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Metric Subregularity for a Piecewise Linear Mapping between two Banach Spaces

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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Observe that the graph of a piecewise linear mapping to an infinite dimensional Banach space is not the union of finitely many convex polyhedra. This and Robinson's theorem on local metric subregularity for a polyhedral mapping motivate us to consider the metric subregualrity for a piecewise linear mapping between two general Banach spaces. We prove that a piecewise linear mapping G between two Banach spaces is boundedly metrically subregular at any point in its graph $\mathrm{gph}(G)$ if and only if G is metrically subregular at some point in $\mathrm{gph}(G)$ if and only if G has the L-closed range property, which complements Robinson's theorem. As an complement of Mordukhovich's criterion on the metric regularity in the finite dimension case, we also provide a dual characterization for a piecewise linear mapping to be metrically regular.

Keywords: Piecewise linear mapping, metric subregularity, metric regularity.

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1. Introduction

Continuous linear operators between Banach spaces enjoy many interesting and beautiful properties. However, the linearity is quite restrictive. The family of all piecewise linear functions is much larger than that of all linear functions and there exists a wide class of functions that can be approximated by piecewise linear functions. Moreover, one may use piecewise linear functions generated by finitely many test data to establish mathematical models for some practical problems. Therefore it is useful to study piecewise linear problems (cf. [3, 5, 6, 11, 15, 16, 17, 18] and the references therein).

Let F be a closed set-valued mapping between two Banach spaces X and Y, that is, it graph $\mathrm{gph}(F) := \{(x, y) \in X \times Y : \ y \in F(x)\}$ is a closed set in the product $X \times Y$. Recall that F is (locally) metrically suregular at $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$ if there exists $\kappa, \delta \in (0, \ +\infty)$ such that $d(x, F^{-1}(\bar{y})) \leq \kappa d(\bar{y}, F(x))$ for all $x \in B(\bar{x}, \delta)$, where $B(\bar{x}, \delta)$ denotes the open ball with center $\bar{x}$ and radius δ. The above metric subregularity as a valuable extension of the error bound in optimization theory has been recognized to be important in variational analysis and studied extensively

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(cf. [2, 4, 9, 10, 13, 17] and the references therein). It is well-known that if the graph gph(F) of F is a convex polyhedron in the product $X \times Y$ then there exists $\kappa \in (0,\ +\infty)$ such that

$$d(x, F^{-1}(y)) \le \kappa d(y, F(x)) \quad \forall x \in X \text{ and } \forall y \in F(X),$$

that is, F is uniformly globally metrically subregular on the range $F(X)$. Without the convexity assumption, the above global metric subregularity result is not necessarily valid (cf. [2, 13]). However Robinson [11] proved that if F is a (nonconvex) polyhedral set-valued mapping between two Banach spaces X and Y (i.e. the graph gph(F) is the union of finitely many convex polyhedra in $X \times Y$) then there exists $\kappa \in (0,\ +\infty)$ such that for any $y \in F(X)$ there exists $\varepsilon_y > 0$ with the following property:

$$d(x, F^{-1}(y)) \le \kappa d(y, F(x)) \quad \forall x \in X \text{ with } d(y, F(x)) < \varepsilon_y,$$

that is, F is locally metrically subregular at every point in the range $F(X)$. Recall (cf.[3, 15, 16, 18]) that a single-valued mapping $G : X \to Y$ is piecewise linear if there exist convex polyhedra P_i in X, $T_i \in \mathcal{L}(X, Y)$ and $b_i \in Y$ $(i = 1, \cdots, n)$ such that

$$X = \bigcup_{i=1}^{n} P_i \quad \text{and} \quad G(x) = T_i(x) + b_i \ \ \forall x \in P_i,\ i = 1, \cdots, n, \tag{1}$$

where $\mathcal{L}(X, Y)$ denotes the space of all continuous linear operators from X to Y. Zheng and Yang [18] proved that if $G : X \to Y$ is a piecewise linear single-valued mapping then its graph is the union of finitely many convex polyhedra in the product $X \times Y$ if and only if Y is finite dimensional. Therefore, a piecewise linear single-valued mapping to a infinite dimensional Banach space must not be a polyhedral mapping. This motivates us to consider the metric subregualrity of a piecewise linear single-valued mapping between two general Banach spaces.

For a continuous linear operator $T : X \to Y$ between two Banach spaces X and Y the closedness of $T(X)$ implies $T(X) \cap rB_Y \subset T(B_X)$ for some $r > 0$ by the famous open mapping theorem, which means the following global metric subregularity:

$$d(x, T^{-1}(y)) \le \frac{1}{r}\|y - T(x)\| \quad \forall (x, y) \in X \times T(X).$$

In this paper, we consider the extension of the above result from a continuous linear operator to a piecewise linear single-valued mapping. In particular, if G is a piecewise linear single-valued mapping between two Banach spaces X and Y, we prove the following properties are equivalent:

(i) G has the L-closed range property (see Section 2 for its definition).

(ii) For any $y \in G(X)$ and any $r > 0$ there exists $\kappa \in (0,\ +\infty)$ such that
$d(x, G^{-1}(y)) \le \kappa\|y - G(x)\| \quad \forall x \in G^{-1}(y) + rB_X$.

(iii) For any $y \in G(X)$ and any $r > 0$ there exists $\kappa \in (0,\ +\infty)$ such that
$d(x, G^{-1}(y)) \le \kappa\|y - G(x)\| \quad \forall x \in rB_X$.

(iv) There exist $(x_0, y_0) \in \mathrm{gph}(G)$ and $\kappa_0, \delta_0 \in (0,\ +\infty)$ such that
$d(x, G^{-1}(y_0)) \le \kappa\|y_0 - G(x)\| \quad \forall x \in B(x_0, \delta_0)$.

We also prove that G is globally metrically subregular at $y_0 \in G(X)$, that is, there exists $\kappa \in (0,\ +\infty)$ such that

$$d(x, G^{-1}(y_0)) \leq \kappa \|y_0 - G(x)\| \quad \forall x \in X$$

if and only if $\lim\limits_{d(x,G^{-1}(y_0))\to+\infty} \|y_0 - G(x)\| = +\infty.$

It is well known that a continuous linear operator T between two Banach spaces X and Y is metrically regular if and only if T is surjective if and only if $T(X)$ is closed and $(T^*)^{-1}(0) = \{0\}$. This paper uses the coderivative to establish the corresponding characterization for a piecewise linear single-valued mapping between two Banach spaces to be metrically regular. In particular, we prove that a piecewise linear single-valued mapping G between two Banach spaces is metrically regular at some point $(\bar{x}, G(\bar{x}))$ if and only if G has the L-closed range property and $(D^*G(\bar{x}))^{-1}(0) = \{0\}$, where $D^*G(\bar{x})$ is the coderivative of G at $\bar{x}$. This can be also regarded as an complement of Mordukhovich's criterion on the metric regularity,

2. Preliminaries

Let X be a normed space with the dual space X^*. Recall (cf.[1, 12]) that a subset P of X is a *convex polyhedron* if there exist $x_1^*, \cdots, x_m^* \in X^*$ and $t_1, \cdots, t_m \in \mathbb{R}$ such that

$$P = \{x \in X : \langle x_i^*, x\rangle \leq t_i,\ i = 1, \cdots, m\}. \tag{2}$$

The following result on polyhedra is known and useful for our analysis (cf. [18, Proposition 2.6]).

