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Convex Analysis in Groups and Semigroups and Applications to Separation and Optimization*

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The aim of this paper is to investigate some fundamental properties of convex sets, cones, and functions in groups and semigroups. In particular, we introduce the concept of quasi core of a convex set, which allows us to provide separation theorems between convex sets in groups where at least one of them has nonempty quasi core. We also analyze optimization problems defined on groups and semigroups and provide saddle point necessary or sufficient optimality conditions by considering suitable Lagrangian functions associated with the given problem.

Keywords: Convex sets, convex functions, groups, semigroups, separation, optimization problems.

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1. Introduction

In the last decades, the development of discrete optimization has widely increased, in particular as regards convexity of sets and functions, (see, e.g., [9, 13, 15–20] and references therein). In this context, at first Volle investigated convex functions with values in complete abelian ordered groups, obtaining, in particular, a characterization of support functions [21, 22]. In 2018 Adivar and Fang developed a new theory for the analysis of discrete convex sets and functions providing applications to duality for integer programming [1]. Parallelly, Borwein and Giladi [3] proposed, in the field of non-divisible groups, a convexity notion for groups and monoids that collapses to the classical one when applied to a vector space, proving that several results of classic convex analysis also hold for functions defined on groups and semigroups and providing suitable examples of applications. Moreover, in the particular setting of locally convex topological groups and monoids, they further proved separation and minimax theorems [2].

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In [14], Li and Mastroeni proved, in finite dimensional integer spaces, the equivalence of the definitions of convex sets and functions introduced by Adivar and Fang, Borwein and Giladi. They further introduced the definition of convex cones, affine sets and analysed the separation between convex sets. By means of the image space analysis, they obtained necessary or sufficient optimality conditions for an integer linear programming problem with inequality constraints, providing suitable computational results.

Following the line of [3, 14], the aim of this paper is to deepen the analysis of the main properties of convex sets, convex cones, and convex functions in groups and semigroups. In particular, we introduce the concept of quasi core of a set which allows us to prove separation theorems between two convex subsets of a commutative group under the assumption of nonemptiness of the quasi core of at least one of them. Moreover, we consider optimization problems defined on groups and semigroups and provide necessary or sufficient optimality conditions under the form of saddle points of the Lagrangian function associated with the given problem.

The paper is organized as follows. In Section 2, we report the preliminary concepts and notations used throughout the paper. In Section 3, we investigate convex sets and cones in groups and semigroups while, in Section 4, we study convex functions in groups and semigroups and we introduce the concept of quasi core of a set in a group. In Subsection 5.1, we provide separation theorems between convex sets in groups where at least one of them has nonempty quasi core. In Subsection 5.2, we obtain saddle points of Lagrangian functions for optimization problems defined on groups and semigroups and provide some necessary or sufficient optimality conditions.

2. Basic concepts and notations

We recall in this section the main concepts and notations for groups and semigroups.

Definition 2.1. (Semigroup and group) A set X endowed with a binary operation $\cdot$ is called a *semigroup*, if the associative law $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ is satisfied for all $a, b, c \in X$. X is called *commutative*, if $a \cdot b = b \cdot a$ for all $a, b \in X$. $e \in X$ is called a *unit element* in X, if it satisfies $e \cdot x = x \cdot e = x$ for all $x \in X$. A set X endowed with a binary operation $\cdot$ is called a *group*, if it is a semigroup with the unique unit element $e \in X$, and the unit inverse of each $x \in X$, say x^*, i.e., $x^* \cdot x = x \cdot x^* = e$.

Definition 2.2. (Monoid) A set X endowed with a binary operation $\cdot$ is called a *monoid*, if $(X, \cdot)$ is a commutative semigroup with unit element.

Definition 2.3. (Semiring and ring) A set X endowed with two binary operations $+$ and $\cdot$ is called a *semiring* (resp., *ring*), if $(X, +)$ is a commutative semigroup (resp., group) with unit element 0, and $(X, \cdot)$ is a semigroup (not necessarily commutative) with unit element 1. The two operations are connected by the distributive laws. We additionally assume for a semigroup X that $r0 = 0r = 0$ for all $r \in X$.

Definition 2.4. (Semimodule and module) A set X endowed with a binary operation $+$ is called a *left semimodule* (resp., *module*) over a semiring (resp., ring) R, if $(X,+)$ is a commutative semigroup (resp., group) with unit element 0, and there is a map $R \times X \to X$, $(r,x) \mapsto rx$ such that for all $r, r_1, r_2 \in R$ and $x, x_1, x_2 \in X$,

(i) $(r_1 r_2)x = r_1(r_2 x)$.

(ii) $r(x_1 + x_2) = rx_1 + rx_2$.

(iii) $(r_1 + r_2)x = r_1 x + r_2 x$.

(iv) $1x = x$.

(v) $r0 = 0 = 0x$.

We observe that a monoid is a semimodule over the non-negative integers $\mathbb{Z}_+$ and a commutative group is a module over the ring $\mathbb{Z}$ and also a monoid (i.e., a semimodule over the non-negative integers $\mathbb{Z}_+$).

Let $(X,+)$ be a semigroup and C, D be subsets of X. The general Minkowski sum of sets in X is given by $C + D := \{c + d : c \in C, d \in D\}$. If $D := \{d\}$, then we set $C + D := C + d$. If $(X,+)$ is a group, then we denote by $C - D := C + (-D) := \{c + (-d) : c \in C,\ d \in D\}$.

Definition 2.5. (Compatible partial order on a semimodule) Let X be a semimodule over a semiring R. A partial order $\leq$ is said to be *compatible* with the module operations, if $rx_1 \leq rx_2$, $x + x_1 \leq x + x_2$ for all $x \in X, r \in R$, whenever $x_1 \leq x_2$.

Definition 2.6. (Semidivisible, divisible and uniquely divisible semimodule (resp., module)) Let X be a semimodule (resp. module) over a semiring (resp., ring) R.

(i) X is called *divisible*, if for every $y \in X$ and $r \in R \setminus \{0\}$, there is $x \in X$ such that $rx = y$.

(ii) X is called *uniquely divisible*, if for every $y \in X$ and $r \in R \setminus \{0\}$, there is a unique $x \in X$ such that $rx = y$.

(iii) Let X be a commutative group, that is, X is a semimodule over the semiring $\mathbb{Z}_+$. X is called *p-semidivisible*, if there is $p \in \mathbb{N}$ prime such that $pX = X$, and X is called *semidivisible*, if it is p-semidivisible for some prime p.

Proposition 2.7. *The following statements are true:*

(i) *Let X be a semimodule (resp. module) over a semiring (resp., ring) R. If X is uniquely divisible, then it is divisible and $rx = 0 \iff x = 0$, where $r \in R \setminus \{0\}$ and $x \in X$.*

(ii) *Let X be a commutative group, that is, X is a semimodule over the semiring $\mathbb{Z}_+$. If X is divisible and $nx = 0 \iff x = 0$, where $n \in \mathbb{N}$ and $x \in X$, then X is uniquely divisible.*

Proof. (i) Assume that X is uniquely divisible. Then it is divisible. Let $r \in R \setminus \{0\}$ and $x \in X$. Then, we have $x = 0$ implies that $rx = 0$. We now show $rx = 0$ implies that $x = 0$. Clearly, $r0 = 0$. Since X is uniquely divisible and $rx = 0$, x is unique such that $rx = 0$. Thus it follows that $x = 0$.

(ii) By contradiction, assume that X is not uniquely divisible, i.e., there are $n \in \mathbb{N}$ and $y, x_1, x_2 \in X$ with $x_1 \neq x_2$ such that $nx_1 = y = nx_2$. In consequence we have $n(x_1 + (-x_2)) = nx_1 + n(-x_2) = nx_1 + (-nx_2) = 0$. From the assumption it follows that $x_1 + (-x_2) = 0$ and so $x_1 = x_2$, which contradicts that $x_1 \neq x_2$. $\square$

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space, where $m \in \mathbb{N}$, the positive integers. Let $\mathbb{R}^m_+ := \{x := (x_1, \dots, x_m) \in \mathbb{R}^m : \text{each } x_i \geq 0\}$, $\mathbb{Z}^m := \{x := (x_1, \dots, x_m) \in \mathbb{R}^m : \text{each } x_i \in \mathbb{Z}\}$, and $\mathbb{Z}^m_+ := \{x := (x_1, \dots, x_m) \in \mathbb{Z}^m : \text{each } x_i \geq 0\}$. We denote by $\mathbb{Q}_+$ the non-negative rational numbers and by $\|\cdot\|$ the Euclidean 2-norm in $\mathbb{R}^m$. All vectors are considered as row vectors.

Recall that $\mathbb{R}^n$ endowed with the addition $+$ is a left semimodule over a semiring $\mathbb{R}_+$, where $n \in \mathbb{N}$. Clearly, the order $\leq$ on $\mathbb{R}^n$ defined by $y_1 \leq y_2 \Longleftrightarrow y_2 - y_1 \in \mathbb{R}^n_+$, is a compatible partial order and it satisfies that for every $r \in \mathbb{R}_+ \setminus \{0\}$, $y_1, y_2 \in \mathbb{R}^n$, $ry_1 \leq ry_2 \Longrightarrow y_1 \leq y_2$. Define $y_1 < y_2 \Longleftrightarrow y_2 - y_1 \in \mathrm{int}\mathbb{R}^n_+$. Then, $\mathbb{R}^n$ endowed with the addition $+$ is a left semimodule over a semiring $\mathbb{Z}_+$ and $\mathbb{Z}^n$ endowed with the addition $+$ induced from $\mathbb{R}^n$ is a left semimodule over a semiring $\mathbb{Z}_+$, where $n \in \mathbb{N}$. Also, the order $\leq$ on $\mathbb{Z}^n$ defined by $y_1 \leq y_2 \Longleftrightarrow y_2 - y_1 \in \mathbb{Z}^n_+$, is a compatible partial order and it satisfies that for every $r \in \mathbb{Z}_+ \setminus \{0\}$, $y_1, y_2 \in \mathbb{Z}^n$, $ry_1 \leq ry_2 \Longrightarrow y_1 \leq y_2$.

Let Y be a semimodule over a semiring R where Y is endowed with a partial order $\leq$. We often adjoint a maximal element in Y, $+\infty$, defining on $Y \cup \{+\infty\}$ the usual operation given on Y plus the following one: $x + (+\infty) = (+\infty) + x := +\infty$ for all $x \in Y \cup \{+\infty\}$; $r \cdot (+\infty) := +\infty$ for all $r \in R \backslash \{0\}$; $0 \cdot (+\infty) := 0$; $x \leq +\infty$ for all $x \in Y \cup \{+\infty\}$. We define $x < +\infty \Longleftrightarrow x \leq +\infty$ and $x \neq +\infty$, where $x \in Y$.

Throughout the paper, we always refer to additive groups and semigroups, unless differently specified.

3. Convex sets in groups and semigroups

In this section, we analyse in details convex sets and cones in groups and semigroups.

3.1. Convex sets

Definition 3.1. (Convex set and convex hull in a semimodule) Let X be a semimodule over a semiring R. A set $A \subseteq X$ is called *convex set*, if for any choice of $x \in X$, $n \in \mathbb{N}$, $r, r_1, \dots, r_n \in R \backslash \{0\}$ and $x_1, \dots, x_n \in A$ such that

$$rx = \sum_{i=1}^{n} r_i x_i \quad \text{and} \quad r = \sum_{i=1}^{n} r_i,$$

one has $x \in A$. The *convex hull* of A, say $\mathrm{conv}A$, is given by

$$\mathrm{conv}A := \bigcap \{B : A \subseteq B \text{ and } B \text{ is convex}\}.$$

Remark 3.2. In [3, Definitions 7 and 14], Borwein and Giladi did not use the assumption $r \neq 0$, which is suggested by a referee. In fact, if

$$rx = \sum_{i=1}^{n} r_i x_i \quad \text{and} \quad 0 = r = \sum_{i=1}^{n} r_i,$$

then $0x = \sum_{i=1}^{n} r_i x_i$, and, since $0x = 0$, $\forall x \in X$, then the convexity of A would imply that $X \subseteq A$ and so $A = X$, since $A \subseteq X$. This contradicts the existence of a convex set $A \neq X$.

The next result shows that conv$A$ is a convex set, i.e., the convex hull is well defined.

Proposition 3.3. *Let X be a semimodule over a semiring R, and $C_i \subseteq X$ $(i \in I)$ be convex, where I is an index set. Then $\bigcap_{i\in I} C_i$ is convex. Furthermore, if X is a uniquely divisible and $I := \{1, \dots, m\}$, where $m \in \mathbb{N}$, then $\sum_{i=1}^m C_i$ is convex.*

Proof. We preliminarily prove that $\bigcap_{i\in I} C_i$ is convex. Let $x \in X$, $n \in \mathbb{N}$, $r, r_1, \dots, r_n \in R\backslash\{0\}$ and $x_1, \dots, x_n \in \bigcap_{i\in I} C_i$ such that we have $rx = \sum_{j=1}^n r_j x_j$, $r = \sum_{j=1}^n r_j$. Since C_i, $(i \in I)$, are convex and $x \in C_i$, $(i \in I)$, we therefore obtain $x \in \bigcap_{i\in I} C_i$.

Assume that X is uniquely divisible and $I := \{1, \dots, m\}$. We now show that $\sum_{i=1}^m C_i$ is convex. To this end, let $x \in X$, $n \in \mathbb{N}$, $r, r_1, \dots, r_n \in R\backslash\{0\}$,

$$x_1 := c_1^1 + \ldots + c_1^m, \dots,\ x_n := c_n^1 + \ldots + c_n^m \in \textstyle\sum_{i=1}^m C_i$$

and $c_j^i \in C_i$, $(i = 1, \dots, m,\ j = 1, \dots, n)$, such that $rx = \sum_{j=1}^n r_j x_j,\ r = \sum_{j=1}^n r_j$. Since X is divisible, there are $c_i \in X$, $(i = 1, \dots, m)$, such that $rc_i = \sum_{j=1}^n r_j c_j^i$. Since C_i, $(i = 1, \dots, m)$ are convex, $c_i \in C_i$, $(i = 1, \dots, m)$, and so

$$r(\textstyle\sum_{i=1}^m c_i) = \sum_{j=1}^n r_j(\sum_{i=1}^m c_j^i) = \sum_{j=1}^n r_j x_j = rx,$$

which implies $x = \sum_{i=1}^m c_i \in \sum_{i=1}^m C_i$ by assumption. This proves that $\sum_{i=1}^m C_i$ is convex. □

Proposition 3.4. *Let X be a semimodule over a semiring R. Then A is convex if and only if $A =$ conv$A$.*

Proof. Necessity. It is clear $A \subseteq$ conv$A$. Since A is convex and $A \subseteq A$, one has $A \supseteq$ conv$A$ and so $A =$ conv$A$.

Sufficiency. This follows immediately from the first assertion of Proposition 3.3. □

In case X is a monoid the following characterization of the convex hull has been proved in [3].

Proposition 3.5. [3, Proposition 1] *Let X be a monoid and $A \subseteq X$. Then*

$$\mathrm{conv}A = \Big\{x \in X : mx = \sum_{i=1}^n m_i x_i,\ x_i \in A,\ n, m_i \in \mathbb{N},\ m = \sum_{i=1}^n m_i\Big\}.$$

Proposition 3.6. *Let X, Y be semimodules over a semiring R, and $C, D \subseteq Y$. Then C and D are convex if and only if $C \times D$ is convex.*

Proof. Let $(x, y) \in X \times Y, n \in \mathbb{N}$, $r, r_1, ..., r_n \in R\backslash\{0\}$, $(x_1, y_1), ..., (x_n, y_n) \in C \times D$ such that $r(x, y) = \sum_{j=1}^n r_j(x_j, y_j)$, $r = \sum_{j=1}^n r_j$, or equivalently, $rx = \sum_{j=1}^n r_j x_j$, $ry = \sum_{j=1}^n r_j y_j$ and $r = \sum_{j=1}^n r_j$. If C, D are convex, then $x \in C$ and $y \in D$, i.e., $(x, y) \in C \times D$. This yields $C \times D$ is convex. If $C \times D$ is convex, then $(x, y) \in C \times D$, that is, $x \in C$ and $y \in D$. This implies that C and D are convex. □

Given a subset A of a monoid X and $m \in \mathbb{N}$. Let

$$
\begin{aligned}
mA &:= \{x \in X : x = \sum_{i=1}^{m} x_i,\ x_i \in A\} \\
&= \{x \in X : x = \sum_{i=1}^{n} m_i x_i,\ x_i \in A,\ \sum_{i=1}^{n} m_i = m,\ m_i, n \in \mathbb{N}\}.
\end{aligned} \tag{1}
$$

We denote by $0A := \{0\}$. Note that (1) can be generalized in the following in the case of convex sets.

