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A Convex, Finite and Lower Semicontinuous Function with Empty Subdifferential

Gerd Wachsmuth
Institute of Mathematics, Brandenburgische Technische Universität Cottbus-Senftenberg, Cottbus, Germany
wachsmuth@b-tu.de

Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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We give an example of a convex, finite and lower semicontinuous function whose subdifferential is everywhere empty. This is possible since the function is defined on an incomplete normed space. The function serves as a universal counterexample to various statements in convex analysis in which completeness is required.

Keywords: Subdifferential, incomplete space, Fenchel duality, convex sum rule.

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1. Introduction

Some results in convex analysis require that the underlying space is complete and that the (convex) functions are lower semicontinuous. As examples, we mention

- the Brøndsted-Rockafellar theorem about the density of the domain of the subdifferential in the domain of the function, see [2, Theorem 2],
- maximal monotonicity of the subdifferential, see [8, Theorem A],
- Ekeland's variational principle, see [3, Theorem 1.1],
- strong Fenchel duality, see [9, Theorems 17, 18] and [7, Corollary 1],
- the formula for the convex conjugate of a sum, [1, Theorem (1.1)].

It is clear that some of these results are closely connected. The Brøndsted-Rockafellar theorem is usually proved via Ekeland's variational principle. Similarly, strong Fenchel duality is intimately related to the convex conjugate of a sum and to the sum rule for the subdifferential.

A natural question is whether the assumptions of completeness and lower semicontinuity in the above results are actually necessary. If one drops the lower semicontinuity, linear unbounded functionals often serve as a counterexample. In absence of completeness, counterexamples are typically harder to construct, although it is stated "It is easy to see that, in general, the conclusion of Theorem (1.1) fails if E is a (non-complete) normed space" in [1], but no concrete example is given.

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We are mainly interested in the question of non-emptiness of the subdifferential. Let X be a normed space and let $f\colon X \to (-\infty, \infty]$ be a convex function. We further assume that f is proper, i.e., that there exists $x \in X$ with $f(x) < \infty$. The subdifferential of f at $x \in X$ is defined via

$$\partial f(x) := \{x^\star \in X^\star \mid \forall y \in X : f(y) \geq f(x) + \langle x^\star, y - x\rangle\}.$$

For many examples, one can check that there always exists some $x \in X$ such that $\partial f(x) \neq \emptyset$. In fact, we already mentioned the Brøndsted-Rockafellar theorem, which ensures that

$$\operatorname{dom}(\partial f) := \{x \in X \mid \partial f(x) \neq \emptyset\}$$

is dense in

$$\operatorname{dom}(f) := \{x \in X \mid f(x) < \infty\},$$

whenever X is complete and f is lower semicontinuous. If f is an unbounded linear functional, it is clear that $\operatorname{dom}(\partial f) = \emptyset$ is not dense in $\operatorname{dom}(f) = X$. Moreover, in [2] an example of a proper, convex and lower semicontinuous function f defined on an incomplete space such that $\operatorname{dom}(\partial f) = \emptyset$ is given, by building upon an example by [4]. This construction, however, is quite involved. We are not aware of similar examples in the literature.

The example in [6], see also [5, Example 3.8], comes close, since it possesses an empty subdifferential on a dense subset of $\operatorname{dom}(f)$. However, this example is posed in the Hilbert space ℓ^2 and, therefore, the subdifferential cannot be empty at every point due to the Brøndsted-Rockafellar theorem.

Finally, we mention that the assumptions of completeness of X and lower semicontinuity of f can often be replaced by continuity at a single point of some involved function and, in this case, completeness of X is not necessary. A famous example is the validity of the sum rule

$$\partial f(x) + \partial g(x) = \partial(f + g)(x),$$

for convex functions $f, g\colon X \to (-\infty, \infty]$ under the Moreau-Rockafellar condition, i.e., whenever there exists a point $x_0 \in X$

$$\operatorname{cont}(f) \cap \operatorname{dom}(g) \neq \emptyset, \tag{1}$$

where $\operatorname{cont}(f) \subset \operatorname{dom}(f)$ is the set of all points at which f is continuous. We emphasize that this result does not require that X is complete or that any of the functions f and g is lower semicontinuous. To complete the picture, we mention that the sum rule also holds provided that

$$0 \in \operatorname{core}(\operatorname{dom}(f) - \operatorname{dom}(g)), \tag{2a}$$

