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A Condition Number Theorem for Variational Inequalities under Operator Perturbations

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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Nonlinear variational inequalities in Banach spaces are considered with fixed right-hand side and constraint set. Condition number and distance between inequalities are defined taking into account perturbations of the operators defining the inequalities. It is shown that the distance of suitably restricted inequalities to those which are ill-conditioned is bounded below and above by multiples of the reciprocal of the condition number. This extends known results for variational inequalities when the right-hand side is perturbed, and for optimisation problems.

Keywords: Variational inequalities, condition number theorems.

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1. Introduction

We consider variational inequalities on Banach spaces, defined by nonlinear operators acting on a fixed constraint set, and with a fixed right-hand side. Aim of this paper is to find a proper setting in order to obtain a version of the condition number theorem under perturbations of the operator which defines the inequality.

The condition number theorem of linear algebra (see [3, Theorem 1.7 and Corollary 1.8]) shows that the distance of a matrix to the set of singular matrices equals the reciprocal of its condition number. Generalisations of this basic result have been obtained in [6, 7, 8, 9] for scalar optimisation problems, and in [10, 11] for variational inequalities with a fixed operator, with respect to the right-hand side, in the form of a pair of inequalities linking from above and below the distance to ill-conditioning with the reciprocal of the condition number. We call *condition number theorem* any such pair of inequalities.

In analogy with such properties, the main results of this paper show that the distance of a given (suitably restricted) variational inequality from ill-conditioning is bounded, below and above, by multiples of the reciprocal of its condition number. It is appropriate to quote [2, p. 232]: " A very general theme in numerical analysis is a relationship between the condition number of a problem and the reciprocal of the distance to the set of ill-conditioned problems."

The distance, we consider here, between two variational inequalities is given by any fixed norm of the difference between the corresponding operators. The (absolute)

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condition number with respect to the operator is defined as the upper limit, as the perturbations vanish, of the ratio between the output in the solution set and the input errors, due to small perturbations acting on the operators. Then we identify suitable classes of variational inequalities within which lower and upper estimates hold, as discussed above.

Such new results are of interest in the analysis of computational complexity (see [2] and [3]), of stability and sensitivity issues for variational inequalities, moreover in the evaluation of the performance of numerical methods of solution.

After the preliminaries in Section 2, we obtain in Section 3 the lower bound, and in Section 4 the upper one. The proof of the lower bound is direct, while Preiss' mean value theorem for Lipschitz functions (see [5]) is used to get the upper bound. The proofs are collected in Section 5.

2. Notations and basic definitions

We consider a real Banach space X with topological dual X^*, a fixed nonempty set $K \subset X$, a fixed point $f \in X^*$, two further real Banach spaces W_1, W_2 of single-valued continuous operators,

W_1 of functions $$A : K \to X^*, \tag{1}$$

W_2 of functions $$A : X \to X^*, \tag{2}$$

and a nonempty subset Ω of W_1 or of W_2. We denote by $\|\cdot\|$ the norm and by $< \cdot, \cdot >$ the duality pairing for any Banach space we consider. We assume that if any sequence $c_n \to c$ strongly in X^* and $A, A_n \in \Omega$ with

$$A_n(x) = c_n,\ A(x) = c \text{ for every } x \in K$$

so that A_n and A are constant operators, then $A_n \to A$ strongly in W_1 or W_2, moreover there exists a fixed constant $q \geq 0$ such that for every constant operator A, say $A(x) = c$ for every x, then $\|A\| = q\|c\|$.

For each $A \in \Omega$ we consider the variational inequality $Q = Q(A)$, to find $u \in K$ such that

$$< A(u), x - u > \geq < f, x - u > \text{ for every } x \in K. \tag{3}$$

For each $A \in \Omega$ we denote by $S(A)$ the (possibly empty) set of all solutions u to (3), so that $S : \Omega \to K$ is the set-valued solution mapping of the variational inequalities we consider. We need firstly a notion of distance among variational inequalities. Given $A_1, A_2 \in \Omega$ with corresponding variational inequalities Q_1, Q_2, the distance between Q_1 and Q_2 is given by