Definition 3.7. (Rational dilation of set) Let C be a subset of a monoid X and let $q \in \mathbb{Q}_+$ be a reduced fraction. Define

$$
qC := \{x \in X : lx = \sum_{i=1}^{n} m_i x_i,\ x_i \in C,\ \sum_{i=1}^{n} m_i = m,\ \frac{m}{l} = q,\ l, n, m_i \in \mathbb{N}\}. \tag{2}
$$

In [2, Definition 2.6], C is assumed to be a convex subset of a monoid X. Observe that if $q \in \mathbb{N}$, then (2) coincides with (1). It is easy to see that $1C \supseteq C$. Moreover, if C is not convex, then the equality $1C = C$ does not necessarily hold, in fact we note that $1C = \operatorname{conv} C$. The following Proposition 3.8 states that for a subset C of a monoid X and a reduced fraction $q \in \mathbb{Q}_+$, qC is always convex.

Proposition 3.8. *Let C be a subset of a monoid X and $q \in \mathbb{Q}_+$ be a reduced fraction. Then the following statements hold:*

(i) *qC is convex.*

(ii) *If $0 \in C$, then $0 \in qC$.*

Proof. Recall that a monoid is a semimodule over the semiring $\mathbb{Z}_+$.

(i) Let $x \in X, n \in \mathbb{N}, r_1, \ldots, r_n \in \mathbb{Z}_+ \setminus \{0\} = \mathbb{N}$ and $x_1, \ldots, x_n \in qC$ such that

$$
rx = \sum_{i=1}^{n} r_i x_i, \quad r = \sum_{i=1}^{n} r_i. \tag{3}
$$

Since $x_1, \ldots, x_n \in qC$, there are $x_{ij} \in C$, $m_{ij}, l_i, k_i \in \mathbb{N}$ $(i = 1, \ldots, n, j = 1, \ldots, k_i)$ such that

$$
l_i x_i = \sum_{j=1}^{k_i} m_{ij} x_{ij},\ \sum_{j=1}^{k_i} m_{ij} = m_i,\ \frac{m_i}{l_i} = q. \tag{4}
$$

From (3) and (4) one has

$$
\begin{aligned}
(r\prod_{i=1}^{n} l_i)x &= (\prod_{i=1}^{n} l_i)(rx) = (\prod_{i=1}^{n} l_i)(\sum_{i=1}^{n} r_i x_i) \\
&= (r_1 \prod_{i\neq 1} l_i)(l_1 x_1) + \cdots + (r_n \prod_{i\neq n} l_i)(l_n x_n) \\
&= (r_1 \prod_{i\neq 1} l_i)(\sum_{j=1}^{k_1} m_{1j} x_{1j}) + \cdots + (r_n \prod_{i\neq n} l_i)(\sum_{j=1}^{k_n} m_{nj} x_{nj}),
\end{aligned}
$$

hence,
$$(r\prod_{i=1}^{n} l_i)x = \sum_{j=1}^{k_1}(r_1\prod_{i\neq 1} l_i)m_{1j}x_{1j} + \cdots + \sum_{j=1}^{k_n}(r_n\prod_{i\neq n} l_i)m_{nj}x_{nj}. \tag{5}$$

Further, we have

$$\begin{aligned}\sum_{j=1}^{k_1}(r_1\prod_{i\neq 1} l_i)m_{1j} + \cdots + \sum_{j=1}^{k_n}(r_n\prod_{i\neq n} l_i)m_{nj}\\ &= (r_1\prod_{i\neq 1} l_i)\sum_{j=1}^{k_1} m_{1j} + \cdots + (r_n\prod_{i\neq n} l_i)\sum_{j=1}^{k_n} m_{nj}\\ &= (r_1\prod_{i\neq 1} l_i)m_1 + \cdots + (r_n\prod_{i\neq n} l_i)m_n\end{aligned}$$

and so

$$\frac{(r_1\prod_{i\neq 1} l_i)m_1 + \cdots + (r_n\prod_{i\neq n} l_i)m_n}{r\prod_{i=1}^{n} l_i} = \frac{r_1m_1}{rl_1} + \cdots + \frac{r_nm_n}{rl_n} = \frac{r_1}{r}q + \cdots + \frac{r_n}{r}q = q,$$

where the second and third equalities follow from (4) and (3), respectively. This along with (5) shows that $x \in qC$. This proves qC is convex.

(ii) Setting $x = x_i = 0 (i = 1, \ldots, n)$ in (2) yields the desired result. □

From Proposition 3.8 (i) we have:

Corollary 3.9. *Let X be a monoid, $C \subseteq X$ and $m \in \mathbb{N}$. Then mC is convex.*

From Propositions 3.3 and 3.8 (i) we have:

Corollary 3.10. *et X be a monoid, $C_i \subseteq X$ and $q_i \in \mathbb{Q}_+ (i = 1, \ldots, m)$ be a reduced fraction. Then $\bigcap_{i=1}^{m} q_iC_i$ is convex. Furthermore, if X uniquely divisible, then $\sum_{i=1}^{m} q_iC_i$ is convex.*

The following results will be needed in the sequel.

Proposition 3.11. *Let X be a monoid, $C, D \subseteq X$ and let $q_1, q_2 \in \mathbb{Q}_+$ be reduced fractions. Then the following statements hold:*

(i) *Assume that $0 \in C$ and C is convex. If $q_1 \le q_2$, then $q_1C \subseteq q_2C$.*

(ii) *If $C \subseteq D$, then $q_1C \subseteq q_1D$.*

(iii) *Then $q_2(q_1C) \subseteq (q_2q_1)C = (q_1q_2)C$, which implies that if $q_2q_1 = 0$, then $q_2(q_1C) = (q_2q_1)C$; if $q_2q_1 \neq 0$, then $q_2(q_2^{-1}((q_2q_1)C)) \subseteq q_2(q_1C)$. Furthermore, if $0 \in C$ and C is convex, then $(q_1q_2)C \subseteq (\frac{q_1^2+q_2^2}{2})C$.*

(iv) *If X is divisible, then $(q_2q_1)C \subseteq q_2(q_1C)$.*

Proof. (i) This follows from [2, Proposition 2.3].

(ii) Follows immediately from Definition 3.7.

(iii) The equality $(q_2q_1)C = (q_1q_2)C$ is clear. Observe that $q_2q_1 \le \frac{q_1^2+q_2^2}{2}$ and from (i) it follows that $(q_2q_1)C \subseteq (\frac{q_1^2+q_2^2}{2})C$ whenever $0 \in C$.

We next show $q_2(q_1C) \subseteq (q_2q_1)C$. To this end, let $x \in q_2(q_1C)$.

Then there are $x_i \in q_1C$, $l, n, m_i \in \mathbb{N}$ $(i = 1, \ldots, n)$ such that

$$lx = \sum_{i=1}^{n} m_i x_i, \quad \sum_{i=1}^{n} m_i = m, \quad \frac{m}{l} = q_2. \tag{6}$$

Since $x_i \in q_1C$ $(i = 1, \ldots, n)$, there are $x'_{ij} \in C$, $l'_i, n'_i, m'_{ij} \in \mathbb{N}$ $(j = 1, \ldots, n'_i)$ with

$$l'_i x_i = \sum_{j=1}^{n'_i} m'_{ij} x'_{ij}, \quad \sum_{j=1}^{n'_i} m'_{ij} = m'_i, \quad \frac{m'_i}{l'_i} = q_1. \tag{7}$$

Then from (6) and (7) it follows that

$$\begin{aligned}(l\prod_{i=1}^{n} l'_i)x &= (\prod_{i=1}^{n} l'_i)lx = \prod_{i=1}^{n} l'_i \sum_{i=1}^{n} m_i x_i \\ &= \prod_{i\neq 1} l'_i m_1(l'_1 x_1) + \cdots + \prod_{i\neq n} l'_i m_n(l'_n x_n) \\ &= \sum_{j=1}^{n'_1}(m_1 \prod_{i\neq 1} l'_i)m'_{1j}x'_{1j} + \cdots + \sum_{j=1}^{n'_n}(m_n \prod_{i\neq n} l'_i)m'_{nj}x'_{nj}\end{aligned}$$

and

$$\begin{aligned}&\frac{\sum_{j=1}^{n'_1}(m_1 \prod_{i\neq 1} l'_i)m'_{1j} + \cdots + \sum_{j=1}^{n'_n}(m_n \prod_{i\neq n} l'_i)m'_{nj}}{l\prod_{i=1}^{n} l'_i} \\ &= \frac{(m_1 \prod_{i\neq 1} l'_i)m'_1 + \cdots + (m_n \prod_{i\neq n} l'_i)m'_n}{l\prod_{i=1}^{n} l'_i} \\ &= \frac{m_1 m'_1}{ll'_1} + \cdots + \frac{m_n m'_n}{ll'_n} = (\frac{m_1}{l} + \cdots + \frac{m_n}{l})q_1 = q_2 q_1,\end{aligned}$$

which yields $x \in (q_2q_1)C$. This proves $q_2(q_1C) \subseteq (q_2q_1)C$.

If $q_2q_1 = 0$, then it is easy to see $q_2(q_1C) = (q_2q_1)C = \{0\}$. Assume that $q_2q_1 \neq 0$. Now applying previous conclusion yields that $q_2^{-1}((q_2q_1)C) \subseteq (q_2^{-1}q_2q_1)C = q_1C$ and as a consequence, it follows from (ii) that $q_2(q_2^{-1}((q_2q_1)C)) \subseteq q_2(q_1C)$.

(iv) If $q_2q_1 = 0$, then we have $(q_2q_1)C = q_2(q_1C) = \{0\}$.

Assume $q_2q_1 \neq 0$. Set $q_2 := \frac{m'}{l'}$, where $m', l' \in \mathbb{N}$. Let $x \in (q_2q_1)C$. Then there are $x_i \in C$, $l, n, m_i \in \mathbb{N}$ $(i = 1, \ldots, n)$ such that $lx = \sum_{i=1}^{n} m_i x_i$, $\sum_{i=1}^{n} m_i = m$ and $\frac{m}{l} = q_2q_1$. Then

$$(l'l)x = \sum_{i=1}^{n}(m_i l')x_i, \quad \frac{ml'}{lm'} = q_1. \tag{8}$$

Observe that X is divisible. Then there exists $y \in X$ such that

$$(m'l)y = \sum_{i=1}^{n}(m_i l')x_i, \quad \frac{\sum_{i=1}^{n}(m_i l')}{m'l} = \frac{ml'}{m'l} = q_1, \tag{9}$$

which yields $y \in q_1C$. Now, both (8) and (9) allow that

$$(l'l)x = (m'l)y, \quad \frac{m'l}{l'l} = \frac{m'}{l'} = q_2,$$

which along with $y \in q_1C$ yields $x \in q_2(q_1C)$. This proves $(q_2q_1)C \subseteq q_2(q_1C)$. $\square$

3.2. Convex cones

In this subsection, we discuss convex cones in groups and semigroups.

Definition 3.12. (Cone in a semimodule) Let X be a semimodule over a semiring R. A set $A \subseteq X$ is called a *cone*, if for any choice $x' \in X$, $r, r' \in R \setminus \{0\}$, $x \in A$ such that $r'x' = rx$, one has $x' \in A$. The set

$$R\backslash\{0\}A := \{x' \in X : r'x' = rx, x \in A, r, r' \in R\backslash\{0\}\}$$

is called the *cone generated by* A.

We observe that, if $X := \mathbb{Z}^n$ and $R := \mathbb{Z}_+$, then the above definition of cones reduces to [1, Definition 2.11], while in [14, Definition 2.5] a cone is similarly defined but, additionally, it contains 0.

Definition 3.13. (Convex cone in a semimodule) Let X be a semimodule over a semiring R. A set $A \subseteq X$ is said to be a *convex cone*, if for any choice $n \in \mathbb{N}$, $x \in X$, $r, r_1, \ldots, r_n \in R\backslash\{0\}$ and $x_1, \ldots, x_n \in A$ such that

$$rx = \sum_{i=1}^{n} r_i x_i \quad \text{and} \quad \sum_{i=1}^{n} r_i \neq 0, \tag{10}$$

one has $x \in A$. The set of the points x such that (10) holds is called the *convex cone* generated by A and is denoted by $\operatorname{cone} A$.

Remark 3.14. In [3, Definition 8], Borwein and Giladi did not use the assumption $\sum_{i=1}^{n} r_i \neq 0$. In this case, a convex cone A always contains 0. In fact, for $x_1 = x_2 = \ldots = x_n$ we have

$$rx = \sum_{i=1}^{n} r_i x_i = 0x_1 = 0,$$

which is fulfilled by $x = 0$, so that $0 \in A$.

Proposition 3.15. *Let X be a semimodule over a semiring R and $C \subseteq X$. Then the following statements hold:*

(i) *C is a convex cone if and only if $C = \operatorname{cone} C$, which implies that C is both a cone and convex.*

(ii) *Let X be divisible. If C is both a cone and convex, then it is a convex cone.*

(iii) *C is a cone if and only if $C = R\backslash\{0\}C$.*

(iv) *If either C is a convex cone or C is convex and X is divisible, then $\operatorname{cone} C = R\backslash\{0\}C$.*

(v) *Let X be a monoid. Then $\operatorname{cone} C = \bigcup_{q\in\mathbb{Q}_+\backslash\{0\}} qC$, which implies that $C = \bigcup_{q\in\mathbb{Q}_+\backslash\{0\}} qC$ if C is a convex cone.*

Proof. (i) Necessity. It is immediate that $C \subseteq \operatorname{cone} C$. Since C is a convex cone, one has $C \supseteq \operatorname{cone} C$ and so $C = \operatorname{cone} C$.

Sufficiency. Let $x \in X, n \in \mathbb{N}, r, r_1, \ldots, r_n \in R\backslash\{0\}$, and $x_1, \ldots, x_n \in C$ such that $rx = \sum_{i=1}^{n} r_i x_i$ and $\sum_{i=1}^{n} r_i \neq 0$. Then $x \in \operatorname{cone} C$ and so $x \in C$, since $C = \operatorname{cone} C$. This shows that C is a convex cone. Clearly, if C is a convex cone, then it is both a cone and convex.

(ii) Let $x \in X$, $n \in \mathbb{N}$, $r, r_1, \ldots, r_n \in R\backslash\{0\}$ and $x_1, \ldots, x_n \in C$ such that $rx = \sum_{i=1}^n r_i x_i$ and $\sum_{i=1}^n r_i \neq 0$. Since X is divisible, then there is $x' \in X$ with $(\sum_{i=1}^n r_i)x' = \sum_{i=1}^n r_i x_i$. Since C is convex, one has $x' \in C$ and thus we have $rx = (\sum_{i=1}^n r_i)x'$. Since C is a cone and $x' \in C$, one has $x \in C$. This proves C is a convex cone.

(iii) The conclusion follows by using a similar proof to that in (i).

(iv) If C is a convex cone, then (i) gives $C = \operatorname{cone} C = R\backslash\{0\}C$ since we have $\operatorname{cone} C \supseteq R\backslash\{0\}C \supseteq C$. Assume that C is convex and X is divisible. It is immediate that $\operatorname{cone} C \supseteq R\backslash\{0\}C$. Conversely, let $x \in \operatorname{cone} C$. Then there are $r_1, \ldots, r_n$, $r \in R\backslash\{0\}$, and $x_1, \ldots, x_n \in C$, where $n \in \mathbb{N}$, such that $rx = \sum_{i=1}^n r_i x_i$ with $\sum_{i=1}^n r_i \neq 0$. Since X is divisible, there is $y \in X$ such that $\sum_{i=1}^n r_i x_i = (\sum_{i=1}^n r_i)y$ and so $y \in C$ since C is convex. As a consequence, $rx = (\sum_{i=1}^n r_i)y$, which yields $x \in R\backslash\{0\}C$. This yields $\operatorname{cone} C \subseteq R\backslash\{0\}C$.

(v) The inclusion $\supseteq$ is obvious. Conversely, let $x \in \operatorname{cone} C$. Then there are $r_1, \ldots, r_n, r, n \in \mathbb{N}$ and $x_1, \ldots, x_n \in C$ such that $rx = \sum_{i=1}^n r_i x_i$, with $\sum_{i=1}^n r_i \neq 0$. Setting $q := \sum_{i=1}^n r_i/r$ yields that $x \in qC$. This proves the inclusion $\subseteq$ holds. If C is a convex cone, then from (i) we have $C = \bigcup_{q \in \mathbb{Q}_+\backslash\{0\}} qC$. □

Proposition 3.16. *Let X be a monoid and $0 \in C$. Then C is a convex cone if and only if C is a cone and $C + C = C$.*

Proof. Since $0 \in C$, we have $C+C \supseteq C$. Then $C+C = C$ if and only if $C+C \subseteq C$.

Necessity. Let $x := y + z \in C + C$, where $y, z \in C$. Then $1x = x = 1y + 1z \in X$. Since C is a convex cone, from Proposition 3.15(i) one has C is a cone and moreover, $x \in C$ and so $C + C \subseteq C$.

