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On Some Uniform Estimates of Gauge Functions with Respect to Domains

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We establish uniform estimates and properties of gauge functions for domains Ω_ε, $\varepsilon \in [0,1]$, defined by the Minkowski sum $\Omega_\varepsilon = \Omega_0 + \varepsilon\Omega$ where Ω_0 and Ω are convex and bounded subsets of $\mathbb{R}^n$. These estimates are in fact needed when one deals with shape derivatives in PDE-constrained shape optimization problems using this Minkowski sum as a deformation as it is done in a recent paper of A. Boulkhemair and A. Chakib [*On a shape derivative formula with respect to convex domains*, J. Convex Analysis 21/1 (2014) 67–87] for example. We first show that this class of domains Ω_ε satisfies the so-called uniform ball property which is equivalent to the positiveness of its reach. Then, we establish the said uniform estimates on the gauge function of Ω_ε and its gradient as well as its hessian, with respect to the parameter ε.

Keywords: Convex domains, gauge functions, support functions, Minkowski sum, uniform ball condition, reach condition, uniform estimates.

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1. Introduction

In convex analysis, the gauge function (called also the Minkowski function), which is a function that recovers a notion of distance on a linear space, is one of the most fundamental concepts, due to its remarkable properties. For example, the gauge function tool is often used to reformulate a purely geometrical problem into an analytical one, such as in the proof of the geometrical Hahn-Banach theorem which is the core result of the theory of separation of convex sets. Also, in the theory of topological vector spaces, gauge functions are called semi-norms and are used to characterize analytically the topology of locally convex spaces. Furthermore, the

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gauge function tool is widely used in many other fields, notably in convex optimization, the gauge duality theory, linear programming, gauge optimization and conic optimization, see for instance [26, 27, 30, 48], the books [20–22, 24, 43, 44] and the references therein. Finally, let us also mention that the gauge functions were used in the recent discussions and developments on the shape derivative formulas established in [3, 4], using the so-called Minkowski deformation of convex domains.

The aim of this work is to establish some uniform estimates and properties of the gauge functions with respect to domains. To be more precise, let D be a large smooth bounded domain in $\mathbb{R}^n$, $n \geq 2$, let Ω_0 and Ω be two open bounded convex sets containing 0 and consider the family of domains $(\Omega_\varepsilon)_\varepsilon$ defined by the Minkowski sum as follows:

$$\Omega_\varepsilon = \Omega_0 + \varepsilon\Omega := \{x + \varepsilon y \;/\; x \in \Omega_0,\ y \in \Omega\}, \quad \varepsilon \in [0, 1[. \tag{1}$$

First, let us note that this family of domains can be considered as a deformation of Ω_0 by Ω. This way of variation of domains has been used in the convexity context in [3, 35] to establish shape derivative formulas which has been applied to solve concrete numerical shape optimization problems in [5, 6, 8, 11–13]. This method of variation of domains is also used for solving Minkowski type problems associated to some geometrical functionals, see for example the references [15, 16, 32, 33]. Now, using this kind of domain variation, the study of the existence of the shape derivative of a cost functional which is constrained by a partial differential equation (called state problem) requires to establish some estimates on the gauge function associated to Ω_ε, which we shall denote by J_ε and is defined as follows:

$$J_\varepsilon(x) = \inf\{\lambda > 0\,,\ x \in \lambda\Omega_\varepsilon\},\ \ x \in \mathbb{R}^n. \tag{2}$$

Indeed, this existence problem strongly depends on the differentiability of the solution of the state problem with respect to the domains Ω_ε, which leads to study the continuity of the function $\varepsilon \mapsto J_\varepsilon$ as well as of its gradient $\varepsilon \mapsto \nabla J_\varepsilon$. In this work, we prove precisely that there exists a constant $C_0 = C_0(D, \Omega_0, \Omega) > 0$ which depends only on Ω_0, Ω and D, such that

$$\max(\|J_\varepsilon - J_0\|_{L^\infty(D)}, \|J_\varepsilon - J_0\|_{L^\infty(\mathbb{S}^{n-1})}) \leq \varepsilon\, C_0,\ \ \forall\varepsilon \in [0, 1[, \tag{3}$$

and

$$\|\nabla J_\varepsilon - \nabla J_0\|_{L^\infty(\mathbb{R}^n)} \leq \varepsilon\, C_0,\ \ \forall\varepsilon \in [0, 1[, \tag{4}$$

where $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$.

To do that, we begin first by showing some crucial results about the geometry of the surfaces $\partial\Omega_\varepsilon = \partial(\Omega_0 + \varepsilon\Omega)$ for $\varepsilon \in [0, 1[$. More precisely, we show that this class of domains satisfies the so-called uniform ball condition in differential geometry, which is defined as follows: let $\varrho > 0$, we say that an open set $\Lambda \subseteq D$ satisfies the ϱ-ball condition if for all $x \in \partial\Lambda$, there exists a unit vector $\mathbf{d}_x$ of $\mathbb{R}^n$ such that

$$B(x - \varrho\mathbf{d}_x, \varrho) \subseteq \Lambda \ \text{ and } \ B(x + \varrho\mathbf{d}_x, \varrho) \subseteq \mathbb{R}^n \backslash \overline{\Lambda}, \tag{5}$$

where $B(z, r)$ denotes the open ball of $\mathbb{R}^n$ with center z and radius r.

At this stage, let us mention that the uniform (exterior/interior) ball condition was considered first by Poincaré [41] in 1890 to avoid the inconveniences of domain singularities due to corners, cuts, or self-intersections. This notion is largely studied in Nonsmooth Variational Analysis. See for example the papers [37, 38, 40], as well as the recent book [46] and the references therein. In fact, it has been known to characterize the $C^{1,1}$-regularity of domains as well as the positiveness of their reach. In this context, let us note that the reach property introduced by Federer in [25] is also known as: proximal smoothness in [14], weak convexity in [31], p-convexity in [9] or prox-regularity in [42] (for more details about these concepts see also the state of art discussed in the book [46]). Finally, let us also mention that some recent discussions and developments about these notions are given in [17, 18].