$$\text{dist } (Q_1, Q_2) = \|A_1 - A_2\|. \tag{4}$$

Next we need a notion of condition number of any variational inequality $Q(A)$ as far as small changes of A within Ω are considered. Define the (absolute) condition number of $Q(A)$ (relative to Ω) as

$$\begin{aligned} \text{cond } Q(A) &= \text{ lim sup}^*_{A',A''\to A} \frac{\text{hausd } [S(A''), S(A')]}{\|A'' - A'\|} \\ &= \lim_{\delta\to 0} \sup \left\{ \frac{\text{hausd } [S(A''), S(A')]}{\|A'' - A'\|} : \|A' - A\| < \delta, \|A'' - A\| < \delta \right\}, \end{aligned} \tag{5}$$

where A', A'' belong to Ω, lim sup* means that (in contrast with the standard definition of lim sup) A', A'' are possibly equal to A, moreover the convergence of A', A'' towards A is the strong one, and hausd denotes the Pompeiu-Hausdorff distance (see [4, p.138] for the definition). Thus cond $Q(A)$ is the Lipschitz modulus at A of the solution mapping S.

Given $A \in \Omega$, the variational inequality $Q(A)$ will be called *well-conditioned* iff cond $Q(A) < +\infty$, and the set of well-conditioned variational inequalities (corresponding to operators from Ω) will be denoted by WC. The variational inequality $Q(A)$ will be called *ill-conditioned* iff cond $Q(A) = +\infty$, and the corresponding set will be denoted by IC. Thus $Q(A) \in WC$ if $Q(B)$ has solutions for each B sufficiently close to A, and the solution mapping is Lipschitz near A, according to the standard notion of Lipschitz continuity of set-valued mappings based on the Pompeiu-Hausdorff distance, see [4, p. 148] (the definition in the infinite-dimensional setting is identical to the one in finite dimensions, see [4, p.275]). We have $Q(A) \in IC$ either if $S(A_n) = \emptyset$ for some sequence $A_n \in \Omega$ with $A_n \to A$, or if $S(B)$ is nonempty for each $B \in \Omega$ sufficiently close to A and S is not Lipschitz near A.

Example 2.1. Let X be a real Hilbert space. Let K be bounded, closed and convex, and let $P > 0$ such that $\|x\| \leq P$ for every $x \in K$. Let W_2 be the space of all linear bounded operators as in (2) equipped with the operator norm. Let $\alpha > 0$ and let $\Omega \subset W_2$ be the class of all operators A which are strongly monotone of constant α, namely

$$< A(x) - A(y), x - y > \geq \ \alpha\|x - y\|^2 \text{ for every } x, y \in K.$$

As well known (see [1, Teorema 3.2]), $Q(A)$ has exactly one solution $u = S(A)$ for every $A \in \Omega$. Moreover $Q(A) \in WC$ for every $A \in \Omega$. Indeed, given any pair $A', A'' \in \Omega$ write $u' = S(A'), u'' = S(A'')$. Then for every $x \in K$

$$< A'(u'), x - u' > \geq < f, x - u' >, \ < A''(u''), x - u'' > \geq < f, x - u'' >$$

hence $\quad 0 \ \leq < A'(u') - A''(u''), u'' - u' > = < A'(u') - A'(u''), u'' - u' > +$

$$+ < A'(u'') - A''(u''), u'' - u' > \leq \ -\alpha\|u'' - u'\|^2 + \|A' - A''\|\|u'' - u'\|\|u''\|.$$

It follows directly that $\|u' - u''\| \leq \dfrac{P}{\alpha}\|A' - A''\|$, and hence well-conditioning with cond $Q(A) \leq P/\alpha$.

Example 2.2. Let $X = R$, $K = [0, 1]$, $f = 1$, $A(x) = a \in [0, 1]$ for every x and let Ω be the set of all constant operators $B(x) = b \in [0, 1]$. Then, if $0 < a < 1$ the variational inequality

$$a(x - u) \geq x - u \text{ for every } x \in [0, 1]$$

has the unique solution $S(a) = 1$. If $a = 1$ then $S(1) = [0, 1]$. We get cond $Q(a) = 0$ for every $a \in (0, 1)$. Moreover

$$\limsup{}^*_{\ a', a'' \to 1-} \frac{\text{hausd } [S(a''), S(a')]}{|a'' - a'|} = \ \text{cond } Q(1) \geq$$

$$\geq \ \limsup_{a \to 1-} \frac{\text{hausd } [S(a), S(1)]}{1 - a} = \ \limsup_{a \to 1-} \frac{1}{1 - a} = +\infty$$

hence cond $Q(1) = +\infty$ and $Q(1) \in IC$.

