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Approximate Fixed Points for a Set-Valued Quasi-Contraction

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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We study an iterative process induced by a set-valued quasi-contraction in a complete metric space and show it generates approximate fixed points under the presence of small computational errors.

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1. Introduction

For more than sixty years, the fixed point theory of nonlinear operators is an important area of nonlinear analysis. One of its main topics is the study of the existence of fixed points of nonexpansive and contractive maps. See [1]-[23] and the references mentioned therein. Many existence results are collected in [7, 8, 18]. This topic is well developed for single-valued mappings [1, 3, 5, 9, 10, 13, 17] as well as for set-valued mappings [4, 15, 16, 18, 19]. In the present paper we study an iterative process induced by a set-valued quasi-contraction in a complete metric space and show it generates approximate fixed points under the presence of small computational errors.

Let (X, ρ) be a complete metric space. For each $x \in X$ and each $r > 0$, set

$$B(x,r) = \{y \in X : \ \rho(x,y) \le r\}, \ B^0(x,r) = \{y \in X : \ \rho(x,y) < r\}.$$

For each $x \in X$ and each $A \subset X$ set

$$\rho(x,A) = \inf\{\rho(x,y) : \ y \in A\}.$$

Note that we assume that the infimum over empty set is ∞. For each $s \in R^1$ set

$$\lfloor s \rfloor = \max\{z : \ z \text{ is an integer and } z \le s\}.$$

For each pair of nonempty sets $A, B \subset X$, set

$$H(A,B) = \max\{\sup\{\rho(x,B) : \ x \in A\}, \ \sup\{\rho(y,A) : \ y \in B\}\}.$$

Assume that $T : X \to 2^X \setminus \{\emptyset\}$, $T(x)$ is closed for each $x \in X$, $c \in [0,1)$ and that for each $x, y \in X$,

$$H(T(x),T(y)) \le c\rho(x,y).$$

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The mapping T is called a *set-valued strict contraction.* We are interested to solve the problem

$$\text{Find } z \in X \text{ such that } z \in T(z).$$

By Nadler's classical theorem [14] the solution of this problem exists.

In practice we can obtain only an approximate solution of this problem $z \in X$ such that $\rho(z, T(z))$ is small.

More precisely, in order to meet our goal we use the following algorithm

Initialization: select an arbitrary point $x_0 \in X$ and a small positive constant δ.

Iterative step: given a current iteration point x_n calculate the next iteration point x_{n+1} such that

$$B(x_{n+1}, \delta) \cap \{\xi \in T(x_n) \ : \ \rho(\xi, x_n) \le \delta + \rho(x_n, T(x_n))\} \ne \emptyset.$$

Clearly, at each iterative step x_{n+1} is an approximate solution of the problem

$$\rho(\xi, x_n) \to \min, \ \xi \in T(x_n).$$

Since the space X is not compact a solution of the problem above does not exist in general. It was shown in [4] (see also [18]) that this algorithm generates approximate fixed points of T. Note that given current iteration point x_n in order to obtain x_{n+1} we should only know the set $T(x_n)$ or even its good approximation. In the present paper we generalize the results of [4] for a large class of set-valued mappings which contains strict contractions and also cyclical mappings (see Example 3.4).

2. The first main result

Let $T : X \to 2^X \setminus \{\emptyset\}$, $T(x)$ be closed for each $x \in X$, $c \in [0, 1)$, $X_0 \subset X$ be a nonempty closed set and let $T(X_0) \subset X_0$. Assume that the following assumption holds.

(A) For each $x \in X_0$ and each $y \in X$, $H(T(x), T(y)) \le c\rho(x, y)$.

By Nadler's theorem [14], there exists $x_* \in X_0$ such that $x_* \in T(x_*)$.

Theorem 2.1. *Let* $\theta \in X$, $\epsilon \in (0, 1)$, $M > 0$,

$$B(\theta, M) \cap X_0 \ne \emptyset, \tag{1}$$

$$M_0 > (1-c)^{-1}(2M+2)(10 + \rho(\theta, T(\theta))), \tag{2}$$

$$\delta = 360^{-1}\epsilon(1-c)^2, \tag{3}$$

$$n_0 = \lfloor 4M_0(1-c)^{-1}\epsilon^{-1} \rfloor + \lfloor 72(2M+1)(1-c)^{-2}\epsilon^{-1} \rfloor + 1. \tag{4}$$

