proof. Let $f_n(\cdot) \in S(L^1), n \geq 1$ converge to $f(\cdot)$ in ω-$L^1(T,H)$. Using (6.21), (6.10) and (1.11), we obtain

$$\int\limits_A \Big\|\frac{dg_n}{d\nu}(t)\Big\| d\nu(t) \leq \frac{1-L}{2}\, d\rho(A), \quad A \in \mathcal{B}, \quad n > 1.$$

From this inequality and Theorem 4.9 it follows that the sequence $\frac{dg_n}{d\nu}(\cdot), n \geq 1$ is a relatively compact subset of the space ω-$L^1_\nu(T,H)$. Since any compact set of the space ω-$L^1_\nu(T,H)$ is metrizable, there exists a subsequence $\frac{dg_{n_k}}{d\nu}(\cdot), k \geq 1$ converging to $q(\cdot)$ in the space ω-$L^1_\nu(T,H)$. With the help of (6.20), (6.21) for $h \in H$ we obtain

$$\langle g_{n_k}(t), h\rangle = \int\limits_{]0,t]} \langle f_{n_k}(\tau), h\rangle \, d\tau = \int\limits_{]0,t]} \langle \frac{dg_{n_k}}{d\nu}(\tau), h\rangle \, d\nu(\tau) \tag{6.22}$$

Passing to the limit in this equality as $k \to \infty$, we arrive at the equality

$$\langle g(t), h\rangle = \int\limits_{]0,t]} \langle f(\tau), h\rangle d\tau = \int\limits_{]0,t]} \langle q(\tau), h\rangle d\nu(\tau).$$

From this equality it follows that

$$g(t) = \int\limits_{]0,t]} f(\tau) d\tau = \int\limits_{]0,t]} q(\tau) d\nu(\tau), \quad t \in T.$$

Then, according to (6.21) we have

$$\frac{dg}{d\nu}(t) = f(t)\frac{d\lambda}{d\nu}(t) = q(t) \quad \nu \text{ a.e.}$$

Therefore, we have shown that if a sequence $f_n(\cdot)$, $n \geq 1$ converges to $f(\cdot)$ in the space ω-$L^1(T,H)$, then there exists a subsequence $\frac{dg_{n_k}}{d\nu}(\cdot)$, $k \geq 1$, converging to $\frac{dg}{d\nu}(\cdot)$ in the space ω-$L^1(T,H)$. If we assume that the sequence $\frac{dg_n}{d\nu}(\cdot)$, $n \geq 1$ itself does not converge to $\frac{dg}{d\nu}(\cdot)$, then there would exist a subsequence $\frac{dg_{n_k}}{d\nu}(\cdot), k \geq 1$, such that any subsequence of this sequence does not converge to $\frac{dg}{d\nu}(\cdot)$ in the space ω-$L^1_\nu(T,H)$. Repeating the reasoning above, we arrive at a contradiction. Consequently, the sequence $\frac{dg_n}{d\nu}(\cdot), n \geq 1$ converges to $\frac{dg}{d\nu}(\cdot)$ in ω-$L^1_\nu(T,H)$. The lemma is proven. □

Consider the measurable sweeping process

$$-dx \in \mathcal{N}_{C(t,u(t))}(x(t)) + f(t), \tag{6.23}$$

$$x(0) = x_0 \in C(0,x_0), \quad u(\cdot) \in S, \quad f(\cdot) \in S(L^1). \tag{6.24}$$

Theorem 6.7. *Let the Hypothesis $H(C)$ hold. Then, for any $u(\cdot) \in S, f(\cdot) \in S(L^1)$ there exists a unique function $x_u(f) : T \to H$ of bounded variation such that*

(1) $x_u(f)(0) = x_0,\ x_u(f)(t) \in C(t,u(t)),\ t \in T;$ (6.25)

(2) *the differential measure $dx_u(f)$ generated by the function $x_u(f)(\cdot)$ is absolutely continuous with respect to any Radon measure ν absolutely continuously equivalent to the measure $\mu + \lambda$;*

(3) $\dfrac{dx_u(f)}{d\nu}(\cdot) \in L^1_\nu(T,H)$ *and we have*

$$-\frac{dx_u(f)}{d\nu}(t) - f(t)\frac{d\lambda}{d\nu}(t) \in \mathcal{N}_{C(t,u(t))}(x_u(f)(t)) \quad \nu \ \textit{a.e.}; \tag{6.26}$$