$$X \text{ is complete and } f, g \text{ are lower semicontinuous} \tag{2b}$$

are satisfied. Here, $\operatorname{core}(A)$ is the algebraic interior of a set $A \subset X$. The condition (2a) is called the Rockafellar-Robinson condition and it is easy to check that it is implied by (1). However, (2b) is not needed for the sum rule if (1) is satisfied. Thus, (2) is not weaker than (1). A natural question is whether the sum-rule still holds provided that only (2a) is satisfied. Again, unbounded linear functionals show that the lower semicontinuity assumption cannot be dropped. The necessity of the completeness of X is slightly harder to verify. One possibility is to use a convex and lower semicontinuous function $f\colon X \to \mathbb{R}$ with an empty subdifferential and check

$$X^\star = \partial(f + \delta_x)(x) \neq \partial f(x) + \partial \delta_x(x) = \emptyset + X^\star = \emptyset$$

for some arbitrary $x \in X$. Due to $\operatorname{dom}(f) = X$, condition (2a) is satisfied. It remains to construct such a function f.

2. The function and its properties

Let us consider the linear space of real-valued, finite sequences

$$c_c := \{(x_n)_{n\in\mathbb{N}} \subset \mathbb{R} \mid \exists N \in \mathbb{N} : \forall n > N : x_n = 0\}.$$

An easy application of Baire's theorem yields that c_c equipped with any norm will be an incomplete space. We equip c_c with the norm of ℓ^2, i.e.,

$$\|x\| := \left(\sum_{n=1}^{\infty} x_n^2\right)^{1/2} \qquad \forall x \in c_c.$$

Note that the dual space of c_c (equipped with this norm) can be canonically identified with ℓ^2. We consider the function $f\colon c_c \to \mathbb{R}$ defined via

$$f(x) := \sum_{n=1}^{\infty} \frac{n^2}{2}\big(x_n - n^{-2}\big)^2 \qquad \forall x \in c_c.$$

This function has some very peculiar properties.

Theorem 2.1. *Consider the space c_c and the function $f\colon c_c \to \mathbb{R}$ as above.*

(a) *The function f is well defined, convex and lower semicontinuous.*

(b) *The function f is nowhere continuous.*

(c) *At every $x \in c_c$ the function f is directionally differentiable and*

$$f'(x; y) = \sum_{n=1}^{\infty} n^2\big(x_n - n^{-2}\big) y_n$$

is the directional derivative in direction $y \in c_c$. In particular, the directional derivative at x is an unbounded, linear functional.

(d) *At every $x \in c_c$, we have $\partial f(x) = \emptyset$.*

Proof. (a): The function f is well defined, since x_n is zero for large n and since $\sum_{n=1}^{\infty} n^{-2}$ converges. The convexity is clear. The lower semicontinuity follows from the Lemma of Fatou or by realizing that f is the supremum of the continuous function

$$x \mapsto \sum_{n=1}^{k} \frac{n^2}{2}\big(x_n - n^{-2}\big)^2, \quad k \in \mathbb{N}.$$

(b): For every $x \in c_c$ and $n \in \mathbb{N}$ large enough, we have

$$f(x + 2n^{-1}e_n) - f(x) = \frac{n^2}{2}\Big(\big(2n^{-1} - n^{-2}\big)^2 - \big(-n^{-2}\big)^2\Big) \geq \frac{n^2}{2}\big(n^{-2} - n^{-4}\big) \geq \frac{1}{4}.$$

Since $x + 2n^{-1}e_n \to x$, the function f cannot be continuous at the arbitrary point $x \in c_c$.

(c): The formula for the directional derivative is clear since x and y are finite sequences. Consequently, $f'(x;\cdot)$ is linear and $f'(x; e_n) = -1$ for large enough n. Thus, $f'(x;\cdot)$ is unbounded.

(d): From the well-known formula

$$\partial f(x) = \{z \in \ell^2 \mid \forall y \in X : \langle z, y\rangle \leq f'(x; y)\}$$

the emptiness follows, since $f'(x;\cdot)$ is linear and unbounded. □

We emphasize that the verification of the properties of f is very elementary.

The theorem already shows that the assertion of the Brøndsted-Rockafellar theorem is not valid for f, thus its completeness assumption is crucial. Similarly, one can check that the assertion of the Ekeland variational principle (which is at the heart of the Brøndsted-Rockafellar theorem) cannot hold, since this would give rise to a non-empty subdifferential of f at some point in c_c. Similarly, the subdifferential ∂f is, of course, not maximally monotone, i.e., the completeness of the space is necessary in [8, Theorem A]. Further, the function f cannot be reconstructed from its subdifferential, since for every linear and continuous functional ℓ, we have $\partial f = \partial(f + \ell)$, but f and $f + \ell$ do not differ by a constant. Again, completeness in [8, Theorem B] is crucial.