The uniform ball property of the family of domains Ω_ε allows us to show that the outward unit normal vector on $\partial\Omega_\varepsilon$, denoted by ν_ε, is a Lipschitz function, for which the Lipschitz constant can be uniformly estimated with respect to ε, that is, there exists $\varrho > 0$ such that:

$$\|\nu_\varepsilon(x) - \nu_\varepsilon(y)\| \leq \frac{1}{\varrho}\|x - y\|, \ \ \forall x, y \in \partial\Omega_\varepsilon, \ \ \forall \varepsilon \in [0, 1[. \tag{6}$$

Note that a result similar to (6) has been established in Theorem 15.34 of [46] for the so-called epi-Lipschitz sets.

The result (6) is then used to show the following uniform estimate: let $\sigma > 0$ be a fixed real number and consider the set $K = D \backslash B(0, \sigma)$. Then there exists a constant $C = C(K, \Omega_0, \Omega) > 0$ depending only on K, Ω_0 and Ω, such that:

$$\|J''_\varepsilon\|_{L^\infty(K)} \leq C, \tag{7}$$

where J''_ε is the Hessian matrix of J_ε. These are the main tools used for the proof of (3) and (4).

The paper is organized as follows. In Section 2, we prove that the family of domains Ω_ε satisfies the uniform ball condition, meanwhile we provide a precise statement about the regularity of Ω_ε in terms of positive reach. Then, in the last section, we show the uniform estimates (3) and (4) on the gauge function J_ε and its gradient with respect to ε.

2. Uniform ball property for Minkowski sum

In this section, we prove some useful results about the geometry of the family of domains $\Omega_\varepsilon = \Omega_0 + \varepsilon\Omega$, for $\varepsilon \in [0, 1[$. We first recall some definitions and results on the so-called uniform ball property and the notion of positive reach in differential geometry [17, 18]. Then, we show that the family of domains $(\Omega_\varepsilon)_\varepsilon$ for $\varepsilon \in [0, 1[$, satisfies the uniform ball property. This important result is then used to establish some crucial properties on gauge functions of Ω_ε.

In all what follows, we denote by D a large smooth bounded convex domain of $\mathbb{R}^n$, $n \geq 2$.

2.1. Uniform ball property and the notion of positive reach

We recall first the definition of an open set satisfying what is called the uniform ball property (see e.g. [17, 18]). Then we establish some useful geometrical properties of the domains $\Omega_\varepsilon = \Omega_0 + \varepsilon\Omega$, for $\varepsilon \in [0, 1[$.

Definition 2.1. Let $\varrho > 0$, we say that an open set $\Omega \subseteq D$ satisfies the *ϱ-ball condition* if for all $x \in \partial\Omega$, there exists a unit vector $\mathbf{d}_x$ of $\mathbb{R}^n$ such that:

$$B(x - \varrho\mathbf{d}_x, \varrho) \subseteq \Omega \text{ and } B(x + \varrho\mathbf{d}_x, \varrho) \subseteq \mathbb{R}^n \backslash \overline{\Omega}. \tag{8}$$

where $B(z, r)$ denotes the open ball of $\mathbb{R}^n$ of center z and radius r.
Denote by $\mathcal{B}_\varrho(D)$ the set of all domains $\Omega \subset D$ satisfying the ϱ-ball condition. □

Let us recall some useful properties verified by domains satisfying the ϱ-ball condition (see e.g. [17]). So, let us first recall the so-called cone-property in differential geometry: an open set A is said to have the σ-cone property ($\sigma > 0$) if

$$\forall x \in \partial A,\ \exists\ \xi \text{ unit vector, such that } \forall y \in \overline{A} \cap B(x, \sigma),\ C(y, \xi, \sigma) \subset A,$$

where $C(y, \xi, \sigma)$ denotes the cone of vertex y, direction ξ and dimension σ, defined by

$$C(y, \xi, \sigma) = \{z \in \mathbb{R}^n \ :\ \langle z - y, \xi\rangle \geq \cos(\sigma)\|z - y\| \text{ and } 0 < \|z - y\| < \sigma\}.$$

So, we have the following properties.

Proposition 2.2. *Let $\varrho > 0$ and Let $\Omega \in \mathcal{B}_\varrho(D)$. Then, we have:*

(i) *Ω satisfies the $g^{-1}(\varrho)$-cone property with $g : \alpha \in]0, \frac{\pi}{2}[\mapsto \alpha/\cos(\alpha) \in]0, +\infty[$.*

(ii) *The map $\mathbf{d} : x \in \partial\Omega \mapsto \mathbf{d}_x \in \mathbb{S}^{n-1}$ is well defined, and $1/\varrho$-Lipschitz, where $\mathbb{S}^{n-1}$ is the unit sphere.*

(iii) *$\mathbf{d} = \nu$, where ν is the outward unit normal vector to $\overline{\Omega}$ at points of $\partial\Omega$.*

Then, let us recall the notion of the positiveness of the reach, which is a characterization of the uniform ball condition, introduced by Federer [25].

Definition 2.3. Consider a non-empty subset A of $\mathbb{R}^n$. First we define for any $x \in A$:

$$\operatorname{Reach}_A(x) = \sup\{r \in \mathbb{R}_+ \ /\ B(x, r) \subseteq \mathrm{U}(A)\},$$

where $\mathrm{U}(A) = \{x \in \mathbb{R}^n \ /\ \text{there exists a unique } \alpha \in A \text{ such that } \|x - \alpha\| = \inf_{y \in A} \|x - y\|\}$.
Then we define the *reach of A* by the following quantity:

$$\operatorname{Reach}(A) = \inf_{x \in A} \operatorname{Reach}_A(x).$$

Finally, we say that A has a positive reach if we have $\operatorname{Reach}(A) > 0$.

Example 2.4. A set Ω is convex and closed if and only if $\operatorname{Reach}(\Omega) = +\infty$.