(4) *the following inequalities are valid*

$$\|x_u(f)(t-0) - x_u(f)(t)\| \leq \frac{1}{1-L}\,\mu(\{t\}), \quad t \in]0,a], \tag{6.27}$$

$$\left\|\frac{dx_u(f)}{d\nu}(t) + f(t)\frac{d\lambda}{d\nu}(t)\right\| \leq \frac{1}{1-L}\,\frac{d\mu}{d\nu}(t) + L\frac{d\rho}{d\nu}(t) + \frac{d\lambda}{d\nu}(t)\|f(t)\| \quad \nu \ \ a.e. \tag{6.28}$$

Proof. Let $u(\cdot) \in S$ and $f(\cdot) \in S(L^1)$. From the inequality (1.3) we see that

$$|d(z, C(t,u(t)) - d(z, C(s,u(s))| \leq \mu(]s,t]) + L\|u(t) - u(s)\|, \ s \leq t, \ s,t \in T.$$

From this inequality and (6.15) it follows that

$$|d(z, C(t,u(t)) - d(z, C(s,u(s))| \leq \frac{1}{1-L}\,\mu(]s,t]) + L\,d\rho(]s,t]) \tag{6.29}$$

for $s \leq t$, $s,t \in T$. Let α be the Radon measure

$$\alpha = \frac{1}{1-L}\,\mu + L\,d\rho \tag{6.30}$$

and

$$Q(t) = C(t,u(t)). \tag{6.31}$$

Then, from (6.30), (1.2), (6.31) and (6.29) we obtain

$$\sup_{s\in]0,a]} \alpha(\{t\}) < \frac{r}{2}, \quad t \in T, \tag{6.32}$$

and

$$|d(z, Q(t)) - d(z, Q(s))| \leq \alpha(]s,t]), \quad s \leq t, \quad s,t \in T. \tag{6.33}$$

Since the measures $\mu + \lambda$ and $\alpha + \lambda$ are absolutely continuously equivalent, the measures ν and $\alpha + \lambda$ are absolutely continuously equivalent as well. Now the theorem follows from (6.32), (6.33), Definition 6.1 and Theorem 6.2. The theorem is proven. □

7. Proof of the main results

In what follows we always assume that Hypotheses $H(C)$ and $H(C_S)$ hold. According to Definition 6.1, the function $x_u(t)(\cdot)$ whose existence is established in Theorem 6.7 is a solution to the inclusion (6.23). Denote by $\mathcal{L} : S \times S(L^1) \to BV_+(T,H)$ the operator which to every $u(\cdot) \in S$ and $f(\cdot) \in S(L^1)$ associates the unique solution of the inclusion (6.23), i.e.

$$x_u(f)(\cdot) = \mathcal{L}(u,f). \tag{7.1}$$

Let

$$\mathcal{M} = \{\mathcal{L}(u,f);\ u(\cdot) \in S,\ f(\cdot) \in S(L^1)\}. \tag{7.2}$$

Lemma 7.1. *The set $\mathcal{M}$ is a uniformly bounded in variation, precompact subset of the space $BV_+(T,H)$ and we have*

$$\mathcal{M} \subset S. \tag{7.3}$$

Proof. From (1.11), (6.10) and Lemma 3.3 we infer that

$$\frac{d\rho}{d\nu} = \frac{2}{1-L}\frac{d\lambda}{d\nu}(t)l(t)(1+\rho(t)) \quad \nu \ \text{ a.e.}$$

From the last equality, (6.28) and (6.19) we obtain

$$\left\|\frac{dx_u(f)}{d\nu}(t)\right\| \le \frac{1}{1-L}\frac{d\mu}{d\nu}(t) + \frac{d\rho}{d\nu}(t) \quad \nu \text{ a.e.} \tag{7.4}$$

This inequality, (6.13) and (6.11) imply the inclusion $x_u(f) \in S$. Consequently, according to (7.1) and (7.2) the inclusion (7.3) holds. Let

$$S(t) = \{\cup u(t); u(\cdot) \in S\}, \;\; t \in T,$$

$$\mathcal{M}(t) = \{\mathcal{L}(u,f)(t);\;\; u(\cdot) \in S,\;\; f(\cdot) \in S(L^1)\},\;\; t \in T.$$