Lemma 2.1. *Let P_i be convex polyhedra in a normed space Z such that* $\mathrm{int}(P_i) \neq \emptyset$ *where $(i = 1, ..., m)$. Then there exist convex polyhedra Q_j in Z with* $\mathrm{int}(Q_j) \neq \emptyset$ *$(j = 1, ..., \nu)$ such that we have $\bigcup_{i=1}^m P_i = \bigcup_{j=1}^{\nu} Q_j$ and $\mathrm{int}(Q_j) \cap Q_{j'} = \emptyset$ for all $j, j' \in \overline{1\nu}$* with $j \neq j'$, *where $\overline{1\nu} := \{1, ..., \nu\}$.*

Based on Lemma 2.1, Zheng and Yang [18] provided the following results on a single-valued piecewise linear mapping.

Lemma 2.2. *Let G be a piecewise linear mapping between two normed spaces X and Y. Then there exist finitely many convex polyhedra $\Lambda_1, \cdots, \Lambda_n$ in the product $X \times Y$ such that* $\mathrm{gph}(G) = \bigcup_{i=1}^n \Lambda_i$ *if and only if Y is finite dimensional.*

Lemma 2.3. *Let G be a piecewise linear mapping between two normed spaces X and Y. Then there exist closed subspaces E_1 and E_2 of X, a subspace $\hat{Y}$ of Y, $\hat{T} \in \mathcal{L}(E_1, Y)$ and a piecewise linear mapping $\hat{g} : E_2 \to \hat{Y}$ such that*

$$X = E_1 + E_2,\ E_1 \cap E_2 = \{0\},\ \dim(E_2) < \infty,\ \dim(\hat{Y}) < \infty \tag{3}$$

and
$$G(x_1 + x_2) = \hat{T}(x_1) + \hat{g}(x_2) \quad \forall (x_1, x_2) \in E_1 \times E_2. \tag{4}$$

With the help of Lemma 2.3, we can establish the following more elaborate expression for a piecewise linear mapping.

Lemma 2.4. *Let G be a piecewise linear mapping between two normed spaces X and Y. Then there exist closed subspaces X_1 and X_2 of X, a subspace $\hat{Y}$ of Y, $T_G \in \mathcal{L}(X_1, Y)$ and a piecewise linear mapping $g : X_2 \to \hat{Y}$ such that*

$$X = X_1 + X_2,\ X_1 \cap X_2 = \{0\},\ \dim(X_2) < \infty,\ \dim(\hat{Y}) < \infty, \tag{5}$$

$$T_G(X_1) \cap \hat{Y} = \{0\} \text{ and } G(x_1 + x_2) = T_G(x_1) + g(x_2) \quad \forall (x_1, x_2) \in X_1 \times X_2. \tag{6}$$

Proof. By Lemma 2.3, there exist closed subspaces E_1 and E_2 of X, a subspace $\hat{Y}$ of Y, $\hat{T} \in \mathcal{L}(E_1, Y)$ and a piecewise linear mapping $\hat{g} : E_2 \to \hat{Y}$ such that (3) and (4) hold. Then $\dim(\hat{T}(E_1) \cap \hat{Y}) \le \dim(\hat{Y}) < \infty$, and hence there exist linearly independent vectors $e_1, \cdots, e_n$ in Y such that

$$\hat{T}(E_1) \cap \hat{Y} = \Big\{ \sum_{i=1}^{n} t_i e_i :\ t_1, \cdots, t_n \in \mathbb{R} \Big\}. \tag{7}$$

For any $i \in \overline{1n} := \{1, \cdots, n\}$, let $\hat{y}_i^* : \hat{T}(E_1) \cap \hat{Y} \to \mathbb{R}$ such that

$$\Big\langle \hat{y}_i^*, \sum_{k=1}^{n} t_k e_k \Big\rangle := t_i \quad \forall t_1, \cdots, t_n \in \mathbb{R}.$$

Then, since $\hat{T}(E_1) \cap \hat{Y}$ is finite dimensional, $\hat{y}_i^*$ is a continuous linear functional on $\hat{T}(E_1) \cap \hat{Y}$. By the Hahn-Banach theorem, there exist $y_1^*, \cdots, y_n^* \in Y^*$ such that $y_1^*|_{\hat{T}(E_1)\cap\hat{Y}} = \hat{y}_1^*, \cdots, y_n^*|_{\hat{T}(E_1)\cap\hat{Y}} = \hat{y}_n^*$. Hence

$$\langle y_k^*, e_k \rangle = 1 \text{ and } \langle y_k^*, e_{k'} \rangle = 0 \quad \forall k \in \overline{1n} \text{ and } \forall k' \in \overline{1n} \setminus \{k\}. \tag{8}$$

Let
$$Y_1 := \big\{ y \in \hat{T}(E_1) : \langle y_1^*, y \rangle = \cdots = \langle y_n^*, y \rangle = 0 \big\}. \tag{9}$$

We claim $Y_1 \cap \hat{Y} = \{0\}$. Indeed, for any $y \in Y_1 \cap \hat{Y}$, one has $y \in Y_1 \cap \hat{T}(E_1) \cap \hat{Y}$. Hence there exist $t_1, \cdots, t_n \in \mathbb{R}$ such that $y = \sum_{k=1}^{n} t_k e_k$ and

$$0 = \langle y_k^*, y \rangle = \sum_{k'=1}^{n} t_{k'} \langle y_k^*, e_{k'} \rangle = t_k \quad \forall k \in \overline{1n}$$

(thanks to (8)). This shows $y = 0$ and so $Y_1 \cap \hat{Y} = \{0\}$. Let

$$X_1 := \hat{T}^{-1}(Y_1) = \{x_1 \in E_1 :\ \hat{T}(x_1) \in Y_1\} \tag{10}$$

and take $u_1, \cdots, u_n \in E_1$ such that

$$\hat{T}(u_k) = e_k,\ k = 1, \cdots, n. \tag{11}$$

Let $E := \big\{ \sum_{k=1}^{n} t_k u_k :\ k = 1, \cdots, n \big\}$. Then

$$\dim(E) = n \text{ and } X_1 + E \subset E_1. \tag{12}$$

On the other hand, let $x_1 \in E_1$. Then, by (7), $y_2 := \sum_{k=1}^{n} \langle y_k^*, \hat{T}(x_1) \rangle e_k \in \hat{T}(E_1) \cap \hat{Y}$. Hence, by (8), one has

$$\langle y_{k'}^*, y_2 \rangle = \sum_{k=1}^{n} \langle y_k^*, \hat{T}(x_1) \rangle \langle y_{k'}^*, e_k \rangle = \langle y_{k'}^*, \hat{T}(x_1) \rangle \quad \forall k' \in \overline{1n}.$$

This and (9) imply that $y_1 := \hat{T}(x_1) - y_2 \in Y_1$.

By (11) we obtain

$$y_2 = \sum_{k=1}^{n} \langle y_k^*, \hat{T}(x_1)\rangle \hat{T}(u_k) \ \text{ and hence } \ y_1 = \hat{T}\Big(x_1 - \sum_{k=1}^{n} \langle y_k^*, \hat{T}(x_1)\rangle u_k\Big) \in Y_1.$$

It follows from (10) that $x_1 - \sum_{k=1}^{n} \langle y_k^*, \hat{T}(x_1)\rangle u_k \in X_1$.

