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On Strong Convergence for the Split Common Null Point Problem in Two Banach Spaces

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Let E be a 2-uniformly convex and uniformly smooth Banach space, F be a smooth, strictly convex and reflexive Banach space and η be a real number with $\eta \in (-\infty, 1)$. Let $T : E \to E$ be a relatively nonexpansive mapping and let $U\colon F \to F$ be an η-demimetric and demiclosed mapping with $F(U) \neq \emptyset$, where $F(U)$ is the set of all fixed points of U. Let $L\colon E \to F$ be a bounded linear operator such that $L \neq 0$ with $F_0 = F(T) \cap L^{-1}F(U) \neq \emptyset$. Then, W. Takahashi [*A strong convergence theorem for solving the split common fixed point problem in two Banach spaces and applications*, Linear and Nonlinear Analysis 6 (2020) 473–495] considered Halpern-type iteration [see B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. 73 (1967) 957–961] and proved a strong convergence theorem for the split common fixed point problem for U and T. From this, we got several new results for the split convex feasibility problem, the split common null point problem and the split common fixed point problem.

In this paper, motivated by the paper of Takahashi cited above, for $p > 1$, when E is a p-uniformly convex and smooth Banach space and F is a strictly convex, reflexive and smooth Banach space, we consider the split common null point problem. We propose another Halpern-type iteration [see K. Nakajo, *Strong convergence for the problem of image recovery by the metric projections in Banach space*, J. Nonlinear Convex Analysis 23 (2022) 357–376] and prove strong convergence theorems, from which we obtain the results for several split problems under weaker conditions than those by Takahashi [loc. cit.].

Keywords: Split common null point problem, split convex feasibility problem, split common fixed point problem, split common equilibrium problem, metric projection.

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1. Introduction

Throughout this paper, let $\mathbf{N}$ and $\mathbf{R}$ be the set of all positive integers and the set of all real numbers, respectively and let E be a real Banach space with norm $\|\cdot\|$ and E^* be the dual of E. For $x \in E$ and $x^* \in E^*$, let $\langle x, x^*\rangle$ be the value of x^* at x. We write $x_n \rightharpoonup x$ to indicate that a sequence $\{x_n\}$ converges weakly to x and similarly, $x_n \to x$ will symbolize strong convergence.

Let H_1 and H_2 be real Hilbert spaces, C and D be nonempty, closed and convex subsets of H_1 and H_2, respectively and $L : H_1 \to H_2$ be a bounded linear operator with $L \neq 0$. Then, the split convex feasibility problem [15] (see also [12, 16]) is formulated as follows: Find $z \in H_1$ such that $z \in C \cap L^{-1}D$.

In general, the split problem is considered to find an element z of H_1 such that z is a solution of the problem P_1 in H_1 and Lz is a solution of the problem P_2 in H_2. If

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P_1 and P_2 are the null point problems (for example, monotone inclusion problems), are the variational inequality problems and are the fixed point problems, the split problem is called the split common null point one [13, 17], the split variational inequality one [17, 31] and the split common fixed point one [18, 30], respectively. Byrne, Censor, Gibali and Reich [13] proposed the split common null point problem which is formulated as follows:

For given set valued mappings $A_1 : H_1 \to 2^{H_1}$ and $A_2 : H_2 \to 2^{H_2}$ and a bounded linear operator $L : H_1 \to H_2$ with $L \neq 0$, find $z \in H_1$ such that $z \in A_1^{-1}0 \cap L^{-1}A_2^{-1}0$, where $A_1^{-1}0$ and $A_2^{-1}0$ are the null point sets of A_1 and A_2, respectively. In fact, let $B_1 \subset H_1 \times H_1$ and $B_2 \subset H_2 \times H_2$ be maximal monotone operators and $L : H_1 \to H_2$ be a bounded linear operator such that $L \neq 0$ and $F_0 = B_1^{-1}0 \cap L^{-1}B_2^{-1}0 \neq \emptyset$ and they [13] considered a sequence $\{x_n\}$ in H_1 generated by

$$\begin{aligned} x_1 &\in H_1 \\ x_{n+1} &= J_r^{B_1}(x_n - \lambda L^*(I - J_r^{B_2})Lx_n) \end{aligned}$$

for every $n \in \mathbf{N}$ and Halpern-type iteration [23] as follows:

$$\begin{aligned} y_1 &\in H_1 \\ y_{n+1} &= \alpha_n y_1 + (1-\alpha_n)J_r^{B_1}(y_n - \lambda L^*(I - J_r^{B_2})Ly_n) \end{aligned}$$

for each $n \in \mathbf{N}$, where $r > 0$, $J_r^{B_1}$ and $J_r^{B_2}$ are the resolvents of B_1 and B_2, respectively, L^* is the adjoint operator of L, $M = \|L^*L\|$, $\lambda \in (0, 2/M)$ and $\{\alpha_n\} \subset [0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. And they proved that $\{x_n\}$ converges weakly to an element of F_0 and $\{y_n\}$ converges strongly to a point of F_0. Later, many researchers studied the split problems in a real Hilbert space, see for example, [3, 14, 32, 45], [48]–[51], [53] and the references therein.

In a real Banach space, Reich and Tuyen [35] and Takahashi [39]–[44] and Takahashi and Yao [46] studied the split convex feasibility problem, the split common null point problem and the split common fixed point problem. Let D be a nonempty, closed and convex subset of a Banach space E. A mapping $U : D \to E$ is called demiclosed if for a sequence $\{x_n\}$ in D such that $x_n \rightharpoonup p$ and $\|x_n - Ux_n\| \to 0$, $p = Up$ holds.

Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E and η be a real number with $\eta \in (-\infty, 1)$. Then, a mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called *η-demimetric* [43] (see also [34]) if

$$\langle x - z,\, J_2(x - Ux)\rangle \geq \frac{1-\eta}{2}\|x - Ux\|^2$$

for every $x \in C$ and $z \in F(U)$, where $F(U)$ is the set of all fixed points of U and J_2 is the normalized duality mapping on E.

Let C be a nonempty, closed and convex subset of a smooth Banach space E.

A mapping $T : C \to E$ is called *relatively nonexpansive* [29] if $F(T) \neq \emptyset$, the mapping T is demiclosed, $\phi(z, Tx) \leq \phi(z, x)$ for every $x \in C$, and $z \in F(T)$, where $\phi : E \times E \to \mathbf{R}$ is a mapping such that $\phi(x, y) = \|x\|^2 - 2\langle x, J_2 y\rangle + \|y\|^2$ for every $x, y \in E$. Takahashi [44] considered Halpern-type iteration [23] and proved the following theorem for the split common fixed point problem.

Theorem 1.1. *Let E be a 2-uniformly convex and uniformly smooth Banach space (these conditions are equivalent to the conditions that E is uniformly convex and uniformly smooth and E^* is 2-uniformly smooth, see* [47, 52]*) and F be a smooth, strictly convex and reflexive Banach space. Let J_E and J_F be the normalized duality mappings on E and F, respectively and η be a real number with $\eta \in (-\infty, 1)$. Let $T : E \to E$ be a relatively nonexpansive mapping and $U : F \to F$ be an η-demimetric and demiclosed mapping with $F(U) \neq \emptyset$. Let $L : E \to F$ be a bounded linear operator such that $L \neq 0$ and L^* be the adjoint operator of L. Suppose that $F_0 = F(T) \cap L^{-1}F(U) \neq \emptyset$. For $u, x_1 = x \in E$, let $\{x_n\} \subset E$ be a sequence generated by*

$$\begin{aligned} y_n &= J_E^{-1}(J_E x_n - r_n L^* J_F(Lx_n - ULx_n)),\\ z_n &= J_E^{-1}(\alpha_n J_E u + (1-\alpha_n) J_E T y_n),\\ x_{n+1} &= J_E^{-1}(\beta_n J_E x_n + (1-\beta_n) J_E z_n) \end{aligned}$$

for every $n \in \boldsymbol{N}$, where $\{r_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset (0, 1)$. If $\{r_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following:

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$\delta \leq r_n \leq \gamma < \frac{(1-\eta)c}{\|L\|^2} \quad \text{and} \quad a \leq \beta_n \leq b < 1 \ (\forall n \in \boldsymbol{N}),$$

where c is a constant for $p = 2$ in Theorem 2.1*, a, b, δ, $\gamma > 0$ with $\delta \leq \gamma$ and $a \leq b$. Then, $\{x_n\}$ converges strongly to a point $z_0 \in F_0$.*

We know that the resolvent of a maximal monotone operator A with $A^{-1}0 \neq \emptyset$ is relatively nonexpansive in a strictly convex, reflexive and uniformly smooth Banach space, the generalized projection is relatively nonexpansive in a strictly convex, reflexive and smooth Banach space (see [1, 2, 7, 24]) and the metric resolvent of a maximal monotone operator B such that $B^{-1}0 \neq \emptyset$ and the metric projection are (-1)-demimetric and demiclosed in a strictly convex, reflexive and smooth Banach space (see [38, 43]). So, we got several new strong convergence theorems for the split convex feasibility problem, the split common null point problem and the split common fixed point problem by Theorem 1.1.