Sufficiency. Let $x \in X$, $n, r, r_1, \ldots, r_n \in \mathbb{N}$ and $x_1, \ldots, x_n \in C$ such that we have $rx = \sum_{i=1}^n r_i x_i$. Since $C + C = C$, one has $rx = 1(\sum_{i=1}^n r_i x_i) = \sum_{i=1}^n r_i x_i \in C$. Since C is a cone, one has $x \in C$, which implies that C is a convex cone. □

In part (i) of the next result, we will assume that the semiring R fulfills the following assumption (S):

$$r_1, r_2 \in R \setminus \{0\} \Longrightarrow r_1 + r_2 \neq 0,$$

or equivalently, $n \in \mathbb{N}, r_1, r_2, \ldots, r_n \in R \setminus \{0\} \Longrightarrow \sum_{i=1}^n r_i \neq 0$.

Moreover, when X is a monoid, we assume that $R = \mathbb{Z}_+$, recalling that a monoid is a semimodule over the semiring $\mathbb{Z}_+$.

Proposition 3.17. *Let X be a semimodule over a semiring R and $C \subseteq X$. Then the following statements hold:*

(i) *Assume that either $0 \in C$ or $0 \notin C$ and $rx = 0 \Longleftrightarrow x = 0$, where $r \in R\backslash\{0\}$ and $x \in X$ (in particular, if X is a commutative group, then this, along with the assumption that X is divisible, equals the fact that X is uniquely divisible, in view of Proposition 2.7). If C is a convex cone and assumption (S) holds, then $C \cup \{0\}$ is also a convex cone.*

(ii) *Assume that either $0 \in C$ or $0 \notin C$ and $\sum_{i=1}^n r_i x_i \neq 0$ for every $r_1, \ldots, r_n \in R\backslash\{0\}$, $x_1, \ldots, x_n \in C$ and $n \in \mathbb{N}$. If $C \cup \{0\}$ is a convex cone, then so is C.*

Proof. (i) If $0 \in C$, then $C = C \bigcup \{0\}$ and so the conclusion is clear. Assume that $0 \notin C$ and $rx = 0 \Longleftrightarrow x = 0$, where $r \in R \backslash \{0\}$ and $x \in X$. Let $x \in X$, $n \in \mathbb{N}$, $r, r_1, \ldots, r_n \in R \backslash \{0\}$, and $x_1, \ldots, x_n \in C \cup \{0\}$ such that $rx = \sum_{i=1}^n r_i x_i$ and $\sum_{i=1}^n r_i \neq 0$.

Case 1. If $x_1, \ldots, x_n \in C$, then $x \in C$, since C is a convex cone.

Case 2. If $x_i = 0$ for every $i = 1, \ldots, n$, then $rx = 0$ and so $x = 0$, in view of the assumption.

Case 3. Let $I := \{i = 1, \ldots, n : x_i \neq 0\} \neq \emptyset$. Then one has $rx = \sum_{i \in I} r_i x_i$, with $\sum_{i \in I} r_i \neq 0$ by assumption (S), which yields that $x \in C$, since C is a convex cone.

The previous three cases show that $x \in C \cup \{0\}$ and thus $C \cup \{0\}$ is a convex cone.

(ii) If $0 \in C$, then $C = C \cup \{0\}$ and so the conclusion is clear. Assume that $0 \notin C$ and $\sum_{i=1}^n r_i x_i \neq 0$ for every $r_1, \ldots, r_n \in R \backslash \{0\}$, $x_1, \ldots, x_n \in C$ and $n \in \mathbb{N}$. Let $x \in X, n \in \mathbb{N}, r, r_1, \ldots, r_n \in R \backslash \{0\}$ and $x_1, \ldots, x_n \in C$ such that $rx = \sum_{i=1}^n r_i x_i$ and $\sum_{i=1}^n r_i \neq 0$. Then $x \in C \cup \{0\}$, since $C \cup \{0\}$ is a convex cone. We declare that $x \in C$, i.e., $x \neq 0$. If not, i.e., $x = 0$, then $\sum_{i=1}^n r_i x_i = rx = 0$, a contradiction with the assumption. This proves C is a convex cone. $\square$

Proposition 3.18. *Let X be a semimodule over a semiring R, and assume $C_i \subseteq X$ $(i = 1, \ldots, m)$. Then the following statements hold:*

(i) *If C_i is a cone for each $i \in I$, where I is an index set, then so are $\bigcup_{i \in I} C_i$ and $\bigcap_{i \in I} C_i$.*

(ii) *If C_i is a convex cone for each $i \in I$, where I is an index set, then so is $\bigcap_{i \in I} C_i$.*

(iii) *If C_i is a convex cone for each $i \in I := \{1, \ldots, m\}$, where $m \in \mathbb{N}$, and X is uniquely divisible, then $\sum_{i \in I} C_i$ is a convex cone.*

Proof. (i) Let $r, r' \in R \backslash \{0\}$ $x \in \bigcup_{i \in I} C_i$ and $x' \in X$ such that $r'x' = rx$. Then $x \in C_{i_0}$ for some $i_0 \in I$. Since C_{i_0} is a cone, one has $x' \in C_{i_0}$ and so $x' \in \bigcup_{i \in I} C_{i_0}$. This yields $\bigcup_{i \in I} C_i$ is a cone. Similarly, we can show $\bigcap_{i \in I} C_i$ is a cone.

(ii) Let $x \in X, n \in \mathbb{N}, r, r_1, \ldots, r_n \in R \backslash \{0\}$, and $x_1, \ldots, x_n \in \bigcap_{i \in I} C_i$ such that $rx = \sum_{j=1}^n r_j x_j$ and $\sum_{j=1}^n r_j \neq 0$. Since $C_i (i \in I)$ are convex cones, one has $x \in C_i (i \in I)$ and so $x \in \bigcap_{i \in I} C_i$, which yields that $\bigcap_{i \in I} C_i$ is a convex cone.

(iii) Let $x \in X$, $n \in \mathbb{N}$, $r, r_j \in R \backslash \{0\}$, $x_j := \sum_{i=1}^m c_j^i \in \sum_{i=1}^m C_i$ and $c_j^i \in C_i$ $(i \in I$, $j = 1, \ldots, n)$ such that $rx = \sum_{j=1}^n r_j x_j$ and $\sum_{j=1}^n r_j \neq 0$. Since X is divisible, there are $c_i \in X (i \in I)$ such that $rc_i = \sum_{j=1}^n r_j c_j^i$. Since $C_i (i \in I)$ are convex cones, $c_i \in C_i (i \in I)$ and so $r(\sum_{i=1}^m c_i) = \sum_{j=1}^n r_j (\sum_{i=1}^m c_j^i) = \sum_{j=1}^n r_j x_j = rx$, which yields that $x = \sum_{i=1}^m c_i \in \sum_{i=1}^m C_i$, since X is uniquely divisible. This proves $\sum_{i=1}^m C_i$ is convex. $\square$

Proposition 3.19. *Let X, Y be semimodules over a semiring R, and $C, D \subseteq Y$. Then C and D are convex cones if and only if $C \times D$ is a convex cone.*

Proof. Let $(x,y)\in X\times Y$, $n\in\mathbb{N}$, $r, r_1, ..., r_n\in R\backslash\{0\}$, and $(x_1,y_1), ..., (x_n,y_n)\in C\times D$ such that $r(x,y)=\sum_{j=1}^n r_j(x_j,y_j)$ and $\sum_{j=1}^n r_j\neq 0$, or equivalently, $rx=\sum_{j=1}^n r_jx_j$ and $ry=\sum_{j=1}^n r_jy_j$ and $\sum_{j=1}^n r_j\neq 0$. If C,D are convex cones, then $x\in C$ and $y\in D$, i.e., $(x,y)\in C\times D$, which yields $C\times D$ is a convex cone. If $C\times D$ is a convex cone, then $(x,y)\in C\times D$, i.e., $x\in C$ and $y\in D$, showing that C and D are convex cones. □

Proposition 3.20. *Let C be a convex cone in a divisible monoid X with $0\in C$ and let $q\in\mathbb{Q}_+$ be a reduced fraction. Then qC is a convex cone.*

Proof. From Proposition 3.8(i) one has qC is convex. From Proposition 3.15(ii), it suffices to show qC is a cone. To this aim, let $x'\in X, r', r\in\mathbb{Z}_+\backslash\{0\}=\mathbb{N}$ and $x\in qC$ such that $r'x'=rx$. Then, there are $l,n,m_i\in\mathbb{N}$ and $x_i\in C$ $(i=1,\dots,n)$ such that

$$lx=\sum_{i=1}^n m_ix_i,\quad \sum_{i=1}^n m_i=m,\quad \frac{m}{l}=q.$$

Then, it follows that

$$\begin{gathered}(lr')x'=l(r'x')=l(rx)=r(lx)=r(\sum_{i=1}^n m_ix_i)=\sum_{i=1}^n m_i(rx_i),\\ \sum_{i=1}^n m_i=m,\quad \frac{m}{lr'}=\frac{q}{r'}.\end{gathered}\tag{11}$$

Since $r\in\mathbb{N}$ and $x_i\in C(i=1,\dots,n)$, from Proposition 3.16 one has $rx_i\in C$ $(i=1,\dots,n)$. Since C is a convex cone, (11) allows that $x'\in C$ and so $x'\in\frac{q}{r'}C$. Since $r'\in\mathbb{N}$, one has $\frac{q}{r'}\le q$. Now, Proposition 3.11 (i) implies that $\frac{q}{r'}C\subseteq qC$ and it follows that $x'\in qC$, which yields that qC is a cone. □

Proposition 3.21. *Let C be a convex cone of a divisible monoid X, $0\in C$ and $q_i\in\mathbb{Q}_+(i=1,\dots,m)$ be reduced fractions. Then $\bigcap_{i=1}^m q_iC=(\min_{1\le i\le m}q_i)C$ and $\bigcup_{i=1}^m q_iC=\sum_{i=1}^m q_iC=(\max_{1\le i\le m}q_i)C$, which implies $\bigcap_{i=1}^m q_iC$, $\bigcup_{i=1}^m q_iC$ and $\sum_{i=1}^m q_iC$ are convex cones.*

Proof. Since $0\in C$, Proposition 3.8(ii) allows $0\in q_iC$ $(i=1,\dots,m)$. Proposition 3.20 implies q_iC $(i=1,\dots,m)$ are convex cones. From Proposition 3.11(i) one has $(\min_{1\le i\le m}q_i)C\subseteq q_iC\subseteq(\max_{1\le i\le m}q_i)C$ for each $i=1,\dots,m$ and as a consequence,

$$\bigcap_{i=1}^m q_iC=(\min_{1\le i\le m}q_i)C$$

and

$$\begin{aligned}\bigcup_{i=1}^m q_iC&=(\max_{1\le i\le m}q_i)C\subseteq\sum_{i=1}^m q_iC\\ &\subseteq\overbrace{(\max_{1\le i\le m}q_i)C+\cdots+(\max_{1\le i\le m}q_i)C}^{m}=(\max_{1\le i\le m}q_i)C,\end{aligned}$$

where the last equality follows from Proposition 3.16. □

4. Convex functions in groups and semigroups

In this section, we study convex functions in groups and semigroups.

Definition 4.1. (Epigraph, effective domain and lower level set) Suppose that X, Y are semimodules over a semiring R and that Y is endowed with a partial order $\leq$. Let $f : X \to Y \cup \{+\infty\}$, where $+\infty$ is a maximal element of Y.

(i) The *epigraph* of f, say epi$f$, is given by epi$f := \{(x, y) \in X \times Y : f(x) \leq y\}$.

(ii) The *effective domain* of f, say dom$f$, is given by dom$f := \{x \in X : f(x) < +\infty\} = \{x \in X : \exists y \in X, (x, y) \in \text{epi}f\}$.

(iii) The *lower level set* of f, say $\text{lev}_{\leq y}f$, is given by $\text{lev}_{\leq y}f := \{x \in X : f(x) \leq y\}$, where $y \in Y$.

Definition 4.2. (Convex function) Suppose that X, Y are semimodules over a semiring R and that Y is endowed with a compatible partial order $\leq$ and $+\infty$ is a maximal element of Y. A function $f : X \to Y \cup \{+\infty\}$ is said to be *convex*, if $f : \text{dom}f \to Y$ is convex, that is, for any choice of $n \in \mathbb{N}$, $r, r_1, \ldots, r_n \in R\backslash\{0\}$ and $x_1, \ldots, x_n \in \text{dom}f$ such that

$$rx = \sum_{i=1}^{n} r_i x_i, \quad r = \sum_{i=1}^{n} r_i,$$

we have $rf(x) \leq \sum_{i=1}^{n} r_i f(x_i)$. f is said to be *concave* if $-f$ is convex.

The definition of the convex function $f : X \to Y$ was given in [3, Definition 16] without the assumption $r \neq 0$.

Proposition 4.3. *Suppose that X, Y are semimodules over a semiring R and that Y is endowed with a compatible partial order $\leq$ and $+\infty$ is a maximal element of Y. Let $f : X \to Y \cup \{+\infty\}$. Then the following statements hold:*

(i) *If f is convex, then* dom$f$ *is convex.*

(ii) *Assume that $(Y, \leq)$ satisfies that for every $r \in R \setminus \{0\}$, $y_1, y_2 \in Y$ we have $ry_1 \leq ry_2 \Longrightarrow y_1 \leq y_2$. If f is convex, then* epi$f$ *and* $\text{lev}_{\leq y}f$ *are convex, where $y \in Y$.*

(iii) *Assume that Y is divisible. If* epi$f$ *is convex, then f is convex.*

Proof. (i) Let $x \in X, n \in \mathbb{N}, x_i \in \text{dom}f$ and $r, r_i \in R \setminus \{0\}$ $(i = 1, \ldots, n)$ such that $rx = \sum_{i=1}^{n} r_i x_i$, $r = \sum_{i=1}^{n} r_i$. Since $x_i \in \text{dom}f$ $(i = 1, \ldots, n)$, one has $f(x_i) \neq +\infty$. Since f is convex, one has

$$rf(x) \leq \sum_{i=1}^{n} r_i f(x_i) \neq +\infty,$$

which yields that $f(x) \neq +\infty$, i.e., $x \in \text{dom}f$. If not, that is, $f(x) = +\infty$, then $rf(x) = +\infty$, which is a contradiction with the previous inequality.

(ii) Let $(x, y) \in X \times Y$, $n \in \mathbb{N}$, $(x_i, y_i) \in \text{epi}f$ and $r, r_i \in R \setminus \{0\}$ $(i = 1, \ldots, n)$ such that $r(x, y) = \sum_{i=1}^{n} r_i(x_i, y_i)$, $r = \sum_{i=1}^{n} r_i$. Since $(x_i, y_i) \in \text{epi}f$ $(i = 1, \ldots, n)$, we have $f(x_i) \leq y_i$ and it follows from the fact that $\leq$ is a compatible partial order that

$$rf(x) \leq \sum_{i=1}^{n} r_i f(x_i) \leq \sum_{i=1}^{n} r_i y_i = ry,$$

where the first inequality follows from the convexity of f. As a consequence, the assumption yields that $f(x) \leq y$, i.e., $(x, y) \in \mathrm{epi} f$. This proves that $\mathrm{epi} f$ is convex. Similarly, we can prove $\mathrm{lev}_{\leq y} f$ is convex.

(iii) Let $x \in X, n \in \mathbb{N}, x_i \in \mathrm{dom} f$ and $r, r_i \in R \setminus \{0\} (i = 1, \dots, n)$ such that $rx = \sum_{i=1}^{n} r_i x_i$, $r = \sum_{i=1}^{n} r_i$. Note that $(x_i, f(x_i)) \in \mathrm{epi} f (i = 1, \dots, n)$. Since Y is divisible, then there exists $y \in Y$ such that $ry = \sum_{i=1}^{n} r_i f(x_i)$. As a consequence, $r(x, y) = \sum_{i=1}^{n} r_i (x_i, f(x_i))$. Since $\mathrm{epi} f$ is convex, one has $(x, y) \in \mathrm{epi} f$, that is, $f(x) \leq y$ and it follows from the fact that $\leq$ is a compatible partial order that

$$rf(x) \leq ry = \sum_{i=1}^{n} r_i f(x_i).$$

This proves f is convex. □

Corollary 4.4. *Let $f : \mathbb{Z}^m \to \mathbb{R}^n$ or $f : \mathbb{R}^m \to \mathbb{R}^n$. Then f is convex if and only if* $\mathrm{epi} f$ *is convex.*

Definition 4.5. (Minkowski functional for groups) [2, Definition 2.7] Assume that X is a monoid and $C \subseteq X$. The Minkowski functional $\rho_C : X \to \mathbb{R}_+ \cup \{+\infty\}$ is given by $\rho_C(x) := \inf\{q \in \mathbb{Q}_+ : x \in qC\}$. If $\{q \in \mathbb{Q}_+ : x \in qC\} = \emptyset$, then define $\rho_C(x) := +\infty$.