Remark 2.5. Note that if A is a non-empty open subset of $\mathbb{R}^n$, so $\mathrm{U}(A) = A$, then

$$\operatorname{Reach}_A(x) = d_{\partial A}(x), \ \ \forall x \in A,$$

hence $\operatorname{Reach}(A) = \inf_{x \in A} d_{\partial A}(x) = 0$ (see also Remark 4.2 in [25]). Moreover, if $\operatorname{Reach}(A) > 0$, then A is closed. Thus, the notion of reach is of little interest for open sets. This is why some authors often assume that A is closed in Definition 2.3 (see for example [18, 19, 25, 46]). In our case, we will always consider the reach of the boundary ∂A of nonempty bounded sets $A \subset \mathbb{R}^n$. □

An example of a set that has positive reach is the boundary of a domain of class $\mathcal{C}^2$ (see Proposition 14, [45]). This is stated in the following proposition.

Proposition 2.6. *Let $\Omega \subset D$ be an open set of class $\mathcal{C}^2$. Then, $\partial\Omega$ has a positive reach.*

The next result gives some important characterizations, of the uniform ball property, in term of positive reach (we refer to [18], for its proof).

Theorem 2.7. *Let $\varrho > 0$. Consider a non-empty open bounded subset Ω of $\mathbb{R}^n$. Then, we have the following assertions:*

(1) *If there exists $\varrho > 0$ such that $\Omega \in \mathcal{B}_\varrho(D)$, then $\partial\Omega$ has a positive reach, and we have $Reach(\partial\Omega) \geq \varrho$;*

(2) *If $\partial\Omega$ has a positive reach, then $\Omega \in \mathcal{B}_\varrho(D)$, for any $\varrho \in]0, Reach(\partial\Omega)[$, and moreover, if $\partial\Omega$ has a finite positive reach, then Ω satisfies the $Reach(\partial\Omega)$-ball condition i.e. $\Omega \in \mathcal{B}_{Reach(\partial\Omega)}(D)$.*

In other words, we have the following characterization:

$$Reach(\partial\Omega) = \sup\{\varrho > 0 \ / \ \Omega \in \mathcal{B}_\varrho(D)\}. \tag{9}$$

2.2. The reach of a Minkowski sum

In this section, we show that the family of domains Ω_ε defined by the Minkowski sum, involving the parameter $\varepsilon \in [0,1[$, as follows $\Omega_\varepsilon = \Omega_0 + \varepsilon\Omega$, satisfies the uniform ball property based on its characterization in term of the positiveness of the reach of their domains. To do that, we first show that the positive reach condition and the uniform ball property are invariant by homotheties. The proofs of these results are summarized in the following two lemmas.

Lemma 2.8. *Let $\varrho > 0$ and let $\Omega \subset D$ be an open in $\mathcal{B}_\varrho(D)$, then for all $\lambda > 0$, such that $\lambda\Omega \subset D$, we have $\lambda\Omega \in \mathcal{B}_{\lambda\varrho}(D)$.*

Proof. Since $\Omega \in \mathcal{B}_\varrho(D)$, using property (iii) of Proposition 2.2, for all $y \in \partial\Omega$, we have:

$$B(y - \varrho\nu_{\partial\Omega}(y), \varrho) \subseteq \Omega; \quad B(y + \varrho\nu_{\partial\Omega}(y), \varrho) \subseteq \mathbb{R}^n \backslash \overline{\Omega}, \tag{10}$$

where $\nu_{\partial\Omega}$ denotes the outward unit normal vector on $\partial\Omega$. Let $\lambda > 0$ and let us show that $\lambda\Omega \in \mathcal{B}_{\lambda\varrho}(D)$. Let $x \in \partial(\lambda\Omega) = \lambda\partial\Omega$, then there exists $y \in \partial\Omega$, such that $x = \lambda y$. Let us first show that $B(x - \lambda\varrho\nu_{\lambda\partial\Omega}(x), \lambda\varrho) \subseteq \lambda\Omega$. Let $z \in B(x - \lambda\varrho\nu_{\lambda\partial\Omega}(x), \lambda\varrho)$, since $\nu_{\lambda\partial\Omega}(\lambda y) = \nu_{\partial\Omega}(y)$, for all $y \in \partial\Omega$, we get $z/\lambda \in B(y - \varrho\nu_{\partial\Omega}(y), \varrho)$. Therefore by using (10), we obtain that $z \in \lambda\Omega$. It remains to show the second inclusion in (10). Let $z \in B(x + \lambda\varrho\nu_{\lambda\partial\Omega}(x), \lambda\varrho)$, so we have that $z/\lambda \in B(y + \varrho\nu_{\partial\Omega}(y), \varrho)$. Thus, by using (10) we get that $z/\lambda \in \mathbb{R}^n \backslash \overline{\Omega}$. Then $z \in \mathbb{R}^n \backslash \lambda\overline{\Omega}$, which completes the proof of the lemma. □

Lemma 2.9. *Let $\varrho > 0$, and let $\Omega \subset D$ be an open such that $Reach(\partial\Omega) > 0$, so for all $\lambda > 0$ we have*

$$Reach(\lambda\partial\Omega) = \lambda Reach(\partial\Omega). \tag{11}$$

Proof. According to Theorem 2.7, we have that $\Omega \in \mathcal{B}_{\mathrm{Reach}(\partial\Omega)}(D)$. Then, by using Lemma 2.8, for all $\lambda > 0$, we have that $\lambda\Omega \in \mathcal{B}_{\lambda\mathrm{Reach}(\partial\Omega)}(D)$. Thus, by applying the characterization (9), we have that $\mathrm{Reach}(\partial(\lambda\Omega)) = \sup\{\varrho > 0 \ / \ \lambda\Omega \in \mathcal{B}_\varrho(D)\}$. Hence $\mathrm{Reach}(\partial(\lambda\Omega)) \geq \lambda\mathrm{Reach}(\partial\Omega)$. On the other hand, let $\varrho > 0$, such that $\lambda\Omega \in \mathcal{B}_\varrho(D)$, according to Lemma 2.8, we have that $\Omega \in \mathcal{B}_{\varrho/\lambda}(D)$. Thus through the characterization of $\mathrm{Reach}(\partial(\Omega))$, for all $\varrho > 0$ such that $\lambda\Omega \in \mathcal{B}_\varrho(D)$, we get that $\mathrm{Reach}(\partial(\Omega)) \geq \frac{\varrho}{\lambda}$. Hence, $\mathrm{Reach}(\partial(\Omega)) \geq \mathrm{Reach}(\partial(\lambda\Omega))/\lambda$. □

We shall need also the following result proved in Proposition 1 of [34].