Motivated by [44], in this paper, for $p > 1$, when E is a p-uniformly convex and smooth Banach space and F is the same as Theorem 1.1, we consider the split common null point problem. We propose another Halpern-type iteration (see [33]) and prove strong convergence theorems, from which we get the results for several split problems under weaker conditions for a Banach space E than those deduced by [44].

2. Preliminaries

We define the *modulus of convexity* δ_E of E as follows: δ_E is a function of $[0, 2]$ into $[0, 1]$ such that

$$\delta_E(\varepsilon) = \inf\{1 - \|x + y\|/2 : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\}$$

for every $\varepsilon \in [0, 2]$. E is called *uniformly convex* if $\delta_E(\varepsilon) > 0$ for all $\varepsilon > 0$.

Let $p > 1$. E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta_E(\varepsilon) \geq c\varepsilon^p$ for each $\varepsilon \in [0, 2]$. It is obviuos that a p-uniformly convex Banach space is uniformly convex. We know that L^p space is p-uniformly convex if $p > 2$ and 2-uniformly convex if $1 < p \leq 2$, see [47]. E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for every $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is known that in a strictly convex Banach space E, for all $p > 1$, $\|\cdot\|^p$ is strictly convex function, that is, $\|\lambda x_1 + (1-\lambda)x_2\|^p < \lambda\|x_1\|^p + (1-\lambda)\|x_2\|^p$ holds for every $\lambda \in (0, 1)$ and $x_1, x_2 \in E$ with $x_1 \neq x_2$. We know that a uniformly convex Banach space is strictly convex and reflexive. It is known that if E is uniformly convex, E has the Kadec-Klee property, that is, for every sequence $\{x_n\} \subset E$, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$. For every $p > 1$, the (generalized) *duality mapping* $J_p : E \longrightarrow 2^{E^*}$ of E is defined by

$$J_p x = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^p, \|f^*\| = \|x\|^{p-1}\}$$

for all $x \in E$. When $p = 2$, J_2 is called the *normalized duality mapping*. We know that for $p, q > 1$, $\|x\|^p J_q x = \|x\|^q J_p x$ for each $x \in E$.

E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1}$$

exists for every $x, y \in S(E)$, where $S(E) = \{x \in E : \|x\| = 1\}$. E is called *uniformly smooth* if the limit (1) is attained uniformly for (x, y) in $S(E) \times S(E)$. It is known that the duality mapping J_p of E is single valued for all $p > 1$ if E is smooth. We also know that if E is strictly convex, the duality mapping J_p of E is one-to-one (in the sense that $x \neq y$ implies $J_p x \cap J_p y = \emptyset$) for each $p > 1$ and if E is reflexive, for every $p > 1$, J_p is surjective and $J_p^{-1} : E^* \longrightarrow 2^E$ is identical to the duality mapping $J_q^* : E^* \longrightarrow 2^E$ of E^* defined by $J_q^* f^* = \{x \in E : \langle x, f^* \rangle = \|f^*\|^q, \|x\| = \|f^*\|^{q-1}\}$ for all $f^* \in E^*$, where $q \in \mathbf{R}$ with $1/p + 1/q = 1$. For $p > 1$, it is known that the duality mapping J_p of a smooth Banach space E is norm to weak* continuous (see [19, 20, 37] for details). The following was proved by Xu [47], see also [8].

Theorem 2.1. *Let E be a smooth Banach space and $p > 1$. Then, E is p-uniformly convex if and only if there exists a constant $c > 0$ such that for every $x, y \in E$, $\|x + y\|^p \geq \|x\|^p + p\langle y, J_p x \rangle + c\|y\|^p$ holds.*

Let E be a smooth Banach space and $p > 1$. The function $\phi_p : E \times E \longrightarrow \mathbf{R}$ is defined by

$$\phi_p(x, y) = \|x\|^p - p\langle x, J_p y \rangle + (p-1)\|y\|^p$$

for all $x, y \in E$. We have $\phi_p(x, y) \geq 0$ for each $x, y \in E$ and $\phi_p(z, x) + \phi_p(x, y) = \phi_p(z, y) + p\langle x - z, J_p x - J_p y \rangle$ for every $x, y, z \in E$. We have the following result by Theorem 2.1, see [25].

Lemma 2.2. *Let $p > 1$ and E be a p-uniformly convex and smooth Banach space. Then, for all $x, y \in E$, we have $\phi_p(x, y) \geq c\|x - y\|^p$ and*

$$\langle x - y, J_p x - J_p y \rangle = \frac{1}{p}(\phi_p(x, y) + \phi_p(y, x)) \geq \frac{2c}{p}\|x - y\|^p,$$

where c is the constant in Theorem 2.1.

And we have the following result.

Lemma 2.3. *Let $p > 1$, C be a nonempty, closed and convex subset of a strictly convex, reflexive and smooth Banach space E and $x \in E$. Then, there exists a unique element $x_0 \in C$ such that $\phi_p(x_0, x) = \inf_{y \in C} \phi_p(y, x)$.*

Proof. $\phi_p(\cdot, x)$ is a proper, lower semicontinuous and convex function of C into $(-\infty, \infty]$. Since E is reflexive and $\|z_n\| \to \infty$ implies $\phi_p(z_n, x) \to \infty$, there exists $x_0 \in C$ such that $\phi_p(x_0, x) = \inf_{y \in C} \phi_p(y, x)$. Since $\|\cdot\|^p$ is a strictly convex function, $\phi_p(\cdot, x)$ is also strictly convex. Therefore, x_0 is unique. $\square$

Let $p > 1$, C be a nonempty, closed and convex subset of a strictly convex, reflexive and smooth Banach space E and $x \in E$. From Lemma 2.3, there exists a unique point $x_0 \in C$ such that

$$\phi_p(x_0, x) = \inf_{y \in C} \phi_p(y, x).$$

We denote x_0 by $\Pi_C^p x$ and call Π_C^p the generalized projection of E onto C; see [1, 2, 24]. Similarly in the result for $p = 2$ in [1, 2, 24], we get the following result.

Lemma 2.4. *Let $p > 1$ and C be a nonempty and convex subset of a smooth Banach space E. For $x \in E$ and $x_0 \in C$, $\phi_p(x_0, x) = \inf\{\phi_p(z, x) : z \in C\}$ if and only if $\langle z - x_0, J_p x_0 - J_p x\rangle \geq 0$ holds for every $z \in C$.*

Proof. Let $x \in E$ and $x_0 \in C$. Suppose that $\phi_p(x_0, x) = \inf\{\phi_p(z, x) : z \in C\}$. Let $z \in C$ and $0 < \lambda < 1$. We have $\lambda x_0 + (1 - \lambda) z \in C$. Using the inequality $\|y\|^p - \|x\|^p \leq p\langle y - x, J_p y\rangle$ for every $x, y \in E$,

$$\begin{aligned}
0 &\leq \phi_p(\lambda x_0 + (1-\lambda)z, x) - \phi_p(x_0, x) \\
&= \|\lambda x_0 + (1-\lambda)z\|^p - \|x_0\|^p - p\langle \lambda x_0 + (1-\lambda)z, J_p x\rangle + p\langle x_0, J_p x\rangle \\
&\leq p\langle \lambda x_0 + (1-\lambda)z - x_0, J_p(\lambda x_0 + (1-\lambda)z)\rangle - p\langle \lambda x_0 + (1-\lambda)z, J_p x\rangle + p\langle x_0, J_p x\rangle \\
&= p(1-\lambda)\langle z - x_0, J_p(\lambda x_0 + (1-\lambda)z) - J_p x\rangle
\end{aligned}$$

which implies $\langle z - x_0, J_p(\lambda x_0 + (1-\lambda)z) - J_p x\rangle \geq 0$ for each $z \in C$ and $\lambda \in (0, 1)$. Since J_p is norm to weak* continuous, for $\lambda \uparrow 1$, we get that $\langle z - x_0, J_p x_0 - J_p x\rangle \geq 0$ holds for every $z \in C$.

Next, assume that $\langle z - x_0, J_p x_0 - J_p x\rangle \geq 0$ holds for every $z \in C$. Let $z \in C$. We obtain

$$\begin{aligned}
\phi_p(z, x) - \phi_p(x_0, x) &= \|z\|^p - \|x_0\|^p - p\langle z - x_0, J_p x\rangle \\
&\geq p\langle z - x_0, J_p x_0\rangle - p\langle z - x_0, J_p x\rangle \\
&= p\langle z - x_0, J_p x_0 - J_p x\rangle \geq 0.
\end{aligned}$$

The proof is complete. $\square$

From Lemmas 2.3 and 2.4, we get the following result.