Remark 4.6. It is easy to see that if $x \in C$, then $\rho_C(x) \leq 1$. If we assume $0 \in C$, then $\rho_C(x) \geq 1$, for any $x \notin C$. Indeed, ab absurdo, suppose that $\rho_C(x) < 1$. Then there is a reduced fraction $q \in \mathbb{Q}_+$ with $q < 1$ such that $x \in qC \subseteq C$, where the inclusion follows from Proposition 3.11(i). This is a contradiction.

Definition 4.7. ($\mathbb{N}$-sublinear function) [3, Definitions 19 and 20] Suppose that X, Y are semimodules over a semiring R, and suppose that Y is endowed with a partial order $\leq$ and $+\infty$ is a maximal element of Y. A function $f : X \to Y \cup \{+\infty\}$ is said to be $\mathbb{N}$-sublinear, if it is both subadditive, i.e., $f(x + y) \leq f(x) + f(y)$ for all $x, y \in X$, and positively homogeneous, that is, $f(kx) = kf(x)$ for all $k \in \mathbb{N} \cup \{0\}$ and $x \in X$.

Proposition 4.8. *Assume that X is a monoid and $f : X \to \mathbb{R} \cup \{+\infty\}$. Then the following statements hold:*

(i) *If f is $\mathbb{N}$-sublinear, then it is convex and positively homogeneous.*

(ii) *If X is semidivisible and f is convex and positively homogeneous, then f is $\mathbb{N}$-sublinear.*

Proof. (i) Let $x \in X$, $n \in \mathbb{N}$, $x_i \in X$ and $r_i \in \mathbb{Z}_+ \setminus \{0\} = \mathbb{N}$ $(i = 1, \dots, n)$ such that $rx = \sum_{i=1}^{n} r_i x_i$, $r = \sum_{i=1}^{n} r_i$. Since f is $\mathbb{N}$-sublinear, it is positively homogeneous and subadditive and it follows that

$$rf(x) = f(rx) = f(\sum_{i=1}^{n} r_i x_i) \leq \sum_{i=1}^{n} f(r_i x_i) = \sum_{i=1}^{n} r_i f(x_i),$$

which implies that f is convex.

(ii) Let $x, y \in X$. Since X is semidivisible, there are $p \in \mathbb{N}$ prime and $z \in X$ such that $pz = \overbrace{x + y + 0 \cdots + 0}^{p}$. Since f is convex and positively homogeneous, one has $f(0) = f(k0) = kf(0)$ for $k = 2$ and so $f(0) = 0$. Then it follows that

$$\begin{aligned} f(x+y) = f(pz) = pf(z) &\leq \overbrace{f(x) + f(y) + f(0) + \cdots + f(0)}^{p} \\ &= f(x) + f(y) + (p-2)f(0) = f(x) + f(y), \end{aligned}$$

which yields f is subadditive. □

Proposition 4.9. *Assume that $C, D \subseteq X$ are convex subsets of a monoid X. Then the following statements hold:*

(i) *ρ_C is $\mathbb{N}$-sublinear and so convex and positively homogeneous.*

(ii) *Assume $C \subseteq D$. Then $\rho_D \leq \rho_C$.*

Proof. (i) It follows immediately from [2, Proposition 2.4] and Proposition 4.8(i).

(ii) Since $C \subseteq D$, one has from Proposition 3.11 (ii) that $qC \subseteq qD$ for $q \in \mathbb{Q}_+$ and so $\rho_D(x) \leq \rho_C(x)$ for each $x \in X$, i.e., $\rho_D \leq \rho_C$. □

Definition 4.10. Assume that X is a monoid. For functions $f, g : X \to \mathbb{R} \cup \{+\infty\}$, the *infimal convolution* (shortly *inf-convolution*) is the function $f \square g : X \to \mathbb{R} \cup \{+\infty\}$ given by

$$(f \square g)(x) := \inf_{u+v=x} \{f(u) + g(v)\}.$$

Proposition 4.11. [2, Proposition 2.2] *Assume that X is a p-semidivisible monoid and $f, g : X \to \mathbb{R} \cup \{+\infty\}$ are convex (resp. $\mathbb{N}$-sublinear). Assume, furthermore, that X has at most unique divisors as holds in the locally convex case. Then the function $f \square g$ is convex (resp. N-sublinear).*

Proposition 4.12. *Suppose that X is a monoid, $f, g : X \to \mathbb{R} \cup \{+\infty\}$ with $f \leq g$ and $f(0) \leq g(0) \leq 0$ and f is subadditive. Then the equality $f \square g = f$ holds. In particular, $f \square f = f$.*

Proof. Since $0 \in X$, $f(0) \leq g(0) \leq 0$ and $f \leq g$, one has $(f \square g)(x) \leq f(x) \leq g(x)$ for any $x \in X$. We now show $(f \square g)(x) = f(x)$ for any $x \in X$. If not, i.e., there is $x_0 \in X$ such that $(f \square g)(x_0) < f(x_0)$, then there are $u_0, v_0 \in X$ with $u_0 + v_0 = x_0$ such that $f(u_0) + g(v_0) < f(x_0)$. Since f is subadditive and $f \leq g$, one has

$$f(x_0) \leq f(u_0) + f(v_0) \leq f(u_0) + g(v_0) < f(x_0),$$

a contradiction. Setting $g := f$ yields $f \square f = f$. □

Corollary 4.13. *Let C, D be convex sets of a monoid X and $C \subseteq D$. Then $\rho_C \square \rho_D = \rho_D$. In particular, $\rho_C \square \rho_C = \rho_C$.*

Proof. From Proposition 4.9 (i) and (ii) one has $\rho_D \leq \rho_C$, $\rho_D(0) = \rho_C(0) = 0$ and ρ_D is subadditive. Setting $f := \rho_D$ and $g := \rho_C$ in Proposition 4.12 allows us to complete the proof. □

Proposition 4.14. *Let X, Y be monoids and $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be convex functions. Then $f + r_0 g$ is convex, where $r_0 \in \mathbb{R}_+$. In particular, $f + g$ and $r_0 g$ are convex.*

Proof. The thesis is obvious if $\mathrm{dom}(f + r_0 g) = \emptyset$. Suppose that $\mathrm{dom}(f + r_0 g) \neq \emptyset$. Let $x \in X, n \in \mathbb{N}, x_i \in \mathrm{dom}(f + r_0 g)$ and $r_i \in \mathbb{Z}_+ \backslash \{0\} = \mathbb{N} (i = 1, \dots, n)$ such that $rx = \sum_{i=1}^n r_i x_i,\ r = \sum_{i=1}^n r_i$. Then $x_i \in \mathrm{dom} f \cap \mathrm{dom} g (i = 1, \dots, n)$. Since f and g are convex, one has $rf(x) \leq \sum_{i=1}^n r_i f(x_i)$ and $rg(x) \leq \sum_{i=1}^n r_i g(x_i)$ and so

$$\begin{aligned} r(f + r_0 g)(x) &= rf(x) + rr_0 g(x) \leq \sum_{i=1}^n r_i f(x_i) + r_0 r g(x) \\ &\leq \sum_{i=1}^n r_i f(x_i) + r_0 \sum_{i=1}^n r_i g(x_i) = \sum_{i=1}^n r_i f(x_i) + \sum_{i=1}^n r_i (r_0 g)(x_i) \\ &= \sum_{i=1}^n r_i (f + r_0 g)(x_i). \end{aligned}$$

This proves that $f + r_0 g$ is convex. Letting $r_0 := 1$ and $f :\equiv 0$ yields $f + g$ and $r_0 g$ are convex, respectively. □

Proposition 4.15. *Assume that X is a monoid and $g_j : X \to \mathbb{R} \cup \{+\infty\} (j \in I)$, where I is an index set. If $g_j (j \in I)$ are convex, then so is $\max_{j \in I} g_j$.*

Proof. If $\mathrm{dom}(\max_{j \in I} g_j) = \emptyset$, then it is clear. Assume that $\mathrm{dom}(\max_{j \in I} g_j) \neq \emptyset$. Let $x \in X, n \in \mathbb{N}, x_i \in \mathrm{dom}(\max_{j \in I} g_i)$ and $r_i \in \mathbb{Z}_+ \backslash \{0\} = \mathbb{N}$ $(i = 1, \dots, n)$ such that $rx = \sum_{i=1}^n r_i x_i,\ r = \sum_{i=1}^n r_i$. Then $x_i \in \mathrm{dom} g_j (i = 1, \dots, n, j \in I)$. Since $g_j (j \in I)$ are convex, one has $rg_j(x) \leq \sum_{i=1}^n r_i g_j(x_i) \leq \sum_{i=1}^n r_i (\max_{j \in I} g_j)(x_i)$ and so $r(\max_{j \in I} g_j)(x) \leq \sum_{i=1}^n r_i (\max_{j \in I} g_j)(x_i)$. This proves $\max_{j \in I} g_j$ is convex. □

Assume that X is a monoid and $C \subseteq X$.

The *indicator function* of C, $i_C : X \to \mathbb{R} \cup \{+\infty\}$, is given by

$$i_C(x) := \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$$

Proposition 4.16. *Assume that X is a monoid and $C \subseteq X$. Then i_C is convex if and only if C is convex.*

Proof. Note that $\mathrm{epi}\ i_C = C \times \mathbb{R}_+$. Then it follows immediately from Propositions 3.6 and 4.3 (ii) and (iii) that the desired conclusion holds. □

Let X and Y be two monoids and let $L(X, Y) := \{a : X \to Y : a \text{ is additive}\}$. Recall that $a \in L(X, Y)$ is additive, if $a(x + y) = a(x) + a(y)$ for any $x, y \in X$. If $Y := Z$ and X, Z are groups, then $L(X, Z)$ is the additive dual group of a group X, denoted by X^*, which is an additive group with the addition being point-wise addition of elements in Z.

Given two monoids X and Y. In [3], the *subdifferential* of $f : X \to Y \cup \{+\infty\}$ at $x_0 \in X$ is given by $\partial f(x_0) := \{a \in L(X, Y) : f(x_0) + a(h) \leq f(x_0 + h), \forall h \in X\}$, where $+\infty$ is a maximal element with respect to the partial order $\leq$ on Y. If X is a group and $Y := \mathbb{R}$, then we can define the subdifferential of $f : X \to \mathbb{R} \cup \{+\infty\}$ at $x_0 \in X$ as $\partial f(x_0) := \{a \in L(X, \mathbb{R}) : f(x_0) + a(x + (-x_0)) \leq f(x), \forall x \in X\}$.

Remark 4.17. Suppose that X is a commutative group, $f : X \to \mathbb{R} \cup \{+\infty\}$ and $x_0 \in X$. It is clear that if $x_0 \notin \mathrm{dom} f$, then $\partial f(x_0) = \emptyset$. Moreover, if $x_0 \in \mathrm{dom} f$, then $\partial f(x_0) := \{a \in L(X, \mathbb{R}) : f(x_0) + a(x) - a(x_0) \leq f(x), \forall x \in \mathrm{dom} f\}$. Indeed, since $a \in L(X, \mathbb{R})$ is additive, we have $a(0) = a(0 + 0) = a(0) + a(0)$ and so $0 = a(0) = a(x_0 + (-x_0)) = a(x_0) + a(-x_0)$ and so $a(-x_0) = -a(x_0)$.

Proposition 4.18. *Let X be a commutative group, $x_0 \in X$ and $f : X \to \mathbb{R} \cup \{+\infty\}$. Then $\partial(\lambda f)(x_0) = \lambda \partial f(x_0)$ for any $\lambda \in \mathbb{Z}_+$ and $\partial f(x_0)$ is convex in $L(X, \mathbb{R})$.*

Proof. Clearly, $\partial(\lambda f)(x_0) = \lambda \partial f(x_0)$ for any $\lambda \in \mathbb{Z}_+$. If $\partial f(x_0) = \emptyset$, then the thesis is obvious. Suppose that $\partial f(x_0) \neq \emptyset$. Let $a \in L(X, \mathbb{R})$, $a_i \in \partial f(x_0)$ and $r_i \in \mathbb{N}$ $(i = 1, \ldots, n)$ such that $ra = \sum_{i=1}^n r_i a_i$ and $r = \sum_{i=1}^n r_i$. Since $a_i \in \partial f(x_0)$ $(i = 1, \ldots, n)$, one has $f(x_0) + a_i(x) - a_i(x_0) \leq f(x)$ for each $x \in X$. Since $r_i \in \mathbb{N}$ $(i = 1, \ldots, n)$, we have for each $x \in X$,

$$\begin{aligned} r(f(x_0) + a(x) - a(x_0)) &= \sum_{i=1}^n r_i f(x_0) + \sum_{i=1}^n r_i(a_i(x) - a_i(x_0)) \\ &\leq \sum_{i=1}^n r_i f(x) = r f(x), \end{aligned}$$

which implies that $f(x_0) + a(x) - a(x_0) \leq f(x)$, i.e., $a \in \partial f(x_0)$. □

Proposition 4.19. *Assume that X is a commutative group, $C \subseteq X$ and $x_0 \in C$. Then $\partial i_C(x_0) = \{a \in L(X, \mathbb{R}) : a(x) - a(x_0) \leq 0, \forall x \in C\}$ and it is a convex cone.*

Proof. It is clear that $\mathrm{dom}\, i_C = C$ and so

$$\partial i_C(x_0) = \{a \in L(X, \mathbb{R}) : a(x) - a(x_0) \leq 0, \forall x \in \mathrm{dom} i_C = C\}.$$

We next show $\partial i_C(x_0)$ is a convex cone. To this end, let $a \in L(X, \mathbb{R})$, $r, r_1, \ldots, r_n$, $n \in \mathbb{N}$, and $a_1, \ldots, a_n \in i_C(x_0)$ such that $ra = \sum_{i=1}^n r_i a_i$. Then, for any $x \in C$,

$$r(a(x) - a(x_0)) = \textstyle\sum_{i=1}^n r_i(a_i(x) - a_i(x_0)) \leq 0,$$

which implies $a(x) - a(x_0) \leq 0$ for each $x \in C$, i.e., $a \in \partial i_C(x_0)$. □

Let $N_C(x_0) := \{a \in L(X, \mathbb{R}) : a(x + (-x_0)) \leq 0, \forall x \in C\}$.
Define $\sigma_C(a) := \sup_{x \in C} a(x)$, where $a \in L(X, \mathbb{R})$. Then $N_C(x_0)$ is the normal cone of C at x_0 and σ_C is the support function of C if X is a vector space.

Proposition 4.20. *Assume that X is a commutative group, $C \subseteq X$ and $x_0 \in C$. Then $N_C(x_0) = \{a \in L(X, \mathbb{R}) : \sigma_C(a) = a(x_0)\}$.*

Proof. By Remark 4.17 it follows that $a \in N_C(x_0)$ if and only if

$$a(x) - a(x_0) = a(x) + a(-x_0) = a(x + (-x_0)) \leq 0 \text{ for all } x \in C,$$

or equivalently, $\sigma_C(a) = a(x_0)$. This completes the proof. □

Let $Y \cup \{+\infty\}$ be a group and $+\infty$ be a maximal element with respect to the partial order $\leq$ on Y. We say that the order $\leq$ is inductive, if every countable chain has an upper bound and the order $\leq$ is a complete order, if every order bounded set has both infimum and supremum.

Definition 4.21. Assume that X, Y are groups, $C \subseteq X$ and $f : X \to Y \cup \{+\infty\}$, where $+\infty$ is a maximal element of Y.

(i) (Core) The *core* of C is defined by
$\operatorname{core} C := \{x \in X : \forall h \in X, \exists n \in \mathbb{N}, \exists g \in X, ng = h, x + g \in C\}$.

(ii) (Core of domain) The *core of the domain* of f is defined by
$\operatorname{core}\operatorname{dom} f := \{x \in X : \forall h \in X, \exists n \in \mathbb{N}, \exists g \in X, ng = h, f(x + g) < +\infty\}$.

(iii) (Quasi core) The *quasi core* of C is defined by
$\operatorname{qcore} C := \{x \in C : \operatorname{cone}(C + (-x)) = X\}$.

(iv) (Directional derivative) Assume that $Y \cup \{+\infty\}$ is a group with an inductive order. For $x \in \operatorname{core}(\operatorname{dom} f)$, define
$f_x(h) := \inf\{n(f(x + g) - f(x)) : ng = h, f(x + g) < +\infty\}$.

If $C \subseteq X$ is convex and $f : X \to Y \cup \{+\infty\}$ is a convex function, then Definition 4.21 (i), (ii) and (v) reduce to [3, Definitions 26, 25 and 27], respectively. Clearly, $\operatorname{core} C \subseteq C$, $\operatorname{qcore} C \subseteq C$ and $\operatorname{core}(\operatorname{dom} f) \subseteq \operatorname{dom} f$. Moreover, $\operatorname{core} C \subseteq \operatorname{core} D$, $\operatorname{qcore} C \subseteq \operatorname{qcore} D$, if $C \subseteq D \subseteq X$. See [4–6, 8] for quasi (relative) interiors related to quasi cores in vector spaces.