3. Lower bound

Aim of this section is to obtain a lower bound of the distance to ill-conditioning in terms of the reciprocal of the condition number. We consider a given nonempty set

$$\Omega^* \subset \{A \in \Omega : Q(A) \in WC\} \tag{6}$$

and the condition that

$$\Omega^* \text{ is strongly compact in } W_1 \text{ or in } W_2. \tag{7}$$

Denote by $B(A, \epsilon)$ the open ball in Ω of center A and radius ϵ. We need the following lemma.

Lemma 3.1. *If* (7) *is fulfilled, then there exist positive constants $\bar{\epsilon}$ and $\bar{L}$ such that, for every $A \in \Omega^*$, the solution mapping S is Lipschitz of constant $\bar{L}$ in the ball $B(A, \bar{\epsilon})$.*

Example 3.2. Let $X = R^N$ and K be compact and convex. Let W_2 be the space of all linear continuous operators on R^N equipped with the operator norm, and, given positive constants α and M, let $\Omega^* \subset W_2$ be the set of all linear operators A which are strongly monotone of constant α and such that $\|A\| \leq M$. Then, by Example 2.1 and Ascoli-Arzelá' s theorem, Ω^* fulfills (7) within W_2.

We remark that (7) cannot be fulfilled by taking equality in the inclusion (6), since the set of all operators in Ω whose corresponding variational inequalities are well-conditioned is open. This fact is a consequence of the following lemma.

Lemma 3.3. *If $A, \bar{A} \in \Omega, \epsilon > 0, S$ is Lipschitz on $B(A, \epsilon)$ and $\|A - \bar{A}\| \leq \epsilon/2$, then S is Lipschitz near $\bar{A}$.*

The following lower estimate is the main result of this section.

Theorem 3.4. *Let Ω^* fulfill* (6) *and* (7)*. Assume that there exists $T > 0$ such that*

$$\textit{cond } Q(A) \geq T \textit{ for every } A \in \Omega^*. \tag{8}$$

Then for every $A \in \Omega^*$ we have

$$\textit{dist } (Q(A), IC) \geq \frac{T\,\bar{\epsilon}}{4 \textit{ cond } Q(A)}, \tag{9}$$

where $\bar{\epsilon}$ is given by Lemma 3.1.

Remark 3.5. Within Theorem 3.4, a sufficient condition for (8) is that

$$\theta(A) = \liminf_{A', A'' \to A} \frac{\text{hausd } [S(A''), S(A')]}{\|A'' - A'\|} > 0$$

for every $A \in \Omega^*$. Indeed, θ is lower semicontinuous (as well known) on WC, hence θ attains its minimum value on the compact set Ω^* at $\bar{A}$ say, whence

$$0 < T = \theta(\bar{A}) \leq \theta(A) \leq \text{cond } Q(A)$$

for every $A \in \Omega^*$.

4. Upper bound

We start with an upper bound of the distance between a given well-conditioned variational inequality and ill-conditioning defined by the set IC. We make use of the following condition (A1) (where q is as in Section 2).

(A1) There exists $M > 0$ such that $\|A\| \leq qM$ for every $A \in \Omega$, Ω contains all constant operators whose norm is $\leq qM$, and $\|f\| < M$.

Then the first bound we obtain is the following.

Theorem 4.1. *Let X be a real separable Hilbert space, and assume* (A1)*. Then for every $A \in \Omega$ we have*

$$\text{dist } (Q(A), IC) \leq q(M + \|f\|). \tag{10}$$

The last step is to find an upper bound of the condition number. We need the following assumptions (A2), (A3).

(A2) The norms of W_1 and of W_2 are Gâteaux differentiable off the origin.