Then for each sequence $\{x_n\}_{n=0}^{\infty} \subset X$ *such that* $\rho(x_0, \theta) \le M$ *and that for each integer* $n \ge 0$,

$$B(x_{n+1}, \delta) \cap \{\xi \in T(x_n) : \ \rho(\xi, x_n) \le \delta + \rho(x_n, T(x_n))\} \ne \emptyset \tag{5}$$

the following relations hold for each integer $n \ge n_0$,

$$\rho(x_n, x_{n+1}) < \epsilon \ \ \textit{and} \ \ B(x_n, \epsilon) \cap T(x_n) \ne \emptyset.$$

Proof. Set
$$\delta_1 = 36^{-1}(1-c)\epsilon. \tag{6}$$
By (3) and (6),
$$\delta \le (1-c)\delta_1/10. \tag{7}$$
Set
$$n_1 = \lfloor 2(2M+1)\delta_1^{-1}(1-c)^{-1}\rfloor + 1, \tag{8}$$
$$n_2 = \lfloor 4M_0(1-c)^{-1}\epsilon^{-1}\rfloor. \tag{9}$$
By (4), (8) and (9),
$$n_0 = n_1 + n_2. \tag{10}$$
Assume that
$$\{x_n\}_{n=0}^{\infty} \subset X, \tag{11}$$
$$\rho(x_0, \theta) \le M \tag{12}$$
and that (5) holds for each integer $n \ge 0$. By (1), there exists
$$\theta_1 \in B(\theta, M) \cap X_0. \tag{13}$$
In view of (12) and (13),
$$\rho(x_0, \theta_1) \le \rho(x_0, \theta) + \rho(\theta, \theta_1) \le 2M. \tag{14}$$
Assumption (A) and equations (12), (13) and (14) imply that
$$\begin{aligned}\rho(x_0, T(x_0)) &\le \rho(x_0, \theta_1) + \rho(\theta_1, T(\theta_1)) + H(T(\theta_1), T(x_0))\\ &\le 2\rho(x_0, \theta_1) + \rho(\theta_1, T(\theta_1))\\ &\le 4M + \rho(\theta_1, \theta) + \rho(\theta, T(\theta)) + H(T(\theta), T(\theta_1))\\ &\le 4M + \rho(\theta, T(\theta)) + 2\rho(\theta, \theta_1) \le 6M + \rho(\theta, T(\theta)).\end{aligned} \tag{15}$$
By (5), (7) and (15),
$$\rho(x_0, x_1) \le \rho(x_0, T(x_0)) + 2\delta \le 6M + \rho(\theta, T(\theta)) + 1. \tag{16}$$
In view of (5), for each integer $n \ge 0$, there exists
$$\xi_n \in B(x_{n+1}, \delta) \cap T(x_n) \tag{17}$$
such that
$$\rho(x_n, \xi_n) \le \delta + \rho(x_n, T(x_n)). \tag{18}$$
For each integer $n \ge 0$ there exists
$$u_n \in X_0 \tag{19}$$
such that
$$\rho(u_n, x_n) \le \rho(x_n, X_0) + \delta. \tag{20}$$
Recall that
$$T(X_0) \subset X_0. \tag{21}$$
Let $n \ge 0$ be an integer. By (A), (17), (20) and (21),
$$\begin{aligned}\rho(u_{n+1}, x_{n+1}) &\le \rho(x_{n+1}, X_0) + \delta \le \rho(x_{n+1}, \xi_n) + \rho(\xi_n, X_0) + \delta\\ &\le 2\delta + \rho(\xi_n, T(u_n)) \le 2\delta + H(T(x_n), T(u_n)) \le 2\delta + c\rho(x_n, u_n).\end{aligned} \tag{22}$$

Assumption (A) and equations (17) and (19) imply that

$$\xi_n \in T(x_n),\ \rho(T(u_n), T(x_n)) \le c\rho(x_n, u_n). \tag{23}$$

In view of (23), there exists
$$v_n \in T(u_n) \tag{24}$$
such that
$$\rho(v_n, \xi_n) \le c\rho(x_n, u_n) + \delta. \tag{25}$$

It follows from (17), (22) and (25) that

$$\begin{aligned}\rho(v_n, u_{n+1}) &\le \rho(v_n, \xi_n) + \rho(\xi_n, u_{n+1}) \\ &\le c\rho(x_n, u_n) + \delta + \rho(\xi_n, x_{n+1}) + \rho(x_{n+1}, u_{n+1}) \\ &\le 2c\rho(x_n, u_n) + 4\delta. \end{aligned} \tag{26}$$