Theorem 2.10. *Let $A, B \subset D$ by two open bounded convex sets of class $\mathcal{C}^1$. If ∂A has a positive reach denoted by ϱ_A and ∂B has a positive reach denoted by ϱ_B, then $\partial(A+B)$ has a positive reach, such that*

$$\mathit{Reach}(\partial(A+B)) \geq \varrho_A + \varrho_B.$$

We can now state the main result of this section.

Theorem 2.11. *Let Ω_0 and Ω two open bounded convex sets of class $\mathcal{C}^2$ and assume that $\Omega_\varepsilon = \Omega_0 + \varepsilon\Omega \subset D$, for all $\varepsilon \in [0,1]$. Then we have the following properties:*

(i) *For all $\varepsilon \in [0,1]$, Ω_ε has a positive reach, furthermore we have*

$$\mathit{Reach}(\partial\Omega_\varepsilon) \geq \mathit{Reach}(\partial\Omega_0) + \varepsilon \mathit{Reach}(\partial\Omega).$$

(ii) *There exists a real $\varrho = \varrho(\Omega_0) > 0$, such that $\Omega_\varepsilon \in \mathcal{B}_\varrho(D)$, $\forall \varepsilon \in [0,1]$.*

(iii) *There exists a real $\varrho > 0$, such that we have*

$$\|\nu_\varepsilon(x) - \nu_\varepsilon(y)\| \leq \frac{1}{\varrho}\|x-y\|, \ \ \forall x, y \in \partial\Omega_\varepsilon, \ \ \forall \varepsilon \in [0,1].$$

where ν_ε is the exterior unit normal vector to $\partial\Omega_\varepsilon$.

Proof. Let $\varepsilon \in [0,1]$. As Ω_0 and Ω are of class $\mathcal{C}^2$, so is $\varepsilon\Omega$. Thus, according to Proposition 2.6, we have $\mathrm{Reach}(\partial\Omega_0) > 0$ and $\mathrm{Reach}(\partial(\varepsilon\Omega)) > 0$. Then by using Theorem 2.10, then we get

$$\mathrm{Reach}(\partial\Omega_\varepsilon) \geq \mathrm{Reach}(\partial\Omega_0) + \mathrm{Reach}(\partial(\varepsilon\Omega)).$$

By using Lemma 2.9, we have $\mathrm{Reach}(\partial(\varepsilon\Omega)) = \varepsilon\mathrm{Reach}(\partial(\Omega))$. This proves the first assertion of this theorem.

Now let us show the second assertion (ii). So according to (i), we have shown that

$$\mathrm{Reach}(\partial\Omega_\varepsilon) \geq \mathrm{Reach}(\partial\Omega_0) + \varepsilon\mathrm{Reach}(\partial\Omega) \geq \mathrm{Reach}(\partial\Omega_0), \ \ \forall \varepsilon \in [0,1].$$

Next, applying Theorem 2.7, we have $\Omega_\varepsilon \in \mathcal{B}_\varrho(D)$, for all $\varrho \in]0, \mathrm{Reach}(\partial\Omega_\varepsilon)[$. In particular, we get

$$\Omega_\varepsilon \in \mathcal{B}_{\mathrm{Reach}(\partial\Omega_0)}(D), \ \ \forall \varepsilon \in [0,1].$$

It follows from Proposition 2.2(iii) that the map $\nu_\varepsilon : \partial\Omega_\varepsilon \mapsto \mathbb{S}^{n-1}$ is $1/\mathrm{Reach}(\partial\Omega_0)$-Lipschitz. □

Remark 2.12. Let us note that the uniform ball condition avoid the inconveniences of domains singularities due to corners, cuts, or self-intersections. In fact, it has been used to characterize the $C^{1,1}$-regularity of hyper-surfaces. Let us also mention that this property is stronger than the well-known ϱ-exterior (and ϱ-interior) sphere condition (for $\varrho > 0$) used in control theory, defined as follows, a set Ω is said to satisfy the ϱ-exterior (resp. ϱ-interior) sphere condition if for any $x \in \partial\Omega$ there exists a unit vector $\mathbf{d}_x$ of $\mathbb{R}^n$ such that $B(x + \varrho\mathbf{d}_x, \varrho) \subseteq \mathbb{R}^n\backslash\overline{\Omega}$ (resp. $B(x - \varrho\mathbf{d}_x, \varrho) \subseteq \Omega$) (see e.g. [10,36,38,39]). In this respect, let us note that, according to Theorem 3.7 of [38], the closed epi-Lipschitz or wedged sets (see [38,46], for the definitions) satisfy a local exterior sphere condition if and only if they have a local positive reach (see also Corollary 3.12 of [38] for a more general result).

3. Some uniform estimates of gauge functions with respect to domains

In this section, we show some uniform estimates of the gauge functions with respect to domains. To do that, we recall first some notions and facts about gauge and support functions, we refer for example to [7, 20, 21, 28, 44, 47] for more details or proofs.

3.1. Preliminaries results on gauge and support functions

In what follows, we shall consider bounded convex domains which are neighborhoods of 0, since they are naturally associated to gauge functions. So, let Θ be a bounded convex which is neighborhood of 0. By definition, the gauge function associated to Θ is the function $J_\Theta : \mathbb{R}^n \to \mathbb{R}_+$ given by

$$J_\Theta(x) = \inf\{\lambda > 0\,,\, x \in \lambda\Theta\}.$$

In the following proposition, we summarize the main properties of gauge functions.