(A3) For every bounded set $Z \subset \Omega$ there exists $C^* > 0$ such that if $B \in Z$, $u \in X$, $w \in S(B)$ with

$$\|u - w\| \leq \text{ dist } [u, S(B)] + 1,$$

and $H \in W_1$ or W_2 with $\|H\| \leq 1$, then there exists a single-valued selection z of $t \to S(B + tH), t \geq 0$ sufficiently small, such that

$$\|z(t) - w\| \leq C^* t \text{ for every } t.$$

Example 4.2. For an example fulfilling (A1) and (A2) consider $X = R^N, K$ a compact set, $W_1 = L^2(K)$, $q =$(meas $K)^{1/2}$, $M > 0$ and Ω the set of all continuous functions $A : K \to R^N$ such that $\|A\| \leq qM$.

Example 4.3. Let $X = W_2 = R$ so that every $A \in W_2$ is a constant operator. Let $f = 0$ and $K = [0, 1]$. Consider $A > 0$, then $A(x - u) \geq 0$ for every $x \in K$ and some $u \in K$ iff $u = 0 = S(A)$. For any $H \in R$ with $|H| \leq 1$ we have $(A + tH)(x - u) \geq 0$ for every x iff $u = 0$ provided $t > 0$ is sufficiently small. Then $z(t) = S(A + tH) = 0$ works in (A3) with any $C^* > 0$.

Example 4.4. Suppose that X is Hilbert, and let W_2 be the Banach space of all operators acting from X in itself which are linear and continuous, equipped with the operator norm. Let K be closed, bounded and convex, and suppose that $B \in W_2$ is strongly monotone on K with constant $\alpha > 0$, namely

$$< B(x) - B(y), x - y > \geq \alpha\|x - y\|^2 \text{ for every } x, y \in K.$$

Let $H \in W_2$ be with $\|H\| \leq 1$ and $t \in (0, \alpha)$. Then $B + tH$ is still strongly monotone with constant $\alpha - t$, hence $S(B + tH)$ is a unique point if $0 \leq t < \alpha$. Let $w = S(B)$ and consider $z(t) = S(B + tH)$. Then for every $x \in K$

$$< (B + tH)[z(t)], x - z(t) > \geq < f, x - z(t) >, \; < B(w), x - w > \geq < f, x - w >$$

hence $$< B(z(t) - w), w - z(t) > \geq \; t < H[z(t)], z(t) - w >$$

whence $$\alpha\|z(t) - w\|^2 \leq tP\|H\|\|z(t) - w\|$$

where P is a constant such that $\|u\| \leq P$ for each $u \in K$, so that $\|z(t)-w\| \leq Pt/\alpha$, and (A3) is fulfilled with $C^* \leq P/\alpha$.

Remark 4.5. As noticed in [11, Remark 5], assumption (A3) is related to differentiability properties of the solution mapping.

The required upper bound is the following.

Theorem 4.6. *Assume* (A2) *and* (A3). *Then for every A interior to Ω such that $Q(A) \in WC$ we have*

$$cond\ Q(A) \leq C^*. \tag{11}$$

Example 4.7. Let W_2 be the Banach space of all linear bounded operators acting from X to X^*, equipped with the operator norm. Let $\Omega \subset W_2$ be the set of all strongly monotone operators. If $A \in \Omega$ is strongly monotone of constant $\alpha > 0$, and $B \in W_2$, $\|B - A\| < \epsilon$ with $0 < \epsilon < \alpha$, then B is still strongly monotone (of constant $\alpha - \epsilon$). Thus every $A \in \Omega$ is interior, as required by Theorem 4.6.

Remark 4.8. An upper estimate of the so called Lipschitz modulus of S related to its Aubin property at given points, under different assumptions, is obtained in [4, Theorem 5E.3].

By (10) and (11) we get

Corollary 4.9. *Assume* (A1), (A2) *and* (A3), *and let X be a real separable Hilbert space. Then for every A interior to Ω with $Q(A) \in WC$ and $cond\ Q(A) > 0$ we have*

$$dist\ (Q(A), IC) \leq \frac{q\, C^*(M + \|f\|)}{cond\ Q(A)}.$$

Summarizing, the condition number theorem for variational inequalities which has been proved is the following.