Lemma 2.5. *Let $p > 1$, C be a nonempty, closed and convex subset of a strictly convex, reflexive and smooth Banach space E and $x \in E$. Then we have*

$$\phi_p(y, \Pi_C^p x) + \phi_p(\Pi_C^p x, x) \leq \phi_p(y, x) \quad \textit{for all } y \in C.$$

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E and let $x \in E$. Then, there exists a unique element $x_0 \in C$ such that $\|x_0 - x\| = \inf_{y\in C} \|y - x\|$. Putting $x_0 = P_C x$, we call P_C the metric projection of E onto C; see [22]. And we have the following result [25] for the metric projection.

Lemma 2.6. *Let $p > 1$ and C be a nonempty, closed and convex subset of a strictly convex, reflexive and smooth Banach space E and let $x \in E$. Then we have:*

(i) *$y = P_C x$ if and only if $\langle y - z,\, J_p(x - y)\rangle \geq 0$ for each $z \in C$;*

(ii) *from* (i), *P_C is (-1)-demimetric.*

An operator $A \subset E \times E^*$ is called *monotone* if $\langle x - y,\, x^* - y^*\rangle \geq 0$ for every (x, x^*), $(y, y^*) \in A$. A monotone operator $A \subset E \times E^*$ is said to be *maximal* if the graph of A is not properly contained in the graph of any other monotone operator. It is known that a monotone operetor $A \subset E \times E^*$ is maximal if and only if for $(u, u^*) \in E \times E^*$, $\langle x - u,\, x^* - u^*\rangle \geq 0$ for every $(x, x^*) \in A$ implies $(u, u^*) \in A$. And we know the following result [10, 36]: Let E be a smooth, strictly convex and reflexive Banach space and let $A \subset E \times E^*$ be a monotone operator. Then, A is maximal if and only if $R(J_2 + rA) = E^*$ for all $r > 0$, where $R(J_2 + rA)$ is the range of $J_2 + rA$. By this, it is also known that if E is a smooth, strictly convex and reflexive Banach space and $A \subset E \times E^*$ is a maximal monotone operator, for any $x \in E$ and $r > 0$, there exists a unique element $x_r \in D(A)$ such that

$$J_2(x_r - x) + rAx_r \ni 0,$$

where $D(A)$ is the domain of A. We define J_r by $J_r x = x_r$ for every $x \in E$ and $r > 0$ and such J_r is called the metric resolvent of A. In a real Hilbert space, the metric resolvent is equal to the resolvent. And we can define Yosida approximation as follows: $A_r x = \frac{1}{r} J_2(x - J_r x)$ for every $r > 0$ and $x \in E$, see [7, 38] for more details. And we have the following result for Yosida approximation (see also [34]).

Lemma 2.7. *Let E be a smooth, strictly convex and reflexive Banach space and let $A \subset E \times E^*$ be a maximal monotone operator such that $A^{-1}0 \neq \emptyset$. Then, $F(J_r) = A^{-1}0$ for all $r > 0$ and $\langle x - z,\, A_r x\rangle \geq r\|A_r x\|^2$ holds for every $r > 0$, $x \in E$ and $z \in A^{-1}0$, that is, the metric resolvent J_r of A is (-1)-demimetric for every $r > 0$.*

Let C be a nonempty subset of a Banach space E and f a function of $E \times E$ into $\mathbf{R} \cup \{-\infty, \infty\}$ with $f(x, x) = 0$ for every $x \in C$. The equilibrium problem with respect to C is formulated as follows: find $z \in C$ such that $f(z, x) \geq 0$ for all $x \in C$. Such a $z \in C$ is said to be a *solution* of the problem and the *solution set* of the problem is defined as $EP(f) = \{z \in C \,:\, f(z, x) \geq 0 \text{ for all } x \in C\}$.

Remark 2.8. Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E and f be a function of $E \times E$ into $\mathbf{R} \cup \{-\infty, \infty\}$ that satisfies the following conditions:

(i) $f(x, x) = 0$ for all $x \in C$;

(ii) f is monotone with respect to C, that is, $f(x, y) + f(y, x) \leq 0$ for every $x, y \in C$;

(iii) $f(x,\cdot)$ is convex and lower semicontinuous for each $x \in C$;

(iv) f is maximal monotone with respect to C, i.e. for all $x \in C$ and $x^* \in E^*$, $f(x,y) + \langle y-x,\, x^*\rangle \geq 0$ for every $y \in C$ holds whenever $\langle z-x,\, x^*\rangle \geq f(z,x)$ for each $z \in C$, see [5, 9].

And, let A_f be a set valued mapping of E into E^* defined by

$$A_f x = \begin{cases} \{x^* \in E^* \,:\, f(x,y) \geq \langle y-x, x^*\rangle \ \text{ for all } \ y \in C\} & \text{if } x \in C\,; \\ \emptyset & \text{if } x \notin C\,, \end{cases}$$

Then, Aoyama, Kimura and Takahashi [5] (see also [9]) proved that A_f is a maximal monotone operator with $A_f^{-1}0 = EP(f)$. So, the monotone inclusion problem contains the equilibrium problem. $\square$

A function $\tau : \mathbf{N} \to \mathbf{N}$ is said to be *eventually increasing* if $\lim_{n\to\infty} \tau(n) = \infty$ and $\tau(n) \leq \tau(n+1)$ for every $n \in \mathbf{N}$. The following was proved by Aoyama, Kimura and Kohsaka [4, Lemma 3.4]; see also [27, Lemma 3.1].

Lemma 2.9. *Let $\{\xi_n\}$ be a sequence of nonnegative real numbers which is not convergent. Then there exist $n_0 \in \boldsymbol{N}$ and an eventually increasing function τ such that $\xi_{\tau(n)} \leq \xi_{\tau(n)+1}$ for all $n \in \boldsymbol{N}$ and $\xi_n \leq \xi_{\tau(n)+1}$ for every $n \geq n_0$.*

3. Main results

Let C be a nonempty, closed and convex subset of a Banach space E and A be a mapping of C into E^* satisfying the conditions:

(I) $A^{-1}0 = \{z \in C \,:\, Az = 0\} \neq \emptyset$ and there exists $\alpha > 0$ such that $\langle x-z, Ax\rangle \geq \alpha\|Ax\|^2$ for every $x \in C$ and $z \in A^{-1}0$;

(II) for any $\{u_n\} \subset C$, $u_n \rightharpoonup u$ and $Au_n \to 0$ imply $u \in A^{-1}0$.

Then, from condition (I), we have that $A^{-1}0$ is closed and convex. In fact, let $z_1,\, z_2 \in A^{-1}0$, $0 < \gamma < 1$ and $z = \gamma z_1 + (1-\gamma)z_2$. From $\langle z - z_1, Az\rangle \geq \alpha\|Az\|^2$ and $\langle z - z_2, Az\rangle \geq \alpha\|Az\|^2$, it follows that

$$0 = \langle z - (\gamma z_1 + (1-\gamma)z_2), Az\rangle = \gamma\langle z - z_1, Az\rangle + (1-\gamma)\langle z - z_2, Az\rangle \geq \alpha\|Az\|^2$$

which implies $Az = 0$, that is, $z \in A^{-1}0$. So, $A^{-1}0$ is convex. Next, let $\{z_n\} \subset A^{-1}0$ be a sequence with $z_n \to z$. By $\langle z - z_n, Az\rangle \geq \alpha\|Az\|^2$ for each $n \in \mathbf{N}$, we get $z \in A^{-1}0$. Therefore, $A^{-1}0$ is closed. Next, we know several examples satisfying the conditions (I) and (II).

Example 3.1. Let $\beta > 0$, C be a nonempty, closed and convex subset of a Banach space E and B be a β-inverse strongly monotone operator [6, 21] of C into E^* with $B^{-1}0 \neq \emptyset$, that is, $\langle x-y, Bx-By\rangle \geq \beta\|Bx-By\|^2$ for every $x, y \in C$. Then, $A = B$ satisfies the conditions (I) and (II) with a constant $\alpha = \beta$. $\square$

Example 3.2. Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E, η be a real number with $\eta \in (-\infty, 1)$ and a mapping $U : C \to E$ with $F(U) \neq \emptyset$ be η-demimetric and demiclosed. Then, $Ax = J_2(x - Ux)$ $(x \in C)$ satisfies the conditions (I) and (II) with $A^{-1}0 = F(U)$ and a constant $\alpha = \frac{1-\eta}{2}$. $\square$

Example 3.3. Let H be a real Hilbert space and C a nonempty, closed and convex subset of H.

(i) Let k be a real number with $0 \leq k < 1$ and $U : C \to H$ a k-strict pseudo-contraction [11] with $F(U) \neq \emptyset$, i.e., we have

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|(x - Ux) - (y - Uy)\|^2 \text{ for all } x, y \in C.$$

Then, U is k-demimetric and demiclosed [28, 43].

(ii) Let $U : C \to H$ be a generalized hybrid [26] with $F(U) \neq \emptyset$, that is, there exist $\alpha, \beta \in \mathbf{R}$ such that we have for all $x, y \in C$

$$\alpha\|Ux - Uy\|^2 + (1 - \alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1 - \beta)\|x - y\|^2.$$

Then, U is 0-demimetric and demiclosed [26, 42].

