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Dynamics of a Convex Combination and Superposition of Non-Volterra Quadratic Stochastic Operators

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We consider a convex combination of non-Volterra quadratic stochastic operators defined on the two-dimensional simplex depending on a parameter λ and study their trajectory behaviours. We show that for any $\lambda \in [0,1]$ and for any initial point the set of limit points of the trajectory is a singleton. We also prove that a superposition of these non-Volterra quadratic stochastic operators is a regular transformation.

Keywords: Quadratic stochastic operator, Volterra and non-Volterra operator, trajectory.

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1. Introduction

Quadratic stochastic operators were first introduced by Bernstein in [1]. They are useful in solving problems arising in mathematical genetics, namely, in the theory of heredity (see [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21]).

A quadratic stochastic operator (QSO) works as a population evolution operator which may arise in population genetics as follows. Consider a biological population, that is, a community of organisms closed with respect to reproduction. Assume that each individual in this population belongs precisely to one of the species (genotype) $1, \dots, m$. The scale of species is such that the species of the parents i and j, unambiguously, determine the probability of every species k for the first generation of direct descendants. Denote this probability, called the heredity coefficient, by $p_{ij,k}$. It is then obvious that $p_{ij,k} \geq 0$ for all i, j, k and that

$$\sum_{k=1}^{m} p_{ij,k} = 1, \quad i, j, k = 1, \dots, m.$$

The state of the population can be described by the tuple $(x_1, x_2, \dots, x_m)$ of species probabilities, that is, x_k is the fraction of the species k in the total population. In the case of panmixia (random interbreeding) the parent pairs i and j arise for a fixed state $\mathbf{x} = (x_1, x_2, \dots, x_m)$ with probability $x_i x_j$.

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Hence, the total probability of the species k in the first generation of direct descendants is defined by

$$x'_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j, \quad k = 1, \ldots, m.$$

The association $\mathbf{x} \mapsto \mathbf{x}'$ defines an evolutionary quadratic operator. Thus evolution of a population can be studied as a dynamical system of a quadratic stochastic operator[17]. We refer the reader to [7] for a review on quadratic stochastic operators and to [13] for a review on dynamical systems generated by convex combination of quadratic stochastic operators. It should be noted that the limit behavior of the trajectories of QSO on one-dimensional space was fully studied by Yu. I. Lyubich [16]. However, the problem is still open even in two-dimensional simplex. It is a motivation to study the trajectory of a QSO which is the main task in the present paper.

The paper is organised as follows. In Section 2 we recall definitions and well known results from the theory of Volterra and non-Volterra QSOs. In Section 3 we studied two concrete non-Volterra QSOs. In Section 4 we consider a class of non-Volterra QSOs which is generated by convex combination of given non-Volterra QSOs and show that each QSO from this class might have infinitely many fixed points. Moreover, we prove that such an operator has the regularity property. In Section 5 we investigate a superposition of given non-Volterra QSOs and we prove that any trajectory of the superposition operator is convergent.

2. Preliminaries

Let $\quad S^{m-1} = \{\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : \text{ for any } i,\ x_i \geq 0, \text{ and } \sum_{i=1}^{m} x_i = 1\}$

be the $(m-1)$-dimensional simplex. A map V of S^{m-1} into itself is called a *quadratic stochastic operator* (QSO) if

$$(V\mathbf{x})_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j$$

for any $\mathbf{x} \in S^{m-1}$ and for all $k = 1, \ldots, m$, where

$$p_{ij,k} \geq 0, \quad p_{ij,k} = p_{ji,k} \text{ for all } i, j, k; \quad \sum_{k=1}^{m} p_{ij,k} = 1.$$

Assume $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$ is the trajectory of the point $\mathbf{x} \in S^{m-1}$, where $\mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)})$ for all $n = 0, 1, 2, \ldots$, with $\mathbf{x}^{(0)} = \mathbf{x}$.

Definition 2.1. A point $\mathbf{x} \in S^{m-1}$ is called a *fixed point* of a QSO V if $V(\mathbf{x}) = \mathbf{x}$. Denote the set of all fixed points of a QSO V by $\mathrm{Fix}(V)$.

Definition 2.2. A QSO V is called *regular* if for any initial point $\mathbf{x} \in S^{m-1}$, the limit $\lim_{n\to\infty} V(\mathbf{x}^{(n)})$ exists.

Note that the limit point is a fixed point of a QSO. Thus, the fixed points of a QSO describe limit or long run behavior of the trajectories for any initial point. The limit behavior of trajectories and fixed points play an important role in many applied problems (see [4, 6, 7, 17, 18]). The biological treatment of the regularity of a QSO is rather clear: in the long run the distribution of species in the next generation coincides with the distribution of species in the previous one, i.e., it is stable.

The following notations will be used throughout this paper:
the *boundary* of S^{m-1} is $\partial S^{m-1}=\{\mathbf{x}\in S^{m-1}: x_i=0 \text{ for at least one } i\in\{1,2,...,m\}\}$;
the *interior* of S^{m-1} is the set $\operatorname{int} S^{m-1}=\{\mathbf{x}\in S^{m-1}: x_1x_2\cdots x_m>0\}$;
$D_{\mathbf{x}}V(\mathbf{x}^*)=(\partial V_i/\partial x_j)(\mathbf{x}^*)$ denotes the Jacobian of V at the point $\mathbf{x}^*$;
$\mathbf{e}_i=(\delta_{1,i},\delta_{2,i},\ldots,\delta_{m,i})\in S^{m-1}$, $i\in\{1,2,...,m\}$, denote the vertices of the simplex S^{m-1}, where $\delta_{i,j}$ is the Kronecker delta.