Theorem 4.10. *Let $A \in \Omega$ satisfy the assumptions of Theorem* 3.4 *and Corollary* 4.9. *Then*

$$\frac{T\bar{\epsilon}}{4\ cond\ Q(A)} \leq\ dist\ (Q(A), IC) \leq \frac{q\, C^*(M + \|f\|)}{cond\ Q(A)}.$$

5. Proofs

Proof of Lemma 3.1. For each $A \in \Omega^*$ there exist positive constants ϵ, L such that

$$\text{hausd}\ [S(A''), S(A')] \leq L\|A'' - A'\|$$

provided $A', A'' \in \Omega$ and $\|A' - A\| < \epsilon$, $\|A'' - A\| < \epsilon$. From the open covering of Ω^* by the balls $B(A, \epsilon/2), A \in \Omega^*$ we can find, by (7), a finite subcovering by the balls

$$B(A_1, \epsilon_1/2), ..., B(A_N, \epsilon_N/2),$$

and there exist positive numbers $L_1, ..., L_N$ such that

$$\text{hausd}\ [S(A''), S(A')] \leq L_j\|A'' - A'\| \tag{12}$$

provided $A', A'' \in \Omega$ and $\|A' - A_j\| < \epsilon_j$, $\|A'' - A_j\| < \epsilon_j$, $j = 1, ..., N$. Given $A \in \Omega^*$ there exists an integer $n \leq N$ such that

$$\|A - A_n\| < \epsilon_n/2. \tag{13}$$

Consider the positive numbers

$$\bar{L} = \max \{L_j : j = 1, ..., N\}, \epsilon^* = \min \{\epsilon_j : j = 1, ..., N\}.$$

Let $C', C'' \in \Omega$ be such that $\|C' - A\| \le \epsilon^*/2, \ \|C'' - A\| \le \epsilon^*/2$.

Then $\|C' - A\| \le \epsilon_n/2, \|C'' - A\| \le \epsilon_n/2$ and by (13)

$$\|C' - A_n\| \le \|C' - A\| + \|A - A_n\| < \epsilon_n, \text{ similarly } \|C'' - A_n\| < \epsilon_n,$$

hence by (12) $\quad \text{hausd}\, [S(C''), S(C')] \le L_n \|C'' - C'\| \le \bar{L}\|C'' - C'\|$.

Thus S is Lipschitz on the ball $B(A, \epsilon^*/2)$ with constant $\bar{L}$, and both $\bar{\epsilon} = \epsilon^*/2, \bar{L}$ are independent of the choice of $A \in \Omega^*$.

Proof of Lemma 3.3. There exists a positive constant L such that

$$\text{hausd}\, [S(A''), S(A')] \le L\|A'' - A'\| \tag{14}$$

provided $A', A'' \in \Omega$ and $\|A' - A\| < \epsilon, \|A'' - A\| < \epsilon$. Let $C', C'' \in \Omega$ be such that $\|C' - \bar{A}\| < \epsilon/2, \|C'' - \bar{A}\| < \epsilon/2$, then

$$\|C' - A\| \le \|C' - \bar{A}\| + \|\bar{A} - A\| < \epsilon$$

and similarly for C''. Then by (14)

$$\text{hausd}\, [S(C''), S(C')] \le L\|C'' - C'\|$$

showing that S is Lipschitz near $\bar{A}$.

Proof of Theorem 3.4. We argue by contradiction. Suppose that some $A \in \Omega^*$ fulfills

$$\text{dist}\, (Q(A), IC) \text{ cond } Q(A) < T\, \bar{\epsilon}/4. \tag{15}$$

Consider positive constants c', c'' such that, by (15)

$$c' > \text{ dist } (Q(A), IC), \ c'' > \text{ cond } Q(A), c'c'' < T\, \bar{\epsilon}/4. \tag{16}$$