So, by Example 3.2, for each U of (i) and (ii), $Ax = x - Ux$ $(x \in C)$ satisfies the conditions (I) and (II). $\square$

Example 3.4. Let C be a nonempty, closed and convex subset of a strictly convex, reflexive and smooth Banach space E. Then, by Lemma 2.6 and Example 3.2, $Ax = J_2(x - P_C x)$ $(x \in E)$ satisfies the conditions (I) and (II) with $A^{-1}0 = C$ and a constant $\alpha = 1$, where P_C is the metric projection of E onto C. $\square$

Example 3.5. Let E be a smooth, strictly convex and reflexive Banach space, $B \subset E \times E^*$ be a maximal monotone operator such that $B^{-1}0 \neq \emptyset$ and $r > 0$. Then, by Lemma 2.7, maximality of B and Example 3.2, $Ax = J_2(x - J_r x)$ $(x \in E)$ satisfies the conditions (I) and (II) with $A^{-1}0 = B^{-1}0$ and a constant $\alpha = 1$, where J_r is the metric resolvent of B. $\square$

And we have the following result for the mapping satisfying the condition (I).

Lemma 3.6. *Let C be a nonempty, closed and convex subset of a strictly convex, reflexive and smooth Banach space E and A a mapping of C into E^* satisfying the condition (I) with a constant $\alpha > 0$. For $p > 1$, $\langle x - z, J_p(J_2^{-1}Ax)\rangle \geq \alpha\|Ax\|^p$ holds for every $x \in C$ and $z \in A^{-1}0$.*

Proof. Let $x \in C$ and $z \in A^{-1}0$. By condition (I), we have $\langle x - z, Ax\rangle \geq \alpha\|Ax\|^2$.

Hence,
$$\langle x - z, J_2(J_2^{-1}Ax)\rangle \geq \alpha\|Ax\|^2.$$

From $\|y\|^2 J_p y = \|y\|^p J_2 y$ for all $y \in E$, we get

$$\langle x - z, J_p(J_2^{-1}Ax)\rangle \geq \|J_2^{-1}Ax\|^{p-2} \cdot \alpha\|Ax\|^2 = \alpha\|Ax\|^p.$$ $\square$

Now, we consider Halpern-type iteration (see [33]) and get the following strong convergence theorems for the split common null point problem.

Theorem 3.7. *Let $p > 1$, C be a nonempty, closed and convex subset of a p-uniformly convex and smooth Banach space E with a constant $c > 0$ in Theorem 2.1, D be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space F and L be a bounded linear operator of E into F such that $L \neq 0$ and $L(C) \subset D$. Let $A_1 : D \to F^*$ and $A_2 : C \to E^*$ satisfy the conditions (I) and (II) with constants $\beta_1 > 0$ and $\beta_2 > 0$, respectively and let $F_0 = A_2^{-1}0 \cap L^{-1}A_1^{-1}0 \neq \emptyset$.*

Let $u \in E$ and $\{x_n\} \subset C$ be a sequence generated by

$$\begin{aligned} x_1 &\in C \\ y_n &= \Pi_C^p J_{p,E}^{-1}(J_{p,E}x_n - r_n L^* J_{p,F} J_{2,F}^{-1} A_1 L x_n - \alpha_n J_{p,E}(x_n - u)) \\ x_{n+1} &= \Pi_C^p J_{p,E}^{-1}(J_{p.E}y_n - \lambda_n J_{p.E} J_{2,E}^{-1} A_2 y_n) \end{aligned}$$

for every $n \in \boldsymbol{N}$, where Π_C^p is the generalized projection of E onto C, $J_{p,E}$ and $J_{p,F}$ are the generalized duality mappings of E and F, respectively, $J_{2,E}$ and $J_{2,F}$ are the normalized duality mappings of E and F, resp., L^* is the adjoint operator of L,

$$a_1 \le r_n \le b_1 < \frac{c}{\|L\|^p}\Big(\frac{\beta_1 p}{p-1}\Big)^{p-1} \ (\forall n \in \boldsymbol{N}),\ a_2 \le \lambda_n \le b_2 < c\Big(\frac{\beta_2 p}{p-1}\Big)^{p-1} \ (\forall n \in \boldsymbol{N})$$

for some $a_1, a_2, b_1, b_2 > 0$ with $a_1 \le b_1$ and $a_2 \le b_2$ and $\{\alpha_n\} \subset (0,1]$ with $\alpha_n \to 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $P_{F_0}u$, where P_{F_0} is the metric projection of E onto F_0.

Proof. Throughout this proof, let $\phi_p(\cdot,\cdot)$ be the function on $E \times E$ such that $\phi_p(x, y) = \|x\|^p - p\langle x,\ J_{p,E}y\rangle + (p-1)\|y\|^p$ for every $x, y \in E$. We have $L^{-1}A_1^{-1}0$ is closed and convex. In fact, let $u, v \in L^{-1}A_1^{-1}0$, $\gamma \in (0,1)$ and $w = \gamma u + (1-\gamma)v$. Since L is linear, $Lu, Lv \in A_1^{-1}0$ and $A_1^{-1}0$ is convex,

$$Lw = L(\gamma u + (1-\gamma)v) = \gamma Lu + (1-\gamma)Lv \in A_1^{-1}0.$$

Hence, $w \in L^{-1}A_1^{-1}0$, that is, $L^{-1}A_1^{-1}0$ is convex. Let $\{u_n\} \subset L^{-1}A_1^{-1}0$ with $u_n \to u$. Since $\{Lu_n\} \subset A_1^{-1}0$, L is continuous and $A_1^{-1}0$ is closed, $Lu \in A_1^{-1}0$, i.e. $u \in L^{-1}A_1^{-1}0$. So, $L^{-1}A_1^{-1}0$ is closed. Therefore, F_0 is nonempty, closed and convex and hence, P_{F_0} is well defined.

By $0 < a_1 \le r_n \le b_1 < \frac{c}{\|L\|^p}(\frac{\beta_1 p}{p-1})^{p-1}$ $(\forall n \in \mathbf{N})$, there exists $\gamma_0 > 0$ such that $(\frac{\beta_1 p}{p-1})^{p-1} > \gamma_0 > \frac{b_1\|L\|^p}{c} \ge \frac{\|L\|^p}{c} r_n$ for all $n \in \mathbf{N}$. So, $d_1 = \inf_{n\in\mathbf{N}}\{c - \frac{r_n\|L\|^p}{\gamma_0}\} > 0$ and $d_2 = \inf_{n\in\mathbf{N}}\Big\{pr_n\Big(\beta_1 - \frac{(p-1)\gamma_0^{\frac{1}{p-1}}}{p}\Big)\Big\} > 0$. Further, there exists $\varepsilon_0 > 0$ with $d = p - 2(p-1)\varepsilon_0^{\frac{1}{p-1}} > 0$ and by $\alpha_n \to 0$, there exists $n_0 \in \mathbf{N}$ such that we get $d_3 = \inf_{n\ge n_0}\{c - \frac{\alpha_n}{\varepsilon_0} - \frac{r_n\|L\|^p}{\gamma_0}\} > 0$. And by $0 < a_2 \le \lambda_n \le b_2 < c(\frac{\beta_2 p}{p-1})^{p-1}$ $(\forall n \in \mathbf{N})$, there exists $\delta_0 > 0$ such that $(\frac{\beta_2 p}{p-1})^{p-1} > \delta_0 > \frac{b_2}{c} \ge \frac{\lambda_n}{c}$ for every $n \in \mathbf{N}$. So, $d_4 = \inf_{n\in\mathbf{N}}\{c - \frac{\lambda_n}{\delta_0}\} > 0$ and $d_5 = \inf_{n\in\mathbf{N}}\Big\{p\lambda_n\Big(\beta_2 - \frac{(p-1)\delta_0^{\frac{1}{p-1}}}{p}\Big)\Big\} > 0$.

