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A Remark on the Nullness of Boundaries of Uniformly Prox-Regular Sets in Banach Spaces

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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We prove that, in any separable uniformly smooth Banach space, the boundary of every a-convex set (in the Efimov-Stechkin sense) is Γ-null (in the Lindenstrauss-Preiss sense). Using well-known results, we obtain that the same is true for proximally smooth sets (in the sense of Clarke, Stern and Wolenski) and for uniformly prox-regular sets (in the sense of Poliquin, Rockafellar and Thibault) in any separable Banach space which is both uniformly convex and uniformly smooth.

Keywords: Proximally smooth set, uniformly prox-regular set, a-convex set, Γ-null set, uniformly smooth Banach space, uniformly convex Banach space.

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1. Introduction

The aim of the present note is to give, in an interesting special case, the positive answer to the following (very unprecise) question.

Question. Let X be a Banach space. Is it true that the boundary of each closed generalized convex (in some sense) set $A \subset X$ is null (in a sense which generalizes Lebesgue nullness in $\mathbb{R}^n$) ?

This question is well-motivated since for *convex* sets A the answer is positive not only in the case of a finite-dimensional X but also for some infinite-dimensional spaces X and some kinds of "nullness".

Namely, if we consider in X *Haar nullness* (in Christensen's sense), then the answer for convex sets A is positive if X is a separable reflexive Banach space. This rather deep interesting result was proved in [11] in the case when A is nowhere dense and the case $\operatorname{int} A \neq \emptyset$ is easy, see e.g. the argument using the Minkowski functional of K on p. 1794 of [10]. (For a counterexample in any separable non-reflexive space X see [12, Theorem 6], cf. [2, Theorem 6.12] and the note after it.)

Further, if we consider in X Γ-*nullness* (in Lindenstrauss-Preiss [8] sense), then the answer is positive for convex sets if X is separable, see [16, Proposition 5.1].

On the other hand, if $X = \ell_2$ and we consider in X Gaussian nullness (or, equivalently, Aronszajn nullness), which is stronger than Haar nullness (cf. [2, pp. 141, 142]), then the answer for convex sets is negative. It follows e.g. from [2, Prop. 6.24].

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(Note that Γ-nullness is uncomparable both with Haar nullness and with Gaussian nullness in any infinite-dimensional separable Banach space; this follows e.g. from [9, Example 5.4.11].)

We are interested in "generalized convex sets" which are, in $X = \mathbb{R}^n$, more general than Federer's sets of positive reach. Using the method of [16], we prove rather easily Theorem 3.3 which gives a positive answer to the question above in the case when X is a separable uniformly smooth Banach space, "nullness" is Γ-nullness and $A \subset X$ is an a-convex set (in the sense of Efimov-Stechkin). Using well-known results, we obtain (see Corollary 3.4) that the same is true for uniformly prox-regular (or, equivalently, proximally smooth) sets in any separable Banach space which is both uniformly convex and uniformly smooth. For definitions of above mentioned types of generalized convex sets see the Preliminaries.

I do not know the answer to the question above in the interesting case when $X = \ell_2$, "nullness" is Haar nullness and $A \subset X$ is a uniformly prox-regular (or, equivalently, proximally smooth) set.

2. Preliminaries

2.1. Basic notation

The norm in any considered (always real) Banach space X is denoted by $|\cdot|$. If $A \subset X$, the closure, the interior and the boundary of A are denoted by $\overline{A}$, $\operatorname{int} A$ and ∂A, respectively. For $A \subset X$, we consider also the *distance function* $d_A(x) = \operatorname{dist}(x, A) := \inf\{|x - a| : \ a \in A\},\ x \in X$, and the *metric projection*

$$P_A(x) = \{a \in A : \ |x - a| = d_A(x)\}, \quad x \in X.$$

We say that P_A is *continuous* on a set $S \subset X$ if P_A is single-valued on S and the corresponding mapping $p : S \to A$ ($P_A(x) = \{p(x)\},\ x \in S$) is continuous. The open ball centered at $c \in X$ with radius $r > 0$ and the open r-neigbourhood of $A \subset X$ are defined as

$$B(c, r) = \{x \in X : |x - c| < r\} \quad \text{and} \quad B(A, r) = \{x \in X : d_A(x) < r\}.$$

2.2. Sets of positive reach and related sets in Banach spaces

H. Federer in his deep paper [6] defined, studied and applied *sets of positive reach* in $\mathbb{R}^n$ which generalize convex sets. A closed set $A \subset \mathbb{R}^n$ has positive reach if:

(PR) The metric projection P_A is single-valued on $B(A, r)$ for some $r > 0$.