From (7.3) and (6.25) we see that

$$\mathcal{M}(t) \subset C(t, S(t)) \cap S(t), \;\; t \in T.$$

This inclusion, (6.14) and Hypothesis $H(C_S)$, imply that the set $\mathcal{M}(t) \subset H$, $t \in T$, is relatively compact. Now the statements of the lemma follow from (7.3), Lemma 6.5 and Theorem 4.4. The lemma is proven. □

Let $\overline{\text{co}}\,\mathcal{M}$ be the closed convex hull of the set $\mathcal{M}$. Then, from Theorem 4.3 it follows that the set $\overline{\text{co}}\,\mathcal{M}$ is a convex compact subset of the space $BV_+(T,H)$. According to Lemma 7.1 and (6) we have

$$\mathcal{M} \subset \overline{\text{co}}\,\mathcal{M} \subset S. \tag{7.5}$$

Theorem 7.2. *The operator $\mathcal{L}$ is continuous from $\overline{\text{co}}\,\mathcal{M} \times \omega\text{-}S(L^1)$ to $\mathcal{M}$.*

Proof. The inclusion

$$\mathcal{L}(u,f) \in \mathcal{M}, \;\; u(\cdot) \in \overline{\text{co}}\,\mathcal{M}, \;\; f(\cdot) \in S(L^1) \tag{7.6}$$

directly follows from (7.2) and (7.3).

Since the set $\omega\text{-}S(L^1)$ is a metrizable compact subset of the space $\omega\text{-}L^1(T,H)$, it is enough to prove the sequential continuity of the operator $\mathcal{L}$.

Let a sequence $u_n(\cdot) \in \overline{\text{co}}\,\mathcal{M}, n \ge 1$ converge to $u(\cdot)$ in the space $BV_+(T,H)$ and a sequence $f_n(\cdot) \in S(L^1), n \ge 1$ converge to $f(\cdot)$ in $\omega\text{-}L^1(T,H)$. Then, $u(\cdot) \in \overline{\text{co}}\,\mathcal{M}$ and $f(\cdot) \in S(L^1)$. According to Theorem 6.7, we can define the solutions $x_n(\cdot) = \mathcal{L}(u_n, f_n)$, $n \ge 1$ and $x(\cdot) = \mathcal{L}(u,f)$ of the sweeping process (6.23). From (7.5), (7.6) and Lemmas 7.1, 6.4 it follows that the sequence $x_n(\cdot)$, $n \ge 1$ is a relatively compact subset of the space $BV_+(T,H)$ and the sequence $\frac{dx_n}{d\nu}(\cdot)$, $n \ge 1$ is a relatively compact subset of the space $\omega\text{-}L^1_\nu(T,H)$. Therefore, there exist subsequences $x_{n_k}(\cdot)$ and $\frac{x_{n_k}}{d\nu}(\cdot), k \ge 1$ converging to $y(\cdot) \in \overline{\text{co}}\,\mathcal{M}$ and $z(\cdot) \in S(L^1_\nu)$ in the spaces $BV_+(T,H)$ and $\omega\text{-}L^1_\nu(T,H)$, respectively.

From the equality $\quad x_{n_k}(t) = x_0 + \int\limits_{]0,t]} \frac{dx_{n_k}}{d\nu}(s)d\nu(s), \;\; t \in T$

it follows that $\quad y(t) = x(0) + \int\limits_{]0,t]} z(s)d\nu(s), \;\; t \in T.$

From this equality and (3.7), (3.8) we obtain

$$\frac{dy}{d\nu}(t) = z(t) \quad \nu \text{ a.e.} \tag{7.7}$$

Let

$$x_k(t) = x_{n_k}(t), \quad f_k(t) = f_{n_k}(t), \quad u_k(t) = u_{n_k}(t). \tag{7.8}$$

Using (7.7), (7.8), (6.25)–(6.28) and Lemma 6.6, we obtain

$$x_k(\cdot) \to y(\cdot) \ \text{ in } \ BV_+(T,H), \tag{7.9}$$

$$\frac{dx_k}{d\nu}(\cdot) \to \frac{dy}{d\nu} \ \text{ in } \ \omega\text{-}L^1_\nu(T,H), \tag{7.10}$$

$$\frac{dg_k}{d\nu}(\cdot) \to \frac{dg}{d\nu}(\cdot) \ \text{ in } \ \omega\text{-}L^1_\nu(T,H), \tag{7.11}$$

where

$$\frac{dg_k}{d\nu}(t) = \frac{d\lambda}{d\nu}(t) f_k(t), \quad \frac{dg}{d\nu}(t) = \frac{d\lambda}{d\nu}(t) f(t),$$