Proposition 3.1. *Let $\Theta \subset \mathbb{R}^n$ be an open convex bounded neighborhood of 0. Then, the gauge function J_Θ is a non negative continuous positively homogeneous function of degree 1. More precisely, we have the following properties:*

(i) $J_\Theta(0) = 0,\ J_\Theta(x) > 0,\ \forall x \neq 0,$

(ii) $J_\Theta(tx) = tJ_\Theta(x),\ \forall x \in \mathbb{R}^n,\ \forall t \in \mathbb{R}_+,$

(iii) $J_\Theta(x + y) \leq J_\Theta(x) + J_\Theta(y)$

(iv) $|J_\Theta(x) - J_\Theta(y)| \leq C\|x - y\|$, $\forall x, y \in \mathbb{R}^n$ *where* $C = \sup_{\mathbb{S}^{n-1}} J_\Theta$,

(v) $\Theta = \{x \in \mathbb{R}^n\,;\ J_\Theta(x) < 1\}$ *and* $\partial\Theta = \{x \in \mathbb{R}^n\,;\ J_\Theta(x) = 1\}$,

(vi) *If Θ' is another domain which is convex contains 0 and $\Theta' \subset \Theta$, then, $J_\Theta \leq J_{\Theta'}$.*

We shall also need the following other properties of gauge functions (see [7] for more details or proofs.)

Proposition 3.2. *Let $\Theta \subset \mathbb{R}^n$ be an open convex bounded neighborhood of 0. Let $r > 0$ such that $B(0, r) \subset \Theta$, then its gauge function J_Θ is Lipschitz continuous and one can take $1/r$ as a Lipschitz constant for J_Θ. Moreover, we have the following properties:*

(i) $\|\nabla J_\Theta\| \leq \dfrac{1}{r}$ *a.e. in* $\mathbb{R}^n$.

(ii) $\nu_{\partial\Theta} = \dfrac{\nabla J_\Theta}{\|\nabla J_\Theta\|}$ *a.e. in* $\partial\Theta$.

(iii) $\langle \nu_{\partial\Theta}(x), x\rangle = \dfrac{1}{\|\nabla J_\Theta(x)\|}$, *for almost every* $x \in \partial\Omega$.

(iv) $\langle \nu_{\partial\Theta}(x), x\rangle \geq r$ *for almost every* $x \in \partial\Omega$.

where $\nu_{\partial\Theta}$ denotes the outward unit normal vector on $\partial\Theta$.

One can also show that the regularity of Θ is characterized by the regularity of its gauge function J_Θ. This is stated in the following lemma (we refer to [7, 29], for its proof.)

Lemma 3.3. *Let $\Theta \subset \mathbb{R}^n$ be an open convex bounded neighborhood of 0. Then, Θ is of class C^k, $k \geq 1$, if and only if its gauge function J_Θ is of class C^k in $\mathbb{R}^n \backslash \{0\}$.*

We will need also the following result (see Proposition 4 of [7]).

Lemma 3.4. *Let $\Theta_1, \Theta_2 \subset \mathbb{R}^n$ be two bounded convex domains, such that $\Theta_1 \subset \Theta_2$ and let $r > 0$ such that $B(0, r) \subset \Theta_1$. Then, we have*

$$\sup_{S^{n-1}} |J_{\Theta_1} - J_{\Theta_2}| \leq \frac{1}{r^2} d^H(\overline{\Theta_1}, \overline{\Theta_2}) \tag{12}$$

where d^H denotes the Hausdorff distance.

Remark 3.5. We note that, the above results and properties on gauge functions (Proposition 3.1, Proposition 3.2 and Lemma 3.3) remain true for star-shaped domains, we refer for example to [7], for more details about these results. □

We shall also use the support function, which is a function characterizing closed convex sets and nonempty open convex sets. Let Θ be an open convex bounded subset of $\mathbb{R}^n$. Then, its support function is defined by

$$P_\Theta(x) = \sup_{y \in \Theta} \langle x, y \rangle = \sup_{y \in \overline{\Theta}} \langle x, y \rangle ,$$

where $\langle x, y \rangle$ denotes the standard scalar product of x and y in $\mathbb{R}^n$.

One can check that the support function P_Θ is continuous and convex. Notably, the support functions satisfy the following properties (see for example [44, 47]).

Proposition 3.6. *Let $\Theta \subset D$ be a bounded convex domain. Then*

(i) *P_Θ is continuous, convex and positively homogeneous of degree 1.*

(ii) *$\|\nabla P_\Theta\|_{L^\infty(D)} \leq c$, where $c = \sup_{\mathbb{S}^{n-1}} P_D$.*

3.2. Uniform estimates on the gauge functions

In all what follows, we denote by J_ε the gauge function of Ω_ε, for $\varepsilon \in [0, 1]$. We establish, in this section, some uniform estimates of J_ε and its gradient as well as its hessian, with respect to the parameter ε. This is stated in the following Proposition.

Proposition 3.7. *Let Ω_0 and Ω two opens bounded convex subset of D, containing 0, which are of class $\mathcal{C}^2$, such that $\Omega_\varepsilon = \Omega_0 + \varepsilon\Omega \subset D$, for all $\varepsilon \in [0, 1]$. Let $\sigma > 0$ a fixed real, consider the set $K = D \backslash B(0, \sigma)$, then there exists a constant $C = C(K, \Omega_0, \Omega) > 0$ depending only on K, Ω_0 and Ω, such that we have the following estimates:*

(i) *For all $\varepsilon \in]0, 1[$, we have*

$$\|\nabla J_\varepsilon(x) - \nabla J_\varepsilon(y)\| \leq C(K, \Omega_0, \Omega)\|y - x\|, \;\; \forall x, y \in D \backslash B(0, \sigma),$$

(ii) *For all $\varepsilon \in]0, 1[$, we have $\|J''_\varepsilon\|_{L^\infty(K)} \leq C(K, \Omega_0, \Omega)$, where J''_ε is the Hessian matrix of J_ε.*

Proof. Let $\varepsilon \in [0, 1]$, to show (i), let $\sigma > 0$ and $x, y \in K = D \backslash B(0, \sigma)$, by using the fact that ∇J_ε is a homogeneous function of degree 0, we have

$$\|\nabla J_\varepsilon(x) - \nabla J_\varepsilon(y)\| = \left\| \nabla J_\varepsilon \left(\frac{x}{J_\varepsilon(x)} \right) - \nabla J_\varepsilon \left(\frac{y}{J_\varepsilon(y)} \right) \right\| .$$