By (18) and (25),

$$\begin{aligned}\rho(u_n, v_n) &\le \rho(x_n, u_n) + \rho(x_n, \xi_n) + \rho(\xi_n, v_n) \\ &\le \rho(u_n, x_n) + \rho(x_n, T(x_n)) + \delta + c\rho(x_n, u_n) + \delta \\ &= (1+c)\rho(x_n, u_n) + \rho(x_n, T(x_n)) + 2\delta. \end{aligned} \tag{27}$$

Assumption (A) implies that

$$\begin{aligned}\rho(x_n, T(x_n)) &\le \rho(x_n, u_n) + \rho(u_n, T(u_n)) + H(T(u_n), T(x_n)) \\ &\le \rho(u_n, T(u_n)) + (1+c)\rho(x_n, u_n). \end{aligned} \tag{28}$$

In view of (27) and (28),

$$\begin{aligned}\rho(u_n, v_n) &\le (1+c)\rho(u_n, x_n) + 2\delta + \rho(x_n, T(x_n)) \\ &\le (1+c)\rho(x_n, u_n) + 2\delta + (1+c)\rho(x_n, u_n) + \rho(u_n, T(u_n)) \\ &\le 2(1+c)\rho(x_n, u_n) + 2\delta + \rho(u_n, T(u_n)). \end{aligned} \tag{29}$$

It follows from (24), (26) and (29) that

$$\begin{gathered} v_n \in T(u_n),\ \rho(v_n, u_{n+1}) \le 2c\rho(x_n, u_n) + 4\delta, \\ \rho(u_n, v_n) \le 2(1+c)\rho(x_n, u_n) + 2\delta + \rho(u_n, T(u_n)). \end{gathered} \tag{30}$$

Let $n \ge 0$ be an integer. Assumption (A) and equations (19) and (30) imply that

$$\begin{aligned} &\rho(u_{n+1}, u_{n+2}) \le \rho(u_{n+1}, v_{n+1}) + \rho(v_{n+1}, u_{n+2}) \\ &\le \rho(u_{n+1}, v_{n+1}) + 2c\rho(x_{n+1}, u_{n+1}) + 4\delta \\ &\le \rho(u_{n+1}, T(u_{n+1})) + 2\delta + 2(1+c)\rho(u_{n+1}, x_{n+1}) + 2c\rho(x_{n+1}, u_{n+1}) + 4\delta \\ &\le \rho(u_{n+1}, T(u_{n+1})) + 2\delta + 2(1+c)\rho(u_{n+1}, x_{n+1}) + 2c\rho(x_{n+1}, u_{n+1}) + 4\delta \\ &\le \rho(u_{n+1}, v_n) + \rho(v_n, T(u_{n+1})) + (2+4c)\rho(u_{n+1}, x_{n+1}) + 6\delta \\ &\le \rho(v_n, T(u_{n+1})) + 2c\rho(x_n, u_n) + 10\delta + \rho(u_{n+1}, x_{n+1})(2+4c) \\ &\le H(T(u_n), T(u_{n+1})) + 2c\rho(u_n, x_n) + (2+4c)\rho(x_{n+1}, u_{n+1}) + 10\delta \\ &\le c\rho(u_n, u_{n+1}) + 2c\rho(u_n, x_n) + (2+4c)\rho(x_{n+1}, u_{n+1}) + 10\delta. \end{aligned} \tag{31}$$

We show that the following claim is true.

(A1) There exists an integer $k \in \{0, \dots, n_1\}$ such that

$$\rho(x_i, u_i) \le \delta_1 \text{ for each integer } i \ge k_1 \tag{32}$$

and for each integer i satisfying $0 \le i < k_1$,

$$\rho(x_i, u_i) > \delta_1 \tag{33}$$

and
$$\rho(x_{i+1}, u_{i+1}) \le \rho(x_i, u_i) - (1-c)\delta_1/2. \tag{34}$$

Assume that $i \ge 0$ is an integer and $\rho(x_i, u_i) > \delta_1$.

By (7) and (22), $\rho(x_{i+1}, u_{i+1}) \le 2\delta + c\rho(x_i, u_i)$

and
$$\begin{aligned}\rho(x_i, u_i) - \rho(x_{i+1}, u_{+1}) &\ge (1-c)\rho(x_i, u_i) - 2\delta \\ &\ge (1-c)\delta_1 - 2\delta \ge (1-c)\delta_1/2.\end{aligned}$$

Thus we have shown that the following property holds:

(i) if $i \ge 0$ is an integer and $\rho(x_i, u_i) > \delta_1$, then

$$\rho(x_i, u_i) - \rho(x_{i+1}, u_{i+1}) \ge (1-c)\delta_1/2.$$

Assume that $i \ge 0$ is an integer and $\rho(x_i, u_i) \le \delta_1$.