Then there exists $\bar{A} \in \Omega$ such that $Q(\bar{A}) \in IC$ and $\|A - \bar{A}\| < c'$, hence by (16)

$$\|A - \bar{A}\| \text{ cond } Q(A) \le c'c'' < T\, \bar{\epsilon}/4.$$

Moreover there exists $\bar{\delta} > 0$ such that, by (16) and (5)

$$c'' > \sup \left\{ \frac{\text{hausd}\, [S(A''), S(A')]}{\|A'' - A'\|} : A', A'' \in \Omega, \|A' - A\| < \bar{\delta}, \|A'' - A\| < \bar{\delta} \right\}$$

hence by (16) $$\|A - \bar{A}\| \frac{\text{hausd}\, [S(A''), S(A')]}{\|A'' - A'\|} \le c'c'' < T\bar{\epsilon}/4$$

whence $$\|A - \bar{A}\| \le \frac{T\bar{\epsilon}\, \|A'' - A'\|}{4 \text{ hausd}\, [S(A''), S(A')]} \tag{17}$$

provided $A', A'' \in \Omega$ with $\|A' - A\| < \bar{\delta}, \|A'' - A\| < \bar{\delta}$. By (8), for every $\delta > 0$:

$$\frac{T}{2} < T \le \sup \left\{ \frac{\text{hausd}\, [S(A''), S(A')]}{\|A'' - A'\|} : A', A'' \in \Omega, \|A' - A\| < \delta, \|A'' - A\| < \delta \right\}. \tag{18}$$

With $\delta=\bar\delta$ we get the existence of $A', A''\in\Omega$ such that $\|A'-A\| < \bar\delta$, $\|A''-A\| < \bar\delta$ and by (18)

$$\frac{\|A''-A'\|}{\text{hausd }[S(A''),S(A')]} \le \frac{2}{T}$$

hence by (17)

$$\|A-\bar A\| \le \bar\epsilon/2. \tag{19}$$

By (6) and Lemma 3.1 we obtain

$$\text{hausd }[S(A''),S(A')] \le \bar L \|A''-A'\|$$

provided $A', A'' \in \Omega$ and $\|A'-A\| < \bar\epsilon, \|A''-A\| < \bar\epsilon$. By (19) and Lemma 3.3, S is Lipschitz near $\bar A$, which is a contradiction. This proves (9).

If X is a real separable Hilbert space, we identify X with X^*, and write

$$x = (x_1,\dots,x_j,x_{j+1},\dots) \in X, f = (f_1,\dots,f_j,f_{j+1},\dots,) \in X^*$$

having fixed some orthonormal basis of X and of X^*. The proof of Theorem 4.1 requires the following example.

Example 5.1. Assume (A1). Given $A \in \Omega$ and $f = (f_1,\dots,f_j,\dots) \in X^*$ we consider

$$A^*(x) = f, A_n(x) = (f_1+1/n, f_2,\dots,f_j,\dots) \text{ for every } x;\ n=1,2,\dots .$$

Thus, by (A1), A^* and A_n are constant operators in Ω if n is sufficiently large. Indeed, by (A1),

$$\|A_n\| = q\Big(\|f\|^2 + \frac{2f_1}{n} + \frac{1}{n^2}\Big)^{1/2} \le q\left[\|f\| + \Big(\frac{2f_1}{n}+\frac{1}{n^2}\Big)^{1/2}\right] \le qM$$

provided n is sufficiently large. Moreover $A_n \to A^*$ as $n\to+\infty$. We have

$$< A^*(u)-f, x-u >= 0 \text{ for every } x \text{ and } u,$$

so that $S(A^*)=K$. Moreover

$$u\in K \text{ with } < A_n(u)-f, x-u >= (1/n)(x_1-u_1) \ge 0 \text{ for every } x\in K$$

$$\text{iff} \qquad u_1 = \inf\{x_1 : (x_1,x_2,\dots,x_j,\dots)\in K \text{ for some } (x_2,\dots,x_j,\dots)\}. \tag{20}$$

Denote by C the set in brackets of (20). If C is not bounded from below, then by (20) $S(A_n)$ is empty for every n and cond $Q(A^*)=+\infty$, hence $Q(A^*)\in IC$. If C is bounded from below and K does not contain any point of the form

$$(u_1,x_2,\dots,x_j,\dots), \tag{21}$$

u_1 as in (20), then again cond $Q(A^*)=+\infty$, hence $Q(A^*)\in IC$. If C is bounded from below and K contains points of the form (21), the set of such points is $S(A_n)$ which is independent of n. Thus hausd $[S(A_n),S(A^*)]$ is a (possibly infinite) constant, hence cond $Q(A^*)=+\infty$, and again $Q(A^*)\in IC$.