Define $x_\varepsilon := x/J_\varepsilon(x)$ and $y_\varepsilon := y/J_\varepsilon(y)$. Using the fact that $\partial\Omega_\varepsilon = \{J_\varepsilon = 1\}$, by homogeneity of the gauge function, one can check that $x_\varepsilon, y_\varepsilon \in \partial\Omega_\varepsilon$. On the other hand, since Ω_0 and Ω are convex domains of class $\mathcal{C}^2$, according to [1], the domain Ω_ε is of class $\mathcal{C}^1$. Thus it follows from Lemma 3.3 that the function J_ε is of class $\mathcal{C}^1$. Moreover, recall that the exterior unit normal vector to $\partial\Omega_\varepsilon$, denoted by ν_ε, can be defined by $\nabla J_\varepsilon/\|\nabla J_\varepsilon\|$ (see Proposition 3.2). Thus, we get

$$\begin{aligned}\|\nabla J_\varepsilon(x) - \nabla J_\varepsilon(y)\| &= \|\nabla J_\varepsilon(x_\varepsilon) - \nabla J_\varepsilon(y_\varepsilon)\| \\ &= \left\| \frac{\nabla J_\varepsilon(x_\varepsilon)}{\|\nabla J_\varepsilon(x_\varepsilon)\|}\|\nabla J_\varepsilon(x_\varepsilon)\| - \frac{\nabla J_\varepsilon(y_\varepsilon)}{\|\nabla J_\varepsilon(y_\varepsilon)\|}\|\nabla J_\varepsilon(y_\varepsilon)\| \right\| \\ &\leq \|\nu_\varepsilon(x_\varepsilon)\|\nabla J_\varepsilon(x_\varepsilon)\| - \nu_\varepsilon(y_\varepsilon)\|\nabla J_\varepsilon(y_\varepsilon)\|\| \\ &\leq \|\nu_\varepsilon(x_\varepsilon) - \nu_\varepsilon(y_\varepsilon)\|\|\nabla J_\varepsilon(x_\varepsilon)\| + \|\nu_\varepsilon(y_\varepsilon)\|\|\|\nabla J_\varepsilon(x_\varepsilon)\| - \|\nabla J_\varepsilon(y_\varepsilon)\|\|.\end{aligned}$$

Using Proposition 3.2, we obtain

$$\langle \nu_\varepsilon(x), x\rangle = \left\langle \frac{\nabla J_\varepsilon(x)}{\|\nabla J_\varepsilon(x)\|}, x \right\rangle = \frac{1}{\|\nabla J_\varepsilon(x)\|}, \quad \text{for all } \ x \in \partial\Omega_\varepsilon.$$

It follows that,

$$\begin{aligned}\|\nabla J_\varepsilon(x) - \nabla J_\varepsilon(y)\| &\leq \|\nu_\varepsilon(x_\varepsilon) - \nu_\varepsilon(y_\varepsilon)\|\|\nabla J_\varepsilon(x_\varepsilon)\| + \left| \frac{1}{\langle \nu_\varepsilon(x_\varepsilon), x_\varepsilon\rangle} - \frac{1}{\langle \nu_\varepsilon(y_\varepsilon), y_\varepsilon\rangle} \right| \\ &\leq \|\nu_\varepsilon(x_\varepsilon) - \nu_\varepsilon(y_\varepsilon)\|\|\nabla J_\varepsilon(x_\varepsilon)\| + \left| \frac{\langle \nu_\varepsilon(x_\varepsilon), x_\varepsilon\rangle - \langle \nu_\varepsilon(y_\varepsilon), y_\varepsilon\rangle}{\langle \nu_\varepsilon(x_\varepsilon), x_\varepsilon\rangle\langle \nu_\varepsilon(y_\varepsilon), y_\varepsilon\rangle} \right|.\end{aligned}$$

Since $0 \in \Omega_0$, let's fix a real $r > 0$ such that $B(0, r) \subset \Omega_0 \subset \Omega_\varepsilon$. Then by convexity of Ω_ε and according to Proposition 3.2, we have that

$$\frac{1}{\|\nabla J_\varepsilon(x)\|} = \langle \nu_\varepsilon(x), x\rangle \geq r, \quad \text{a.e. in } \ x \in \partial\Omega_\varepsilon.$$

Then, we obtain

$$\begin{aligned}&\|\nabla J_\varepsilon(x) - \nabla J_\varepsilon(y)\| \leq \frac{1}{r}\|\nu_\varepsilon(x_\varepsilon) - \nu_\varepsilon(y_\varepsilon)\| + \frac{1}{r^2}\,|\langle \nu_\varepsilon(x_\varepsilon), x_\varepsilon\rangle - \langle \nu_\varepsilon(y_\varepsilon), y_\varepsilon\rangle| \\ &\leq \frac{1}{r}\|\nu_\varepsilon(x_\varepsilon) - \nu_\varepsilon(y_\varepsilon)\| + \frac{1}{r^2}(|\langle \nu_\varepsilon(x_\varepsilon) - \nu_\varepsilon(y_\varepsilon), y_\varepsilon\rangle| + |\langle \nu_\varepsilon(x_\varepsilon), y_\varepsilon - x_\varepsilon\rangle|) \\ &\leq \frac{1}{r}\|\nu_\varepsilon(x_\varepsilon) - \nu_\varepsilon(y_\varepsilon)\| + \frac{1}{r^2}(\|\nu_\varepsilon(x_\varepsilon) - \nu_\varepsilon(y_\varepsilon)\|\|y_\varepsilon\| + \|y_\varepsilon - x_\varepsilon\|).\end{aligned}$$