By (7) and (22), $\rho(x_{i+1}, u_{i+1}) \le 2\delta + c\rho(x_i, u_i) \le 2\delta + c\delta_1 \le \delta_1$.

Thus the following property holds:

(ii) if $i \ge 0$ is an integer and $\rho(x_i, u_i) \le \delta_1$, then $\rho(x_{i+1}, u_{i+1}) \le \delta_1$.

Assume that $k \ge 0$ is an integer and that

$$\rho(x_k, u_k) > \delta_1. \tag{35}$$

Property (ii) implies that

$$\rho(x_i, u_i) > \delta_1 \text{ for each } i \in \{0, \dots, k\}. \tag{36}$$

Property (i) and (36) imply that for each $i \in \{0, \dots, k\}$,

$$\rho(x_i, u_i) - \rho(x_{i+1}, u_{i+1}) \ge (1-c)\delta_1/2. \tag{37}$$

By (12), (13) and (20),

$$\rho(u_0, x_0) < \rho(x_0, X_0) + 1 \le \rho(x_0, \theta) + 1 \le 2M + 1. \tag{38}$$

In view of (8), (36) and (38),

$$\begin{aligned}2M + 1 \ge \rho(x_0, u_0) &\ge \rho(x_0, u_0) - \rho(x_{k+1}, u_{k+1}) \\ &= \sum_{i=0}^{k} (\rho(x_i, u_i) - \rho(x_{i+1}, u_{i+1})) \ge (k+1)(1-c)\delta_1/2\end{aligned}$$

and
$$k \le 2(2M+1)(1-c)^{-1}\delta_1^{-1} - 1 \ge n_1 - 1.$$

This implies that claim (A1) holds. In view of (19),

$$\{u_i\}_{i=k}^{\infty} \subset X_0.$$

Claim (A1), (7) and (30) imply that for each integer $i \geq n_1$,

$$v_i \in T(u_i),\ \rho(v_i, u_{i+1}) \leq 2c\rho(x_i, u_i) + 4\delta \leq 2c\delta_1 + 4\delta \leq 2\delta_1, \tag{39}$$

$$\rho(u_i, v_i) \leq \rho(u_i, T(u_i)) + 2\delta + 2(1+c)\delta_1 \leq \rho(u_i, T(u_i)) + 6\delta_1. \tag{40}$$

By (12), (13), (20), (33) and (34), for each $i \in \{0, \dots, k\} \setminus \{k\}$,

$$\rho(x_{i+1}, u_{i+1}) \leq \rho(x_i, u_i) \leq \rho(x_0, u_0) \leq \rho(x_0, X_0) + 1$$
$$\leq \rho(x_0, \theta_1) + 1 \leq 2M + 1$$

and

$$\rho(x_i, u_i) \leq 2M + 1,\ i = 0, \dots, k. \tag{41}$$

Claim (A1) and equations (31) and (41) imply that

$$\rho(u_{n+1}, u_{n+2}) \leq c\rho(u_n, u_{n+1}) + 8(M+2). \tag{42}$$

It follows from assumption (A) and equations (30) and (41) that

$$\begin{aligned}
\rho(u_0, u_1) &\leq \rho(u_0, v_0) + \rho(v_0, u_1) \\
&\leq \rho(u_0, v_0) + 2c\rho(x_0, u_0) + 4\delta \\
&\leq \rho(u_0, T(u_0)) + 2\delta + 2(1+c)\rho(u_0, x_0) + 2c\rho(x_0, u_0) + 4\delta \\
&\leq \rho(u_0, T(u_0)) + 6(2M+2) \\
&\leq 6(2M+2) + \rho(u_0, \theta) + \rho(\theta, T(\theta)) + H(T(\theta), T(u_0)) \\
&\leq 6(2M+2) + 2\rho(u_0, \theta) + \rho(\theta, T(\theta)) \\
&\leq 10(2M+2) + \rho(\theta, T(\theta)) < M_0.
\end{aligned}$$