Hence $x_1 \in X_1 + \sum_{k=1}^{n} \langle y_k^*, \hat{T}(x_1)\rangle u_k \subset X_1 + E$ (thanks to the definition of E). Since x_1 is arbitrary in E_1, one has $E_1 \subset X_1 + E$. This and the inclusion in (12) show that $E_1 = X_1 + E$. Next let $u \in X_1 \cap E$. Then $\hat{T}(u) \in Y_1 \cap \hat{T}(E)$ and there exist $t_1, \cdots, t_n \in \mathbb{R}$ such that $u = \sum_{k=1}^{n} t_k u_k$. Hence

$$\hat{T}(u) = \sum_{k=1}^{n} t_k \hat{T}(u_k) = \sum_{k=1}^{n} t_k e_k \in Y_1 \cap \hat{Y} = \{0\}.$$

Since $e_1, \cdots, e_n$ are linearly independent, $t_1 = \cdots = t_n = 0$ and so $u = 0$. This implies $X_1 \cap E = \{0\}$. Letting $X_2 := E + E_2$, it follows from (3) that (5) holds. By (3), one also has $E \cap E_2 = \{0\}$. Define $T_G : X_1 \to Y$ and $g : X_2 \to \hat{Y}$ as follows

$$T_G(x_1) = \hat{T}(x_1) \quad \forall x_1 \in X_1$$

and

$$g(u + v) = \hat{T}(u) + \hat{g}(v) \quad \forall (u, v) \in E \times E_2.$$

Then $T_G \in \mathcal{L}(X_1, Y)$, g is a piecewise linear mapping between X_2 and $\hat{Y}$, and

$$T_G(X_1) \cap \hat{Y} = \hat{T}(X_1) \cap \hat{Y} = Y_1 \cap \hat{Y} = \{0\}.$$

Let $(x_1, x_2) \in X_1 \times X_2$. Then it only needs to show $G(x_1 + x_2) = T_G(x_1) + g(x_2)$. To show this, take $(u, v) \in E \times E_2$ such that $x_2 = u + v$. Note that we have $x_1 + u \in X_1 + E = E_1$. It follows from (4) that

$$G(x_1+x_2) = G(x_1+u+v) = \hat{T}(x_1+u)+\hat{g}(v) = \hat{T}(x_1)+\hat{T}(u)+\hat{g}(v) = T_G(x_1)+g(x_2).$$

The proof is complete. □

Lemma 2.5. *Let X_1 and X_2 be closed subspaces of a normed space X such that $X_1 \cap X_2 = \{0\}$ and $\dim(X_2) < \infty$. Then there exists $\tau \in (0, +\infty)$ such that*

$$\|x_1\| + \|x_2\| \le \tau \|x_1 + x_2\| \quad \forall (x_1, x_2) \in X_1 \times X_2.$$

Proof. Suppose to the contrary that there exists a sequence $\{(u_n, v_n)\}$ in $X_1 \times X_2$ such that $\|u_n\| + \|v_n\| > n\|u_n + v_n\|$ for all $n \in \mathbb{N}$.

Let

$$\tilde{u}_n := \frac{u_n}{\|u_n\| + \|v_n\|} \quad \text{and} \quad \tilde{v}_n := \frac{v_n}{\|u_n\| + \|v_n\|}.$$

Then

$$\|\tilde{u}_n + \tilde{v}_n\| < \frac{1}{n} \text{ and } \|\tilde{u}_n\| + \|\tilde{v}_n\| = 1 \quad \forall n \in \mathbb{N}.$$

Moreover, since X_2 is finite dimensional, we can assume without loss of generality that $\tilde{v}_n \to v \in X_2$. Hence $-\tilde{u}_n \to v \in X_1 \cap X_2$ (thanks to the closedness of X_1) and $\|v\| = \frac{1}{2}$. This contradicts $X_1 \cap X_2 = \{0\}$. □

For a closed set A in a Banach space X and $\varepsilon \in [0, +\infty)$, let

$$\hat{N}_\varepsilon(A, \bar{a}) := \left\{ x^* \in X^* : \limsup_{a \xrightarrow{A} \bar{a}} \frac{\langle x^*, a - \bar{a} \rangle}{\|a - \bar{a}\|} \leq \varepsilon \right\} \quad \forall \bar{a} \in A.$$

Let $N(A, \bar{a})$ denote the limiting normal cone to A at $\bar{a}$, that is, $x^* \in N(A, \bar{a})$ if and only if there exists a sequence $\{(a_n, x_n^*, \varepsilon_n)\}$ in $A \times X^* \times \mathbb{R}_+$ such that

$$a_n \to \bar{a}, \; x_n^* \xrightarrow{w^*} x^*, \; \varepsilon_n \to 0 \quad \text{and} \quad x_n^* \in \hat{N}_{\varepsilon_n}(A, a_n) \;\; \forall n \in \mathbb{N}$$

(cf. [9, 13]). For a set-valued mapping $F : X \rightrightarrows Y$, let $D^*F(x, y) : Y^* \rightrightarrows X^*$ denote the coderivative of F at $(x, y) \in \mathrm{gph}(F)$, that is,

$$D^*F(x, y)(y^*) := \{x^* \in X^* : \; (x^*, -y^*) \in N(\mathrm{gph}(F), (x, y))\} \quad \forall y^* \in Y^*$$

(cf. [7, 9]). For convenience, we use $D^*F(x)$ to denote $D^*F(x, F(x))$ when F is single-valued. It is known and easy to verify that if T is a continuous linear operator between two Banach spaces X and Y then

$$D^*T(x)(y^*) = T^*(y^*) \quad \forall (x, y^*) \in X \times Y^*,$$

where T^* is the conjugate operator of T.

In the finite dimensional case, Mordukhovich [8] used the coderivative to establish the following criterion on the metric regularity of a closed set-valued mapping F between two finite dimensional Banach spaces at $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$: there exist numbers $\kappa, \delta \in (0, +\infty)$ such that

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)) \quad \forall (x, y) \in B(\bar{x}, \delta) \times B(\bar{y}, \delta).$$

Lemma 2.6. *Let F be a closed set-valued mapping between two finite dimensional Banach spaces. Then F is metrically regular at $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$ if and only if $D^*F(\bar{x}, \bar{y})^{-1}(0) = \{0\}$.*

Under the piecewise linearity assumption, we will extend Mordukhovich's criterion to the infinite dimension case. To do this, we need the following lemma.