Let $z \in F_0$. Define $z_n = J_{p,E}^{-1}(J_{p,E}x_n - r_n L^* J_{p,F} J_{2,F}^{-1} A_1 L x_n - \alpha_n J_{p,E}(x_n - u))$ and $w_n = J_{p,E}^{-1}(J_{p.E}y_n - \lambda_n J_{p.E} J_{2,E}^{-1} A_2 y_n)$. From $y_n = \Pi_C^p z_n$ and Lemma 2.5, we obtain

$$\phi_p(z, y_n) \le \phi_p(z, z_n) - \phi_p(y_n, z_n) \tag{2}$$

for every $n \in \mathbf{N}$. We get for all $n \in \mathbf{N}$

$$\begin{aligned} \phi_p(z, z_n) &= \phi_p(z, x_n) - \phi_p(z_n, x_n) + p\langle z_n - z, J_{p,E}z_n - J_{p,E}x_n\rangle \\ &= \phi_p(z, x_n) - \phi_p(z_n, x_n) - pr_n\langle z_n - z, L^* J_{p,F} J_{2,F}^{-1} A_1 L x_n\rangle \\ &\quad - p\alpha_n\langle z_n - z, J_{p,E}(x_n - u)\rangle. \end{aligned} \tag{3}$$

By $Lz \in A_1^{-1}0$ and Lemma 3.6,

$$\begin{aligned}
&\langle z_n - z, L^* J_{p,F} J_{2,F}^{-1} A_1 L x_n \rangle \\
&= \langle z_n - x_n, L^* J_{p,F} J_{2,F}^{-1} A_1 L x_n \rangle + \langle x_n - z, L^* J_{p,F} J_{2,F}^{-1} A_1 L x_n \rangle \\
&= \langle L(z_n - x_n), J_{p,F} J_{2,F}^{-1} A_1 L x_n \rangle + \langle L(x_n - z), J_{p,F} J_{2,F}^{-1} A_1 L x_n \rangle \\
&\geq -\|L(z_n - x_n)\| \cdot \|J_{p,F} J_{2,F}^{-1} A_1 L x_n\| + \beta_1 \|A_1 L x_n\|^p \\
&\geq -\|L\| \cdot \|z_n - x_n\| \cdot \|A_1 L x_n\|^{p-1} + \beta_1 \|A_1 L x_n\|^p \qquad (4)
\end{aligned}$$

for every $n \in \mathbf{N}$. We have that

$$st \leq \frac{s^p}{\beta p} + \beta^{\frac{1}{p-1}} \cdot \frac{(p-1)t^{\frac{p}{p-1}}}{p}$$

for all $p > 1$, $\beta > 0$ and s, $t \geq 0$. So, by (4),

$$\begin{aligned}
&\langle z_n - z, L^* J_{p,F} J_{2,F}^{-1} A_1 L x_n \rangle \\
&\qquad \geq -\Big(\frac{\|L\|^p \cdot \|z_n - x_n\|^p}{\gamma_0 p} + \gamma_0^{\frac{1}{p-1}} \frac{(p-1)\|A_1 L x_n\|^p}{p} \Big) + \beta_1 \|A_1 L x_n\|^p
\end{aligned}$$

for every $n \in \mathbf{N}$. From this, (2)–(4) and Lemma 2.2, we have

$$\begin{aligned}
\phi_p(z, y_n) \leq \phi_p(z, x_n) - c\|y_n - z_n\|^p - \Big(c - \frac{r_n \|L\|^p}{\gamma_0} \Big) \|z_n - x_n\|^p \\
- p r_n \Big(\beta_1 - \frac{(p-1)\gamma_0^{\frac{1}{p-1}}}{p} \Big) \|A_1 L x_n\|^p - p\alpha_n \langle z_n - z,\ J_{p,E}(x_n - u) \rangle \qquad (5)
\end{aligned}$$

for all $n \in \mathbf{N}$. And we get

$$\begin{aligned}
&\langle z_n - z, J_{p,E}(x_n - u) \rangle \\
&= \langle z_n - x_n, J_{p,E}(x_n - u) \rangle + \langle x_n - u, J_{p,E}(x_n - u) \rangle + \langle u - z, J_{p,E}(x_n - u) \rangle \\
&\geq -\|z_n - x_n\| \cdot \|x_n - u\|^{p-1} + \|x_n - u\|^p - \|u - z\| \cdot \|x_n - u\|^{p-1}.
\end{aligned}$$

By $$\|z_n - x_n\| \cdot \|x_n - u\|^{p-1} \leq \frac{1}{p\varepsilon_0} \|z_n - x_n\|^p + \frac{p-1}{p} \varepsilon_0^{\frac{1}{p-1}} \|x_n - u\|^p$$

and $$\|u - z\| \cdot \|x_n - u\|^{p-1} \leq \frac{1}{p\varepsilon_0} \|u - z\|^p + \frac{p-1}{p} \varepsilon_0^{\frac{1}{p-1}} \|x_n - u\|^p,$$

we obtain

$$\begin{aligned}
&\langle z_n - z, J_{p,E}(x_n - u) \rangle \\
&\geq -\frac{1}{p\varepsilon_0} (\|z_n - x_n\|^p + \|u - z\|^p) + \Big(1 - \frac{2(p-1)}{p} \varepsilon_0^{\frac{1}{p-1}} \Big) \|x_n - u\|^p \qquad (6)
\end{aligned}$$

for every $n \in \mathbf{N}$. From (5) and (6), we have for each $n \in \mathbf{N}$

$$\phi_p(z, y_n) \leq \phi_p(z, x_n) - c\|y_n - z_n\|^p - \Big(c - \frac{\alpha_n}{\varepsilon_0} - \frac{r_n \|L\|^p}{\gamma_0} \Big) \|z_n - x_n\|^p \qquad (7)$$

$$-p r_n \Big(\beta_1 - \frac{(p-1)\gamma_0^{\frac{1}{p-1}}}{p} \Big) \|A_1 L x_n\|^p - \alpha_n (p - 2(p-1)\varepsilon_0^{\frac{1}{p-1}}) \|x_n - u\|^p + \frac{\alpha_n}{\varepsilon_0} \|u - z\|^p.$$

From (5) and (7), we get

$$\begin{aligned}\phi_p(z, y_n) \;&\leq\; \phi_p(z, x_n) - c\|y_n - z_n\|^p - d_1\|z_n - x_n\|^p - d_2\|A_1 L x_n\|^p \\ &\quad - p\alpha_n\langle z_n - z,\ J_{p,E}(x_n - u)\rangle\end{aligned} \tag{8}$$

for all $n \in \mathbf{N}$ and

$$\begin{aligned}\phi_p(z, y_n) \;&\leq\; \phi_p(z, x_n) - c\|y_n - z_n\|^p - d_3\|z_n - x_n\|^p - d_2\|A_1 L x_n\|^p \\ &\quad - d\alpha_n\|x_n - u\|^p + \frac{\alpha_n}{\varepsilon_0}\|u - z\|^p\end{aligned} \tag{9}$$

for every $n \geq n_0$. We also have

$$\phi_p(z, x_{n+1}) \leq \phi_p(z, w_n) - \phi_p(x_{n+1}, w_n) \leq \phi_p(z,\ w_n) - c\|x_{n+1} - w_n\|^p \tag{10}$$

for each $n \in \mathbf{N}$ by $x_{n+1} = \Pi_C^p w_n$ and Lemmas 2.2 and 2.5. And from Lemmas 2.2 and 3.6 we obtain

$$\begin{aligned}\phi_p(z, w_n) &= \phi_p(z, y_n) - \phi_p(w_n, y_n) + p\langle w_n - z, J_{p,E}w_n - J_{p,E}y_n\rangle \\ &= \phi_p(z,\ y_n) - \phi_p(w_n,\ y_n) - p\lambda_n\langle w_n - z, J_{p,E}J_{2,E}^{-1}A_2 y_n\rangle \\ &\leq \phi_p(z,\ y_n) - c\|w_n - y_n\|^p - p\lambda_n\langle w_n - y_n, J_{p,E}J_{2,E}^{-1}A_2 y_n\rangle \\ &\qquad - p\lambda_n\langle y_n - z, J_{p,E}J_{2,E}^{-1}A_2 y_n\rangle \\ &\leq \phi_p(z,\ y_n) - c\|w_n - y_n\|^p + p\lambda_n\|w_n - y_n\| \cdot \|A_2 y_n\|^{p-1} - p\lambda_n\beta_2\|A_2 y_n\|^p.\end{aligned}$$

By $\|w_n - y_n\| \cdot \|A_2 y_n\|^{p-1} \leq \frac{\|w_n - y_n\|^p}{p\delta_0} + \frac{(p-1)\delta_0^{\frac{1}{p-1}}}{p}\|A_2 y_n\|^p$, we get

$$\phi_p(z, w_n) \leq \phi_p(z,\ y_n) - \Big(c - \frac{\lambda_n}{\delta_0}\Big)\|w_n - y_n\|^p - p\lambda_n\Big(\beta_2 - \frac{(p-1)\delta_0^{\frac{1}{p-1}}}{p}\Big)\|A_2 y_n\|^p \tag{11}$$

for all $n \in \mathbf{N}$. By (11),

$$\phi_p(z, w_n) \leq \phi_p(z,\ y_n) - d_4\|w_n - y_n\|^p - d_5\|A_2 y_n\|^p \tag{12}$$

for every $n \in \mathbf{N}$. From (8)–(10) and (12), we have

$$\begin{aligned}\phi_p(z, x_{n+1}) \;&\leq\; \phi_p(z, x_n) - c\|x_{n+1} - w_n\|^p - c\|y_n - z_n\|^p - d_1\|z_n - x_n\|^p \\ &\quad - d_2\|A_1 L x_n\|^p - d_4\|w_n - y_n\|^p - d_5\|A_2 y_n\|^p \\ &\quad - p\alpha_n\langle z_n - z,\ J_{p,E}(x_n - u)\rangle\end{aligned} \tag{13}$$

for all $n \in \mathbf{N}$ and

$$\begin{aligned}\phi_p(z, x_{n+1}) \;&\leq\; \phi_p(z, x_n) - c\|x_{n+1} - w_n\|^p - c\|y_n - z_n\|^p - d_3\|z_n - x_n\|^p \\ &\quad - d_2\|A_1 L x_n\|^p - d_4\|w_n - y_n\|^p - d_5\|A_2 y_n\|^p \\ &\quad - d\alpha_n\|x_n - u\|^p + \frac{\alpha_n}{\varepsilon_0}\|u - z\|^p\end{aligned} \tag{14}$$

for every $n \geq n_0$.