Definition 2.3. [3]A fixed point $\mathbf{x}^*$ is called *hyperbolic* if its Jacobian $D_{\mathbf{x}}V(\mathbf{x}^*)$ has no eigenvalues on the unit circle.

Definition 2.4. [3]A hyperbolic fixed point $\mathbf{x}^*$ is called:
(i) *attracting* if all the eigenvalues of the Jacobian $D_{\mathbf{x}}V(\mathbf{x}^*)$ are less than 1 in absolute value;
(ii) *repelling* if all the eigenvalues of the Jacobian $D_{\mathbf{x}}V(\mathbf{x}^*)$ are greater than 1 in absolute value;
(iii) a *saddle* otherwise.

3. Dynamics of non-Volterra QSOs

Consider the following two non-Volterra QSOs on the two-dimensional simplex

$$V_1:\begin{cases} x_1'=x_1^2,\\ x_2'=x_2^2+x_3^2+2x_1x_2,\\ x_3'=2x_1x_3+2x_2x_3,\end{cases}\qquad(1)\qquad V_2:\begin{cases} x_1'=x_1^2+x_3^2+2x_1x_2,\\ x_2'=x_2^2,\\ x_3'=2x_1x_3+2x_2x_3.\end{cases}\qquad(2)$$

First we need the following results.

Proposition 3.1. [3] *For the function* $f(x)=2x(1-x)$ *we have* $\operatorname{Fix}(f)=\{0,1/2\}$, $f(1)=f(0)=0$ *and* $\lim\limits_{n\to\infty} f^n(x)=1/2,\quad \forall x\in(0,1)$.

In the following theorem we describe the asymptotical behavior of the quadratic operators (1) and (2).

Theorem 3.2. *For the QSOs* (1) *and* (2) *the following assertions hold:*
(i) *the face* $\Gamma_{\{1,2\}}=\{\mathbf{x}\in S^2: x_3=0\}$ *is an invariant set;*
(ii) $\operatorname{Fix}(V_1)=\{\mathbf{e}_1,\mathbf{e}_2,(0,1/2,1/2)\}$;
(iii) $\operatorname{Fix}(V_2)=\{\mathbf{e}_1,\mathbf{e}_2,(1/2,0,1/2)\}$;
(iv) *if* $\mathbf{x}^{(0)}\in\Gamma_{\{1,2\}}\backslash\{\mathbf{e}_1\}$ *then* $\lim\limits_{n\to\infty} V_1^n\left(\mathbf{x}^{(0)}\right)=\mathbf{e}_2$;
(v) *if* $\mathbf{x}^{(0)}\in\Gamma_{\{1,2\}}\backslash\{\mathbf{e}_2\}$ *then* $\lim\limits_{n\to\infty} V_2^n\left(\mathbf{x}^{(0)}\right)=\mathbf{e}_1$;
(vi) $\lim\limits_{n\to\infty} V_1^n\left(\mathbf{x}^{(0)}\right)=\left(0,\frac{1}{2},\frac{1}{2}\right)$ *for any* $\mathbf{x}^{(0)}\in S^2\backslash\Gamma_{\{1,2\}}$;
(vii) $\lim\limits_{n\to\infty} V_2^n\left(\mathbf{x}^{(0)}\right)=\left(\frac{1}{2},0,\frac{1}{2}\right)$ *for any* $\mathbf{x}^{(0)}\in S^2\backslash\Gamma_{\{1,2\}}$.

Proof. (i) Obvious.

(ii) The equation $V_1(\mathbf{x}) = \mathbf{x}$ has the form

$$\begin{cases} x_1 = x_1^2, \\ x_2 = x_2^2 + x_3^2 + 2x_1x_2, \\ x_3 = 2x_1x_3 + 2x_2x_3, \end{cases} \tag{3}$$

From the first equation of the system (3) one can see that either $x_1 = 0$ or $x_1 = 1$. If $x_1 = 1$ then from the $x_1 + x_2 + x_3 = 1$ we get a fixed point $\mathbf{e}_1$.

Let $x_1 = 0$. From the third equation of system (3) one has either $x_3 = 0$ or $x_3 = 1/2$. If $x_3 = 0$ then from $x_1 + x_2 + x_3 = 1$ we get the fixed point $\mathbf{e}_2$. If $x_3 = 1/2$ then we obtain a fixed point $(0, 1/2, 1/2)$. Thus $\mathrm{Fix}\,(V_1) = \{\mathbf{e}_1, \mathbf{e}_2, (0, 1/2, 1/2)\}$.

(iii) This can be proved similarly.

(iv) Let $\mathbf{x}^{(0)} \in \Gamma_{\{1,2\}} \backslash \{\mathbf{e}_1\}$ be an initial point. Since $\Gamma_{\{1,2\}}$ is an invariant set it follows $x_3^{(n)} = 0$ for all $n = 0, 1, 2, \ldots$ Then using $x_1 + x_2 = 1$ from the second equation of (1) one has $x_2' = x_2\,(2 - x_2)$, that is a trajectory $x_2^{(n)}$ is given by the dynamical system $\theta(x) = x(2 - x)$. One has that $\mathrm{Fix}\,(\theta) = \{0, 1\}$ and $x = 0$ is a repelling point and $x = 1$ is an attracting point. Further, since for any $x \in [0, 1]$ it holds $\theta(x) \geq x$ we have $\lim\limits_{n\to\infty} \theta^n(x) = 1$ for all $x \neq 0$. Therefore it follows $\lim\limits_{n\to\infty} x_2^{(n)} = 1$ for any $x_2^{(0)} \neq 0$. Thus we obtain $\lim\limits_{n\to\infty} V_1^n\left(\mathbf{x}^{(0)}\right) = \mathbf{e}_2$ for all $\mathbf{x}^{(0)} \in \Gamma_{\{1,2\}} \backslash \{\mathbf{e}_1\}$.