Lemma 2.7. *Let X and Y be Banach spaces, $G : X \to Y$ be a piecewise linear mapping, and let X_1 and X_2 be closed subspaces of X, $\hat{Y}$ be a subspace of Y, $T_G : X_1 \to Y$ be a continuous linear operator and $g : X_2 \to \hat{Y}$ be a piecewise linear mapping such that (5) and (6) hold. Further suppose that $T_G(X_1)$ is closed. Then there exists a closed subspace Y_0 of Y with the following properties:*

(i) *$Y = Y_0 + \hat{Y}$, $Y_0 \cap \hat{Y} = \{0\}$ and $T_G(X_1) \subset Y_0$.*

(ii) *For any $x \in X$ and $(x_1, x_2) \in X_1 \times X_2$ with $x = x_1 + x_2$,*

$$D^*G(x)(y_0^* \oplus \hat{y}^*) = \{T_G^*(y_0^*) \oplus x_2^* : \; x_2^* \in D^*g(x_2)(\hat{y}^*)\} \quad \forall (y_0^*, \hat{y}^*) \in Y_0^* \times \hat{Y}^*,$$

where $T_G^* : Y_0^* \to X_1^*$ is the conjugate operator of T_G as a continuous linear operator from X_1 to Y_0 and $y_0^* \oplus \hat{y}^* \in Y^*$ defined by

$$\langle y_0^* \oplus \hat{y}^*, y_0 + \hat{y} \rangle = \langle y_0^*, y_0 \rangle + \langle \hat{y}^*, \hat{y} \rangle \quad \forall (y_0, \hat{y}) \in Y_0 \times \hat{Y}.$$

Proof. Since $\hat{Y}$ is finite dimensional, there exist linearly independent vectors $\hat{y}_1, ..., \hat{y}_n$ in Y such that

$$\hat{Y} = \mathrm{span}\{\hat{y}_1, \cdots, \hat{y}_n\} = \Big\{\sum_{i=1}^{n} t_k \hat{y}_k : \ t_1, \cdots, t_n \in \mathbb{R}\Big\}.$$

For any $k \in \overline{1n}$, let $Y_k := T_G(X_1) + \mathrm{span}\left(\{\hat{y}_1, \cdots, \hat{y}_n\} \setminus \{\hat{y}_k\}\right)$. Then Y_k does not contain $\hat{y}_k$ (thanks to the first equality in (6)) and is a closed subspace of Y (thanks to the closedness of $T_G(X_1)$ and [14, Theorem 1.42]). Hence, by the Hahn-Banach theorem, there exists $y_k^* \in Y^*$ such that $\langle y_k^*, \hat{y}_k\rangle = 1$ and $\langle y_k^*, y\rangle = 0$ for all $y \in Y_k$.

Letting $Y_0 := \{y \in Y : \ \langle y_k^*, y\rangle = 0, \ k = 1, \cdots, n\}$, it follows that (i) holds.

To prove (ii), let $x \in X$ and $(x_1, x_2) \in X_1 \times X_2$ be such that $x = x_1 + x_2$, and let $(y_0^*, \hat{y}^*) \in Y_0^* \times \hat{Y}^*$. Let x^* be an arbitrary element in $D^*G(x)(y_0^* \oplus \hat{y}^*)$. Then $(x^*, -(y_0^* \oplus \hat{y}^*)) \in N(\mathrm{gph}(G), (x, G(x)))$. Noting that the piecewise linear mapping G is globally Lipschitzian, it follows there exists a sequence $\{(x_n, x_n^*, y_n^*, \delta_n)\}$ in the product $X \times X^* \times Y^* \times (0, +\infty)$ such that $x_n \to x$, $x_n^* \stackrel{w^*}{\to} x^*$, $y_n^* \stackrel{w^*}{\to} y_0^* \oplus \hat{y}^*$ and

$$\langle x_n^*, x - x_n\rangle - \langle y_n^*, G(x) - G(x_n)\rangle \le \frac{1}{n}\|x - x_n\| \quad \forall x \in B(x_n, \delta_n). \tag{13}$$

Set $x_1^* := x^*|_{X_1}$, $x_2^* := x^*|_{X_2}$, $u_n^* := x_n^*|_{X_1}$, $v_n^* := x_n^*|_{X_2}$, $z_n^* := y_n^*|_{Y_0}$ and $\hat{y}_n^* := y_n^*|_{\hat{Y}}$, and take $(u_n, v_n) \in X_1 \times X_2$ such that $x_n = u_n + v_n$. Then

$$u_n^* \stackrel{w^*}{\to} x_1^*, \quad v_n^* \stackrel{w^*}{\to} x_2^*, \quad z_n^* \stackrel{w^*}{\to} y_0^*, \quad \hat{y}_n^* \stackrel{w^*}{\to} \hat{y}^*, \quad u_n \to x_1 \ \text{ and } \ v_n \to x_2$$

(thanks to Lemma 2.5). Moreover, by (5), (6) and (13), one has

$$\langle u_n^*, u - u_n\rangle - \langle z_n^*, T_G(u) - T_G(u_n)\rangle \le \frac{1}{n}\|u - u_n\| \quad \forall u \in B_{X_1}(u_n, \delta_n)$$

and $$\langle v_n^*, v - v_n\rangle - \langle \hat{y}_n^*, g(v) - g(v_n)\rangle \le \frac{1}{n}\|v - v_n\| \quad \forall v \in B_{X_2}(v_n, \delta_n).$$

Hence $\|u_n^* - T_G^*(z_n^*)\| \le \frac{1}{n}$ and $(v_n^*, -\hat{y}_n^*) \in \hat{N}_{\frac{1}{n}}(\mathrm{gph}(g), (v_n, g(v_n)))$. Letting $n \to \infty$ and noting that the conjugate operator T_G^* is weak*-weak* continuous, it follows that $x_2^* \in D^*g(x_2)(\hat{y}^*)$ and $x^* = x_1^* \oplus x_2^* = T_G^*(y_0^*) \oplus x_2^*$. This shows that

$$D^*G(x)(y_0^* \oplus \hat{y}^*) \subset \{T_G^*(y_0^*) \oplus x_2^* : \ x_2^* \in D^*g(x_2)(\hat{y}^*)\}. \tag{14}$$

To prove the inverse inclusion, let $v^* \in X_2^*$ be such that $v^* \in D^*g(x_2)(\hat{y}^*)$. Then there exists a sequence $\{(v_n, v_n^*, \hat{y}_n^*)\}$ in $X_2 \times X_2^* \times \hat{Y}^*$ such that

$$v_n \to x_2, \ v_n^* \xrightarrow{w^*} v^*, \ \hat{y}_n^* \xrightarrow{w^*} \hat{y}^* \ \text{ and } \ (v_n^*, -\hat{y}_n^*) \in \hat{N}_{\frac{1}{n}}(\mathrm{gph}(g), (v_n, g(v_n))) \ \ \forall n \in \mathbb{N}.$$

Hence for each $n \in \mathbb{N}$ there exists $r_n > 0$ such that

$$\langle v_n^*, v - v_n\rangle - \langle \hat{y}_n^*, g(v) - g(v_n)\rangle \le \frac{2}{n}\|v - v_n\| \quad \forall v \in B_{X_2}(v_n, r_n).$$

Noting that $\langle T_G^*(y_0^*), u - x_1\rangle - \langle y_0^*, T_G(u) - T_G(x_1)\rangle = 0$ for all $u \in X_1$, it follows that

$$\langle T_G^*(y_0^*) \oplus v_n^*, u + v - x_1 - v_n\rangle - \langle y_0^* \oplus \hat{y}_n^*, T_G(u) + g(v) - T_G(x_1) - g(v_n)\rangle \le \frac{2}{n}\|v - v_n\|$$

for all $(u, v) \in X_1 \times B_{X_2}(v_n, r_n)$.