$$u_k(\cdot) \to u(\cdot) \ \text{ in } \ BV_+(T,H), \tag{7.12}$$

$$x_k(t) \in C(t, u_k(t)), \quad t \in T, \tag{7.13}$$

$$-\frac{dx_k}{d\nu}(t) - \frac{dg_k}{d\nu}(t) \in \mathcal{N}_{C(t,u_k(t))}(x_k(t)) \quad \nu \text{ a.e.}, \tag{7.14}$$

$$\Big\| \frac{dx_k}{d\nu}(t) + \frac{dg_k}{d\nu}(t) \Big\| \le \frac{1}{1-L} \frac{d\mu}{d\nu}(t) + \frac{L+1}{2} \frac{d\rho}{d\nu}(t) \ \ \nu \text{ a.e.} \tag{7.15}$$

$$\|x_k(t) - x_k(t-0)\| \le \frac{1}{1-L} \mu(\{t\}), \ t \in]0,a]. \tag{7.16}$$

Let

$$v_k(t) = -\frac{dx_k}{d\nu}(t) - \frac{dg_k}{d\nu}(t), \quad k \ge 1, \tag{7.17}$$

$$v(t) = -\frac{dy}{d\nu}(t) - \frac{dg}{d\nu}(t). \tag{7.18}$$

From (7.10), (7.11), (7.15), (7.17), (7.18) it follows that

$$v_k(\cdot) \to v(\cdot) \ \text{ in } \ \omega\text{-}L^1_\nu(T,H), \tag{7.19}$$

$$\|v_k(t)\| \le \frac{1}{1-L} \frac{d\mu}{d\nu}(t) + \frac{L+1}{2} \frac{d\rho}{d\nu}(t) \ \ \nu \text{ a.e.} \tag{7.20}$$

Using (7.12), (1.3), (2.1), we obtain

$$C(t, u_k(t)) \to C(t, u(t)), \quad t \in T \tag{7.21}$$

in the Hausdorff metric. From (7.9), (7.13), (7.21) and (7.16) we have

$$y(t) \in C(t, u(t)), \quad t \in T, \tag{7.22}$$

$$\|y(t) - y(t-0)\| \le \frac{1}{1-L} \mu(\{t\}), \ t \in]0,a]. \tag{7.23}$$

From (7.20) it follows that for ν almost all $t \in T$ the set $\omega\text{-}\operatorname*{Ls}_{k\to\infty}(\{v_k(t)\})$ is not empty. Using Definition 2.2, (7.14), (7.17), (7.9), (7.21), and Theorem 4.8, we obtain the inclusion

$$\omega\text{-}\operatorname*{Ls}_{k\to\infty}(\{v_k(t)\}) \subset \mathcal{N}_{C(t,u(t))}(y(t)) \ \ \nu \text{ a.e.}$$

Since the set $\mathcal{N}_{C(t,u(t))}(y(t))$ is convex and closed, we have

$$\overline{\text{co}}\,\omega\text{-}\operatorname*{Ls}_{k\to\infty}(\{v_k(t)\}) \subset \mathcal{N}_{C(t,u(t))}(y(t)) \ \ \nu \ \text{ a.e.} \tag{7.24}$$

From (7.17), (7.18), (7.24) and Proposition 4.10 we infer that

$$\frac{dy}{d\nu}(t) + \frac{d\lambda}{d\nu}(t)f(t) \in \mathcal{N}_{C(t,u(t))}(y(t)) \ \ \nu \ \text{ a.e.} \tag{7.25}$$

From (6.13), the inclusion $y(t) \in S, t \in T$, (7.22), (7.25) and Definition 6.1 it follows that the function $y(\cdot)$ is a solution of (6.23), i.e. $y(\cdot) = \mathcal{L}(u, f)$.