Note that property (PR) is in $\mathbb{R}^n$ equivalent (cf. e.g. [6, Theorem 4.8, (4)]) with the following "formally stronger" condition:

(PR)* The metric projection P_A is *continuous* on $B(A, r)$ for some $r > 0$.

Both (PR) and (PR)* can be considered in each Banach space, but it seems that it is not known whether (PR) implies (PR)* in ℓ_2 (this question is closely related to the well-known open "Chebyshev set problem", cf. [14, p. 524]).

F. H. Clarke, R. J. Stern and P. R. Wolenski in [4] defined *proximally smooth* subsets of a Hilbert space X as closed sets $A \subset X$ with the following property:

(PS) The distance function d_A is continuously differentiable on $B(A,r)\setminus A$ for some $r>0$.

Among other things, they proved that (PS) implies (PR).

R. A. Poliquin, R. T. Rockafellar and L. Thibault in [14] defined (via normal vectors) the notions of *prox-regular* and *uniformly prox-regular* subsets of a Hilbert space. These two types of sets were subsequently defined in [3] in Banach spaces which are both uniformly convex with modulus of convexity of power type and uniformly smooth with modulus of smoothness of power type and in [15] in Banach spaces which are both uniformly convex and uniformly smooth.

For definitions of Banach spaces X which are uniformly convex (resp. uniformly convex with modulus of convexity of power type, resp. uniformly smooth, resp. uniformly smooth with modulus of smoothness of power type) see e.g. [3]. We recall here only the definition of uniformly smooth spaces which is needed in the proof of Theorem 3.3.

Definition 2.1. A Banach space is called *uniformly smooth* if its modulus of smoothness $\rho_X(\tau) := \sup\left\{\frac{|x+y|}{2}+\frac{|x-y|}{2}-1 :\ |x|=1, |y|\le\tau\right\}$ satisfies $\lim\limits_{\tau\to 0+}\frac{\rho_X(\tau)}{\tau}=0$.

We will use the following fact.

Fact 2.2. *Let X be a Banach space which is both uniformly convex and uniformly smooth. Then, for a closed subset $A\subset X$, the uniform prox-regularity, the proximal smoothness and property* (PR)* *are pairwise equivalent properties.*

This fact immediately follows from [15, Theorem 18.102]. Its versions for less general Banach spaces are contained in [14] and in [3]. The equivalence (PS)⇔(PR)* follows easily from [1, Theorem 2.4].

Finally recall that a subset A of a Banach space is called *a-convex* ($a>0$), see [5] or [17, Definition 1.1], if $X\setminus A$ is a union of open balls with radius a.

Remark 2.3. (1) a-convex sets are called *weakly convex sets in the sense of Efimov and Stechkin with constant a* in [7] and [1]. Note also that the notion of a-convex sets appears already in [13].

(2) Obviously, each a-convex set is closed. Further, it is easy to show (see [7, Lemma 3.2]) that each a-convex set is $\tilde{a}$-convex for each $0<\tilde{a}<a$. □

Using [1, Theorem 2.4, (2) ⇒ (1)] and [1, Theorem 2.5], we immediately obtain the following:

Fact 2.4. *Let X be both uniformly convex and uniformly smooth and let $A\subset X$ be a closed proximally smooth set. Then A is a-convex for some $a>0$.*

3. New results

In the following, we will work, for $m\in\mathbb{N}$ and a fixed Banach space X, with the m-dimensional Hausdorff measure $\mathcal{H}^m$ on X. We will use only the facts that $\mathcal{H}^m$ is a Borel meeasure on X invariant with respect to isometries and that $0<\mathcal{H}^m(B)<\infty$ for each open ball B in every m-dimensinal subspace V of X.

The following definition is a slight reformulation of the basic part of [16, Def. 3.2].

Definition 3.1. Let $S \subset X$ be a subset of the Banach space X, let $s \in S$ and $\lambda \in [0, 1)$. We say that S is *P_λ-small at the point $s \in S$* if the following property holds: For every finite dimensional subspace $V \subset X$ there are sequences $(z_k)_{k\in\mathbb{N}}$ and $(r_k)_{k\in\mathbb{N}}$ of points of X and positive reals, respectively, such that

(1) $r_k \searrow 0$;

(2) $|z_k - s| = o(r_k)$, $k \to \infty$, and

(3) for every k, $\mathcal{H}^m\big(B(z_k, r_k) \cap (z_k + V) \cap S\big) \le \lambda\, \mathcal{H}^m\big(B(z_k, r_k) \cap (z_k + V)\big)$, where $m = \dim V \ge 1$.