(v) This can be proved similarly to part (iv).

(vi) Let $\mathbf{x}^{(0)} \in S^2 \backslash \Gamma_{\{1,2\}}$ be an initial point. Then from the first equation of the (1) one has $x_1' = x_1^2 \leq x_1$. So the sequence $x_1^{(n)}$ is a decreasing and bounded from the below sequence and we obtain

$$\lim_{n\to\infty} x_1^{(n+1)} = \lim_{n\to\infty} \left(x_1^{(0)}\right)^{2^n} = 0.$$

The last equation of (1) can be rewritten in the form $x_3' = 2x_3(1 - x_3)$. Due to Proposition 3.1 we obtain $\lim\limits_{n\to\infty} x_3^{(n)} = 1/2$. Thus we have $\lim\limits_{n\to\infty} V_1^n\left(\mathbf{x}^{(0)}\right) = (0, 1/2, 1/2)$ for any $\mathbf{x}^{(0)} \in S^2 \backslash \Gamma_{\{1,2\}}$.

(vii) This can be proved as in part (vi). □

Corollary 3.3. *The QSOs* (1) *and* (2) *are regular transformations.*

4. Convex combination of the operators

Let us consider the convex combination of given QSOs (1) and (2), that is convex combination of

$$V_\lambda = \lambda V_1 + (1 - \lambda)V_2, \quad \lambda \in [0, 1].$$

It is easy to verify that V_λ has the form

$$V_\lambda : \begin{cases} x_1' = x_1^2 + (1 - \lambda)x_3^2 + 2(1 - \lambda)x_1x_2, \\ x_2' = x_2^2 + \lambda x_3^2 + 2\lambda x_1x_2, \\ x_3' = 2x_1x_3 + 2x_2x_3. \end{cases} \tag{4}$$

Since the QSOs (1) and (2) are regular transformations it is natural to expect that the QSO V_λ also is a regular transformation. Below we will check this hypothesis.

Setting
$$\lambda = \frac{1}{2} + \tau, \quad \tau \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$
one can rewrite the QSO (4) in the following form
$$V_\tau : \begin{cases} x_1' = x_1^2 + \left(\frac{1}{2} - \tau\right) x_3^2 + (1 - 2\tau)x_1x_2, \\ x_2' = x_2^2 + \left(\frac{1}{2} + \tau\right) x_3^2 + (1 + 2\tau)x_1x_2, \\ x_3' = 2x_1x_3 + 2x_2x_3. \end{cases} \tag{5}$$

Theorem 4.1. *Let $\tau = 0$. Then one has*

(i) $\mathrm{Fix}\,(V_0) = \{(1/4, 1/4, 1/2)\} \cup \Gamma_{\{1,2\}}$, *where* $\Gamma_{\{1,2\}} = \{\mathbf{x} \in S^2 : x_3 = 0\}$*;*

(ii) *the fixed point $(1/4, 1/4, 1/2)$ is an attracting point and arbitrary fixed point $\mathbf{x} \in \Gamma_{\{1,2\}}$ is a non-hyperbolic point;*

(iii) $\lim\limits_{n\to\infty} V_0^n\left(\mathbf{x}^{(0)}\right) = (1/4, 1/4, 1/2)$ *for any* $\mathbf{x}^{(0)} \in S^2 \backslash \Gamma_{\{1,2\}}$.

Proof. (i) Let $\tau = 0$. Then the equation $V_0(\mathbf{x}) = \mathbf{x}$ has the form
$$\begin{cases} x_1 = x_1^2 + \frac{1}{2}x_3^2 + x_1x_2, \\ x_2 = x_2^2 + \frac{1}{2}x_3^2 + x_1x_2, \\ x_3 = 2x_1x_3 + 2x_2x_3. \end{cases} \tag{6}$$

From the last equation of system (6) we get
$$x_3 = 2x_3(x_1 + x_2) \quad \Rightarrow \quad x_3 = 2x_3(1 - x_3) \quad \Rightarrow \quad x_3 = 0 \;\text{ and }\; x_3 = \frac{1}{2}.$$

Evidently that any point from the set $\Gamma_{\{1,2\}} = \{\mathbf{x} \in S^2 : x_3 = 0\}$ is a fixed point of the QSO V_0.

From the difference of the first and second equations of the system (6) one has
$$x_1 - x_2 = x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2) \quad \Rightarrow \quad x_1 = x_2.$$

So in the case $x_3 = 1/2$ we obtain $x_1 = x_2 = 1/4$.

Thus $\mathrm{Fix}\,(V_0) = \{(1/4, 1/4, 1/2)\} \cup \Gamma_{\{1,2\}}$.

(ii) To find the type of the fixed point we rewrite the operator V_0 in the form
$$\begin{cases} x_1' = \frac{3x_1^2}{2} + \frac{x_2^2}{2} - x_1 - x_2 + 2x_1x_2 + \frac{1}{2}, \\ x_2' = \frac{x_1^2}{2} + \frac{3x_2^2}{2} - x_1 - x_2 + 2x_1x_2 + \frac{1}{2}, \end{cases}$$
where $(x_1, x_2) \in \{(x_1, x_2) : x_1 \geq 0,\, x_2 \geq 0,\, x_1 + x_2 \in [0, 1]\}$, and x_1, x_2 are the first two coordinates of a point lying in the simplex S^2.