Thus, by (5), (6) and Lemma 2.5, there exists a positive constant τ such that

$$\big(T_G^*(y_0^*) \oplus v_n^*, -(y_0^* \oplus \hat{y}_n^*)\big) \in \hat{N}_{\frac{\tau}{n}}\big(\mathrm{gph}(G), (x_1 + v_n, G(x_1 + v_n))\big) \quad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$, one has

$$\big(T_G^*(y_0^*) \oplus v^*, -(y_0^* \oplus \hat{y}^*)\big) \in N\big(\mathrm{gph}(G), (x_1 + x_2, G(x_1 + x_2))\big),$$

that is, $T_G^*(y_0^*) \oplus v^* \in D^*G(x_1 + x_2)(y_0^* \oplus \hat{y}^*)$. Since v^* is arbitrary in $D^*g(x_2)(\hat{y}^*)$, we have

$$D^*G(x)(y_0^* \oplus \hat{y}^*) = D^*G(x_1 + x_2)(y_0^* \oplus \hat{y}^*) \supset \{T_G^*(y_0^*) \oplus x_2^* : \ x_2^* \in D^*g(x_2)(\hat{y}^*)\}.$$

This and (14) show that (ii) hold. The proof is complete. $\square$

3. Main results

To establish the main results in this section, we introduce the following L-closed range property of a piecewise linear mapping between two Banach spaces: *Let G be a piecewise linear mapping between two Banach spaces X and Y, and let E_1, E_2, $\hat{Y}$, $\hat{T}$ and $\hat{g}$ be as in Lemma 2.3, We say that G has the L-closed range property if $\hat{T}(E_1)$ is a closed subspace of Y.*

The following proposition shows that the L-closed range property of a piecewise linear mapping between two Banach spaces is well-defined.

Proposition 3.1. *Let G be a piecewise linear mapping between two Banach spaces X and Y. Suppose that E_1, $\tilde{E}_1$, E_2 and $\tilde{E}_2$ are closed subspaces of X, $\hat{Y}$ and $\tilde{Y}$ are subspaces of Y, $\hat{T} \in \mathcal{L}(E_1, Y)$, $\tilde{T} \in \mathcal{L}(\tilde{E}_1, Y)$ and that $\hat{g} : E_2 \to \hat{Y}$ and $\tilde{g} : \tilde{E}_2 \to \tilde{Y}$ are piecewise linear mappings such that*

$$X = E_1 + E_2 = \tilde{E}_1 + \tilde{E}_2, \ E_1 \cap E_2 = \tilde{E}_1 \cap \tilde{E}_2 = \{0\}, \tag{15}$$

$$\max\big\{\dim(E_2), \dim(\tilde{E}_2), \dim(\hat{Y}), \dim(\tilde{Y})\big\} < \infty, \tag{16}$$

$$G(x_1 + x_2) = \hat{T}(x_1) + \hat{g}(x_2) \quad \forall (x_1, x_2) \in E_1 \times E_2 \tag{17}$$

and

$$G(\tilde{x}_1 + \tilde{x}_2) = \tilde{T}(\tilde{x}_1) + \tilde{g}(\tilde{x}_2) \quad \forall (\tilde{x}_1, \tilde{x}_2) \in \tilde{E}_1 \times \tilde{E}_2. \tag{18}$$

Then $\hat{T}(E_1)$ is closed if and only if $\tilde{T}(\tilde{E}_1)$ is closed.

Proof. From (15) and (16), it is easy to verify that $E_1 \cap \tilde{E}_1$ is a closed subspace of X whose codimension is less than or equal to $\dim(E_2) + \dim(\tilde{E}_2)$. Hence there exist finite dimensional subspaces Z_1 and $\tilde{Z}_1$ of X such that

$$E_1 = (E_1 \cap \tilde{E}_1) \oplus Z_1 \ \text{ and } \ \tilde{E}_1 = (E_1 \cap \tilde{E}_1) \oplus \tilde{Z}_1. \tag{19}$$

It follows that there exist finite dimensional subspaces H_1 and $\tilde{H}_1$ of X such that

$$(E_1 \cap \tilde{E}_1) + E_1 \cap \hat{T}^{-1}(0) = (E_1 \cap \tilde{E}_1) + H_1 \text{ and } (E_1 \cap \tilde{E}_1) + \tilde{E}_1 \cap \tilde{T}^{-1}(0) = (E_1 \cap \tilde{E}_1) + \tilde{H}_1.$$

Hence $(E_1 \cap \tilde{E}_1) + E_1 \cap \hat{T}^{-1}(0)$ and $(E_1 \cap \tilde{E}_1) + \tilde{E}_1 \cap \tilde{T}^{-1}(0)$ are closed subspaces of E_1 and $\tilde{E}_1$, respectively. For any $u \in E_1 \cap \tilde{E}_1$, by (17) and (18), we have

$$G(u) = \hat{T}(u) + \hat{g}(0) = \tilde{T}(u) + \tilde{g}(0) \ \text{ and } \ G(0) = \hat{g}(0) = \tilde{g}(0).$$

Hence $\hat{T}(E_1 \cap \tilde{E}_1) = \tilde{T}(E_1 \cap \tilde{E}_1)$, and so

$$\hat{T}(E_1) = \hat{T}(E_1 \cap \tilde{E}_1) + \hat{T}(Z_1) \text{ and } \tilde{T}(\tilde{E}_1) = \hat{T}(E_1 \cap \tilde{E}_1) + \tilde{T}(\tilde{Z}_1).$$

Noting that $\max\{\dim(\hat{T}(Z_1)), \dim(\tilde{T}(\tilde{Z}_1))\} < \infty$, it follows from [14, Theorem 1.42] that both $\hat{T}(E_1)$ and $\tilde{T}(\tilde{E}_1)$ are closed subspaces of Y whenever $\hat{T}(E_1 \cap \tilde{E}_1)$ is closed. Thus, it suffices to show that the closedness of either $\hat{T}(E_1)$ or $\tilde{T}(\tilde{E}_1)$ implies the closseness of $\hat{T}(E_1 \cap \tilde{E}_1)$. To do this, suppose without loss of generality that $\tilde{T}(\tilde{E}_1)$ is closed. Since $\tilde{E}_1$ is a closed subspace of the Banach space X, $\tilde{T}$ can be regarded as a surjective continuous linear operator between Banach spaces $\tilde{E}_1$ and $\tilde{T}(\tilde{E}_1)$. By the open mapping theorem, since $E_1 \cap \tilde{E}_1 + \tilde{E}_1 \cap \tilde{T}^{-1}(0)$ is a closed subspace of $\tilde{E}_1$, $\tilde{T}\big(\tilde{E}_1 \setminus \big(E_1 \cap \tilde{E}_1 + \tilde{E}_1 \cap \tilde{T}^{-1}(0)\big)\big)$ is an open set in $\tilde{T}(\tilde{E}_1)$. Noting that

$$\tilde{T}\Big(\tilde{E}_1 \setminus \big(E_1 \cap \tilde{E}_1 + \tilde{E}_1 \cap \tilde{T}^{-1}(0)\big)\Big) = \tilde{T}(\tilde{E}_1) \setminus \tilde{T}\big(E_1 \cap \tilde{E}_1 + \tilde{E}_1 \cap \tilde{T}^{-1}(0)\big),$$

it follows that $\tilde{T}(E_1 \cap \tilde{E}_1) = \tilde{T}\big(E_1 \cap \tilde{E}_1 + \tilde{E}_1 \cap \tilde{T}^{-1}(0)\big)$ is closed in $\tilde{T}(\tilde{E}_1)$. This and closedness of $\tilde{T}(\tilde{E}_1)$ show that $\tilde{T}(E_1 \cap \tilde{E}_1)$ is closed. The proof is complete. □