Lemma 4.22. *Let X be a commutative group and $C \subseteq X$. Then the following statements hold:*

(i) $\operatorname{core} C \subseteq \operatorname{qcore} C$.

(ii) *Assume that* $0 \in \operatorname{qcore} C$. *Then one has* $\rho_C : X \to \mathbb{R}_+$, *that is,* $\rho_C(x) < +\infty$ *for each* $x \in X$.

Proof. (i) If $\operatorname{core} C = \emptyset$, then it is clear that (i) holds. Assume that $\operatorname{core} C \neq \emptyset$. Let $\bar{x} \in \operatorname{core} C$. Then for any $h \in X, \exists n \in \mathbb{N}, \exists g \in X$ with $ng = h$ such that $\bar{x} + g \in C$. It follows that $h = n(x + (-\bar{x}))$, for some $x \in C$. Therefore, $h \in \mathbb{N}(C + (-\bar{x}))$ and so $\mathbb{N}(C + (-\bar{x})) = X$, being $h \in X$ arbitrary. Since $\mathbb{N}(C + (-\bar{x})) \subseteq \operatorname{cone}(C + (-\bar{x}))$, one has $\operatorname{cone}(C + (-\bar{x})) = X$ and moreover, $\bar{x} \in C$, since $\bar{x} \in \operatorname{core} C$ and $\operatorname{core} C \subseteq C$. This yields $\bar{x} \in \operatorname{qcore} C$.

(ii) Since $0 \in \operatorname{qcore} C$, one has $\operatorname{cone} C = X$. By Proposition 3.15 (v), we have $\operatorname{cone} C = \bigcup_{q \in \mathbb{Q}_+ \setminus \{0\}} qC = X$ and thus for every $x \in X$, there exists $q_x \in \mathbb{Q}_+ \setminus \{0\}$ such that $x \in q_x C$. This implies $\rho_C(x) \leq q_x < +\infty$. $\square$

The assumption that the core of a subset is nonempty in non-divisible groups and semigroups is hard to be fulfilled. But, it is not the case for the quasi core of a subset in non-divisible groups and semigroups. The following remark and the subsequent example show that the nonemptiness of the quasi core of a convex subset is a mild assumption compared with that of the nonemptiness of the core.

Remark 4.23. If $C \subseteq \mathbb{Z}^n$ is a bounded set, then $\operatorname{core} C = \emptyset$. This can be seen from the following facts. Since $C \subseteq \mathbb{Z}^n$ is bounded, there exists $K > 0$ such that $\|x\| \leq K$ for each $x \in C$. Take $x \in C$. Let $m \in \mathbb{Z}$ be such that $m^2 + (m + 1)^2 > 4K^2$. Let $h := (m, m + 1, 0, \ldots, 0) \in \mathbb{Z}^n$. Then $\|h\| = \sqrt{m^2 + (m + 1)^2} > 2K$ and there are unique $n_0 := 1 \in \mathbb{N}$ and $g := h \in \mathbb{Z}^n$ such that $n_0 g = h$.

By the triangle inequality it follows that

$$\|x+g\| \geq \|g\| - \|x\| > 2K - K = K,$$

which implies $x+g \notin C$. This shows $x \notin \text{core}C$. Since $x \in C$ is arbitrary, one has $\text{core}\, C = \emptyset$.

Example 4.24. Let e_i $(i = 1, \dots, n)$ be the standard basis of $\mathbb{Z}^n$ and define $C := \{e_1, ..., e_n, -e_1, \dots, -e_n, 0\}$. Then C is bounded. For any $u := (u_1, ..., u_n) \in \mathbb{Z}^n$, let $I^+ := \{i = 1, \dots, n : u_i > 0\}$, $I^- := \{i = 1, \dots, n : u_i < 0\}$ and finally $I^= := \{i = 1, \dots, n : u_i = 0\}$. Then

$$1u = u = \textstyle\sum_{i \in I^+} u_i(e_i - 0) + \sum_{i \in I^-}(-u_i)(-e_i - 0) + \sum_{i \in I^=} 1(0-0),$$

which yields $u \in \text{cone}\,(C-0)$ and so $\text{cone}\,(C-0) = \mathbb{Z}^n$ since $u \in \mathbb{Z}^n$ is arbitrary. This implies $0 \in \text{qcore}C$.

Lemma 4.25. *Assume that X is a commutative group, $\bar{x} \in X$ and $C \subseteq X$. Then the following statements hold:*

(i) $\text{core}(\bar{x} + C) = \bar{x} + \text{core}\, C$.

(ii) $\text{qcore}(\bar{x} + C) = \bar{x} + \text{qcore}\, C$.

Proof. (i) Observe that

$$\begin{aligned} x \in \text{core}(\bar{x}+C) &\iff \forall h \in X, \exists n \in \mathbb{N}, \exists g \in X, ng = h, x + g \in \bar{x} + C \\ &\iff \forall h \in X, \exists n \in \mathbb{N}, \exists g \in X, ng = h, x + (-\bar{x}) + g \in C \\ &\iff x + (-\bar{x}) \in \text{core}\, C \\ &\iff x \in \bar{x} + \text{core}\, C. \end{aligned}$$

(ii) Observe that

$$\begin{aligned} y \in \text{qcore}(C+\bar{x}) &\iff y \in C + \bar{x} : \text{cone}\,(C + \bar{x} + (-y)) = X \\ &\iff y + (-\bar{x}) \in C : \text{cone}\,(C + (-(y + (-\bar{x})))) = X \\ &\iff y + (-\bar{x}) \in \text{qcore}\, C \\ &\iff y \in \bar{x} + \text{qcore}\, C. \end{aligned}$$

This completes the proof. □

Proposition 4.26. *Let X be a commutative group, $C \subseteq X$ and $\emptyset \neq D \subseteq X$. Then the following statements hold:*

(i) *If* $\text{core}\, C = C$ *(respectively,* $\text{qcore}\, C = C$*), then* $\text{core}(C+D) = \text{core}\, C + D$ *(respectively,* $\text{qcore}(C+D) = \text{qcore}\, C + D$*).*

(ii) *If C is a convex cone with $0 \in C$, then we have* $\text{core}\, C + C = \text{core}\, C$ *and* $\text{qcore}\, C + C = \text{qcore}\, C$.

Proof. (i) Observe that

$$\text{core}\, C + D = \bigcup_{d \in D} (\text{core}C + d) = \bigcup_{d \in D} \text{core}(C+d) \subseteq \text{core}(C+D) \subseteq C + D,$$

where the second equality follows immediately from Lemma 4.25 (i). This yields $\text{core}(C+D) = \text{core}\, C + D$ if $\text{core}\, C = C$. Similarly, we can prove the other case.

(ii) The inclusion $\operatorname{core} C + C \supseteq \operatorname{core} C$ is clear since $0 \in C$. Observe that

$$\operatorname{core} C + C = \bigcup_{c\in C}(\operatorname{core} C + c) = \bigcup_{c\in C} \operatorname{core}(C + c) \subseteq \operatorname{core}(C + C) = \operatorname{core} C,$$

where the second and the third equalities follow immediately from Lemma 4.25 (i) and Proposition 3.16, respectively. As a consequence, $\operatorname{core} C + C = \operatorname{core} C$. Similarly, we can prove that $\operatorname{qcore} C + C = \operatorname{qcore} C$ by using Lemma 4.25 (ii). □

Proposition 4.27. *Assume that C is a convex subset of a divisible commutative group X. Then* $\operatorname{core} C$ *and* $\operatorname{qcore} C$ *are convex. In particular,* $\operatorname{core}(\operatorname{dom} f)$ *and* $\operatorname{qcore}(\operatorname{dom} f)$ *are convex, where Y is a commutative group endowed with a compatible partial order $\leq$, $f : X \to Y \cup \{+\infty\}$ is convex, $+\infty$ is a maximal element of Y.*

Proof. We first show $\operatorname{core} C$ is convex. If $\operatorname{core} C = \emptyset$, then it is clear. Suppose that $\operatorname{core} C \neq \emptyset$. Let $x \in X$, $n, r_1, \ldots, r_n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \operatorname{core} C$ such that $rx = \sum_{i=1}^n r_i x_i, r = \sum_{i=1}^n r_i$. Given $h \in X$. Since $x_i \in \operatorname{core} C (i = 1, \ldots, n)$, there are $n_i \in \mathbb{N}$ and $g_i \in X$ with $n_i g_i = h (i = 1, \ldots, n)$ such that $x_i + g_i \in C$. Let $n_0 := \max_{1\leq i\leq n} n_i$. Since X is divisible, there is $g \in X$ such that $n_0 g = h$. Let $I := \{i = 1, \ldots, n : n_0 > n_i\}$. If $I = \emptyset$, then $n_0 = n_i$ for every $i = 1, \ldots, n$ and it follows that

$$\begin{aligned}\sum_{i=1}^n n_i r_i(x_i + g_i) &= \sum_{i=1}^n n_0 r_i x_i + \sum_{i=1}^n n_0 r_i g_i = n_0 \sum_{i=1}^n r_i x_i + \sum_{i=1}^n r_i(n_0 g_i)\\ &= n_0 r x + r h = (n_0 r)x + (r n_0)g = (n_0 r)(x + g),\end{aligned}$$

which leads to $x+g\in C$, since $x_i+g_i \in C$ $(i = 1, \ldots, n)$, $\sum_{i=1}^n n_i r_i = \sum_{i=1}^n n_0 r_i = n_0 r$ and C is convex. If $I \neq \emptyset$, then

$$\begin{aligned}&\sum_{i=1}^n n_i r_i(x_i + g_i) + \sum_{i\in I}(n_0 - n_i)r_i x_i = \sum_{i=1}^n n_i r_i(x_i + g_i) + \sum_{i=1}^n (n_0 - n_i)r_i x_i\\ &= \sum_{i=1}^n n_i r_i(x_i + g_i) + \sum_{i=1}^n n_0 r_i x_i + \sum_{i=1}^n n_i r_i(-x_i) = \sum_{i=1}^n n_0 r_i x_i + \sum_{i=1}^n n_i r_i g_i\\ &= n_0 \sum_{i=1}^n r_i x_i + \sum_{i=1}^n r_i h = n_0 r x + r h = (n_0 r)x + (r n_0)g,\end{aligned}$$

hence,
$$\sum_{i=1}^n n_i r_i(x_i + g_i) + \sum_{i\in I}(n_0 - n_i)r_i x_i = (n_0 r)(x + g),$$

which implies that $x + g \in C$, since $x_i, x_i + g_i \in C$ $(i = 1, \ldots, n)$,

$$\sum_{i=1}^n n_i r_i + \sum_{i\in I}(n_0 - n_i)r_i = \sum_{i=1}^n n_i r_i + \sum_{i=1}^n (n_0 - n_i)r_i = n_0 r$$

and C is convex. As a consequence, previous two cases show $x \in \operatorname{core} C$ and so $\operatorname{core} C$ is convex.

We now prove that $\operatorname{qcore} C$ is convex. To this aim, let $x \in X$, $n, r_i \in \mathbb{N}$, $x_i \in \operatorname{qcore} C$ $(i = 1, \ldots, n)$ such that $rx = \sum_{i=1}^n r_i x_i$ with $r = \sum_{i=1}^n r_i$. Since $x_i \in \operatorname{qcore} C$ $(i = 1, \ldots, n)$, i.e., $x_i \in C$ and $\operatorname{cone}(C + (-x_i)) = X$, for any $h \in X$, there are $s^i_j, \alpha_i \in \mathbb{N}$ and $y^i_j \in C (j = 1, \ldots, m)$ such that $\alpha_i h = \sum_{j=1}^m s^i_j(y^i_j + (-x_i))$.

Then $\sum_{j=1}^{m} s_j^i x_i = \sum_{j=1}^{m} s_j^i y_j^i + (-\alpha_i h)$ and it follows that

$$\begin{aligned} r(\prod_{i=1}^{n}(\sum_{j=1}^{m} s_j^i))x &= (\prod_{i=1}^{n}(\sum_{j=1}^{m} s_j^i))rx = (\prod_{i=1}^{n}(\sum_{j=1}^{m} s_j^i))(\sum_{i=1}^{n} r_i x_i) \\ &= (\prod_{i=1}^{n}(\sum_{j=1}^{m} s_j^i))(r_1 x_1 + \cdots + r_n x_n) \\ &= (r_1 \prod_{i\neq 1}(\sum_{j=1}^{m} s_j^i))(\sum_{j=1}^{m} s_j^1 x_1) + \cdots + (r_n \prod_{i\neq n}(\sum_{j=1}^{m} s_j^i))(\sum_{j=1}^{m} s_j^n x_n), \end{aligned}$$

hence,

$$\begin{aligned} & r(\prod_{i=1}^{n}(\sum_{j=1}^{m} s_j^i))x \\ &= (r_1 \prod_{i\neq 1}(\sum_{j=1}^{m} s_j^i))(\sum_{j=1}^{m} s_j^1 y_j^1 + (-\alpha_1 h)) + \cdots + (r_n \prod_{i\neq n}(\sum_{j=1}^{m} s_j^i))(\sum_{j=1}^{m} s_j^n y_j^n + (-\alpha_n h)). \end{aligned}$$

In consequence,

$$\begin{aligned} & (r_1 \prod_{i\neq 1}(\sum_{j=1}^{m} s_j^i)\alpha_1 + \cdots + r_n \prod_{i\neq n}(\sum_{j=1}^{m} s_j^i)\alpha_n)h \\ &= \sum_{j=1}^{m}(r_1 \prod_{i\neq 1}(\sum_{j=1}^{m} s_j^i)s_j^1)y_j^1 + \cdots + \sum_{j=1}^{m}(r_n \prod_{i\neq n}(\sum_{j=1}^{m} s_j^i)s_j^n)y_j^n + (-r(\prod_{i=1}^{n}(\sum_{j=1}^{m} s_j^i)x) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}(r_i \prod_{t\neq i}(\sum_{j=1}^{m} s_j^t)s_j^i)y_j^i + (-r(\prod_{i=1}^{n}(\sum_{j=1}^{m} s_j^i)x). \end{aligned} \tag{12}$$

Observe that $\sum_{i=1}^{n}\sum_{j=1}^{m}(r_i \prod_{t\neq i}(\sum_{j=1}^{m} s_j^t)s_j^i) = r(\prod_{i=1}^{n}(\sum_{j=1}^{m} s_j^i)$.
Since X is divisible, there is $c \in X$ such that

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(r_i \prod_{t\neq i}(\sum_{j=1}^{m} s_j^t)s_j^i)y_j^i = r(\prod_{i=1}^{n}(\sum_{j=1}^{m} s_j^i)c$$

and it follows that $c\in C$ since C is convex and $y_j^i \in C$ $(i = 1, \ldots, n,\ j = 1, \ldots, m)$. From (12) it follows that

$$(r_1 \prod_{i\neq 1}(\sum_{j=1}^{m} s_j^i)\alpha_1 + \cdots + r_n \prod_{i\neq n}(\sum_{j=1}^{m} s_j^i)\alpha_n)h = r(\prod_{i=1}^{n}(\sum_{j=1}^{m} s_j^i)(c + (-x)),$$

which yields $h \in \operatorname{cone}(C + (-x))$ and so $\operatorname{cone}(C + (-x)) = X$ since $h \in X$ is arbitrary. Since C is convex, $x_i \in C$ $(i = 1, \ldots, n)$ and $(\sum_{i=1}^{n} r_i)x = \sum_{i=1}^{n} r_i x_i$, one has $x \in C$ and therefore, $x \in \mathrm{qcore}C$. This proves $\mathrm{qcore}C$ is convex.