We show by induction that for each integer $i \geq 0$,

$$\rho(u_i, u_{i+1}) \leq M_0. \tag{43}$$

We already showed that for $i = 0$ (43) holds. Assume that $i \geq 0$ is an integer and (43) is true. By (2), (42) and (43),

$$\rho(u_i, u_{i+1}) \leq cM_0 + 8(M+2) \leq M_0.$$

Thus our assumption holds for $i + 1$ too. Therefore we showed by induction that (43) holds for each integer $i \geq 0$. Claim (A1) and equations (7) and (31) imply that

$$\rho(u_{i+1}, u_{i+2}) \leq c\rho(u_i, u_{i+1}) + 8\delta_1 + 10\delta \leq c\rho(u_i, u_{i+1}) + 9\delta_1. \tag{44}$$

Assume that $i \geq k$ is an integer and that $\rho(u_i, u_{i+1}) \leq \epsilon/2$.

By (6) and (44), $\rho(u_{i+1}, u_{i+2}) \leq c\epsilon/2 + 9\delta_1 \leq \epsilon/2$.

Thus we have shown that the following property holds:

(iii) if $i \geq 0$ is an integer and $\rho(u_i, u_{i+1}) \leq \epsilon/2$, then $\rho(u_{i+1}, u_{i+2}) \leq \epsilon/2$.

We show that there exists $j \in \{k, \dots, k+n_2\}$ such that $\rho(u_j, u_{j+1}) \leq \epsilon/2$.

Assume the contrary. Then for each $i \in \{k, \dots, k+n_2\}$,

$$\rho(u_i, u_{i+1}) > \epsilon/2. \tag{45}$$

In view of (6), (44) and (45), for each $i \in \{k, \dots, k+n_2\}$,

$$\begin{aligned}\rho(u_i, u_{i+1}) - \rho(u_{i+1}, u_{i+2}) &\geq (1-c)\rho(u_i, u_{i+1}) - 9\delta_1 \\ &\geq (1-c)\epsilon/2 - 9\delta_1 \geq (1-c)\epsilon/4. \end{aligned} \tag{46}$$

By (43) and (46),

$$\begin{aligned} M_0 \geq \rho(u_k, u_{k+1}) &\geq \rho(u_k, u_{k+1}) - \rho(u_{k+n_2+2}, x_{n_2+k+1}) \\ &\geq \sum_{i=k}^{n+n_2} (\rho(u_i, u_{i+1}) - \rho(u_{i+1}, u_{i+2})) \geq (n_2+1)(1-c)\epsilon/4 \end{aligned}$$

and
$$n_2 \leq \lfloor 4M_0(1-c)^{-1}\epsilon^{-1} \rfloor - 1.$$

This contradicts (9). The contradiction we have reached proves that there exists

$$p \in \{k, \dots, k+n_2\} \text{ such that } \rho(u_p, u_{p+1}) \leq \epsilon/2.$$

Together with Claim A1, property (iii) and (10) this implies that

$$\rho(u_i, u_{i+1}) \leq \epsilon/2$$

for each integer $i \geq n_0 \geq n_1 + n_2$.

Combined with Claim A1 and (6) this implies that for each integer $i \geq n_0$,

$$\rho(x_i, x_{i+1}) \leq \rho(x_i, u_i) + \rho(u_i, u_{i+1}) + \rho(u_{i+1}, x_{i+1}) \leq \epsilon/2 + 2\delta_1 \leq \epsilon$$

and
$$\xi_n \in B(x_{n+1}, \epsilon/2 - 2\delta_1) \subset B(x_n, \epsilon) \cap T(x_n).$$

Theorem 2.1 is proved. □

3. The second an third main results

We continue to assume all the assumptions made in Sections 1 and 2. In particular, we assume that (A) holds.

The following result was obtained in [4].

Proposition 3.1. *Let $\epsilon > 0$. Then for each $x \in X_0$ satisfying*

$$\rho(x, T(x)) \leq 6^{-1}(1-c)\epsilon$$

there exists $\bar{x} \in X_0$ such that $\bar{x} \in T(\bar{x}) \cap B(x, \epsilon)$.

Theorem 3.2. *Let $\epsilon \in (0,1)$ and*

$$\delta = 64^{-1}(1-c)^{-2}\epsilon. \tag{47}$$

Then for each $x \in X$ satisfying $\rho(x, T(x)) \leq \delta$ there exists $\bar{x} \in X_0$ such that

$$\bar{x} \in T(\bar{x}) \cap B(x, \epsilon).$$

Proof. Assume that $x \in X$ and

$$\rho(x, T(x)) \leq \delta. \tag{48}$$

We show that there exists $\bar{x} \in T(\bar{x}) \cap B(x, \epsilon)$.