Remark 3.2. It is known that if $\dim(Y) \geq 2$ then there exist an open mapping T in $\mathcal{L}(X, Y)$ and a closed set A in X such that $T(A)$ is not closed in Y. However, by the proof of Proposition 3.1, we have the following result: *If $T \in \mathcal{L}(X, Y)$ is an open mapping and E is a closed subspace of X with* $\mathrm{codim}(E) < \infty$*, then $T(E)$ is a closed subspace of Y.* □

Theorem 3.3. *Let G be a piecewise linear mapping between two Banach spaces X and Y. Then the following statements are equivalent.*

(i) *G has the L-closed range property.*

(ii) *For any $y \in G(X)$ and any $r \in (0, +\infty)$, there exists $\kappa \in (0, +\infty)$ such that*
$d(x, G^{-1}(y)) \leq \kappa \|y - G(x)\| \quad \forall x \in G^{-1}(y) + rB_X,$
where B_X denotes the closed unit ball of X.

(iii) *For any $y \in G(X)$ and any $r \in (0, +\infty)$, there exists $\kappa \in (0, +\infty)$ such that*
$d(x, G^{-1}(y)) \leq \kappa \|y - G(x)\| \quad \forall x \in rB_X.$

(iv) *There exists $x_0 \in X$ and $\delta, \kappa \in (0, +\infty)$ such that*
$d(x, G^{-1}(G(x_0))) \leq \kappa \|G(x_0) - G(x)\| \quad \forall x \in B(x_0, \delta).$

Proof. By Lemma 2.4, take closed subspaces X_1 and X_2 of X, a subspace $\hat{Y}$ of Y, $T_G \in \mathcal{L}(X_1, Y)$ and a piecewise linear mapping $g : X_2 \to \hat{Y}$ such that (5) and (6) hold. Then

$$G^{-1}(y_1 + \hat{y}) = T_G^{-1}(y_1) + g^{-1}(\hat{y}) \quad \forall (y_1, \hat{y}) \in T_G(X_1) \times g(X_2). \tag{20}$$

First suppose that (i) holds. Then, by Proposition 3.1, $T_G(X_1)$ is a closed subspace of Y and $T_G \in \mathcal{L}(X_1, T_G(X_1))$ is surjective. Then, by the open mapping theorem, there exists $r_0 > 0$ such that $T_G(X_1) \cap r_0 B_Y \subset T_G(B_{X_1})$.

Let $x_1 \in X_1$ and $y_1 \in T_G(X_1) \setminus \{T_G(x_1)\}$. Then $\frac{r_0(T_G(x_1) - y_1)}{\|T_G(x_1) - y_1\|} \in T(B_{X_1})$, and so there exists $u_1 \in B_{X_1}$ such that $T_G(u_1) = \frac{r_0(T_G(x_1) - y_1)}{\|T_G(x_1) - y_1\|}$, that is,

$$y_1 = T_G(x_1) - \frac{\|T_G(x_1) - y_1\|}{r_0} T_G(u_1) = T_G\left(x_1 - \frac{\|T_G(x_1) - y_1\| u_1}{r_0}\right).$$

Hence $x_1 - \frac{\|T_G(x_1)-y_1\|u_1}{r_0} \in T_G^{-1}(y_1)$. Noting that $\|u_1\| \le 1$, it follows that

$$d\left(x_1, T_G^{-1}(y_1)\right) \le \frac{\|T_G(x_1) - y_1\|}{r_0} \quad \forall (x_1, y_1) \in X_1 \times T_G(X_1). \tag{21}$$

Since $g : X_2 \to \hat{Y}$ is piecewise linear and $\hat{Y}$ is finite dimensional, gph(g) is the union of finitely many convex polyhedra in the product $X_2 \times \hat{Y}$ (thanks to Lemma 2.2). Hence, by the Robinson theorem, there exists $\kappa_1 \in (0, +\infty)$ such that for any $\hat{y} \in g(X_2)$ there exists $\delta_{\hat{y}} > 0$ such that

$$d\left(x_2, g^{-1}(\hat{y})\right) \le \kappa_1 \|\hat{y} - g(x_2)\| \quad \forall x_2 \in X_2 \text{ with } \|\hat{y} - g(x_2)\| \le \delta_{\hat{y}}. \tag{22}$$

Let $y \in G(X)$. Then, by the second equality in (6), there exist $y_1 \in T_G(X_1)$ and $\hat{y} \in g(X_2)$ such that $y = y_1 + \hat{y}$. It follows from (5), (20) and Lemma 2.5 there exists $\tau \in (0, +\infty)$ such that

$$G^{-1}(y) + rB_X \subset T_G^{-1}(y_1) + \tau r B_{X_1} + g^{-1}(\hat{y}) + \tau r B_{X_2} \quad \forall r \in (0, +\infty)$$

and $d(x_1 + x_2, G^{-1}(y)) \le \tau(d(x_1, T_G^{-1}(y_1)) + d(x_2, g^{-1}(\hat{y})) \quad \forall (x_1, x_2) \in X_1 \times X_2.$

Take some $r \in (0, +\infty)$ and $x \in G^{-1}(y) + rB_X$. Then there exist $x_1 \in X_1$ and $x_2 \in g^{-1}(\hat{y}) + \tau r B_{X_2}$ such that $x = x_1 + x_2$. Letting $\kappa_2 := \max\left\{\frac{\tau}{r_0}, \tau\kappa_1, \frac{\delta_{\hat{y}}}{\tau r}\right\}$ and noting that $d(x_2, g^{-1}(\hat{y})) \le \tau r$, it follows from (21) and (22) that

$$d(x, G^{-1}(y)) \le \kappa_2(\|y_1 - T_G(x_1)\| + \|\hat{y} - g(x_2)\|). \tag{23}$$

On the other hand, since $T_G(X_1)$ is a closed subspace of Y and $\hat{Y}$ is a finite dimensional subspace disjoint with $T_G(X_1)$, Lemma 2.5 implies there exists a positive constant κ_3 such that

$$\begin{aligned}&\|y_1 - T_G(x_1)\| + \|\hat{y} - g(x_2)\| \le \kappa_3 \|y_1 - T_G(x_1) + \hat{y} - g(x_2)\| \\ &= \kappa_3 \|y - T_G(x_1) - g(x_2)\| = \kappa_3 \|y - G(x_1 + x_2)\|.\end{aligned}$$

Therefore, by (23), one has $d(x, G^{-1}(y)) \le \kappa_2\kappa_3 \|y - G(x)\|$. This shows that (i) implies (ii).