Since for fixed $u(\cdot)$ and $f(\cdot)$ the inclusion (6.23) has a unique solution, we have

$$y(\cdot) = x(\cdot) = \mathcal{L}(u, f).$$

Therefore, we have proven that if a sequence $u_n(\cdot)$, $n \geq 1$ converges to $u(\cdot)$ in $BV_+(T, H)$ and a sequence $f_n(\cdot), n \geq 1$ converges to $f(\cdot)$ in ω-$L^1(T, H)$, then there exists a subsequence $x_{n_k}(\cdot) = \mathcal{L}(y_{n_k}, f_{n_k})$, $k \geq 1$ converging to $x(\cdot) = \mathcal{L}(u, f)$ in $BV_+(T, H)$. If we assume that the sequence $x_n(\cdot), n \geq 1$ itself does not converge to $x(\cdot)$, then there would exist a subsequence $x_{n_m}(\cdot)$, $m \geq 1$ such that any subsequence of this sequence does not converge to $x(\cdot)$. Repeating the reasoning above to the subsequence $x_{n_m}(\cdot)$, $m \geq 1$ and taking into account that for $u(\cdot), f(\cdot)$ the inclusion (6.23) has a unique solution, we arrive at a contradiction. Hence, we have proven that if $u_n(\cdot) \in \overline{\text{co}}\,\mathcal{M}$, $n \geq 1$ converges to $u(\cdot)$ in the space $BV_+(T, H)$, and a sequence $f_n(\cdot) \in S(L^1)$, $n \geq 1$ converges to $f(\cdot)$ in ω-$L^1(T, H)$, then the sequence $x_n(\cdot) = \mathcal{L}(u_n, f_n) \in \overline{\text{co}}\,\mathcal{M}$, $n \geq 1$ converges to $x(\cdot) = \mathcal{L}(u, f)$ in the space $BV_+(T, H)$. The theorem is proven. □

Proof of Theorem 1.1

From Theorem 7.1 it follows that the set

$$\mathcal{K} = \{\mathcal{L}(u, f); \ \ u(\cdot) \in \overline{\text{co}}\,\mathcal{M}, \ \ f(\cdot) \in S(L^1)\} \tag{7.26}$$

is a compact subset of the space $\Pi_+(T, H)$ for which the inclusion (5.2) holds. Hence, on the set $\mathcal{K}$ we can define the mapping $g : \mathcal{K} \to L^1(T, H)$ and the multivalued mapping $\mathcal{G}_{\tilde{V}} : \mathcal{K} \rightrightarrows \omega$-$L^1(T, H)$ with the properties established in Theorems 5.2 and 5.4.

Define the multivalued mapping $\mathcal{F} : \mathcal{K} \rightrightarrows \omega$-$L^1(T, H)$

$$\mathcal{F}(x) = g(x) + \mathcal{G}_{\tilde{V}}(x), \tag{7.27}$$

whose values are convex weakly compact sets in the space $L^1(T, H)$. From Theorem 7.2, (7.5), (6.14) it follows that for any $x(\cdot) \in \mathcal{K}$ it holds $\|x(t)\| \leq \rho(t)$, $t \in T$. Using (5.6), (5.18), (5.12), (1.13), we obtain the inequality

$$\|f(t)\| \leq l(t)(1 + \rho(t)) \ \text{ a.e.}, \quad f(\cdot) \in \mathcal{F}(x).$$

From this inequality and (6.19) we infer that

$$\mathcal{F}(x) \subset S(L^1), \ \ x(\cdot) \in \mathcal{K}. \tag{7.28}$$

From the properties of mappings $g, \mathcal{G}_{\tilde{V}}$ given in Theorems 5.2, 5.4, it follows that the mapping $\mathcal{F} : \mathcal{K} \rightrightarrows \omega\text{-}S(L^1)$ is weakly upper semicontinuous with convex weakly compact values.

Consider the multivalued mapping

$$\mathcal{F}(\mathcal{L}) : \overline{\text{co}}\,\mathcal{M} \times \omega\text{-}S(L^1) \rightrightarrows \omega\text{-}S(L^1), \tag{7.29}$$

$$\mathcal{F}(\mathcal{L})(u, f) = \mathcal{F}(\mathcal{L}(u, f)), \tag{7.30}$$

where $\mathcal{L} : \overline{\text{co}}\,\mathcal{M} \times \omega\text{-}S(L^1) \to \mathcal{K}$ is the operator defined in (7.1).

From Theorem 7.2 it follows that the multivalued mapping $(u(\cdot), f(\cdot)) \to \mathcal{F}(\mathcal{L})(u, f)$ is weakly upper semicontinuous with convex weakly compact values. Define

$$Y = \Pi_+(T, H) \times \omega\text{-}L^1(T, H) \ \text{ and } \ Z = \overline{\text{co}}\,\mathcal{M} \times \omega\text{-}S(L^1) \subset Y. \tag{7.31}$$

The set Z is a convex compact subset of the space Y. Let

$$z(\cdot) = (u(\cdot), f(\cdot)), \tag{7.32}$$

$$\Phi(z) = \mathcal{L}(z) \times \mathcal{F}(\mathcal{L})(z), \quad z(\cdot) \in Z. \tag{7.33}$$

Considering the mapping $\mathcal{L}(u, f)$ as a multivalued mapping with singleton values and taking into account (7.5), (7.28), (7.6), we see that $\Phi(z) \subset Z$ for every$z(\cdot) \in Z$.