Proof of Theorem 4.1. Given $A\in\Omega$ we consider $A^*\in\Omega$ given by Example 5.1. Then, as shown there, $Q(A^*)\in IC$ and, by (4) and (A1),

$$\text{dist }(Q(A),IC) \le \text{ dist }[Q(A),Q(A^*)] = \|A-A^*\| \le \|A\|+\|A^*\| \le q\,(M+\|f\|)$$

as required.

Proof of Theorem 4.6. Since $Q(A) \in WC$, the solution mapping S is Lipschitz near A. Then there exists a constant $L > 0$ such that, for every $A', A'' \in \Omega$ sufficiently close to A,

$$S(A') \subset S(A'') + L\|A' - A''\|U, \tag{22}$$

where U is the closed unit ball of X. Write

$$\rho(y, B) = \text{ dist } [y, S(B)].$$

Let $u \in S(A)$. By [11, Theorem 16], $\rho(y, B)$ is Lipschitz with respect to B near A, uniformly with respect to y near u. Then there exist positive numbers δ_1, δ_2, with δ_1 sufficiently small, such that if $A', A'' \in \Omega$ and $\|A' - A\| < \delta_1$, $\|A'' - A\| < \delta_1$, $\|y - u\| < \delta_2$, for some constant $\bar{L}$ we have

$$\rho(y, A'') - \rho(y, A') \le \bar{L}\|A'' - A'\|.$$

Then by [5, Theorem 2.4], $\rho(y, \cdot)$ is (by (A2)) Gâteaux differentiable in some set D dense in the ball $B(A, \delta_1)$, and the mean value theorem holds. Then, if we have $A', A'' \in B(A, \delta_1)$ and $\|y - u\| < \delta_2$ we obtain

$$\rho(y, A'') - \rho(y, A') \le \|A'' - A'\| \text{ sup } \{\|\nabla\rho(y, B)\| : B \in D\} \tag{23}$$

where ∇ denotes the Gâteaux gradient with respect to B. Given $B \in D$, $0 < t < 1$, $0 < \epsilon < 1$, there exists $w \in S(B)$ such that

$$\|y - w\| \le \rho(y, B) + \epsilon t. \tag{24}$$

We have $S(B + tH) \neq \emptyset$ if $H \in W_1$ or W_2, $\|H\| \le 1$ and δ_1 and t are sufficiently small, since $Q(A) \in WC$. Moreover

$$\frac{1}{t}[\rho(y, B + tH) - \rho(y, B)] \to < \nabla\rho(y, B), H > \text{ as } t \to 0+. \tag{25}$$

If t is sufficiently small, by (A3) and (24) we get

$$\begin{aligned}\rho(y, B + tH) - \rho(y, B) &\le \|w - y\| + \text{ dist } [w, S(B + tH)] - \rho(y, B) \\ &\le \epsilon t + \text{ dist } [w, S(B + tH)] \le \epsilon t + \|w - z(t)\| \le (\epsilon + C^*)t.\end{aligned}$$

It follows that, by (25),

$$< \nabla\rho(y, B), H > \le \epsilon + C^* \text{ provided } \|H\| \le 1,$$

hence

$$\|\nabla\rho(y, B)\| \le C^* \text{ for every } B \in D. \tag{26}$$

Then by (23) and (26)

$$\rho(y, A'') - \rho(y, A') \le C^*\|A'' - A'\| \tag{27}$$

provided

$$\|A' - A\| < \delta_1, \|A'' - A\| < \delta_1, \|y - u\| < \delta_2. \tag{28}$$

Now let A', A'' be as in (28) and let $y \in S(A')$. By (22)

$$S(A') \subset S(A) + L\|A' - A\|U$$

hence there exists $u \in S(A)$ such that

$$\|y - u\| \leq L\|A' - A\| \leq L\delta_1 < \delta_2$$

provided δ_1 is sufficiently small. Then by (27)

$$\rho(y, A'') \leq C^* \|A'' - A'\| \text{ for every } y \in S(A'),$$

hence the excess $$e[S(A'), S(A'')] \leq C^* \|A' - A''\|$$

and similarly for $e[S(A''), S(A')]$. It follows that

$$\text{hausd } [S(A'), S(A'')] \leq C^* \|A' - A''\| \text{ near } A$$

proving (11).

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