Let $y \in G(X)$, and take $x' \in G^{-1}(y)$. Then $rB_X \subset G^{-1}(y) + (\|x'\| + r)B_X$ for any $r \in (0, +\infty)$. Hence (ii) implies (iii). Since (iii) implies trivially (iv), it remains to show that (iv) implies (i). Next suppose that (iv) holds. By (5) and (6), take $\bar{x}_1 \in X_1$ and $\bar{x}_2 \in X_2$ such that $x_0 = \bar{x}_1 + \bar{x}_2$ and $y_0 := G(x_0) = T_G(\bar{x}_1) + g(\bar{x}_2)$. Then, by (iv) and Lemma 2.5, there exists $\tau_0 \in (0, +\infty)$ such that

$$\begin{aligned}&\tau_0(d(x_1, T_G^{-1}(T_G(\bar{x}_1))) + d(x_2, g^{-1}(g(\bar{x}_2)))) \le d(x_1 + x_2, G^{-1}(y_0)) \\ &\le \kappa\|y_0 - G(x_1 + x_2)\| = \kappa\|T_G(\bar{x}_1) + g(\bar{x}_2) - T_G(x_1) - g(x_2)\|\end{aligned}$$

for all $(x_1, x_2) \in X_1 \times X_2$ with $\|x_1 + x_2 - x_0\| \le \delta_0$. Letting $x_2 = \bar{x}_2$, then

$$d\left(x_1, T_G^{-1}(T_G(\bar{x}_1))\right) \le \frac{\kappa}{\tau_0}\|T_G(\bar{x}_1) - T_G(x_1)\| \quad \forall x_1 \in B_{X_1}(\bar{x}_1, \delta_0).$$

Since T_G is a linear operator, $T_G^{-1}(T_G(\bar{x}_1)) = \bar{x}_1 + T_G^{-1}(0)$ and hence

$$d(x_1, T_G^{-1}(0)) \leq \frac{\kappa}{\tau_0}\|T_G(x_1)\| \quad \forall x_1 \in X_1. \tag{24}$$

Let $\{v_n\}$ be a Cauchy sequence in $T_G(X_1)$, and take a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that

$$\|v_{n_{k+1}} - v_{n_k}\| < \frac{1}{2^{k+1}} \quad \forall k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, take $x_k \in X_1$ such that $T_G(x_k) = v_{n_{k+1}} - v_{n_k}$. By (24), for each $k \in \mathbb{N}$ there exists $u_k \in T_G^{-1}(0)$ such that

$$\|x_k - u_k\| \leq \frac{2\kappa}{\tau_0}\|T_G(x_k - u_k)\| = \frac{2\kappa}{\tau_0}\|T_G(x_k)\| = \frac{2\kappa}{\tau_0}\|v_{n_{k+1}} - v_{n_k}\| < \frac{\kappa}{\tau_0 2^k}.$$

Since X_1 is a closed subspace of the Banach space X, there exists $\bar{u}_1 \in X_1$ such that $\sum_{k=1}^{\infty}(x_k - u_k) = \bar{u}_1$. It follows that

$$\sum_{k=1}^{\infty}(v_{n_{k+1}} - v_{n_k}) = \sum_{k=1}^{\infty} T_G(x_k - u_k) = T_G(\bar{u}_1) \in T_G(X_1),$$

that is, $\lim_{k\to\infty} v_{n_k} = T_G(\bar{u}_1) + v_{n_1} \in T_G(X_1)$. Hence the Cauchy sequence $\{v_n\}$ in $T_G(X_1)$ converges to $T_G(\bar{u}_1) + v_{n_1}$. This shows that $T_G(X_1)$ is complete and hence closed in Y. The proof is complete. $\square$

Recall that a set-valued mapping F between two Banach spaces X and Y is globally metrically subregular at $\bar{y} \in F(X)$ if there exists $\kappa \in (0, +\infty)$ such that

$$d(x, F^{-1}(\bar{y})) \leq \kappa\|\bar{y} - F(x)\| \quad \forall x \in X.$$

The following theorem provides a characterization for a piecewise linear mapping to be globally metrically subregular.

Theorem 3.4. *Let X and Y be Banach spaces, and let $G : X \to Y$ be a piecewise linear mapping. Then G is globally metrically subregular at $\bar{y} \in G(X)$ if and only if* $\lim\limits_{d(x,G^{-1}(\bar{y}))\to+\infty} \|\bar{y} - G(x)\| = +\infty$.

Proof. Since the necessity part is trivially true, we only need to prove the suffciency part. To prove this, suppose that

$$\lim_{d(x,G^{-1}(\bar{y}))\to+\infty} \|\bar{y} - G(x)\| = +\infty. \tag{25}$$

By Lemma 2.4, there exist closed subspaces X_1 and X_2 of X, a closed subspace $\hat{Y}$ of Y, a continuous linear operator $T_G : X_1 \to Y$ and a piecewise linear mapping $g : X_2 \to \hat{Y}$ such that (5) and (6) hold. It follows that $G(X) = T_G(X_1) + g(X_2)$. Hence there exist $\bar{y}_1 \in T_G(X_1)$ and $\bar{y}_2 \in g(X_2)$ such that $\bar{y} = \bar{y}_1 + \bar{y}_2$. By (5) and Lemma 2.5, $G^{-1}(\bar{y}) = T_G^{-1}(\bar{y}_1) + g^{-1}(\bar{y}_2)$ and there exists $\tau > 0$ such that

$$d(x_1 + x_2, G^{-1}(\bar{y})) \geq \tau(d(x_1, T_G^{-1}(\bar{y}_1)) + d(x_2, g^{-1}(\bar{y}_2))) \quad \forall (x_1, x_2) \in X_1 \times X_2.$$

Thus, taking $\bar{x}_1 \in T_G^{-1}(\bar{y}_1)$ and $\bar{x}_2 \in g^{-1}(\bar{y}_2)$, one has

$$d(x_1 + \bar{x}_2, G^{-1}(\bar{y})) \geq \tau d(x_1, T_G^{-1}(\bar{y}_1)) \quad \forall x_1 \in X_1 \tag{26}$$

and $d(\bar{x}_1 + x_2, G^{-1}(\bar{y})) \geq \tau d(x_2, g^{-1}(\bar{y}_2))$ for all $x_2 \in X_2$. Noting that

$$\|\bar{y} - G(\bar{x}_1 + x_2)\| = \|\bar{y} - T_G(\bar{x}_1) - g(x_2)\| = \|\bar{y}_2 - g(x_2)\| \quad \forall x_2 \in X_2,$$

it follows from (25) that $\lim\limits_{d(x_2, g^{-1}(\bar{y}_2)) \to +\infty} \|\bar{y}_2 - g(x_2)\| = +\infty$. Since $g : X_2 \to \hat{Y}$ is piecewise linear and $\hat{Y}$ is finite dimensional, Proposition 2.1 implies that gph(g) is the union of finitely many convex polyhedra in the product $X_2 \times \hat{Y}$. Hence, by [17, Theorem 3.2], there exists $\kappa_1 \in (0, +\infty)$ such that

$$d(x_2, g^{-1}(\bar{y}_2)) \leq \kappa_1 \|\bar{y}_2 - g(x_2)\| \quad \forall x_2 \in X_2.$$

Noting that $G^{-1}(\bar{y}) = T_G^{-1}(\bar{y}_1) + g^{-1}(\bar{y}_2)$ and that there exists $\tau > 0$ such

$$\begin{aligned}\|\bar{y} - G(x_1 + x_2)\| &= \|\bar{y}_1 + \bar{y}_2 - T_G(x_1) - g(x_2)\| \\ &\geq \tau(\|\bar{y}_1 - T_G(x_1)\| + \|\bar{y}_2 - g(x_2)\|\end{aligned}$$

for all $(x_1, x_2) \in X_1 \times X_2$ (thanks to (5), (6) and Lemma 2.5), it suffices to show that there exists $\kappa_2 \in (0, +\infty)$ such that

$$d(x_1, T_G^{-1}(\bar{y}_1)) \leq \kappa_2 \|\bar{y}_1 - T_G(x_1)\| \quad \forall x_1 \in X_1.$$