After some algebra one has that $\mu_1 = 0$ and $\mu_2 = 1/2$ are eigenvalues of Jacobian $D_{\mathbf{x}}V_0(\mathbf{x})$ of the the operator V_0 at the point $(1/4, 1/4, 1/2)$. Similarly we have that $\mu_1 = 1$ and $\mu_2 = 2$ are eigenvalues of Jacobian $D_{\mathbf{x}}V_0(\mathbf{x})$ for any fixed point $\mathbf{x} \in \Gamma_{\{1,2\}}$. This implies immediately the statements in the theorem.

(iii) The last equation of system (5) has the form $x_3' = 2x_3(1-x_3)$. Due to Proposition 3.1 we have

$$\lim_{n\to\infty} x_3^{(n)} = \frac{1}{2}, \quad \forall\, x_3^{(0)} \in (0,1).$$

From the first and second equations of the (5) one has

$$|x_1' - x_2'| = |x_1^2 - x_2^2| = |x_1 - x_2|(x_1 + x_2) \leq |x_1 - x_2|.$$

Consequently we have

$$\left|x_1^{(n+1)} - x_2^{(n+1)}\right| \leq \left|x_1^{(n)} - x_2^{(n)}\right|, \quad n = 0, 1, 2, \ldots$$

It follows that the sequence $\left\{\left|x_1^{(n)} - x_2^{(n)}\right|\right\}_{n=0,1,2,\ldots}$ is decreasing and bounded from the below. Therefore there is $\lim\limits_{n\to\infty} \left|x_1^{(n)} - x_2^{(n)}\right| = \xi$. We claim that $\xi = 0$. Indeed, if $\xi > 0$ then it follows

$$1 = \lim_{n\to\infty} \frac{\left|x_1^{(n+1)} - x_2^{(n+1)}\right|}{\left|x_1^{(n)} - x_2^{(n)}\right|} = \lim_{n\to\infty} \left(x_1^{(n)} + x_2^{(n)}\right).$$

But this contradicts $\lim\limits_{n\to\infty} x_3^{(n)} = 1/2$. Therefore we have

$$\lim_{n\to\infty} \left|x_1^{(n)} - x_2^{(n)}\right| = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \left(x_1^{(n)} - x_2^{(n)}\right) = 0.$$

Further using $\lim\limits_{n\to\infty} \left(x_1^{(n)} + x_2^{(n)}\right) = \lim\limits_{n\to\infty} \left(1 - x_3^{(n)}\right) = 1/2$ one has

$$\lim_{n\to\infty} x_1^{(n)} = \lim_{n\to\infty} x_2^{(n)} = \frac{1}{4}.$$

Thus $\lim\limits_{n\to\infty} V_0^n\left(\mathbf{x}^{(0)}\right) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ for any $\mathbf{x}^{(0)} \in S^2 \backslash \Gamma_{\{1,2\}}$, completing the proof. $\square$

Theorem 4.2. *Let $\tau \neq 0$. Then one has*

(i) *the face $\Gamma_{\{1,2\}}$ is invariant and* $\mathrm{Fix}\,(V_\tau) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{x}^*\}$*;*

(ii) *The vertex $\mathbf{e}_2$ is* $\begin{cases} \textit{saddle}, & \textit{if} \quad \tau > 0, \\ \textit{repelling}, & \textit{if} \quad \tau < 0; \end{cases}$

The vertex $\mathbf{e}_1$ is $\begin{cases} \textit{saddle}, & \textit{if} \quad \tau \in \left[-\frac{1}{2}, 0\right) \cup \left(0, \frac{\sqrt{3}-1}{4}\right), \\ \textit{non-hyperbolic}, & \textit{if} \quad \tau = \frac{\sqrt{3}-1}{4}, \\ \textit{repelling}, & \textit{if} \quad \tau \in \left(\frac{\sqrt{3}-1}{4}, \frac{1}{2}\right]; \end{cases}$

(iii) *if $\tau > 0$ then $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{e}_2$ for any $\mathbf{x}^{(0)} \in \Gamma_{\{1,2\}} \backslash \{\mathbf{e}_1\}$;*

(iv) *if $\tau < 0$ then $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{e}_1$ for any $\mathbf{x}^{(0)} \in \Gamma_{\{1,2\}} \backslash \{\mathbf{e}_2\}$;*

(v) $\lim\limits_{n\to\infty} V_\tau^n\left(\mathbf{x}^{(0)}\right) = \mathbf{x}^*$ *for any* $\mathbf{x}^{(0)} \in S^2 \backslash \Gamma_{\{1,2\}}$.

Proof. (i) Let $\tau > 0$ (the case $\tau < 0$ can be considered similarly). Then the equation $V_\tau(\mathbf{x}) = \mathbf{x}$ has the form

$$\begin{cases} x_1 = x_1^2 + \left(\frac{1}{2} - \tau\right) x_3^2 + (1-2\tau)x_1x_2, \\ x_2 = x_2^2 + \left(\frac{1}{2} + \tau\right) x_3^2 + (1+2\tau)x_1x_2, \\ x_3 = 2x_1x_3 + 2x_2x_3. \end{cases}$$

Using $x_1+x_2+x_3 = 1$ from the last equation one has the equation $x_3 = 2x_3(1-x_3)$ and by Proposition 3.1 we have its solutions $x_3 = 0$ and $x_3 = 1/2$.

From the first and second equations we have

$$x_2 - x_1 = (x_2 - x_1)(x_2 + x_1) + 2\tau x_3^2 + 4\tau x_1x_2.$$

If $x_3 = 0$ then from the last equation we get $x_1x_2 = 0$, that is the fixed points $\mathbf{e}_1 = (1, 0, 0)$ and $\mathbf{e}_2 = (0, 1, 0)$.