Now by applying assertion Theorem 2.11(iii) there exists a constant $C = C(\Omega_0, D)$, such that $\|\nu_\varepsilon(x_\varepsilon) - \nu_\varepsilon(y_\varepsilon)\| \leq C\|x_\varepsilon - y_\varepsilon\|$. Therefore, we obtain

$$\|\nabla J_\varepsilon(x) - \nabla J_\varepsilon(y)\| \leq \frac{C}{r}\|x_\varepsilon - y_\varepsilon\| + \frac{C}{r^2}\|x_\varepsilon - y_\varepsilon\|\|y_\varepsilon\| + \frac{C}{r^2}\|y_\varepsilon - x_\varepsilon\|.$$

It remains to estimating the term $\|x_\varepsilon - y_\varepsilon\|$. Indeed, we have

$$\begin{aligned}\|x_\varepsilon - y_\varepsilon\| = \left\| \frac{x}{J_\varepsilon(x)} - \frac{y}{J_\varepsilon(y)} \right\| &= \frac{1}{J_\varepsilon(x)}\,\|x - y\| + \|y\| \left| \frac{1}{J_\varepsilon(x)} - \frac{1}{J_\varepsilon(y)} \right| \\ &\leq \frac{1}{J_\varepsilon(x)}\,\|x - y\| + \frac{1}{J_\varepsilon(x)}\frac{1}{J_\varepsilon(y/\|y\|)}|J_\varepsilon(y) - J_\varepsilon(x)|.\end{aligned}$$

On the other hand, since $B(0,r) \subset \Omega_\varepsilon \subset D$, for all $\varepsilon \in [0,1]$, we have from Proposition 3.2, that

$$|J_\varepsilon(y) - J_\varepsilon(x)| \leq \frac{1}{r}\|y-x\|, \quad \forall x, y \in \mathbb{R}^n, \tag{13}$$

and $1/J_\varepsilon \leq 1/J_D$. Thus, we get

$$\begin{aligned} \|x_\varepsilon - y_\varepsilon\| &\leq \frac{1}{J_D(x)}\|x-y\| + \frac{1}{r}\frac{1}{J_D(x)}\frac{1}{J_D(y/\|y\|)}\|y-x\| \\ &\leq (\sup_K \frac{1}{J_D} + \frac{1}{r}\sup_K \frac{1}{J_D}\sup_{\mathbb{S}^{n-1}}\frac{1}{J_D})\|y-x\|. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\nabla J_\varepsilon(x) - \nabla J_\varepsilon(y)\| &\leq \frac{C}{r}\|x_\varepsilon - y_\varepsilon\| + \frac{C}{r^2}\|x_\varepsilon - y_\varepsilon\| \left\| \frac{y}{J_{\Omega_\varepsilon}(y)} \right\| + \frac{C}{r^2}\|y_\varepsilon - x_\varepsilon\| \\ &\leq \left(\frac{C}{r} + \frac{C}{r^2}\frac{1}{J_D(y/\|y\|)} + \frac{C}{r^2}\right)\|x_\varepsilon - y_\varepsilon\| \\ &\leq \left(\frac{C}{r} + \frac{C}{r^2}\sup_{\mathbb{S}^{n-1}}\frac{1}{J_D} + \frac{C}{r^2}\right)\left(\sup_K \frac{1}{J_D} + \frac{1}{r}\sup_K \frac{1}{J_D}\sup_{\mathbb{S}^{n-1}}\frac{1}{J_D}\right)\|y-x\|. \end{aligned}$$

Thereby there exists a constant $C = C(K, \Omega_0, \Omega) > 0$, such that

$$\|\nabla J_\varepsilon(x) - \nabla J_\varepsilon(y)\| \leq C(K,\Omega_0,\Omega)\|y-x\|, \;\; \forall x, y \in D\backslash B(0,\sigma), \;\; \forall \varepsilon \in]0,1[. \tag{14}$$

Let us now prove (ii) using estimation (14). By Rademacher's theorem [24] the hessian matrix of J_ε is defined almost everywhere in $D\backslash B(0,\sigma)$. Furthermore we have the uniform estimation:

$$\|J''_\varepsilon(x)\| \leq C(K,\Omega_0,\Omega), \;\; \forall x \in D\backslash B(0,\sigma), \;\; \forall \varepsilon \in]0,1[. \qquad \square$$

Now, we establish the continuity of the gauge function J_ε as well as of its gradient, with respect to the parameter ε. The proof of this result is based on the fact that the hessian J''_ε is uniformly bounded, which is proved in Proposition 3.7, as well as on the derivative formulas for the functions $\varepsilon \mapsto J_\varepsilon$ and $\varepsilon \mapsto \nabla J_\varepsilon$, which are stated in the following Proposition (see Lemma 2 in [2]).

Proposition 3.8. *Let Ω_0 and Ω tow open bounded convex which are neighborhoods of* 0 *and are of class $\mathcal{C}^2$, such that $\Omega_\varepsilon = \Omega_0 + \varepsilon\Omega \subset D$, for all $\varepsilon \in [0,1]$. Then, we have the following formulas:*

(i) *For all $\varepsilon \in [0,1]$ and for almost all $x \in \mathbb{R}^n\backslash\{0\}$, we have*

$$J_\varepsilon(x) - J_0(x) = -\int_0^\varepsilon J_s(x)P_\Omega(\nabla J_s(x))ds.$$

(ii) *For all $i = 1, ..., n$, for all $\varepsilon \in [0,1]$ and for almost all $x \in \mathbb{R}^n\backslash\{0\}$, we have*

$$\partial_i J_\varepsilon(x) - \partial_i J_0(x) = -\int_0^\varepsilon [\partial_i J_s(x)P_\Omega(\nabla J_s(x)) + J_s(x)\partial_i \nabla J_s(x)^\top \nabla P_\Omega(\nabla J_s(x))]ds.$$

Let us now prove the following result.