To show this, suppose to the contrary that there exists a sequence $\{u_n\}$ in X_1 such that

$$d(u_n, T_G^{-1}(\bar{y}_1)) > n\|\bar{y}_1 - T_G(u_n)\| \quad \forall n \in \mathbb{N}.$$

Since $d(u_n, T_G^{-1}(\bar{y}_1)) = d(0, T_G^{-1}(\bar{y}_1) - u_n) = d(0, T_G^{-1}(\bar{y}_1 - T_G(u_n)))$, we have

$$d\Big(0, T_G^{-1}\Big(\frac{\bar{y}_1 - T_G(u_n)}{\|\bar{y}_1 - T_G(u_n)\|}\Big)\Big) > n \quad \forall n \in \mathbb{N}.$$

For $n \in \mathbb{N}$, take $v_n \in T_G^{-1}\left(\frac{\bar{y}_1 - T_G(u_n)}{\|\bar{y}_1 - T_G(u_n)\|}\right)$. Then

$$\|T_G(v_n)\| = 1 \tag{27}$$

and
$$T_G^{-1}\left(\frac{\bar{y}_1 - T_G(u_n)}{\|\bar{y}_1 - T_G(u_n)\|}\right) = v_n + T_G^{-1}(0) = v_n - \bar{x}_1 + T_G^{-1}(\bar{y}_1)$$

(thanks to $\bar{x}_1 \in T_G^{-1}(\bar{y}_1)$). Therefore

$$d(\bar{x}_1 - v_n, T_G^{-1}(\bar{y}_1)) = d\Big(0, T_G^{-1}\Big(\frac{\bar{y}_1 - T_G(u_n)}{\|\bar{y}_1 - T_G(u_n)\|}\Big)\Big) > n \quad \forall n \in \mathbb{N}.$$

This and (26) imply that $d(\bar{x}_1 - v_n + \bar{x}_2, G^{-1}(\bar{y})) \to +\infty$. Hence, by (25), one has $\lim\limits_{n \to +\infty} \|\bar{y} - G(\bar{x}_1 - v_n + \bar{x}_2)\| = +\infty$. Noting that $\bar{x}_1 \in T_G^{-1}(\bar{y}_1) \subset X_1$ and $v_n \in T_G^{-1}\left(\frac{\bar{y}_1 - T_G(u_n)}{\|\bar{y}_1 - T_G(u_n)\|}\right) \subset X_1$,

$$G(\bar{x}_1 - v_n + \bar{x}_2) = T_G(\bar{x}_1 - v_n) + g(\bar{x}_2) = \bar{y}_1 - T_G(v_n) + \bar{y}_2 = \bar{y} - T_G(v_n)$$

(thanks to (6)). It follows that $\lim\limits_{n \to \infty} \|T_G(v_n)\| = +\infty$, contradicting (27). The proof is complete. $\square$

The following corollary is immediate from Theorems 3.1 and 3.2.

Corollary 3.5. *Let $G : X \to Y$ be a piecewise linear mapping between two Banach spaces X and Y, and let $\bar{y} \in G(X)$ be such that* $\lim\limits_{d(x,G^{-1}(\bar{y}))\to+\infty} \|\bar{y} - G(x)\| = +\infty$. *Then G has the L-closed range property.*

Next we consider the metric regularity, which is stronger than the metric subregularity. The following proposition provides a dual characterization for a piecewise linear mapping to be metrically regular.

Proposition 3.6. *Let X and Y be Banach spaces, and let $G : X \to Y$ be a piecewise linear mapping. Then G is metrically regular at $(\bar{x}, G(\bar{x}))$ if and only if G has the L-closed range property and $D^*G(\bar{x})^{-1}(0) = \{0\}$.*

Proof. By Theorem 3.1, the metric regularity of G at $(\bar{x}, G(\bar{x}))$ implies that G has the L-closed range property. Thus, by Lemmas 2.4 and 2.7 and Proposition 3.1, we can assume without loss of generality that there exist closed subspaces X_1 and X_2 of X, closed subspaces Y_0 and $\hat{Y}$ of Y, $T_G \in \mathcal{L}(X_1, Y_0)$ and a piecewise linear mapping $g : X_2 \to \hat{Y}$ such that (5) and (6) hold, $T_G(X_1)$ is closed,

$$Y = Y_0 + \hat{Y},\ Y_0 \cap \hat{Y} = \{0\},\ \ T_G(X_1) \subset Y_0 \tag{28}$$

and

$$D^*G(\bar{x})(y_0^* \oplus \hat{y}^*) = \{T_G^*(y_0^*) \oplus x_2^* :\ x_2^* \in D^*g(\bar{x}_2)(\hat{y}^*)\} \quad \forall (y_0^*, \hat{y}^*) \in Y_0^* \times \hat{Y}^* \tag{29}$$

where $\bar{x}_2 \in X_2$ such that $\bar{x}_1 := \bar{x} - \bar{x}_2 \in X_1$.

First suppose that G is metrically regular at $(\bar{x}, G(\bar{x}))$. Then, from (5), (6), (28) and Lemma 2.5, it is easy to verify that T_G and g are metrically regular at $(\bar{x}_1, T_G(\bar{x}_1))$ and $(\bar{x}_2, g(\bar{x}_2))$, respectively. Hence $T_G(X_1) = Y_0$ and $D^*g(\bar{x}_1)^{-1}(0) = \{0\}$ (thanks to Lemma 2.6). By [14, Theorem 4.12], one has $(T_G^*)^{-1}(0) = \{0\}$. It follows from (29) that $D^*G(\bar{x})^{-1}(0) = \{0\}$. This shows the necessity part.

To prove the sufficiency part, suppose that $D^*G(\bar{x})^{-1}(0) = \{0\}$. Then, by (29), $D^*g(\bar{x}_2)^{-1}(0) = \{0\}$ and $(T_G^*)^{-1}(0) = \{0\}$. Hence, by Lemma 2.6, there exist numbers $\kappa_1, \delta \in (0,\ +\infty)$ such that

$$d(x_2, g^{-1}(\hat{y})) \le \kappa_1\|\hat{y} - g(x_2)\| \quad \forall (x_2, \hat{y}) \in B_{X_2}(\bar{x}_2, \delta) \times B_{\hat{Y}}(g(\bar{x}_2), \delta).$$

By the closedness of $T_G(X_1)$ and [14, Theorems 4.12 and 4.14], one also has $T_G(X_1) = Y_0$. Noting that X_1 and Y_0 are respectively closed subspaces of Banach spaces X and Y, it follows from the open mapping theorem that there exists $\kappa_2 \in (0,\ +\infty)$ such that

$$d(x_1, T_G^{-1}(y_0)) \le \kappa_2\|y_0 - T_G(x_1)\| \quad \forall (x_1, y_0) \in X_1 \times Y_0.$$

Thus, from (5), (6), (28) and Lemma 2.5, it is easy to verify that G is metrically regular at $(\bar{x}, G(\bar{x}))$. This shows the sufficiency part. □

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