If $x_3 = 1/2$ then using $x_1 + x_2 = 1/2$ from the last equation one has

$$x_2 - x_1 = \tau + 8\tau x_1x_2 \quad \Rightarrow \quad 16\tau x_1^2 - 4(1 + 2\tau)x_1 + 1 - 2\tau = 0$$

Consequently we get the solutions

$$x_1^{(1)} = \frac{1 + 2\tau + \sqrt{1 + 12\tau^2}}{8\tau}, \quad x_1^{(2)} = \frac{1 + 2\tau - \sqrt{1 + 12\tau^2}}{8\tau} \quad \text{and}$$

$$x_2^{(1)} = \frac{2\tau - 1 - \sqrt{1 + 12\tau^2}}{8\tau}, \quad x_2^{(2)} = \frac{2\tau - 1 + \sqrt{1 + 12\tau^2}}{8\tau}$$

Some algebra shows that $x_1^{(2)}, x_2^{(2)} \in (0, 1/2)$. Therefore we get the fixed point $\mathbf{x}^* = (x_1^*, x_2^*, 1/2)$ of the operator V_τ, where $x_1^* = x_1^{(2)}$ and $x_2^* = x_2^{(2)}$.

(ii) To find the type of the fixed point we rewrite the operator V_0 in the form

$$\begin{cases} x_1' = \frac{3-2\tau}{2}x_1^2 + \frac{1-2\tau}{2}x_2^2 - (1-2\tau)x_1 - (1-2\tau)x_2 + 2(1-2\tau)x_1x_2 + \frac{1}{2} - \tau, \\ x_2' = \frac{1+2\tau}{2}x_1^2 + \frac{3+2\tau}{2}x_2^2 - (1+2\tau)x_1 - (1+2\tau)x_2 + 2(1+2\tau)x_1x_2 + \frac{1}{2} + \tau, \end{cases}$$

where $(x_1, x_2) \in \{(x_1, x_2) : x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \in [0, 1]\}$, and x_1, x_2 are the first two coordinates of a point lying in the simplex S^2.

After some algebra one has that

$$\nu_1 = 2(1 + \tau) + \sqrt{2 - 4\tau^2} \quad \text{and} \quad \nu_2 = 2(1 + \tau) - \sqrt{2 - 4\tau^2}$$

are eigenvalues of Jacobian $D_{\mathbf{x}}V_\tau(\mathbf{x})$ of the operator V_τ at the vertex $\mathbf{e}_1$. Similarly we have that $\nu_1 = 2$ and $\nu_2 = 1 - 2\tau$ are eigenvalues of Jacobian $D_{\mathbf{x}}V_\tau(\mathbf{x})$ at the vertex $\mathbf{e}_2$. Consequently from this immediately follow the statements in the theorem.

(iii) Let $\mathbf{x}^{(0)} \in \Gamma_{\{1,2\}} \backslash \{\mathbf{e}_1\}$ be an initial point and $\tau > 0$. Since the face $\Gamma_{\{1,2\}}$ is invariant it follows $x_3^{(n)} = 0$, $n = 0, 1, 2, \ldots$ Using $0 < \tau \leq 1/2$ from (5) we have

$$x_1' = x_1(1 - 2\tau x_2) \leq x_1, \quad x_2' = x_2(1 + 2\tau x_1) \geq x_2 > 0$$

and consequently it follows

$$x_1^{(n+1)} = x_1^{(n)}\big(1-2\tau x_2^{(n)}\big) \le x_1^{(n)}, \ \ x_2^{(n+1)} = x_2^{(n)}\big(1+2\tau x_1^{(n)}\big) \ge x_2^{(n)} > 0, \ \ n = 0, 1, 2, ...$$

Therefore we obtain there are $\lim\limits_{n\to\infty} x_1^{(n)} = \alpha$ and $\lim\limits_{n\to\infty} x_2^{(n)} = 1-\alpha$.

We claim that $\alpha = 0$. Indeed, if $\alpha > 0$ then

$$1 = \lim_{n\to\infty} \frac{x_1^{(n+1)}}{x_1^{(n)}} = \lim_{n\to\infty} \Big(1 - 2\tau x_2^{(n)}\Big) \quad \Rightarrow \quad \lim_{n\to\infty} x_2^{(n)} = 0.$$

But the latter contradicts to the relation $x_2^{(n+1)} \ge x_2^{(n)} > 0, \quad n = 0, 1, 2, \ldots$.
Hence we have $\lim\limits_{n\to\infty} x_2^{(n)} = 1$ and it follows $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{e}_2$ for any $\mathbf{x}^{(0)} \in \Gamma_{\{1,2\}} \backslash \{\mathbf{e}_1\}$.

(iv) This can be proved similarly.

(v) Let $\mathbf{x}^{(0)} \in S^2 \backslash \Gamma_{\{1,2\}}$. The last equation of (5) has the form $x_3' = 2x_3(1-x_3)$. Due to Proposition 3.1 we have

$$\lim_{n\to\infty} x_3^{(n)} = \frac{1}{2}, \quad \forall\, x_3^{(0)} \in (0,1). \tag{7}$$

Consider the function $\quad f_\tau(x) = -2\tau x^2 + \dfrac{1+2\tau}{2} x + \dfrac{1+2\tau}{8}.$

One can verify that $\mathrm{Fix}\,(f_\tau) \cap [0, 1/2] = \{x_2^*\}$.

Definition 4.3. Let $f : A \to A$ and $g : B \to B$ be two maps. f and g are called *topologically conjugate* if there exists a homeomorphism $h : A \to B$ such that, $h \circ f = g \circ h$.