In particular, previous conclusions and Proposition 4.3 (i) imply that core(dom$f$) and qcore(dom$f$) are convex. □

Lemma 4.28. (Sum rule for subdifferentials) [3, Theorem 6] *Assume that X_1, X_2 are commutative groups, $(Z, \leq)$ is an order complete group, $x_0 \in X_1$ and $+\infty$ is a maximal element of Z. Suppose $f : X_1 \to Z \cup \{+\infty\}, g : X_2 \to Z \cup \{+\infty\}$ and $T : X_1 \to X_2$ is additive. Then we have*

$$\partial(f + g \circ T)(x_0) \supseteq \partial f(x_0) + T^* \partial g(Tx_0).$$

Furthermore, if X_1 is semidivisible and $0 \in \operatorname{core}(\operatorname{dom} g - T\operatorname{dom} f)$, while f and g are convex, then equality holds. Here $T^* : X_2^* \to X_1^*$ denotes the adjoint operator given by

$$T^*(x_2^*)x_1 := x_2^*(Tx_1),\ x_1 \in X_1, x_2^* \in X_2^*.$$

Lemma 4.29. (Subdifferentials for maximum of finite functions) [3, Theorem 9] *Assume that X is a semidivisible commutative group, $g_i : X \to \mathbb{R} \cup \{+\infty\} (i \in I)$ are convex, where I is a finite index set. Assume that $g := \max_{i \in I} g_i$ and that $x_0 \in \bigcap_{i \in I(x_0)} \operatorname{core}(\operatorname{dom} g_i)$, where $I(\bar{x}) = \{i \in I : g(\bar{x}) = g_i(\bar{x})\}$. Then we have*

$$\partial g(x_0) = \operatorname{conv}(\bigcup_{i \in I(x_0)} \partial g_i(x_0)).$$

Lemma 4.30. (Max formula) [3, Theorem 3] *Let X be a p-semidivisible group and let $(Y, \leq)$ be an additive and inductive order group and let $f : X \to Y \cup \{+\infty\}$ be a convex function, where $+\infty$ is a maximal element of Y. Suppose also that there is some $x_0 \in \operatorname{core}(\operatorname{dom} f)$ such that $f_{x_0}(x_0) + f_{x_0}(-x_0) \leq 0$. Then we have*

$$f_{x_0}(h) = \max\{a(h) : a \in \partial f(x_0)\} \quad \textit{and} \quad \partial f(x_0) \neq \emptyset.$$

Lemma 4.31. (Hahn-Banach theorem for groups) [3, Theorem 7] *Assume that X is a group, $E \subseteq X$ is a subgroup and $(Y, \leq)$ is an order complete group. Suppose that $f : X \to Y$ is $\mathbb{N}$-sublinear and $g : E \to Z$ is additive such that $g \leq f$ on E. Then there is an additive map $\bar{g} : X \to Y$ such that $\bar{g} \leq f$ and $\bar{g} = g$ on E.*

5. Applications to separation between convex sets and optimization problems

In this section, the results obtained in Section 4 will be applied to separation between two convex subsets of a commutative group and to optimization problems.

5.1. Applications to separation between convex sets

Proposition 5.1. *Let X be a group and $\Omega \subseteq X$ be such that $\operatorname{qcore} \Omega \neq \emptyset$, and let $a : X \to \mathbb{R}$ be a nonzero additive function. Then the function a is not a constant function on Ω.*

Proof. Suppose to the contrary that $a(x) = c$ for any $x \in \Omega$, where c is a constant. Observe that we have $0 = a(0) = a(x + (-x)) = a(x) + a(-x)$ for any $x \in X$ and so $a(-x) = -a(x)$. Let $x_0 \in \operatorname{qcore} \Omega$, i.e., $x_0 \in \Omega$ and $\operatorname{cone}(\Omega + (-x_0)) = X$. In consequence for any $x \in X$ there are $r, r_i, n \in \mathbb{N}$, $x_i \in \Omega$, $i = 1, \ldots, n$, such that $rx = \sum_{i=1}^n r_i(x_i + (-x_0))$.

Since a is additive, one has

$$a(rx) = ra(x) = \sum_{i=1}^{n} r_i a(x_i + (-x_0)) = \sum_{i=1}^{n} r_i (a(x_i) - a_i(x_0)) = 0,$$

which yields, since x is arbitrary, that $a \equiv 0$, a contradiction. This completes the proof. □

Let C and D be two nonempty subsets of a group X. Recall that C and D are separable if there is a nonzero additive function $a : X \to \mathbb{R}$ such that

$$\sup_{x \in C} a(x) \leq \inf_{x \in D} a(x).$$

If, in addition,

$$\inf_{x \in C} a(x) < \sup_{x \in D} a(x),$$

i.e. there is $x_1 \in C$ and $x_2 \in D$ such that $a(x_1) < a(x_2)$, then C and D are properly separable.

Proposition 5.2. *Let X be a group, $C \subseteq X$ such that* $\operatorname{qcore} C \neq \emptyset$, *and let $x_0 \notin C$. Then C and x_0 are separable if and only if they are properly separable.*

Proof. It is enough to show that if C and x_0 are separable, then they are properly separable. Assume that C and x_0 are separable. Then there exists a nonzero additive function $a : X \to \mathbb{R}$ such that $a(x) \leq a(x_0)$, $\forall x \in C$. We declare that there is $\bar{x} \in C$ such that $a(\bar{x}) < a(x_0)$. If not, then $a(x) = a(x_0)$ for all $x \in C$, i.e., a is constant on C. Since $\operatorname{qcore} C \neq \emptyset$, Proposition 5.1 yields that $a \equiv 0$, a contradiction. □

Theorem 5.3. (Proper separation between a point and a convex set I) *Let C be a convex subset of a commutative group X with $0 \in C$ and* $\operatorname{qcore} C \neq \emptyset$, *and let $x_0 \notin C$. Then C and x_0 are properly separable.*

Proof. We first assume that $0 \in \operatorname{qcore} C$. Define the subgroup $Y := \{rx_0 : r \in \mathbb{Z}\}$ and the function $h : Y \to \mathbb{R}$ by $h(rx_0) := r$. We now show h is additive. In fact, let $r_1 x_0, r_2 x_0 \in Y$, where $r_1, r_2 \in \mathbb{Z}$. Then

$$h(r_1 x_0 + r_2 x_0) = h((r_1 + r_2)x_0) = r_1 + r_2 = h(r_1 x_0) + h(r_2 x_0),$$

which implies h is additive. From Proposition 4.9 (i) one has ρ_C is $\mathbb{N}$-sublinear. Since $0 \in \operatorname{qcore} C$, Lemma 4.22 (ii) yields that $\rho_C(x) < +\infty$ for every $x \in X$. Let $y := rx_0 \in Y$, where $r \in \mathbb{Z}$. If $r \leq 0$, then $h(y) = r \leq 0 \leq \rho_C(y)$. Assume $r > 0$. Since $0 \in C$ and $x_0 \notin C$, from Remark 4.6 we have $\rho_C(x_0) \geq 1$ and so

$$h(y) = r \leq r\rho_C(x_0) = \rho_C(rx_0) = \rho_C(y).$$

Now Lemma 4.31 yields that there is an additive function $\bar{h} : X \to \mathbb{R}$ such that $\bar{h} \leq \rho_C$ and $\bar{h} = h$ on Y. Observe that $\bar{h}$ is nonzero since $\bar{h}(x_0) = 1$. Therefore,

$$\bar{h}(y) \leq \rho_C(y) \leq 1 = \bar{h}(x_0), \ \forall y \in C.$$

This proves that C and x_0 are separable and so properly separable in view of Proposition 5.2.

In the case $0 \notin \text{qcore}C$, let $\bar{x} \in \text{qcore}C$ and consider the set $C' := C + (-\bar{x})$. Then $0 \in C'$ and $x_0 + (-\bar{x}) \notin C'$ since $x_0 \notin C$. Since $\bar{x} \in \text{qcore}\, C$, Lemma 4.25 (ii) yields

$$0 \in \text{qcore}\, C + (-\bar{x}) = \text{qcore}(C + (-\bar{x})) = \text{qcore}\, C'.$$

Then by previous argument, we have C' and $x_0 + (-\bar{x})$ are properly separable, and so C and x_0 are properly separable. □

Theorem 5.4. (Proper separation between a point and a convex set II) *Let C be a convex subset of a commutative group X with* $\text{qcore}\, C \neq \emptyset$*, and let $x_0 \notin C$. Then C and x_0 are properly separable.*

Proof. If $0 \in C$, then the thesis follows immediately from Theorem 5.3. Suppose that $0 \notin C$ and let $\bar{x} \in C$. Then $0 \in C + (-\bar{x})$, $x_0 + (-\bar{x}) \notin C + (-\bar{x})$ and from Lemma 4.25 (ii) one has $\text{qcore}(C + (-\bar{x})) = \text{qcore}\, C + (-\bar{x}) \neq \emptyset$. It is easy to check that $C + (-\bar{x})$ is a convex set. By Theorem 5.3, $C + (-\bar{x})$ and $x_0 + (-\bar{x})$ are properly separable, i.e., there is a nonzero additive function $a : X \to \mathbb{R}$ such that $a(x + (-\bar{x})) \leq a(x_0 + (-\bar{x}))$, $\forall x \in C$, and, in addition, there is $x_1 \in C$ with $a(x_1 + (-\bar{x})) < a(x_0 + (-\bar{x}))$. Since a is additive, for any $x \in X$, we have

$$0 = a(0) = a(x + (-x)) = a(x) + a(-x)$$

so that $a(-x) = -a(x)$. It thus follows that $a(x) + a(-\bar{x}) \leq a(x_0) + a(-\bar{x})$, for all $x \in C$, i.e., $a(x) \leq a(x_0)$ for all $x \in C$, and similarly, $a(x_1) < a(x_0)$. This proves that C and x_0 are properly separable.. □

Theorem 5.5. (Proper separation between a point and a convex cone) *Let C be a convex cone of a commutative group X with* $0 \notin \text{qcore}\, C \neq \emptyset$*. Then C and 0 are properly separable.*

Proof. If $0 \notin C$, then the thesis follows immediately from Theorem 5.4. Let $0 \in C$. Since $0 \notin \text{qcore}\, C$, we have $C = \text{cone}\, C \neq X$, where the equality follows from Proposition 3.15 (i). Then there is $x_0 \in X$ such that $x_0 \notin C$. From Theorem 5.4, C and x_0 are properly separable, i.e., there is a nonzero additive function $a : X \to \mathbb{R}$ such that

$$a(x) \leq a(x_0), \forall x \in C, \tag{13}$$

and $a(x_1) < a(x_0)$ for some $x_1 \in C$. Since $0 \in C$, from Proposition 3.16 we have $mx \in C$ for every $x \in C$ and $m \in \mathbb{N}$. As a consequence, the inequality (13) yields that $a(mx) \leq a(x_0), \forall x \in C$, $\forall m \in \mathbb{N}$, and so $a(x) \leq \frac{1}{m} a(x_0), \forall x \in C, \forall m \in \mathbb{N}$, since a is additive, which allows $a(x) \leq 0, \forall x \in C$, as $m \to \infty$. Since $a(0) = 0$, this shows C and 0 are separable. We next prove that the separation is proper. Since $\text{qcore}\, C \neq \emptyset$, there is $\bar{x} \in C$ such that $\text{cone}\,(C - \bar{x}) = X$, i.e., for every $x \in X$, there exist $r, r_i, n \in \mathbb{N}, x_i \in C, i = 1, \ldots, n$, such that $rx = \sum_{i=1}^{n} r_i(x_i + (-\bar{x}))$. Suppose to the contrary that $a(x) = 0$ for every $x \in C$. Since a is additive, one has

$$a(rx) = ra(x) = \sum_{i=1}^{n} r_i a(x_i + (-\bar{x})) = \sum_{i=1}^{n} r_i (a(x_i) - a_i(\bar{x})) = 0,$$

which yields that $a \equiv 0$, since $x \in X$ is arbitrary, a contradiction. This proves that C and x_0 are properly separable. □

Theorem 5.6. (Proper separation between sets I) *Let C, D be subsets of a commutative group X such that $C \bigcap D = \emptyset$, $C - D$ is convex and either* $\operatorname{qcore} C \neq \emptyset$ *or* $\operatorname{qcore} D \neq \emptyset$. *Then C and D are properly separable.*

Proof. We preliminarily prove that $\operatorname{qcore}(C - D) \neq \emptyset$. If $\operatorname{qcore} C \neq \emptyset$, then letting $x_0 \in D$ yields that

$$\emptyset \neq \operatorname{qcore} C + (-x_0) = \operatorname{qcore}(C + (-x_0)) \subseteq \operatorname{qcore}(C - D),$$

where the equality follows from Lemma 4.25 (ii). If $\operatorname{qcore} D \neq \emptyset$, then by similar argument we have $\operatorname{qcore}(C - D) \neq \emptyset$.

Observe that $C \bigcap D = \emptyset$ if and only if $0 \notin C - D$. Since $C - D$ is convex and $\operatorname{qcore}(C - D) \neq \emptyset$, Theorem 5.4 yields that $C - D$ and 0 are properly separable, i.e., there is a nonzero additive function $a : X \to \mathbb{R}$ such that $\sup_{x \in C-D} a(x) \leq a(0)$ and in addition, there are $x_1 \in C$ and $x_2 \in D$ such that $a(x_1 + (-x_2)) < 0$. From similar proof to that in Theorem 5.4, one has $a(-x) = -a(x)$ for any $x \in X$. Therefore, $\sup_{x \in C} a(x) \leq \inf_{x \in D} a(x)$ and in addition, $a(x_1) < a(x_2)$, which yields that the sets C and D are properly separable. □

Remark 5.7. The corresponding results [7, Theorem 3.3] are extended by Theorems 5.3 and 5.4 from vector spaces to groups. In [2, Theorem 2.1], the separation between two convex sets is obtained under the assumptions that X is a semidivisible topological group and C has nonempty interior. In [2, Theorem 2.3], a strict separation between a point and a convex set is provided under the assumptions that X is a connected, locally convex topological group, $C \subseteq X$ is closed and convex. Compared with [2, Thm. 2.1 and 2.3], we remove the topological structure, i.e., the non emptiness of the interior and the closedness of C, in Theorems 5.3, 5.4 and 5.6.

Corollary 5.8. *Let C and D be convex subsets of a uniquely divisible commutative group X such that $C \bigcap D = \emptyset$. Assume that either* $\operatorname{qcore} C \neq \emptyset$ *or* $\operatorname{qcore} D \neq \emptyset$. *Then C and D are properly separable.*

Proof. Similar to the proof of Theorem 5.6 we can show that $\operatorname{qcore}(C - D) \neq \emptyset$. It suffices to show $C - D$ is convex. Since D is convex, it is easy to show $-D$ is convex. Now Proposition 3.3 allows that $C - D$ is convex.

The desired conclusion follows immediately from Theorem 5.6. □

As a further development of the analysis we can show the following propositions.

Proposition 5.9. *Let X be a commutative group X, let $C \subseteq X$ and $x_0 \in X$. Then C and x_0 are properly separable if and only if* 0 *and* $\operatorname{cone}(C + (-x_0))$ *are properly separable.*

Proof. It suffices to show that if C and x_0 are separable, then 0 and $\operatorname{cone}(C+(-x_0))$ are separable. Assume that C and x_0 are separable. Then we can find a nonzero additive function $a : X \to \mathbb{R}$ such that

$$\begin{aligned} a(x) \leq a(x_0),\ \forall x \in C &\iff a(x + (-x_0)) \leq 0,\ \forall x \in C \\ &\iff a(y) \leq 0,\ \forall y \in C + (-x_0). \end{aligned}$$

Let $x \in \operatorname{cone}(C + (-x_0))$. Then there exist $x_i \in C + (-x_0)$, $r, r_i \in \mathbb{N}$ $(i = 1, \ldots, n)$ such that $rx = \sum_{i=1}^n r_i x_i$. Then $ra(x) = \sum_{i=1}^n r_i a(x_i) \leq 0$ and so $a(x) \leq 0 = a(0)$, which implies that 0 and $\operatorname{cone}(C + (-x_0))$ are separable. □

Proposition 5.10. *Let C be a subset of a commutative group X.*
Let $\operatorname{cone}_0(C+(-x_0)) := \operatorname{cone}(C+(-x_0)) \setminus \{0\}$*. Then the following statements hold:*

(i) 0 *and* $\operatorname{cone}(C + (-x_0))$ *are properly separable if and only if* 0 *and* $\operatorname{cone}_0(C + (-x_0))$ *are properly separable.*

(ii) *Suppose that* $0 \notin \operatorname{qcore}(\operatorname{cone}(C + (-x_0))) \neq \emptyset$*. Then* 0 *and* $\operatorname{cone}(C + (-x_0))$ *are properly separable and consequently, C and x_0 are properly separable.*

Proof. (i) This is clear.

(ii) From Theorem 5.5 we have that 0 and $\operatorname{cone}(C + (-x_0))$ are properly separable and so C and x_0 are properly separable, in view of Proposition 5.9. □

Theorem 5.11. (Proper separation between a point and a convex set III) *Let C be a convex subset of a commutative group X with* $\operatorname{qcore} C \neq \emptyset$*, and let* $x_0 \notin \operatorname{qcore} C$*. Then C and x_0 are properly separable.*

Proof. Observe that

$$\begin{aligned} x_0 \notin \operatorname{qcore} C \iff & \text{ either } x_0 \notin C \text{ or } x_0 \in C,\ \operatorname{cone}(C + (-x_0)) \neq X \\ \iff & \text{ either } x_0 \notin C \text{ or } x_0 \in C,\ \operatorname{cone}(\operatorname{cone}(C + (-x_0)) + (-0)) \neq X \\ & \text{ (by Proposition 3.15 (i))} \\ \iff & \text{ either } x_0 \notin C \text{ or } x_0 \in C,\ 0 \notin \operatorname{qcore}(\operatorname{cone}(C + (-x_0))). \end{aligned}$$

We notice that $\operatorname{qcore} C \neq \emptyset$ implies $\operatorname{qcore}(\operatorname{cone}(C + (-x_0))) \neq \emptyset$. In fact, from Lemma 4.25 (ii) it follows immediately that $\operatorname{qcore}(C + (-x_0)) \neq \emptyset$ and so we get $\operatorname{qcore}(\operatorname{cone}(C + (-x_0))) \neq \emptyset$.