Our results are based on the following fact which is an immediate consequence of [16, Proposition 3.6].

Fact 3.2. *Let X be a separable Banach space, $S \subset X$ a closed set and $0 \le \lambda < 1$. If S is P_λ-small at every point $s \in S$, then S is Γ-null.*

Theorem 3.3. *Let X be a separable uniformly smooth Banach space, $a > 0$ and let $A \subset X$ be an a-convex set. Then the boundary of A is Γ-null.*

Proof. By Remark 2.3(2), we can suppose that $0 < a < 1$. The set A is closed by Remark 2.3(2) and so $S := \partial A \subset A$. Since S is closed, by Fact 3.2 it is sufficient to prove that

$$S \text{ is } P_{1/2} - \text{small at every } s \in S. \tag{1}$$

So fix an arbitrary $s \in S$ and an arbitrary finite dimensional subspace $V \subset X$ with $\dim V =: m \ge 1$. We define sequences $(z_k)_{k\in\mathbb{N}}$ and $(r_k)_{k\in\mathbb{N}}$ (from Definition 3.1) as follows: By Definition 2.1, we can easily define inductively a sequence

$$\frac{a}{2} > \tau_1 > \tau_2 > .... > 0 \tag{2}$$

such that

$$\tau_k \to 0 \quad \text{and} \quad \rho_X(\tau_k) \le \frac{\tau_k}{2k},\ k \in \mathbb{N}. \tag{3}$$

Now consider an arbitrary $k \in \mathbb{N}$. As $s \in S = \partial A$, we can choose $w \in X \setminus A$ with $|s - w| < \tau_k/k$. Since A is an a-convex set, we can choose $c \in X$ such that $w \in B(c, a) \subset X \setminus A$. By the definitions of w and c we obtain

$$a \le |s - c| \le |c - w| + |s - w| < a + \frac{\tau_k}{k}. \tag{4}$$

Set

$$x := \frac{s - c}{|s - c|} \tag{5}$$

and

$$z_k := c + \Big(a - \frac{\tau_k}{k}\Big)x, \quad r_k := \frac{a\tau_k}{2}. \tag{6}$$

Then $r_k \searrow 0$ and thus condition (1) of Definition 3.1 holds. By (4) we have

$$|s - z_k| = |s - c| - |z_k - c| = |s - c| - \Big(a - \frac{\tau_k}{k}\Big) < 2\frac{\tau_k}{k} = o(r_k),\ \ k \to \infty, \tag{7}$$

and so also condition (2) of Definition 3.1 holds.

Thus, to finish the proof of theorem, it is sufficient to show that condition (3) of Definition 3.1 holds with $\lambda = 1/2$, i.e. that

$$\mathcal{H}^m\big(B(z_k, r_k) \cap (z_k + V) \cap S\big) \leq \frac{1}{2}\, \mathcal{H}^m\big(B(z_k, r_k) \cap (z_k + V)\big). \tag{8}$$

Set $M := B(z_k, r_k) \cap (z_k + V)$ and let $\sigma : M \to M$ be the restriction of the symmetry map with respect to the point z_k, i.e.

$$\sigma(t) = 2z_k - t, \quad t \in M.$$

Observe that, to prove (8), it is sufficient to show that

$$M = (M \setminus S) \cup \sigma(M \setminus S). \tag{9}$$

Indeed, since σ is the restriction of an isometry on X, we have

$$\mathcal{H}^m(\sigma(M \setminus S)) = \mathcal{H}^m(M \setminus S)$$

and so (9) implies $\mathcal{H}^m(M) \leq 2\mathcal{H}^m(M \setminus S)$. Consequently $\mathcal{H}^m(M \setminus S) \geq \frac{1}{2}\mathcal{H}^m(M)$ and since $M \setminus S$ is clearly $\mathcal{H}^m$-measurable, (8) follows.