We show that $\{x_n\}$ is bounded. If there exists $\lim_{n\to\infty}\phi_p(z,x_n)$, it is obvious that $\{x_n\}$ is bounded from Lemma 2.2. If not so, by Lemma 2.9, there exist $n_1 \in \mathbf{N}$ and an eventually increasing function τ such that $\phi_p(z,x_{\tau(n)}) \le \phi_p(z,x_{\tau(n)+1})$ for all $n \in \mathbf{N}$ and $\phi_p(z,x_n) \le \phi_p(z,x_{\tau(n)+1})$ for each $n \ge n_1$. From (14), we obtain

$$\phi_p(z,x_{\tau(n)}) \le \phi_p(z,x_{\tau(n)+1}) \le \phi_p(z,x_{\tau(n)}) - d\alpha_{\tau(n)}\|x_{\tau(n)} - u\|^p + \frac{\alpha_{\tau(n)}}{\varepsilon_0}\|u-z\|^p$$

which implies
$$d\|x_{\tau(n)} - u\|^p \le \frac{1}{\varepsilon_0}\|u-z\|^p$$

for all $n \in \mathbf{N}$ with $\tau(n) \ge n_0$. Hence, $\{x_{\tau(n)}\}$ is bounded.

By $\phi_p(z,x_n) \le \phi_p(z,x_{\tau(n)+1})$ and (14), we get

$$\phi_p(z,x_n) \le \phi_p(z,x_{\tau(n)+1}) \le \phi_p(z,x_{\tau(n)}) + \frac{\alpha_{\tau(n)}}{\varepsilon_0}\|u-z\|^p \le \phi_p(z,x_{\tau(n)}) + \frac{1}{\varepsilon_0}\|u-z\|^p$$

for each $n \ge n_1$ with $\tau(n) \ge n_0$. So, we obtain $\{x_n\}$ is bounded. Next, we show that $\{y_n\}$ and $\{z_n\}$ are bounded. The estimation (14) implies

$$c\|y_n - z_n\|^p + d_3\|z_n - x_n\|^p \le \phi_p(z,x_n) + \frac{\alpha_n}{\varepsilon_0}\|u-z\|^p \le \phi_p(z,x_n) + \frac{1}{\varepsilon_0}\|u-z\|^p$$

for every $n \ge n_0$. Hence, $\{y_n\}$ and $\{z_n\}$ are bounded.

(i). If $\{\phi_p(P_{F_0}u,x_n)\}$ is not convergent, by Lemma 2.9, there exist $n_2 \in \mathbf{N}$ and an eventually increasing function i such that $\phi_p(P_{F_0}u,x_{i(n)}) \le \phi_p(P_{F_0}u,x_{i(n)+1})$ for all $n \in \mathbf{N}$ and $\phi_p(P_{F_0}u,x_n) \le \phi_p(P_{F_0}u,x_{i(n)+1})$ for every $n \ge n_2$. From (14), we have

$$\begin{aligned}
\phi_p(P_{F_0}u,x_{i(n)}) &\le \phi_p(P_{F_0}u,x_{i(n)+1}) \\
&\le \phi_p(P_{F_0}u,x_{i(n)}) - c\|x_{i(n)+1} - w_{i(n)}\|^p - c\|y_{i(n)} - z_{i(n)}\|^p \\
&\quad - d_2\|A_1Lx_{i(n)}\|^p - d_3\|z_{i(n)} - x_{i(n)}\|^p - d_4\|w_{i(n)} - y_{i(n)}\|^p \\
&\quad - d_5\|A_2y_{i(n)}\|^p + \frac{\alpha_{i(n)}}{\varepsilon_0}\|u - P_{F_0}u\|^p
\end{aligned}$$

which implies

$$\begin{aligned}
&c\|x_{i(n)+1} - w_{i(n)}\|^p + c\|y_{i(n)} - z_{i(n)}\|^p + d_3\|z_{i(n)} - x_{i(n)}\|^p \\
&+d_4\|w_{i(n)} - y_{i(n)}\|^p + d_2\|A_1Lx_{i(n)}\|^p + d_5\|A_2y_{i(n)}\|^p \le \frac{\alpha_{i(n)}}{\varepsilon_0}\|u - P_{F_0}u\|^p
\end{aligned}$$

for every $n \in \mathbf{N}$ with $i(n) \ge n_0$. By $\alpha_n \to 0$, we get

$$\begin{aligned}
&\|x_{i(n)+1} - w_{i(n)}\| \to 0,\ \|y_{i(n)} - z_{i(n)}\| \to 0,\ \|z_{i(n)} - x_{i(n)}\| \to 0, \\
&\|w_{i(n)} - y_{i(n)}\| \to 0,\ \|A_1Lx_{i(n)}\| \to 0 \ \text{ and } \ \|A_2y_{i(n)}\| \to 0.
\end{aligned} \tag{15}$$

From condition (II) for A_1 and A_2, $\omega_w(x_{i(n)}) \subset A_2^{-1}0$ and $\omega_w(Lx_{i(n)}) \subset A_1^{-1}0$, where $\omega_w(x_{i(n)})$ is the set of all weak cluster points of $\{x_{i(n)}\}$. Since L is bounded and linear, we obtain $\omega_w(x_{i(n)}) \subset L^{-1}A_1^{-1}0$ and hence, $\omega_w(x_{i(n)}) \subset F_0$. In fact, let $\{x_{i(n_j)}\}$ be a subsequence of $\{x_{i(n)}\}$ such that $x_{i(n_j)} \rightharpoonup z$. We have

$$\langle Lx_{i(n_j)}, f^*\rangle = \langle x_{i(n_j)}, L^*f^*\rangle \to \langle z, L^*f^*\rangle = \langle Lz, f^*\rangle,$$

for every $f^* \in F^*$, hence $Lx_{i(n_j)} \rightharpoonup Lz$. So, $z \in L^{-1}A_1^{-1}0$.

Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$. From (13),

$$\begin{aligned}\phi_p(P_{F_0}u, x_{i(n_j)+1}) &\leq \phi_p(P_{F_0}u, x_{i(n_j)}) - p\alpha_{i(n_j)}\langle z_{i(n_j)} - P_{F_0}u, J_{p,E}(x_{i(n_j)} - u)\rangle \\ &\leq \phi_p(P_{F_0}u, x_{i(n_j)+1}) - p\alpha_{i(n_j)}\langle z_{i(n_j)} - P_{F_0}u, J_{p,E}(x_{i(n_j)} - u)\rangle\end{aligned}$$

which implies

$$\langle z_{i(n_j)} - P_{F_0}u, J_{p,E}(x_{i(n_j)} - u)\rangle \leq 0 \tag{16}$$

for every $j \in \mathbf{N}$. By (6), we have

$$\|x_{i(n_j)} - u\| \leq \|z_{i(n_j)} - x_{i(n_j)}\| + \|u - P_{F_0}u\|$$

for all $j \in \mathbf{N}$. From $\|z_{i(n_j)} - x_{i(n_j)}\| \to 0$ in (15), we get

$$\liminf_{j\to\infty} \|x_{i(n_j)} - u\| \leq \|u - P_{F_0}u\|. \tag{17}$$

There exists a subsequence $\{x_{i(n_{j_k})}\}$ of $\{x_{i(n_j)}\}$ such that $x_{i(n_{j_k})} \rightharpoonup v \in F_0$ and $\liminf_{j\to\infty} \|x_{i(n_j)} - u\| = \lim_{k\to\infty} \|x_{i(n_{j_k})} - u\|$. So, from (17) and weak lower semi-continuity of norm, we have

$$\|v - u\| \leq \lim_{k\to\infty} \|x_{i(n_{j_k})} - u\| \leq \|P_{F_0}u - u\|$$

which implies $v = P_{F_0}u$ and hence, $\lim_{k\to\infty} \|x_{i(n_{j_k})} - u\| = \|P_{F_0}u - u\|$. From Kadec Klee property, $x_{i(n_{j_k})} \to P_{F_0}u$. By $\|x_{i(n)+1} - x_{i(n)}\| \to 0$ in (15), $x_{i(n_{j_k})+1} \to P_{F_0}u$. Since $J_{p,E}$ is norm to weak* continuous,

$$\begin{aligned}\phi_p(P_{F_0}u, x_{i(n_{j_k})+1}) &= \|P_{F_0}u\|^p - p\langle P_{F_0}u, J_{p,E}(x_{i(n_{j_k})+1})\rangle + (p-1)\|x_{i(n_{j_k})+1}\|^p \\ &\to \|P_{F_0}u\|^p - p\langle P_{F_0}u, J_{p,E}(P_{F_0}u)\rangle + (p-1)\|P_{F_0}u\|^p = 0.\end{aligned}$$

By $\phi_p(P_{F_0}u, x_n) \leq \phi_p(P_{F_0}u, x_{i(n)+1})$ for each $n \geq n_2$

$$\phi_p(P_{F_0}u, x_{n_{j_k}}) \to 0$$

which implies $x_{n_{j_k}} \to P_{F_0}u$ from Lemma 2.2. Therefore, $x_n \to P_{F_0}u$.