If $x_0 \notin C$, then the conclusion follows from Theorem 5.4. If otherwise $x_0 \in C$ and $0 \notin \operatorname{qcore}(\operatorname{cone}(C + (-x_0))) \neq \emptyset$, then the conclusion follows from Proposition 5.10 (ii). □

The following remark provides a further illustration of Proposition 5.10 (ii).

Remark 5.12. Note that

$$C\backslash\{x_0\} + (-x_0) = (C + (-x_0))\backslash\{0\} \subseteq \operatorname{cone}_0(C + (-x_0)) \subseteq \operatorname{cone}(C + (-x_0)).$$

If $\operatorname{qcore}(C\backslash\{x_0\}) \neq \emptyset$, then similarly to the proof in Theorem 5.6, from Lemma 4.25 (ii) we have $\operatorname{qcore}(\operatorname{cone}(C + (-x_0))) \neq \emptyset$. Moreover, assume that $\sum_{i=1}^n r_i x_i \neq 0$ for every $r_1, \ldots, r_n, n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \operatorname{cone}_0(C + (-x_0))$. Since $\operatorname{cone}(C + (-x_0))$ is a convex cone, so is $\operatorname{cone}_0(C + (-x_0))$, in view of Proposition 3.17 (ii).

Theorem 5.13. (Proper separation between sets II) *Assume that A and B are subsets of a commutative group X. Suppose that* $0 \notin \operatorname{qcore}(\operatorname{cone}(A - B)) \neq \emptyset$*. Then A and B are properly separable.*

Proof. From Proposition 5.10 (ii) we know that 0 and $A - B$ are properly separable. Therefore, as in the proof of Theorem 5.6, A and B are properly separable. □

The next example clarifies the assumptions of Theorem 5.13.

Example 5.14. Let $A := \{(1,0),(0,1)\}$ and $B := \{(0,0),(1,1)\}$ be two subsets in $\mathbb{Z}^2$. Then we have $A - B = \{(1,0),(0,1),(0,-1),(-1,0)\}$. As in Example 4.24, we can prove that $\operatorname{cone}(A-B) = \mathbb{Z}^2$, which yields that $0 \in \operatorname{qcore}(\operatorname{cone}(A-B))$. Indeed, it is simple to show that A and B are not (properly) separable.

Remark 5.15. From Lemmas 4.22 (i) and 4.25 (i) it follows that, in the previous results, we can replace the assumption of nonemptiness of the quasi core of the involved sets with the one of nonemptiness of the core.

5.2. Applications to constrained optimization problems

Let X be a monoid. Consider the following optimization problem with a set constraint (for short, OP):

$$\inf f(x) \quad \text{s. t.} \quad x \in S,$$

where $S \subseteq X$ and $f : X \to \mathbb{R} \cup \{+\infty\}$. Denote by S_f the solution set of OP and by $V_{\text{OP}} := \inf_{x \in S} f(x)$ the optimal value of OP. We always assume that $S_f \neq \emptyset$. It is easy to see that OP is equivalent to the following optimization problem without constraints (for short, OPWC):

$$\inf f(x) + i_S(x) \quad \text{s. t.} \quad x \in X,$$

Proposition 5.16. *Let X be a monoid. If f and S are convex, then so is S_f.*

Proof. Let $x \in X$, $x_i \in S_f$ and $n, r_i \in \mathbb{N}$ $(i = 1, ..., n)$ such that $rx = \sum_{i=1}^n r_i x_i$, $r = \sum_{i=1}^n r_i$. Since $x_i \in S_f$ $(i = 1, \ldots, n)$, one has $x_i \in S$ and $f(x_i) \le f(y)$ for every $y \in S$. Since f and S are convex, one has $x \in S$ and

$$rf(x) \le \sum_{i=1}^n r_i f(x_i) \le \sum_{i=1}^n r_i f(y) = rf(y), \forall y \in S,$$

and so $f(x) \le f(y), \forall y \in S$. This shows that $x \in S_f$. □

Proposition 5.17. *Let X be a monoid. Let $S := \{x \in X : g_i(x) \le 0,\ i = 1, \ldots, k\}$, where $g_i : X \to \mathbb{R} \cup \{+\infty\}$ $(i = 1, ..., k)$. If g_i $(i = 1, ..., k)$ are convex, then so is S.*

Proof. Observe that $S = \bigcap_{i=1}^k \operatorname{lev}_{\le 0} g_i$, $\mathbb{R} \cup \{+\infty\}$ is endowed with a compatible partial order $\le$ and $(\mathbb{R} \cup \{+\infty\}, \le)$ satisfies that for every $r \in \mathbb{N}, y_1, y_2 \in \mathbb{R} \cup \{+\infty\}$, $ry_1 \le ry_2 \Longrightarrow y_1 \le y_2$. Now Propositions 3.3 and 4.3 (ii) yield S is convex. □

Proposition 5.18. *Assume that X is a semidivisible commutative group, S and f are convex and $0 \in \operatorname{core}(S - \operatorname{dom} f)$. Then $\bar{x} \in S$ solves OP if and only if $0 \in \partial f(\bar{x}) + N_S(\bar{x})$.*

Proof. Let $g := i_S$. Then $\operatorname{dom} g = S$. Since OP is equivalent to OPWC, $\bar{x} \in S$ solves OP if and only if $0 \in \partial(f + i_S)(\bar{x}) = \partial f(\bar{x}) + N_S(\bar{x})$, where the equality follows from Lemma 4.28, Propositions 4.16 and 4.19. □

Let $I := \{1, ..., k\}$ and $S := \{x \in X : g_i(x) \leq 0, \ i \in I\}$, where $g_i : X \to \mathbb{R} \cup \{+\infty\}$ $(i \in I)$. Let $g := \max_{1 \leq i \leq k} g_i$. Then $S := \{x \in X : g(x) \leq 0\}$. Define the Lagrangian function $L : X \times \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ by

$$L(x, \lambda) := \begin{cases} f(x) + \lambda g(x), & x \in \mathrm{dom} f, \lambda \in \mathbb{R}_+, \\ +\infty, & x \notin \mathrm{dom} f, \lambda \notin \mathbb{R}_+. \end{cases}$$

Then
$$\sup_{\lambda \in \mathbb{R}_+} L(x, \lambda) = \begin{cases} f(x), & x \in S, \\ +\infty, & \text{otherwise}, \end{cases} = f(x) + i_S(x),$$

and so
$$V_{\mathrm{OP}} = \inf_{x \in S} f(x) = \inf_{x \in X} \sup_{\lambda \in \mathbb{R}_+} L(x, \lambda) = \inf_{x \in X} f(x) + i_S(x).$$

The optimal value of the dual problem (for short, DP) of OP is defined by

$$V_{\mathrm{DP}} := \sup_{\lambda \in \mathbb{R}_+} \inf_{x \in X} L(x, \lambda) =: \sup_{\lambda \in \mathbb{R}_+} h(x).$$

Clearly, one always has the weak duality relation $V_{\mathrm{OP}} \geq V_{\mathrm{DP}}$, but in general, we do not have the equality $V_{\mathrm{OP}} = V_{\mathrm{DP}}$. But this can be characterized by saddle points.

Theorem 5.19. $(\bar{x}, \bar{\lambda}) \in X \times \mathbb{R}_+$ *is a saddle point for* L *on* $X \times \mathbb{R}_+$*, i.e.,*

$$L(\bar{x}, \lambda) \leq L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda}), \forall (x, \lambda) \in X \times \mathbb{R}_+, \tag{14}$$

if and only if $V_{\mathrm{OP}} = L(\bar{x}, \bar{\lambda}) = V_{\mathrm{DP}}$ *with* $\bar{x}$ *an optimal solution of OP and* $\bar{\lambda}$ *an optimal solution of DP.*

Theorem 5.20. *Assume that* X *is a semidivisible commutative group and that the functions* $f, g_i : X \to \mathbb{R} \cup \{+\infty\} (i \in I)$ *are convex. Let*

$$\bar{x} \in \mathrm{dom} f \bigcap (\bigcap_{i \in I(\bar{x})} \mathrm{core}(\mathrm{dom} g_i)),$$

where $I(\bar{x}) = \{i \in I : g(\bar{x}) = g_i(\bar{x})\}$. *Then* $(\bar{x}, \bar{\lambda}) \in X \times \mathbb{R}_+$ *is a saddle point for* L *on* $X \times \mathbb{R}_+$ *if and only if*

$$0 \in \partial f(\bar{x}) + \bar{\lambda} \mathrm{conv}(\bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x})) \tag{15}$$

and
$$\bar{\lambda} g(\bar{x}) = 0, \ g(\bar{x}) \leq 0. \tag{16}$$

Proof. Necessity. We first note that $\bar{x} \in \mathrm{dom} f \bigcap (\bigcap_{i \in I(\bar{x})} \mathrm{core}(\mathrm{dom} g_i))$ implies

$$0 \in \mathrm{core}(\mathrm{dom}(\bar{\lambda} g) - \mathrm{dom} f) \ \text{ for } \ \bar{\lambda} \in \mathbb{R}_+.$$

Indeed, let $\bar{x} \in \mathrm{dom} f \bigcap (\bigcap_{i \in I(\bar{x})} \mathrm{core}(\mathrm{dom} g_i))$. Then

$$\bar{x} \in \bigcap_{i \in I(\bar{x})} \mathrm{core}(\mathrm{dom} g_i) \subseteq \mathrm{core}(\mathrm{dom} g).$$

If $\bar{\lambda} > 0$, then $\mathrm{core}(\mathrm{dom} g) = \mathrm{core}(\mathrm{dom}(\bar{\lambda} g))$;

if $\bar{\lambda} = 0$, then $\mathrm{core}(\mathrm{dom} g) \subseteq \mathrm{core}(\mathrm{dom}(\bar{\lambda} g))$. It follows that

$$0 \in \mathrm{core}(\mathrm{dom}(\bar{\lambda} g)) - \bar{x} = \mathrm{core}(\mathrm{dom}(\bar{\lambda} g) - \bar{x}) \subseteq \mathrm{core}(\mathrm{dom}(\bar{\lambda} g) - \mathrm{dom} f),$$

where the equality follows from Lemma 4.25 (i).

The first inequality in (14) implies $\lambda g(\bar{x}) \leq \bar{\lambda} g(\bar{x})$ for any $\lambda \in \mathbb{R}_+$, which yields $g(\bar{x}) \leq 0$, i.e., $\bar{x} \in S$. From the second inequality in (14) we have for any $x \in X$ $f(\bar{x}) + \bar{\lambda} g(\bar{x}) \leq f(x) + \bar{\lambda} g(x)$. From the assumptions, Propositions 4.14 and 4.15 we get that $\bar{\lambda} g$ and $f + \bar{\lambda} g$ are convex. Then the previous inequality is equivalent to

$$\begin{aligned} 0 \in \partial(f + \bar{\lambda} g)(\bar{x}) &= \partial f(\bar{x}) + \partial(\bar{\lambda} g)(\bar{x}) = \partial f(\bar{x}) + \bar{\lambda} \partial g(\bar{x}) \\ &= \partial f(\bar{x}) + \bar{\lambda} \mathrm{conv}(\bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x})), \end{aligned} \tag{17}$$

where the first, second and third equality follow from Lemma 4.28, Proposition 4.18 and Lemma 4.29, respectively. This yields (15) holds. Since $\bar{x} \in S$, $g(\bar{x}) \leq 0$. From the first inequality in (14) we have $f(\bar{x}) + \lambda g(\bar{x}) \leq f(\bar{x}) + \bar{\lambda} g(\bar{x})$ for any $\lambda \in \mathbb{R}_+$ and so $\bar{\lambda} g(\bar{x}) \geq 0$. Since $g(\bar{x}) \leq 0$ and $\bar{\lambda} \in \mathbb{R}_+$, one has $\bar{\lambda} g(\bar{x}) \leq 0$. This implies (16) is true.

Sufficiency. From (17), (15) is equivalent to

$$L(\bar{x}, \bar{\lambda}) = f(\bar{x}) + \bar{\lambda} g(\bar{x}) \leq f(x) + \bar{\lambda} g(x) = L(x, \bar{\lambda}) \text{ for any } x \in X.$$

Also, (16) yields that

$$L(\bar{x}, \lambda) = f(\bar{x}) + \lambda g(\bar{x}) \leq f(\bar{x}) + \bar{\lambda} g(\bar{x}) = L(\bar{x}, \bar{\lambda}) \text{ for any } \lambda \in \mathbb{R}_+.$$

This proves (16) holds. □

Proposition 5.21. *The inclusion* (15) *is equivalent to*

$$0 \in \partial f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda'_i \partial g_i(\bar{x}),$$

for some $\lambda'_i = \lambda_i \bar{\lambda}$, $\lambda_i \in \mathbb{Q}_+$, $i \in I(\bar{x})$ *with* $\sum_{i \in I(\bar{x})} \lambda_i = 1$.

Proof. If $\bar{\lambda} = 0$, then the assertion is obviously true. Assume $\bar{\lambda} > 0$. Then, for any $a \in L(X, \mathbb{R})$ and $\lambda > 0$, we have $a = \lambda \frac{a}{\lambda}$ and so $L(X, \mathbb{R})$ is a divisible monoid. From Proposition 3.5 one has

$$\begin{aligned} &\mathrm{conv}(\bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x})) \\ &= \Big\{ a \in L(X, \mathbb{R}) : ma = \sum_{j=1}^{n} m_j a_j,\ a_j \in \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x}),\ n, m_j \in \mathbb{N},\ m = \sum_{j=1}^{n} m_j \Big\} \\ &= \Big\{ a \in L(X, \mathbb{R}) : a = \sum_{j=1}^{n} \frac{m_j}{m} a_j,\ a_j \in \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x}),\ n, m_j \in \mathbb{N},\ m = \sum_{j=1}^{n} m_j \Big\} \\ &= \Big\{ a \in L(X, \mathbb{R}) : a = \sum_{j=1}^{n} \lambda_j a_j,\ a_j \in \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x}),\ n \in \mathbb{N}, \lambda_j \in \mathbb{Q}_+, \sum_{j=1}^{n} \lambda_j = 1 \Big\}, \\ &\supseteq B, \quad \text{after defining} \end{aligned}$$

$$B := \Big\{a \in L(X,\mathbb{R}) : a = \sum_{i\in I(\bar{x})} \lambda_i a_i, a_i \in \partial g_i(\bar{x}),\ i \in I(\bar{x}),\ \lambda_i \in \mathbb{Q}_+,\ \sum_{i\in I(\bar{x})} \lambda_i = 1\Big\},$$

where the third equality follows from similar proof of [14, Proposition 2.3].