To prove (9), suppose to the contrary that a point $m \in M \setminus ((M \setminus S) \cup \sigma(M \setminus S))$ exists. Then $m = z_k + \tilde{y}$, where $\tilde{y} \in W$, $|\tilde{y}| < r_k$ and both $m = z_k + \tilde{y} \in S$ and $\sigma(m) = z_k - \tilde{y} \in S$. Since $B(c, a) \cap S = \emptyset$, we have

$$|z_k + \tilde{y} - c| \geq a \quad \text{ and } \quad |z_k - \tilde{y} - c| \geq a. \tag{10}$$

Set
$$y := \frac{\tilde{y}}{|z_k - c|}.$$

Using (6) and (2) we obtain

$$|z_k - c| = a - \frac{\tau_k}{k} \geq \frac{a}{2}, \qquad |\tilde{y}| \leq r_k = \frac{a\tau_k}{2}$$

and consequently
$$|y| \leq \tau_k. \tag{11}$$

Further, using the definitions of x and z_k we obtain $|x| = 1$ and $x = (z_k - c)/|z_k - c|$. Using also (10) and $a < 1$ we obtain

$$|x \pm y| = \left| \frac{z_k - c}{|z_k - c|} \pm \frac{\tilde{y}}{|z_k - c|} \right| \geq \frac{a}{|z_k - c|} = \frac{a}{a - \frac{\tau_k}{k}} \geq 1 + \frac{\tau_k}{k},$$

and consequently
$$\frac{|x + y|}{2} + \frac{|x - y|}{2} - 1 \geq \frac{\tau_k}{k}.$$

So, using also $|x| = 1$ and (11), we obtain $\rho_X(\tau_k) \geq \tau_k/k$ which contradicts (3). $\square$

Using Theorem 3.3, Fact 2.4 and Fact 2.2, we immediately obtain the following result.

Corollary 3.4. *Let X be a separable Banach space which is both uniformly convex and uniformly smooth. Let $A \subset X$ be a closed uniformly prox-regular (or, equivalently, proximally smooth) set. Then the boundary of A is Γ-null.*

References

[1] M. V. Balashov, G. E. Ivanov: *Weakly convex and proximally smooth sets in Banach spaces*, Izv. Math. 73 (2009) 455–499.

[2] Y. Benyamini, J. Lindenstrauss: *Geometric Nonlinear Functional Analysis, Vol. 1*, AMS Colloquium Publications Vol. 48, American Mathematical Society, Providence (2000).

[3] F. Bernard, L. Thibault, N. Zlateva: *Characterizations of prox-regular sets in uniformly convex Banach spaces*, J. Convex Analysis 13 (2006) 525–559.

[4] F. H. Clarke, R. J. Stern, P. R. Wolenski: *Proximal smoothness and the lower-C^2 property*, J. Convex Analysis 2 (1995) 117–144.

[5] N. V. Efimov, S. B. Stečkin: *Support properties of sets in Banach spaces and Čebyšev sets (Russian)*, Dokl. Akad. Nauk SSSR 127 (1959) 254–257.

[6] H. Federer: *Curvature measures*, Trans. Amer. Math. Soc. 93 (1959) 418–491.

[7] G. E. Ivanov: *Sets that are weakly convex in the sense of Vial and Efimov-Stechkin (Russian)*, Izv. Ross. Akad. Nauk Ser. Mat. 69/6 (2005) 35–60; English translation in Izv. Math. 69/6 (2005) 1113–1135.

[8] J. Lindenstrauss, D. Preiss: *On Fréchet differentiability of Lipschitz maps between Banach spaces*, Annals Math. 157 (2003) 257–288.

[9] J. Lindenstrauss, D. Preiss, J. Tišer: *Fréchet Differentiability of Lipschitz Maps and Porous Sets in Banach Spaces*, Princeton University Press, Princeton (2012).

[10] E. Matoušková: *Convexity and Haar null sets*, Proc. Amer. Math. Soc. 125 (1997) 1793–1799.

[11] E. Matoušková: *Translating finite sets into convex sets*, Bull. London Math. Soc. 33 (2001) 711–714.

[12] E. Matoušková, Ch. Stegall: *A characterization of reflexive Banach spaces*, Proc. Amer. Math. Soc. 124 (1996) 1083–1090.

[13] J. Perkal: *Sur les ensembles ε-convexes*, Colloq. Math. 4 (1956) 1–10.

[14] R. A. Poliquin, R. T. Rockafellar, L. Thibault: *Local differentiability of distance functions*, Trans. Amer. Math. Soc. 352 (2000) 5231–5249.

[15] L. Thibault: *Unilateral Variational Analysis in Banach Spaces. II: Special Classes of Functions and Sets*, World Scientific, Singapore (2023).

[16] J. Tišer, L. Zajíček: *A criterion of Γ-nullness and differentiability of convex and quasiconvex functions*, Studia Math. 227 (2015) 149–164.

[17] L. P. Vlasov: *Approximative properties of sets in normed linear spaces (Russian)*, Uspehi Mat. Nauk 28/6 (1973) 3–66; English translation in Russian Math. Surveys 28/6 (1973) 1–66.