Note that mappings which are topologically conjugate are completely equivalent in terms of their dynamics. In particular, h gives a one-to-one correspondence between the set of limit points of f and g.

Taking $h(x) = ax + b$ one can see that the function $f_\tau(x)$ is topologically conjugate to the well-known logistic map $g(x) = \mu x(1-x)$ with

$$\mu = 1 + \frac{\sqrt{1 + 12\tau^2}}{2}.$$

Using $-1/2 \le \tau \le 1/2$ we have $\mu \in (1,3)$. For the logistic map the following is known (see [3]):

If $\mu \in (1,3)$ then g has an attracting fixed point $\widetilde{x} = (\mu-1)/\mu$ and repelling fixed point $x = 0$. The all trajectories (except when started at the fixed point) will approach fixed point $\widetilde{x}$.

From this fact, by the conjugacy argument, it follows that all trajectories of $f_\tau(x)$ started at an initial point will approach fixed point x_2^*.

Using the relation $x_1 = 1 - x_2 - x_3$ from the second equation of (5) we obtain

$$x_2' = x_2^2 + \left(\frac{1}{2} + \tau\right) x_3^2 + (1+2\tau)(1-x_2-x_3)x_2 = -2\tau x_2^2 - (1+2\tau)x_2x_3 + \left(\frac{1}{2}+\tau\right) x_3^2.$$

Consequently we have

$$x_2^{(n+1)} = -2\tau\left(x_2^{(n)}\right)^2 - (1+2\tau)x_2^{(n)}x_3^{(n)} + \left(\frac{1}{2}+\tau\right)\left(x_3^{(n)}\right)^2, \quad n = 0, 1, 2, \ldots$$

From (7) for arbitrarily small $\varepsilon > 0$ it follows that for enough large n it holds

$$\frac{1}{2} - \varepsilon < x_3^{(n)} < \frac{1}{2} + \varepsilon.$$

Using the latter we get

$$-2\tau\left(x_2^{(n)}\right)^2 - (1+2\tau)\left(\frac{1}{2}+\varepsilon\right)x_2^{(n)} + \left(\frac{1}{2}+\tau\right)\left(\frac{1}{2}-\varepsilon\right)^2 < x_2^{(n+1)},$$

$$x_2^{(n+1)} < -2\tau\left(x_2^{(n)}\right)^2 - (1+2\tau)\left(\frac{1}{2}-\varepsilon\right)x_2^{(n)} + \left(\frac{1}{2}+\tau\right)\left(\frac{1}{2}+\varepsilon\right)^2.$$

So for sufficiently large n we have

$$f_\tau\left(x_2^{(n)}\right) - \frac{(1+2\tau)\varepsilon}{2}\left(2x_2^{(n)} + 1 - \varepsilon\right) < x_2^{(n+1)} < f_\tau\left(x_2^{(n)}\right) + \frac{(1+2\tau)\varepsilon}{2}\left(2x_2^{(n)} + 1 + \varepsilon\right)$$

Since $\varepsilon > 0$ is arbitrarily small it follows that

$$\lim_{n\to\infty} x_2^{(n+1)} = \lim_{n\to\infty} f_\tau\left(x_2^{(n)}\right) = x_2^*.$$

Consequently we have

$$\lim_{n\to\infty} x_1^{(n)} = \lim_{n\to\infty}\left(1 - x_2^{(n)} - x_3^{(n)}\right) = 1 - x_2^* - \frac{1}{2} = x_1^*.$$

Thus $\lim\limits_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{x}^*$ for any $\mathbf{x}^{(0)} \in S^2 \backslash \Gamma_{\{1,2\}}$. $\square$

5. Superposition of the operators

In section 3 we showed that any trajectory of the QSOs (1) and (2) converges to its fixed point, respectively. It is a natural question: what about the asymptotic behaviour of the trajectories of a superposition of the QSOs (1) and (2)? In this section we will solve this question.

It is easy to check that the operator $V_{12} = V_1 \circ V_2$ has the form

$$V_{12}: \begin{cases} x_1' = \left(x_1^2 + x_3^2 + 2x_1x_2\right)^2, \\ x_2' = x_2^4 + 4\left(x_1x_3 + x_2x_3\right)^2 + 2x_2^2\left(x_1^2 + x_3^2 + 2x_1x_2\right), \\ x_3' = 4\left(x_1^2 + x_3^2 + 2x_1x_2\right)\left(x_1x_3 + x_2x_3\right) + 4x_2^2\left(x_1x_3 + x_2x_3\right). \end{cases} \tag{8}$$

First we need some auxiliary results. Consider the following function

$$s(x) = x^4 - 2x^3 + \frac{x^2}{2} + \frac{x}{2} + \frac{1}{16}, \qquad x \in [0, 1/2].$$

Proposition 5.1. *The point $x = 1 - \sqrt{3}/2$ is a unique fixed point of the function $s(x)$ in $[0, 1/2]$ and $\lim_{n\to\infty} s^n(x) = 1 - \sqrt{3}/2$ for any $x \in [0, 1/2]$.*

Proof. One has that

$$s(x) - x = x^4 - 2x^3 + \frac{x^2}{2} + \frac{x}{2} + \frac{1}{16} = \left(x^2 + \frac{1}{4}\right)\left(x^2 - 2x + \frac{1}{4}\right).$$

Therefore it follows that $x = 1 - \sqrt{3}/2$ is a unique fixed point in $[0, 1/2]$.