We next show that B is convex. Let $a \in L(X,\mathbb{R})$, $a^j \in B$ and $n, r^j \in \mathbb{N}$ $(j = 1,\dots,n)$ such that $ra = \sum_{j=1}^n r^j a^j$, $r = \sum_{j=1}^n r^j$. Then $a^j = \sum_{i\in I(\bar{x})} \lambda_i^j a_i^j$, where $a_i^j \in \partial g_i(\bar{x})$, $i \in I(\bar{x})$, $\lambda_i^j \in \mathbb{Q}_+$, $\sum_{i\in I(\bar{x})} \lambda_i^j = 1 (j = 1,\dots,n)$. From Proposition 4.18, $\partial g_i(\bar{x})$ is convex for each $i \in I(\bar{x})$. Since $L(X,\mathbb{R})$ is a divisible monoid, there are some $b_i \in L(X,\mathbb{R})$ $(i \in I(\bar{x}))$ such that $\sum_{j=1}^n r^j \lambda_i^j a_i^j = (\sum_{j=1}^n r^j \lambda_i^j) b_i$ and so $b_i \in \partial g_i(\bar{x})$ since $\partial g_i(\bar{x})$ is convex. In consequence,

$$ra = \sum_{j=1}^n r^j (\sum_{i\in I(\bar{x})} \lambda_i^j a_i^j) = \sum_{i\in I(\bar{x})} (\sum_{j=1}^n r^j \lambda_i^j a_i^j) = \sum_{i\in I(\bar{x})} (\sum_{j=1}^n r^j \lambda_i^j) b_i$$

and therefore
$$a = \sum_{i\in I(\bar{x})} \frac{\sum_{j=1}^n r^j \lambda_i^j}{r} b_i.$$

Since

$$\frac{\sum_{j=1}^n r^j \lambda_i^j}{r} \in \mathbb{Q}_+, i \in I(\bar{x}) \quad \text{and} \quad \sum_{i\in I(\bar{x})} \frac{\sum_{j=1}^n r^j \lambda_i^j}{r} = \frac{\sum_{j=1}^n r^j \sum_{i\in I(\bar{x})} \lambda_i^j}{r} = 1,$$

we have $a \in B$, which proves that B is convex. Moreover, for any $i \in I(\bar{x})$, $\partial g_i(\bar{x}) \subseteq B$ and so $\mathrm{conv}(\bigcup_{i\in I(\bar{x})} \partial g_i(\bar{x})) \subseteq \mathrm{conv}B = B$, where the equality follows from Proposition 3.4, since B is convex. This proves $\mathrm{conv}(\bigcup_{i\in I(\bar{x})} \partial g_i(\bar{x})) = B$. Therefore,

$$\begin{aligned} & y \in \bar{\lambda}\mathrm{conv}(\bigcup_{i\in I(\bar{x})} \partial g_i(\bar{x})) \\ \Longleftrightarrow\ & y \in \bar{\lambda} \sum_{i\in I(\bar{x})} \lambda_i \partial g_i(\bar{x}), \lambda_i \in \mathbb{Q}_+, i \in I(\bar{x}),\ \text{with} \sum_{i\in I(\bar{x})} \lambda_i = 1 \\ \Longleftrightarrow\ & y \in \sum_{i\in I(\bar{x})} \lambda_i' \partial g_i(\bar{x}), \lambda_i' = \lambda_i \bar{\lambda}, \lambda_i \in \mathbb{Q}_+, i \in I(\bar{x}),\ \text{with} \sum_{i\in I(\bar{x})} \lambda_i = 1. \end{aligned}$$

This proves the desired equivalence. □

Define also the Lagrangian function $L' : X \times \mathbb{R}_+^k \to \mathbb{R} \cup \{+\infty\}$ by

$$L'(x,\lambda) := \begin{cases} f(x) + \sum_{i=1}^k \lambda_i g_i(x), & x \in \mathrm{dom}f, \lambda \in \mathbb{R}_+^k, \\ +\infty, & x \notin \mathrm{dom}f, \lambda \in \mathbb{R}_+^k. \end{cases}$$

Then
$$\sup_{\lambda\in\mathbb{R}_+^k} L'(x,\lambda) = \begin{cases} f(x), & x \in S, \\ +\infty, & \text{otherwise}, \end{cases} = f(x) + i_S(x),$$

and so
$$V_{\mathrm{OP}} = \inf_{x\in S} f(x) = \inf_{x\in X} \sup_{\lambda\in\mathbb{R}_+^k} L'(x,\lambda) = \inf_{x\in X} f(x) + i_S(x).$$

Similarly, the optimal value of the dual problem (for short, DP′) of OP is defined by

$$V_{\mathrm{DP}'} := \sup_{\lambda\in\mathbb{R}^k_+} \inf_{x\in X} L'(x,\lambda) =: \sup_{\lambda\in\mathbb{R}^k_+} h'(x).$$

Also, one always has the weak duality relation $V_{\mathrm{OP}} \geq V_{\mathrm{DP}'}$. Similar to Theorem 5.19, we have:

Theorem 5.22. *The point $(\bar{x},\bar{\lambda}) \in X\times\mathbb{R}^k_+$ is a saddle point for L' on $X\times\mathbb{R}^k_+$, i.e.,*

$$L'(\bar{x},\lambda) \leq L'(\bar{x},\bar{\lambda}) \leq L'(x,\bar{\lambda}), \quad \forall (x,\lambda)\in X\times\mathbb{R}^k_+,$$

if and only if $V_{\mathrm{OP}} = L'(\bar{x},\bar{\lambda}) = V_{\mathrm{DP}'}$ with $\bar{x}$ an optimal solution of OP and $\bar{\lambda}$ an optimal solution of DP′.

Similar to Theorem 5.20, we have:

Theorem 5.23. *Assume that X is a semidivisible commutative group and that the functions $f, g_i : X \to \mathbb{R}\cup\{+\infty\}$ $(i\in I)$ are convex. Assume*

$$\bar{x} \in \mathrm{dom} f \bigcap (\textstyle\bigcap_{i=1}^k \mathrm{core}(\mathrm{dom} g_i)).$$

Then $(\bar{x},\bar{\lambda}) \in X\times\mathbb{R}^k_+$ is a saddle point for L' on $X\times\mathbb{R}^k_+$ if and only if

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^k \bar{\lambda}_i \partial g_i(\bar{x}).$$

and

$$\bar{\lambda}_i g_i(\bar{x}) = 0,\ g_i(\bar{x}) \leq 0, \quad i = 1,\ldots,k.$$

Proof. We first show $\bar{x} \in \mathrm{dom} f \bigcap (\bigcap_{i=1}^k \mathrm{core}(\mathrm{dom} g_i))$ implies that

$$0 \in \mathrm{core}(\mathrm{dom}(\bar{\lambda}_k g_k) - \mathrm{dom}(f + \textstyle\sum_{i=1}^{k-1} \bar{\lambda}_i g_i)),$$

$$0 \in \mathrm{core}(\mathrm{dom}(\bar{\lambda}_{k-1} g_{k-1}) - \mathrm{dom}(f + \textstyle\sum_{i=1}^{k-2} \bar{\lambda}_i g_i)), \ldots,$$

$$0 \in \mathrm{core}(\mathrm{dom}(\bar{\lambda}_1 g_1) - \mathrm{dom} f),$$

where $\bar{\lambda} := (\bar{\lambda}_1,\ldots,\bar{\lambda}_k) \in \mathbb{R}^k_+$. Indeed, let $\bar{x} \in \mathrm{dom} f \bigcap (\bigcap_{i=1}^k \mathrm{core}(\mathrm{dom} g_i))$. Then

$$\bar{x} \in \textstyle\bigcap_{i=1}^k \mathrm{core}(\mathrm{dom} g_i) \subseteq \mathrm{core}(\mathrm{dom} g_i)\ (i = 1,\ldots,k)$$

and $\bar{x} \in \mathrm{dom}(f + \sum_{i=1}^{k-1} \bar{\lambda}_i g_i)$. If $\bar{\lambda}_i > 0$, then $\mathrm{core}(\mathrm{dom} g_i) = \mathrm{core}(\mathrm{dom}(\bar{\lambda}_i g_i))$; if $\bar{\lambda}_i = 0$, then $\mathrm{core}(\mathrm{dom} g_i) \subseteq \mathrm{core}(\mathrm{dom}(\bar{\lambda}_i g_i))$, where $i = 1,\ldots,k$. It follows that

$$\begin{aligned} 0 \in \mathrm{core}(\mathrm{dom}(\bar{\lambda}_k g_k)) - \bar{x} &= \mathrm{core}(\mathrm{dom}(\bar{\lambda}_k g_k) - \bar{x}) \\ &\subseteq \mathrm{core}(\mathrm{dom}(\bar{\lambda}_k g_k) - \mathrm{dom}(f + \sum_{i=1}^{k-1} \bar{\lambda}_i g_i)), \end{aligned}$$

where the equality follows from Lemma 4.25 (i). By using a similar argument, we can show that

$$0 \in \mathrm{core}(\mathrm{dom}(\bar{\lambda}_{k-1} g_{k-1}) - \mathrm{dom}(f + \textstyle\sum_{i=1}^{k-2} \bar{\lambda}_i g_i)), \ldots, 0 \in \mathrm{core}(\mathrm{dom}(\bar{\lambda}_1 g_1) - \mathrm{dom} f).$$

The desired result can be proved by using similar proof to that in Theorem 5.20. $\square$

Suppose that X is a divisible commutative group, $f, g_i : X \to \mathbb{R} \cup \{+\infty\}(i \in I)$ are convex with $\emptyset \neq \mathrm{dom} f \subseteq \bigcap_{i=1}^k \mathrm{dom} g_i$ and $\inf_{x \in X} f(x) > -\infty$. The value function $v : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$v(b) := \inf\{f(x) : x \in X, g(x) \leq b\},$$

where $g := \max_{1 \leq i \leq k} g_i$ and $b \in \mathbb{R}$. Clearly, $v(0) = P$ and $v(b) > -\infty$ for any $b \in \mathbb{R}$.

Proposition 5.24. *The value function $v : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is convex.*

Proof. Let $b \in \mathbb{R}$, $b^1, \ldots, b^n \in \mathbb{R}$ and $n, r^1, \ldots, r^n \in \mathbb{N}$ such that $rb = \sum_{i=1}^n r^j b^j$, $r = \sum_{j=1}^n r^j$. Let $x^j \in C^j := \{x \in X : g(x) \leq b^j\}$ $(j = 1, \ldots, n)$. Since X is divisible, there is $x \in X$ such that $rx = \sum_{j=1}^n r^j x^j$. Since $g_i (i = 1, \ldots, k)$ are convex, from Proposition 4.15 it follows that g is convex and

$$rg(x) \leq \sum_{j=1}^n r^j g(x^j) \leq \sum_{j=1}^n r^j b^j = rb$$

and so $g(x) \leq b$. Since f is convex, one gets $rf(x) \leq \sum_{j=1}^n r^j f(x^j)$. Consequently,

$$rv(b) = r \inf\{f(y) : y \in X, g(y) \leq b\} \leq rf(x) \leq \sum_{j=1}^n r^j f(x^j).$$

Taking the infimum $\inf_{x^j \in C^j} (j = 1, \ldots, n)$ over the right side yields the inequality $rv(b) \leq \sum_{j=1}^n r^j v(b^j)$, which implies that v is convex. □

Consider the following constraint qualification (Slater) for OP:

$$\text{There is } \hat{x} \in \mathrm{dom} f \text{ with } g_i(\hat{x}) < 0 \text{ for } i = 1, \ldots, k. \tag{18}$$

Theorem 5.25. *Assume that X is a divisible commutative group. Let the functions $f, g_i : X \to \mathbb{R} \cup \{+\infty\}$ $(i \in I)$ be convex with $\emptyset \neq \mathrm{dom} f \subseteq \bigcap_{i=1}^k \mathrm{dom} g_i$ and $\inf_{x \in X} f(x) > -\infty$ and assume that the Slater constraint qualification (18) holds. If $\bar{x} \in S$ solves OP, then there is $\bar{\lambda} \in \mathbb{R}_+$ such that $(\bar{x}, \bar{\lambda})$ is a saddle point for L on $X \times \mathbb{R}_+$ and consequently $\bar{\lambda}$ solves DP.*

Proof. From Proposition 5.24 one has v is convex. Clearly, for any given $b \in \mathbb{R}_+$ we have $g(\bar{x}) \leq b$. Since the constraint qualification (18) holds, $0 \in \mathrm{core}(\mathrm{dom} v)$ and also $v_0(0) + v_0(0) \leq 0$. Now Lemma 4.30 yields there is $-\bar{\lambda} \in \partial v(0)$, that is, $f(\bar{x}) = v(0) \leq v(b) + \bar{\lambda} b \leq f(\bar{x}) + \bar{\lambda} b$ and so $0 \leq \bar{\lambda} b$. This yields $\bar{\lambda} \in \mathbb{R}_+$ since $b \in \mathbb{R}_+$ is arbitrary. For any $x \in \mathrm{dom} f$ we have

$$f(x) \geq v(g(x)) \geq v(0) - \bar{\lambda} g(x) = f(\bar{x}) - \bar{\lambda} g(x). \tag{19}$$

Letting $x := \bar{x}$ in (19) allows $\bar{\lambda} g(\bar{x}) \geq 0$. Since $\bar{\lambda} \geq 0$ and $g(\bar{x}) \leq 0$, we have $\bar{\lambda} g(\bar{x}) \leq 0$ and as a consequence, $\bar{\lambda} g(\bar{x}) = 0$. This proves (16) is true. Thus, from (19) we get

$$L(x, \bar{\lambda}) = f(x) + \bar{\lambda} g(x) \geq f(\bar{x}) + \bar{\lambda} g(\bar{x}) = L(\bar{x}, \bar{\lambda}), \ \forall x \in X,$$

showing the second inequality in (14) holds. The first inequality in (14) can be proved as that in Theorem 5.20. From Theorem 5.19, $\bar{\lambda}$ solves DP. □

Suppose that X is a divisible commutative group, $f, g_i : X \to \mathbb{R} \cup \{+\infty\}$ $(i \in I)$ are convex with $\emptyset \neq \mathrm{dom} f \subseteq \bigcap_{i=1}^{k} \mathrm{dom} g_i$ and $\inf_{x \in X} f(x) > -\infty$. Define the value function $v' : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$ by

$$v'(b) := \inf\{f(x) : x \in X, g'(x) \leq b\},$$

where $g' := (g_1, \ldots, g_k)$ and $b \in \mathbb{R}^k$. Clearly, $v'(0) = P$ and $v'(b) > -\infty$ for any $b \in \mathbb{R}^k$.

Similarly to Proposition 5.24 and Theorem 5.25 we can prove the following results:

Proposition 5.26. *The value function $v' : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$ is convex.*

Theorem 5.27. *Let X be a divisible commutative group. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g' : X \to \mathbb{R}^k \cup \{+\infty\}$ be convex with $\emptyset \neq \mathrm{dom} f \subseteq \mathrm{dom}\, g'$ and $\inf_{x \in X} f(x) > -\infty$ and assume that there exists $\hat{x} \in \mathrm{dom} f$* with $g'(\hat{x}) < 0$. *If $\bar{x} \in S$ solves OP, then there exists $\bar{\lambda} \in \mathbb{R}^k_+$ such that $(\bar{x}, \bar{\lambda})$ is a saddle point for L' on $X \times \mathbb{R}^k_+$ and consequently $\bar{\lambda}$ solves DP.*

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References

[1] M. Adivar, S. C. Fang: *Convex analysis and duality over discrete domains*, J. Oper. Res. Soc. China 6 (2018) 189–247.

[2] J. M. Borwein, O. Giladi: *Some remarks on convex analysis in topological groups*, J. Convex Analysis 23 (2016) 313–332.

[3] J. M. Borwein, O. Giladi: *Convex analysis in groups and semigroups: a sampler*, Math. Program., Ser. B 168 (2018) 11–53.

[4] J. M. Borwein, R. Goebel: *Notions of relative interior in Banach spaces*, J. Math. Sci. (N.Y.) 115 (2003) 2542–2553.

[5] J. M. Borwein, A. S. Lewis: *Partially finite convex programming. I: Quasi relative interiors and duality theory*, Math. Program., Ser. B 57 (1992) 15–48.

[6] R. I. Boţ, E. R. Csetnek, G. Wanka: *Regularity conditions via quasi-relative interior in convex programming*, SIAM J. Optim. 19 (2008) 217–233.

[7] D. V. Cuong, B. S. Mordukhovich, N. M. Nam, A. Cartmell: *Algebraic core and convex calculus without topology*, Optimization 71 (2022) 309–335.

[8] F. Flores-Bazán, G. Mastroeni: *Strong duality in cone constrained nonconvex optimization*, SIAM J. Optim. 23 (2013) 153–169.

[9] S. Fujishige, K. Murota: *Notes on L-M-convex functions and the separation theorems*, Math. Program., Ser. A 88 (2000) 129–146.

[10] R. E. Gomory: *Outline of an algorithm for integer solutions to linear programs*, Bull. Amer. Math. Soc. 64 (1958) 275–278.

[11] R. E. Gomory, W. J. Baumol: *Integer programming and pricing*, Econometrica 28 (1960) 521–550.

[12] J. B. Hiriart-Urruty, C. Lemaréchal: *Convex Analysis and Minimization Algorithms I and II*, Springer, Berlin (1993).

[13] Y. Iwamasa, K. Takazawa: *Optimal matroid bases with intersection constraints: valuated matroids, M-convex functions, and their applications*, Math. Program., Ser. A 194 (2022) 229–256.

[14] J. Li, G. Mastroeni: *Convex analysis in Z^n and applications to integer linear programming*, SIAM J. Optim. 30 (2020) 2809–2840.

[15] B. L. Miller: *On minimizing nonseparable functions defined on the integers with an inventory application*, SIAM J. Appl. Math. 21 (1971) 166–185.

[16] K. Murota: *Convexity and Steinitz's exchange property*, Adv. Math. 124 (1996) 272–311.

[17] K. Murota: *Discrete convex analysis*, Math. Program., Ser. A 83 (1998) 313–371.

[18] K. Murota: *Discrete Convex Analysis*, SIAM Monographs on Discrete Mathematics and Applications, Society for Industrial and Applied Mathematics, Philadelphia (2003).

[19] K. Murota, A. Shioura: *M-convex function on generalized polymatroid*, Math. Oper. Res. 24 (1999) 95–105.

[20] G. L. Nemhauser, L. A. Wolsey: *Integer and Combinatorial Optimization*, Wiley & Sons, New York (1988).

[21] M. Volle: *Regularizations in abelian complete ordered groups*, J. Math. Anal. Appl. 84 (1981) 418–430.

[22] M. Volle: *Sur les applications regularisées á valeurs dans un groupe*, Math. Oper. Statist. Ser. Optim. 13 (1982) 493–500.