Proposition 3.9. *Let Ω_0 and Ω tow open bounded convex which are neighborhoods of* 0 *and are of class $\mathcal{C}^2$, such that $\Omega_\varepsilon = \Omega_0 + \varepsilon\Omega \subset D$, for all $\varepsilon \in [0,1]$. Then, we have the following estimations:*

(i) *There exists a constant $C_0 > 0$ that depends only on Ω_0, Ω and D, such that*

$$\max(\|J_\varepsilon - J_0\|_{L^\infty(D)}, \|J_\varepsilon - J_0\|_{L^\infty(\mathbb{S}^{n-1})}) \le \varepsilon\, C_0(D, \Omega_0, \Omega), \ \ \forall \varepsilon \in]0,1[.$$

(ii) *There exists a constant $C_1 > 0$ that depends only on Ω_0, Ω and D, such that*

$$\|\nabla J_\varepsilon - \nabla J_0\|_{L^\infty(\mathbb{R}^n)} \le \varepsilon\, C_1(D, \Omega_0, \Omega), \ \ \forall \varepsilon \in]0,1[.$$

Proof. To show (i), since the domains Ω_0 and Ω are neighborhoods of 0, let us fix a real number $r > 0$ such that $B(0,r) \subset \Omega_0 \subset \Omega_\varepsilon$. By applying Lemma 3.4, we get

$$\|J_\varepsilon - J_0\|_{L^\infty(\mathbb{S}^{n-1})} \le \frac{1}{r^2}\, d^H(\overline{\Omega}_0, \overline{\Omega}_\varepsilon).$$

By using the properties of the Hausdorff distance (see page 64 of [44] or Lemma 9 of [7]), we get

$$\|J_\varepsilon - J_0\|_{L^\infty(\mathbb{S}^{n-1})} \le \frac{1}{r^2}\, d^H(\overline{\Omega}_0, \overline{\Omega}_0 + \varepsilon\overline{\Omega}) \le \varepsilon C_0(D, \Omega_0, \Omega),$$

where $C(\Omega_0, \Omega) = \frac{1}{r^2} d^H(\{0\}, \overline{\Omega})$. Then, by the homogeneity of J_ε, we obtain

$$\|J_\varepsilon - J_0\|_{L^\infty(D)} \le \sup_{x\in D} \|x\|\, \|J_\varepsilon - J_0\|_{L^\infty(\mathbb{S}^{n-1})} \le \varepsilon C_0(D, \Omega_0, \Omega),$$

where $C_0(D, \Omega_0, \Omega) = \sup\limits_{x\in D} \|x\|\, C(\Omega_0, \Omega)$.

To prove (ii): since Ω_0 is a convex domain of class $\mathcal{C}^2$ and Ω is convex domain, according to [1], the domain Ω_ε is of class $\mathcal{C}^1$, thus it follows from Lemma 3.3 that the function J_ε is of class $\mathcal{C}^1$. On the other hand, using the estimation (i) of Proposition 3.7, the function ∇J_ε is Lipschitz. Thus J_ε is of class $\mathcal{C}^{1,1}$. Then, according to Proposition 3.8, for almost all $x \in \mathbb{R}^n$, we have

$$\nabla J_\varepsilon(x) - \nabla J_0(x) = -\int_0^\varepsilon [\nabla J_s(x) P_\Omega(\nabla J_s(x)) + J_s(x) J_s''(x)^\top \nabla P_\Omega(\nabla J_s(x))] ds.$$

Since the gauge function J_s is homogeneous of degree 1, the hessian J_s'' is homogenous of degree -1, so we obtain $J_s(x)J_s''(x)^\top = J_s(x/\|x\|)J_s''(x/\|x\|)^\top$. On the other hand, using Proposition 3.6, we have $\|\nabla P_\Omega\|_{L^\infty(\mathbb{R}^n)} \le c$, where $c = \sup_{\mathbb{S}^{n-1}} P_D$. Therefore, we get

$$\begin{aligned}\|\nabla J_\varepsilon(x) - \nabla J_0(x)\| &\le \int_0^\varepsilon [\|\nabla J_s(x)\|^2 |P_\Omega(\nabla J_s(x)/\|\nabla J_s(x)\|)| \\ &\quad + \|J_s(x/\|x\|) J_s''(x/\|x\|)^\top \|\nabla P_\Omega\|_{L^\infty(\mathbb{R}^n)}] ds \\ &\le \int_0^\varepsilon [c\|\nabla J_s(x)\|^2 \sup_{\mathbb{S}^{n-1}} P_D + c \sup_{\mathbb{S}^{n-1}} J_s \|J_s''(x/\|x\|)\|] ds\end{aligned}$$

As $0 \in \Omega_0$, let us fix a real $r > 0$ such that $B(0,r) \subset \Omega_0 \subset \Omega_\varepsilon$. By convexity of Ω_ε, according to Proposition 3.2, we have that

$$\|\nabla J_s(x)\| \le \frac{1}{r} \quad \forall x \in \mathbb{R}^n, \quad \forall s \in [0, \varepsilon]. \tag{15}$$

On the other hand, we have $B(0,r) \subset \Omega_\varepsilon \subset D$, for almost all $\varepsilon \in [0,1]$, so that $J_s \leq J_{B(0,r)}$, $\forall s \in [0,\varepsilon]$. Hence, we obtain

$$\|\nabla J_\varepsilon(x) - \nabla J_0(x)\| \leq \varepsilon \, \frac{c}{r^2} \sup_{\mathbb{S}^{n-1}} P_D + c \sup_{\mathbb{S}^{n-1}} J_{B(0,r)} \int_0^\varepsilon \|J_s''(x/\|x\|)\| ds.$$

Let $K = D \backslash B(0,\sigma)$, $(\sigma > 0)$ a neighborhood of the sphere $\mathbb{S}^{n-1}$, so it follows from assertion (ii) of Proposition 3.7, that there exists a constant $C > 0$ that depends on K, Ω_0 and Ω, such that $\|J_\varepsilon''\|_{L^\infty(K)} \leq C$, $\forall \varepsilon \in]0,1[$. Therefore, for a.e. $x \in \mathbb{R}^n$, we obtain

$$\|\nabla J_\varepsilon(x) - \nabla J_0(x)\| \leq \varepsilon \, \frac{c}{r^2} \sup_{\mathbb{S}^{n-1}} P_D + c \sup_{\mathbb{S}^{n-1}} J_{B(0,r)} \, \varepsilon \, C. \tag{16}$$

which completes the proof. □

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