It is easy to see that

$$s'(x) = 4x^3 - 6x^2 + x + \frac{1}{2} \quad \text{and} \quad s'\left(1 - \sqrt{3}/2\right) = 4 - 2\sqrt{3} < 1.$$

Using $s(x) \geq x$ for any $x \in [0, 1 - \sqrt{3}/2]$ and $s(x) \leq x$ for any $x \in [1 - \sqrt{3}/2, 1]$ and

$$s'(x) = 4x^3 - 6x^2 + x + \frac{1}{2} = \left(x - \frac{1}{2}\right)(4x^2 - 4x - 1) \geq 0, \quad x \in [0, 1/2]$$

we obtain

$$\lim_{n\to\infty} s^n(x) = 1 - \sqrt{3}/2 \quad \text{for any} \quad x \in [0, 1/2].$$

The proof is completed. □

Theorem 5.2. *For the operator V_{12} given by (8) the following assertions are hold:*

(i) *the face $\Gamma_{\{1,2\}}$ is invariant and* $\mathrm{Fix}\,(V_{12}) = \{\mathbf{e}_1, \mathbf{e}_2, \widetilde{\mathbf{x}}, \widehat{\mathbf{x}}\}$, *where*

$$\widetilde{\mathbf{x}} = \left(\frac{3-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, 0\right), \quad \widehat{\mathbf{x}} = \left(\frac{2-\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2}, \frac{1}{2}\right);$$

(ii) *for any $\mathbf{x}^{(0)} \in \Gamma_{\{1,2\}} \setminus \{\mathbf{e}_1\}$ we have*

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \begin{cases} \mathbf{e}_2, & \text{if } x_1^{(0)} < (3-\sqrt{5})/2, \\ \widetilde{\mathbf{x}} & \text{if } x_1^{(0)} = (3-\sqrt{5})/2, \\ \mathbf{e}_1, & \text{if } x_1^{(0)} > (3-\sqrt{5})/2. \end{cases}$$

(iii) $\lim_{n\to\infty} V_{12}^n\left(\mathbf{x}^{(0)}\right) = \widehat{\mathbf{x}}$ *for any* $\mathbf{x}^{(0)} \in S^2 \setminus (\Gamma_{\{1,2\}} \cup \mathrm{Fix}\,(V_{12}))$.

Proof. (i) One has

$$x_3' = 4x_3(1-x_3)\left(2x_3^2 - 2x_3 + 1\right) = 2\left(2x_3(1-x_3)\right)\left(1 - 2x_3(1-x_3)\right) = f^2(x_3).$$

Since $\mu = 2$ in the logistic function $f(x) = 2x(1-x)$ by Proposition 3.1 we have $x_3 = 0$ and $x_3 = 1/2$ are solutions of the equation $f^2(x_3) = x_3$.

The equation $V_{12}(\mathbf{x}) = \mathbf{x}$ has the form

$$\begin{cases} x_1 = \left(x_1^2 + x_3^2 + 2x_1x_2\right)^2, \\ x_2 = x_2^4 + 4\left(x_1x_3 + x_2x_3\right)^2 + 2x_2^2\left(x_1^2 + x_3^2 + 2x_1x_2\right), \\ x_3 = 4\left(x_1^2 + x_3^2 + 2x_1x_2\right)\left(x_1x_3 + x_2x_3\right) + 4x_2^2\left(x_1x_3 + x_2x_3\right). \end{cases} \tag{9}$$

If we take $x_3 = 0$ then from the first equation of (9) it follows that

$$x_1 = x_1^2 (2 - x_1)^2 \quad \Rightarrow \quad x_1^{(1)} = 0, \quad x_1^{(2)} = 1, \quad x_1^{(3)} = \frac{3-\sqrt{5}}{2}, \quad x_1^{(4)} = \frac{3+\sqrt{5}}{2} > 1.$$

Therefore we have $x_2^{(1)} = 1, \quad x_2^{(2)} = 0$ and $x_2^{(3)} = \frac{-1+\sqrt{5}}{2}$.

So we obtain the fixed points

$$\mathbf{e}_1, \quad \mathbf{e}_2, \quad \widetilde{\mathbf{x}} = \left(\frac{3-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, 0 \right).$$

If we take $x_3 = 1/2$ then from the first equation of (9) it follows that

$$x_1 = (x_1(1-x_1) + 1/4)^2 = s(x_1)$$

and by Proposition 5.1 we have $x_1 = 1 - \sqrt{3}/2$. Therefore we have

$$x_2 = \frac{\sqrt{3}-1}{2}.$$

So we obtain the fixed point $\widehat{\mathbf{x}} = \left(\frac{2-\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2}, \frac{1}{2} \right)$.

Thus we have that $\mathrm{Fix}\,(V_{12}) = \{\mathbf{e}_1, \mathbf{e}_2, \widetilde{\mathbf{x}}, \widehat{\mathbf{x}}\}$.

(ii) Let $\mathbf{x}^{(0)} \in \Gamma_{\{1,2\}} \setminus \{\mathbf{e}_1\}$ be an initial point. Then we get oviously $x_3^{(n)} = 0$ for all $n = 0, 1, 2, \ldots$ Using $x_1^{(n)} + x_2^{(n)} = 1$ from the first equation of (8) one has

$$x_1^{(n+1)} = \left(x_1^{(n)}\right)^2 \left(2 - x_1^{(n)}\right)^2, \quad n = 0, 1, 2, \ldots$$