It is also well known, see, e.g., [9, Corollary 8B], that convex and lower semicontinuous functions on a Banach space are continuous in the interior of their domain. The function f demonstrates that this is not true in incomplete spaces.

Next, we check that the duality results are no longer valid. Indeed, we just use $g = \delta_{\{0\}}$, i.e., the indicator function of the origin. Then, the Fenchel dual of

$$\text{Minimize} \qquad f(x) + g(x) \qquad \text{with respect to } x \in c_c$$

is

$$\text{Maximize} \qquad -f^\star(y) - g^\star(-y) \qquad \text{with respect to } y \in \ell^2.$$

Note that $g^\star \equiv 0$. If the dual problem would possess a solution $y \in \ell^2$, we would have

$$0 \in \partial f^\star(y)$$

and, consequently, $y \in \partial f(0)$, but this is impossible. Thus, the dual problem does not have a solution. However, one can check that no duality gap occurs.

Similarly, one can check that

$$\partial(f + g)(0) = \ell^2 \neq \emptyset = \partial f(0) + \partial g(0),$$

i.e., the sum rule fails.

Finally,

$$(f+g)^\star = f^\star \oplus g^\star \equiv \frac{\pi^2}{12}$$

holds, where "$\oplus$" indicates infimal convolution, but the infimal convolution is not exact, since the function $f^\star$ does not possess a minimizer.

3. Dual pair with positive duality gap

We have seen that a very simple function g is sufficient to get a dual problem without a solution. We show that using an operator $A\colon c_c \to c_c$ yields a pair of problems with positive duality gap.

We define $A\colon c_c \to c_c$ via

$$(Ax)_n := \begin{cases} x_1 & \text{if } n = 1, \\ x_n - x_{n-1} & \text{if } n > 1 \end{cases}$$

for all $x \in c_c$ and $n \in \mathbb{N}$. It is clear that A is linear and the boundedness is easy to check, since

$$\|Ax\|^2 = \sum_{n=1}^\infty (Ax_n)^2 \le x_1^2 + \sum_{n=2}^\infty \big(2x_n^2 + 2x_{n-1}^2\big) \le 4\|x\|^2.$$

Thus $\|A\| \le 2$ and one can check that we actually have $\|A\| = 2$.

Further, let $g\colon c_c \to \mathbb{R}$ be the zero function. We start by investigating the primal problem

$$\text{Minimize} \quad f(Ax) + g(x) \qquad \text{with respect to } x \in c_c.$$

We check that 0 is the solution of this problem. In fact, for every $y \in c_c$, the directional derivative of $f \circ A$ at 0 in direction y equals

$$(f \circ A)'(0; y) = f'(0; Ay) = -y_1 + \sum_{n=2}^\infty -(y_n - y_{n-1}) = 0,$$

see Theorem 2.1(c). Since f is convex and $g \equiv 0$, it follows that 0 is a solution of the primal problem and the primal infimal value is $\pi^2/12$.

It is also interesting to note that this implies

$$\partial(f \circ A)(0) = \{0\} \ne A^\star \partial f(A0) = \emptyset,$$

i.e., the failure of the chain rule.

Now, we consider the dual problem

$$\text{Maximize} \qquad -f^\star(y) - g^\star(-A^\star y) \qquad \text{with respect to } y \in \ell^2.$$

Note that $g^\star = \delta_{\{0\}}$ and a short calculation shows

$$f^\star(y) = \sum_{n=1}^\infty \frac{1}{n^2}\Big(\frac{1}{2}y^2 + y\Big),$$

since we can argue coefficient-wise.

Since the operator A has a dense range, the operator $A^\star$ is injective (which can also be shown directly). Consequently, $y = 0$ is the only point with $g^\star(-A^\star y) < \infty$. Thus, this is the solution of the dual problem and the objective value is 0. Due to

$$0 < \frac{\pi^2}{12},$$

this pair of problems possesses a positive duality gap.

We mention that $\operatorname{dom} f = c_c$ implies

$$0 \in \operatorname{core}(A \operatorname{dom} g - \operatorname{dom} f).$$

Further, the functions f and g are lower semicontinuous. Only the non-completeness of c_c causes problems. This prevents [9, Theorem 18] from being applied.

There also does not exist a point $x_0 \in \operatorname{dom} g$ such that f is continuous at Ax_0, since f is not continuous at 0 and $\operatorname{dom} g = \{0\}$. Consequently, [9, (8.25)] is not satisfied.

4. Conclusion

Using elementary arguments, we have verified in Theorem 2.1 that the subdifferential of the function f is everywhere empty. Consequently, this function shows that the completeness assumption in many theorems of convex analysis cannot be dropped.

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