In view of Proposition 3.1, we may assume without loss of generality that $x \notin X_0$.

Let $\gamma > 0$. There exists
$$x_0 \in X_0 \tag{49}$$
such that
$$\rho(x, x_0) < \rho(x, X_0) + \gamma. \tag{50}$$

By (A), the relation $T(X_0) \subset X_0$ and the relations (48)–(50),

$$\rho(x, X_0) \leq \rho(x, T(x_0)) \leq \rho(x, T(x)) + H(T(x), T(x_0))$$
$$\leq \delta + c\rho(x, x_0) \leq \delta + c\rho(x, X_0) + \gamma$$

and
$$(1-c)\rho(x, X_0) \leq \delta + \gamma.$$

Since γ is any positive number we conclude that

$$\rho(x, X_0) \leq \delta(1-c)^{-1}. \tag{51}$$

By (51), there exists $x_0 \in X_0$ such that

$$\rho(x, X_0) \leq 2\delta(1-c)^{-1}. \tag{52}$$

Assumption (A) and equations (48) and (52) imply that

$$\rho(x_0, T(x_0)) \leq \rho(x_0, x) + \rho(x, T(x)) + H(T(x), T(x_0))$$
$$\leq 2\rho(x_0, x) + \delta \leq \delta(4(1-c)^{-1} + 1).$$

Together with Proposition 3.1 this implies that there exists $\bar{x} \in X_0$ such that

$$\bar{x} \in T(\bar{x}), \quad \text{and} \quad \rho(\bar{x}, x_0) \leq 6\delta(4(1-c)^{-1} + 1)(1-c)^{-1}. \tag{53}$$

By (52),
$$\begin{aligned}\rho(x, \bar{x}) &\leq \rho(x, x_0) + \rho(x_0, \bar{x}) \\ &\leq 2\delta(1-c)^{-1} + 6\delta(4(1-c)^{-1} + 1)(1-c)^{-1} \\ &\leq \delta((1-c)^{-1}(8 + 24(1-c)^{-1})) \leq 32(1-c)^{-2}\delta \leq \epsilon.\end{aligned}$$

Theorem 3.2 is proved. □

Theorems 2.1 and 3.2 imply the following result.

Theorem 3.3. *Let* $\theta \in X$, $\epsilon \in (0,1)$, $M > 0$,

$$\epsilon_0 = 64^{-1}\epsilon(1-c)^{-2}, \quad \delta = 360^{-1}\epsilon_0(1-c)^2,$$
$$B(\theta, M) \cap X_0 \neq \emptyset,$$
$$M_0 > (1-c)^{-1}((2M+2)10 + \rho(\theta, T(\theta))),$$
$$n_0 = \lfloor 4M_0(1-c)^{-1}\epsilon^{-1}\rfloor + \lfloor 2 \cdot 36^2(2M+1)(1-c)^{-2}\epsilon^{-1}\rfloor + 1.$$

Assume that $\{x_i\}_{i=0}^{\infty} \subset X$, $\rho(x_0, \theta) \leq M$, *and that for each integer* $n \geq 0$,

$$B(x_{n+1}, \delta) \cap \{\xi \in T(x_n) : \ \rho(\xi, x_n) \leq \delta + \rho(x_n, T(x_n))\} \neq \emptyset.$$

Then for each integer $n \geq n_0$,

$$\rho(x_n, x_{n+1}) < \epsilon_0, \ B(x_n, \epsilon_0) \cap T(x_n) \neq \emptyset$$

and there exists $y_n \in X_0$ *such that* $y_n \in T(y_n) \cap B(x_n, \epsilon)$.

Example 3.4. Let $n \geq 2$ be an integer and let $X_i \subset X$, $i = 1, \ldots, n$ be nonempty closed sets satisfying

$$X_0 = \bigcap_{i=1}^{n} X_i \neq \emptyset \ \text{ and } \ X_{n+1} = X_1.$$

Assume $T : X \to 2^X \setminus \{\emptyset\}$, $c \in [0,1)$, for each $i \in \{1, \ldots, n\}$ $T(X_i) \subset X_{i+1}$, and let for each $x, y \in X_i$,

$$H(T(x), T(y)) \leq c\rho(x, y).$$

The set-valued map T is called *cyclical* [12]. Clearly it belongs to the class of quasi-contractions studied in the paper. □

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