Thus, we can define the multivalued mapping $\Phi : Z \rightrightarrows Z$ with convex compact values, which is upper semicontinuous in the topology of the space

$$Y = \Pi_+(T, H) \times \omega\text{-}L^1(T, H).$$

Using Theorem 1 in [13], we infer that the mapping $z \mapsto \Phi(z)$ has a fixed point $z^*(\cdot) \in Z$, i.e. $z^*(\cdot) \in \Phi(z^*), z^*(\cdot) = (u^*(\cdot), f^*(\cdot))$. From this inclusion, (7.33), (7.30) we have

$$u^*(\cdot) = \mathcal{L}(u^*, f^*), \tag{7.34}$$

$$f^*(\cdot) \in \mathcal{F}(\mathcal{L})(u^*, f^*). \tag{7.35}$$

From (7.30), (7.34), (7.35) we obtain

$$f^*(\cdot) \in \mathcal{F}(u^*). \tag{7.36}$$

Then, according to (7.36), (7.27)

$$f^*(\cdot) \in g(u^*) + \mathcal{G}_{\tilde{V}}(u^*).$$

Consequently, there exists

$$f^*_V(\cdot) \in \mathcal{G}_{\tilde{V}}(u^*) \tag{7.37}$$

such that

$$f^*(\cdot) = g(u^*) + f_{\tilde{V}}(\cdot). \tag{7.38}$$

From (7.37), (7.38), (5.5), (5.18), (5.12), (5.19) we have the inclusion

$$f^*_U(t) = g(u^*)(t) \in U(t, u^*(t)) \ \text{ a.e.}, \tag{7.39}$$

$$f^*_V(t) \in V(t, u^*(t)) \ \text{ a.e.} \tag{7.40}$$

Let
$$x^*(\cdot) = \mathcal{L}(u^*, f^*) \tag{7.41}$$
be a solution of the inclusion (6.23). Since the inclusion (6.23) has a unique solution, from (7.34), (7.41) we have
$$x^*(\cdot) = u^*(\cdot). \tag{7.42}$$
Consequently,
$$x^*(\cdot) = \mathcal{L}(x^*, f^*). \tag{7.43}$$

Using (7.42), (7.43), (6.25) and (6.26), we obtain
$$x^*(0) = x_0, \;\; x^*(t) \in C(t, x^*(t)), \; t \in T, \tag{7.44}$$
$$-\frac{dx^*}{d\nu}(t) - f^*(t)\frac{d\lambda}{d\nu}(t) \in \mathcal{N}_{C(t,x^*(t))}(x^*(t)) \;\; \nu \;\text{ a.e.} \tag{7.45}$$

Then, from (7.38), (7.39), (7.40), (7.44), (7.45) and Definition 1.1 it follows that the pair $(x^*(f)(\cdot), f^*(\cdot))$, $x^*(f^*) = x^*(\cdot)$ is a solution to inclusion (1.4). The inequalities (1.15)–(1.19) follow from the inclusion $x^*(\cdot) \in \overline{\mathrm{co}}\,\mathcal{M}$, $f^*(\cdot) \in S(L^1)$ and (6.27), (6.11), (6.10), (6.13), (6.14), (7.5), (1.9), (1.10). The theorem is proven.

Corollary 1.3 follows from Theorem 1.2. To show this, invoking the inequality (1.2), one needs to take the measure
$$\mu(A) = \int_A |\dot{\beta}(t)| dt$$

Since the measures $\mu+\lambda$ and λ are absolutely continuously equivalent, as the measure ν we take the measure λ. Using the equality
$$\frac{dx}{d\nu}(t) = \frac{dx(t)}{dt} \;\;\text{a.e.},$$
where we have the usual derivative on the right-hand side, we arrive at the corollary's statements, as the inclusion $\frac{dx(t)}{dt}(\cdot) \in L^1(T, H)$ means that the function $t \mapsto x(t)$ is absolutely continuous.

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