Therefore it follows that in this case a trajectory $x_1^{(n)}$ is given by the dynamical system $g(x) = x^2(2-x)^2$. This implies that $\mathrm{Fix}\,(g) = \{0, 1, (3-\sqrt{5})/2\}$ and $x = 0$, $x = 1$ are attracting and $\xi = (3-\sqrt{5})/2$ is repelling. As $g'(x) = 4x(2-x)(1-x)$ it is easy to check that $g'(0) = g'(1) = 0 < 1$ and $g'(\xi) = 2(3-\sqrt{5}) > 1$. Also it hold that $g(x) \leq x$ for any $x \in (0, \xi)$ and $g(x) \geq x$ for any $x \in (\xi, 1)$. Using these facts together we get

$$\lim_{n\to\infty} x_1^{(n)} = \lim_{n\to\infty} g\left(x_1^{(0)}\right) = \begin{cases} 0, & \text{if} \quad x_1^{(0)} < \xi, \\ \xi, & \text{if} \quad x_1^{(0)} = \xi, \\ 1, & \text{if} \quad x_1^{(0)} > \xi. \end{cases}$$

From the latter for any $\mathbf{x}^{(0)} \in \Gamma_{\{1,2\}} \setminus \{\mathbf{e}_1\}$ we have

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \begin{cases} \mathbf{e}_2, & \text{if} \quad x_1^{(0)} < \xi, \\ \widetilde{\mathbf{x}} & \text{if} \quad x_1^{(0)} = \xi, \\ \mathbf{e}_1, & \text{if} \quad x_1^{(0)} > \xi. \end{cases}$$

(iii) Due to Proposition 3.1 it follows that

$$\lim_{n\to\infty} x_3^{(n)} = \frac{1}{2}, \quad \forall\, x_3^{(0)} \in (0, 1).$$

From the latter for arbitrarily small $\varepsilon > 0$ we again have that for sufficiently large n it holds

$$\frac{1}{2} - \varepsilon < x_3^{(n)} < \frac{1}{2} + \varepsilon.$$

Using it from the first equation of (8) we get

$$\left(2x_1^{(n)} + \left(\frac{1}{2} - \varepsilon\right)\left(2x_1^{(n)} + \frac{1}{2} - \varepsilon\right) - \left(x_1^{(n)}\right)^2\right)^2 < x_1^{(n+1)},$$

$$x_1^{(n+1)} < \left(2x_1^{(n)} + \left(\frac{1}{2} + \varepsilon\right)\left(2x_1^{(n)} + \frac{1}{2} + \varepsilon\right) - \left(x_1^{(n)}\right)^2\right)^2$$

$$s\left(x_1^{(n)}\right) - \varepsilon\widetilde{h}\left(x_1^{(n)}, \varepsilon\right) < x_1^{(n+1)} < s\left(x_1^{(n)}\right) + \varepsilon\widehat{h}\left(x_1^{(n)}, \varepsilon\right)$$

where $$s(x) = \left(x - x^2 + 1/4\right)^2,$$

$$\widetilde{h}(x, \varepsilon) = (1 - \varepsilon - 2x)\left(2\sqrt{s(x)} + \varepsilon(1 - \varepsilon - 2x)\right)$$

and $$\widehat{h}(x, \varepsilon) = (1 + \varepsilon + 2x)\left(2\sqrt{s(x)} + \varepsilon(1 + \varepsilon + 2x)\right).$$

Since $\varepsilon > 0$ is an arbitrary small it follows that

$$\lim_{n\to\infty} x_1^{(n+1)} = \lim_{n\to\infty} s\left(x_1^{(n)}\right) = 1 - \frac{\sqrt{3}}{2}.$$

Hence we have $\lim\limits_{n\to\infty} x_2^{(n)} = 1 - \lim\limits_{n\to\infty} x_1^{(n)} - \lim\limits_{n\to\infty} x_3^{(n)} = \dfrac{\sqrt{3} - 1}{2}.$

Consequently we obtain that

$$\lim_{n\to\infty} V_{12}^n\left(\mathbf{x}^{(0)}\right) = \widehat{\mathbf{x}}, \quad \text{for any } \mathbf{x}^{(0)} \in S^2 \setminus \left(\Gamma_{\{1,2\}} \cup \mathrm{Fix}\,(V_{12})\right). \qquad \square$$

Similarly the following result can be proved

Theorem 5.3. *For the operator $V_{21} = V_2 \circ V_1$ the following assertions are hold:*

(i) *the face $\Gamma_{\{1,2\}}$ is invariant and $\mathrm{Fix}\,(V_{12}) = \left\{\mathbf{e}_1, \mathbf{e}_2, \widetilde{\widetilde{\mathbf{x}}}, \widehat{\widehat{\mathbf{x}}}\right\}$, where*

$$\widetilde{\widetilde{\mathbf{x}}} = \left(\frac{-1+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, 0\right), \quad \widehat{\widehat{\mathbf{x}}} = \left(\frac{-1+\sqrt{3}}{2}, \frac{2-\sqrt{3}}{2}, \frac{1}{2}\right);$$

(ii) *for any $\mathbf{x}^{(0)} \in \Gamma_{\{1,2\}} \setminus \{\mathbf{e}_2\}$ we have*

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \begin{cases} \mathbf{e}_1, & \text{if } x_2^{(0)} < (3-\sqrt{5})/2, \\ \widetilde{\mathbf{x}} & \text{if } x_2^{(0)} = (3-\sqrt{5})/2, \\ \mathbf{e}_2, & \text{if } x_2^{(0)} > (3-\sqrt{5})/2. \end{cases}$$

(iii) $\lim\limits_{n\to\infty} V_{21}^n\left(\mathbf{x}^{(0)}\right) = \widehat{\widehat{\mathbf{x}}}$ *for any $\mathbf{x}^{(0)} \in S^2 \setminus \left(\Gamma_{\{1,2\}} \cup \mathrm{Fix}\,(V_{21})\right)$.*

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