(ii). Assume that $\lim_{n\to\infty} \phi_p(P_{F_0}u, x_n)$ exists. By (14) and $\alpha_n \to 0$, we get

$$\|y_n - z_n\| \to 0,\ \|z_n - x_n\| \to 0,\ \|A_1Lx_n\| \to 0 \text{ and } \|A_2y_n\| \to 0. \tag{18}$$

So, from the condition (II) for A_1 and A_2, $\omega_w(x_n) \subset F_0$. Suppose that for every $\varepsilon_1 > 0$ and $N_1 \in \mathbf{N}$, there exists $N \in \mathbf{N}$ with $N > N_1$ such that $\|x_N - u\| < \varepsilon_1$. So, a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $x_{n_i} \to u \in F_0$ exists. Since $J_{p,E}$ is norm to weak* continuous,

$$\begin{aligned}\phi_p(P_{F_0}u, x_{n_i}) &= \phi_p(u, x_{n_i}) \\ &= \|u\|^p - p\langle u.J_{p,E}x_{n_i}\rangle + (p-1)\|x_{n_i}\|^p \\ &\to \|u\|^p - p\langle u, J_{p,E}u\rangle + (p-1)\|u\|^p = 0.\end{aligned}$$

So, we obtain $\phi_p(P_{F_0}u, x_n) \to 0$. Hence, $x_n \to P_{F_0}u$. Assume that there exist $\varepsilon_2 > 0$ and $N_2 \in \mathbf{N}$ such that $\|x_n - u\| \geq \varepsilon_2$ for all $n > N_2$. Then, we have

$$\limsup_{n\to\infty} \frac{\langle P_{F_0}u - z_n, J_{p,E}(x_n - u)\rangle}{\|x_n - u\|^{p-1}} \geq 0. \tag{19}$$

In fact, suppose that $\limsup_{n\to\infty} \frac{\langle P_{F_0}u - z_n, J_{p,E}(x_n - u)\rangle}{\|x_n - u\|^{p-1}} = l < 0$. There exists $N_3 \in \mathbf{N}$ with $N_3 > N_2$ such that

$$\langle P_{F_0}u - z_n, J_{p,E}(x_n - u)\rangle < (l/2)\|x_n - u\|^{p-1}$$

for every $n \geq N_3$. Since $\|x_n - u\| \geq \varepsilon_2$ $(\forall n > N_2)$, we have

$$\langle P_{F_0}u - z_n, J_{p,E}(x_n - u)\rangle < (l/2)\varepsilon_2^{p-1}$$

for all $n \geq N_3$. So, from (13), we get

$$\begin{aligned} -((lp\varepsilon_2^{p-1})/2)\alpha_n &< \alpha_n p\langle z_n - P_{F_0}u, J_{p,E}(x_n - u)\rangle \\ &\leq \phi_p(P_{F_0}u, x_n) - \phi_p(P_{F_0}u, x_{n+1}) \end{aligned}$$

for every $n \geq N_3$. Therefore we obtain

$$-((lp\varepsilon_2^{p-1})/2)\sum_{n=N_3}^{\infty} \alpha_n \leq \phi_p(P_{F_0}u, x_{N_3}) < \infty.$$

This is a contradiction by $\sum_{n=N_3}^{\infty} \alpha_n = \infty$. So, (19) holds. From (6),

$$\frac{\langle P_{F_0}u - z_n, J_{p,E}(x_n - u)\rangle}{\|x_n - u\|^{p-1}} \leq \|z_n - x_n\| - \|x_n - u\| + \|u - P_{F_0}u\|$$

for each $n \geq N_2$. By $\|z_n - x_n\| \to 0$ in (18), we get

$$0 \leq -\liminf_{n\to\infty} \|x_n - u\| + \|u - P_{F_0}u\|. \tag{20}$$

There exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w \in F_0$ and

$$\liminf_{n\to\infty} \|x_n - u\| = \lim_{i\to\infty} \|x_{n_i} - u\|.$$

So, from (20) and weak lower semicontinuity of norm, we obtain

$$\|w - u\| \leq \lim_{i\to\infty} \|x_{n_i} - u\| \leq \|P_{F_0}u - u\|$$

which implies $w = P_{F_0}u$ and $\lim_{i\to\infty} \|x_{n_i} - u\| = \|P_{F_0}u - u\|$. By Kadec-Klee property, $x_{n_i} \to P_{F_0}u$. Since J_p is norm to weak* continuous,

$$\phi_p(P_{F_0}u, x_{n_i}) \to 0.$$

Therefore, we obtain $\phi_p(P_{F_0}u, x_n) \to 0$ which implies $x_n \to P_{F_0}u$ from Lemma 2.2. So, the proof is complete. □

Theorem 3.8. *Assume that* p, C, E, c, D, F, L, A_1, A_2, β_1, β_2, F_0, Π_C^p, $J_{p,E}$, $J_{p,F}$, $J_{2,E}$, $J_{2,F}$, L^*, $\{r_n\}$, $\{\lambda_n\}$, $\{\alpha_n\}$, a_1, a_2, b_1, b_2 *and* P_{F_0} *are the same as Theorem 3.7. Let* $u \in E$ *and* $\{x_n\} \subset C$ *be a sequence generated by*

$$\begin{aligned} x_1 &\in C \\ y_n &= \Pi_C^p J_{p,E}^{-1}(J_{p,E}x_n - r_n L^* J_{p,F} J_{2,F}^{-1} A_1 L x_n) \\ x_{n+1} &= \Pi_C^p J_{p,E}^{-1}(J_{p.E}y_n - \lambda_n J_{p.E} J_{2,E}^{-1} A_2 y_n - \alpha_n J_{p,E}(y_n - u)) \end{aligned}$$

for every $n \in \boldsymbol{N}$. *Then,* $\{x_n\}$ *converges strongly to* $P_{F_0}u$.

Proof. Similarly in Theorem 3.7, we have F_0 is nonempty, closed and convex and hence, P_{F_0} is well defined. And there exist $\gamma_0 > 0$ and $\delta_0 > 0$ such that

$$d_1 = \inf_{n\in\mathbf{N}}\Big\{c - \frac{r_n\|L\|^p}{\gamma_0}\Big\} > 0, \quad d_2 = \inf_{n\in\mathbf{N}}\Big\{pr_n\Big(\beta_1 - \frac{(p-1)\gamma_0^{\frac{1}{p-1}}}{p}\Big)\Big\} > 0,$$

$$d_4 = \inf_{n\in\mathbf{N}}\Big\{c - \frac{\lambda_n}{\delta_0}\Big\} > 0, \quad \text{and} \quad d_5 = \inf_{n\in\mathbf{N}}\Big\{p\lambda_n\Big(\beta_2 - \frac{(p-1)\delta_0^{\frac{1}{p-1}}}{p}\Big)\Big\} > 0.$$

Further, there exist $\varepsilon_0 > 0$ and $m_0 \in \mathbf{N}$ such that $d = p - 2(p-1)\varepsilon_0^{\frac{1}{p-1}} > 0$ and $d_3 = \inf_{n\geq m_0}\{c - \frac{\lambda_n}{\delta_0} - \frac{\alpha_n}{\varepsilon_0}\} > 0$.

Let $z \in F_0$. Define

$$z_n = J_{p,E}^{-1}(J_{p,E}x_n - r_n L^* J_{p,F} J_{2,F}^{-1} A_1 L x_n)$$

and

$$w_n = J_{p,E}^{-1}(J_{p.E}y_n - \lambda_n J_{p.E} J_{2,E}^{-1} A_2 y_n - \alpha_n J_{p,E}(y_n - u)).$$

As in the proof of (2)–(5) in Theorem 3.7, we get

$$\begin{aligned} \phi_p(z, y_n) \;\leq\; & \phi_p(z, x_n) - c\|y_n - z_n\|^p - \Big(c - \frac{r_n\|L\|^p}{\gamma_0}\Big)\|z_n - x_n\|^p \\ & - pr_n\Big(\beta_1 - \frac{(p-1)\gamma_0^{\frac{1}{p-1}}}{p}\Big)\|A_1 L x_n\|^p \end{aligned} \tag{21}$$

for all $n \in \mathbf{N}$. And similarly in (10) and (11) in Theorem 3.7, we obtain

$$\phi_p(z, x_{n+1}) \leq \phi_p(z,\, w_n) - c\|x_{n+1} - w_n\|^p \tag{22}$$

for each $n \in \mathbf{N}$ and

$$\begin{aligned} \phi_p(z, w_n) \leq\; & \phi_p(z,\, y_n) - \Big(c - \frac{\lambda_n}{\delta_0}\Big)\|w_n - y_n\|^p \\ & - p\lambda_n\Big(\beta_2 - \frac{(p-1)\delta_0^{\frac{1}{p-1}}}{p}\Big)\|A_2 y_n\|^p - p\alpha_n\langle w_n - z,\, J_{p,E}(y_n - u)\rangle \end{aligned} \tag{23}$$

for all $n \in \mathbf{N}$. As in the proof of (6) in Theorem 3.7, we have

$$\begin{aligned} & \langle w_n - z, J_{p,E}(y_n - u)\rangle \\ & \geq\; -\|w_n - y_n\| \cdot \|y_n - u\|^{p-1} + \|y_n - u\|^p - \|u - z\| \cdot \|y_n - u\|^{p-1} \\ & \geq\; -\frac{1}{p\varepsilon_0}(\|w_n - y_n\|^p + \|u - z\|^p) + \Big(1 - \frac{2(p-1)}{p}\varepsilon_0^{\frac{1}{p-1}}\Big)\|y_n - u\|^p \end{aligned} \tag{24}$$

for all $n \in \mathbf{N}$.

By (21) and (23), we get

$$\phi_p(z, y_n) \ \leq \ \phi_p(z, x_n) - c\|y_n - z_n\|^p - d_1\|z_n - x_n\|^p - d_2\|A_1 L x_n\|^p \qquad (25)$$

for each $n \in \mathbf{N}$ and

$$\begin{aligned} \phi_p(z, w_n) \ &\leq \ \phi_p(z,\ y_n) - d_4\|w_n - y_n\|^p - d_5\|A_2 y_n\|^p \\ &\quad - p\alpha_n\langle w_n - z,\ J_{p,E}(y_n - u)\rangle \end{aligned} \qquad (26)$$

for every $n \in \mathbf{N}$. By (23) and (24), we have

$$\begin{aligned} &\phi_p(z, w_n) \\ &\leq \phi_p(z,\ y_n) - d_3\|w_n - y_n\|^p - d_5\|A_2 y_n\|^p - d\alpha_n\|y_n - u\|^p + \frac{\alpha_n}{\varepsilon_0}\|u - z\|^p \end{aligned} \qquad (27)$$

for each $n \geq m_0$. From (22) and (25)–(27), we obtain

$$\begin{aligned} &\phi_p(z, x_{n+1}) \leq \phi_p(z, x_n) - c\|x_{n+1} - w_n\|^p - c\|y_n - z_n\|^p - d_1\|z_n - x_n\|^p \\ &- d_2\|A_1 L x_n\|^p - d_4\|w_n - y_n\|^p - d_5\|A_2 y_n\|^p - p\alpha_n\langle w_n - z,\ J_{p,E}(y_n - u)\rangle \end{aligned} \qquad (28)$$

for every $n \in \mathbf{N}$ and

$$\begin{aligned} &\phi_p(z, x_{n+1}) \leq \phi_p(z, x_n) - c\|x_{n+1} - w_n\|^p - c\|y_n - z_n\|^p - d_1\|z_n - x_n\|^p \\ &- d_2\|A_1 L x_n\|^p - d_3\|w_n - y_n\|^p - d_5\|A_2 y_n\|^p - d\alpha_n\|y_n - u\|^p + \frac{\alpha_n}{\varepsilon_0}\|u - z\|^p \end{aligned} \qquad (29)$$

for all $n \geq m_0$.

As in the proof of Theorem 3.7, we can prove that $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are bounded and $x_n \to P_{F_0} u$. □

4. Deduced results

By Theorems 3.7 and 3.8, we get several results under weaker conditions for a Banach space E than those deduced by [44]. At, first, we have the following results for the split common fixed point problem from Example 3.2.

Theorem 4.1. *Let $p > 1$, C be a nonempty, closed and convex subset of a p-uniformly convex and smooth Banach space E with a constant $c > 0$ in Theorem 2.1, D be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space F and L be a bounded linear operator of E into F such that $L \neq 0$ and $L(C) \subset D$. Let $\eta_1,\ \eta_2 \in (-\infty,\ 1)$, $U_1 :\ D \to F$ be an η_1-demimetric and demiclosed mapping and $U_2 :\ C \to E$ be an η_2-demimetric and demiclosed mapping such that $F_0 = F(U_2) \cap L^{-1}F(U_1) \neq \emptyset$. Let $u \in E$ and $\{x_n\} \subset C$ be a sequence generated by*

$$\begin{aligned} x_1 \ &\in \ C \\ y_n \ &= \ \Pi_C^p J_{p,E}^{-1}(J_{p,E}x_n - r_n L^* J_{p,F}(Lx_n - U_1 L x_n) - \alpha_n J_{p,E}(x_n - u)) \\ x_{n+1} \ &= \ \Pi_C^p J_{p,E}^{-1}(J_{p.E}y_n - \lambda_n J_{p.E}(y_n - U_2 y_n)) \end{aligned}$$

for every $n \in \boldsymbol{N}$, where Π_C^p is the generalized projection of E onto C, $J_{p,E}$ and $J_{p,F}$ are the generalized duality mappings of E and F, respectively, L^* is the adjoint operator of L,

$$a_1 \leq r_n \leq b_1 < \frac{c}{\|L\|^p}\Big(\frac{(1-\eta_1)p}{2(p-1)}\Big)^{p-1} \ (\forall n \in \boldsymbol{N}),$$

$$a_2 \leq \lambda_n \leq b_2 < c\Big(\frac{(1-\eta_2)p}{2(p-1)}\Big)^{p-1} \ (\forall n \in \boldsymbol{N})$$

for some $a_1, a_2, b_1, b_2 > 0$ with $a_1 \leq b_1$ and $a_2 \leq b_2$ and $\{\alpha_n\} \subset (0,1]$ with $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $P_{F_0}u$, where P_{F_0} is the metric projection of E onto F_0.

Theorem 4.2. *Assume that p, C, E, c, D, F, L, η_1, η_2, U_1, U_2, F_0, Π_C^p, $J_{p,E}$, $J_{p,F}$, L^*, $\{r_n\}$, $\{\lambda_n\}$, $\{\alpha_n\}$, a_1, a_2, b_1, b_2 and P_{F_0} are the same as Theorem 4.1. Let $u \in E$ and $\{x_n\} \subset C$ be a sequence generated by*

$$\begin{aligned} x_1 &\in C \\ y_n &= \Pi_C^p J_{p,E}^{-1}(J_{p,E}x_n - r_n L^* J_{p,F}(Lx_n - U_1 L x_n)) \\ x_{n+1} &= \Pi_C^p J_{p,E}^{-1}(J_{p.E}y_n - \lambda_n J_{p.E}(y_n - U_2 y_n) - \alpha_n J_{p.E}(y_n - u)) \end{aligned}$$

for every $n \in \boldsymbol{N}$. Then, $\{x_n\}$ converges strongly to $P_{F_0}u$.

Remark 4.3. By Example 3.3 and Theorems 4.1 and 4.2, we also have strong converegnce theorems for the split common fixed point problem in Hilbert spaces.

Next, we get the following results for the split feasibility problem from Example 3.4.

Theorem 4.4. *Let $p > 1$, C be a nonempty, closed and convex subset of a p-uniformly convex and smooth Banach space E with a constant $c > 0$ in Theorem 2.1, D be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space F and L be a bounded linear operator of E into F with $L \neq 0$. Let P_D^F and P_C^E be the metric projections of F onto D and of E onto C, respectively, and $F_0 = C \cap L^{-1}D \neq \emptyset$. Let $u \in E$ and $\{x_n\} \subset E$ be a sequence generated by*

$$\begin{aligned} x_1 &\in E \\ y_n &= J_{p,E}^{-1}(J_{p,E}x_n - r_n L^* J_{p,F}(Lx_n - P_D^F L x_n) - \alpha_n J_{p,E}(x_n - u)) \\ x_{n+1} &= J_{p,E}^{-1}(J_{p.E}y_n - \lambda_n J_{p.E}(y_n - P_C^E y_n)) \end{aligned}$$

for every $n \in \boldsymbol{N}$, where $J_{p,E}$ and $J_{p,F}$ are the generalized duality mappings of E and F, respectively, L^* is the adjoint operator of L,

$$a_1 \leq r_n \leq b_1 < \frac{c}{\|L\|^p}\big(\frac{p}{p-1}\big)^{p-1} \ (\forall n \in \boldsymbol{N}),$$

$$a_2 \leq \lambda_n \leq b_2 < c\big(\frac{p}{p-1}\big)^{p-1} \ (\forall n \in \boldsymbol{N})$$

for some $a_1, a_2, b_1, b_2 > 0$ with $a_1 \leq b_1$ and $a_2 \leq b_2$ and $\{\alpha_n\} \subset (0,1]$ with $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $P_{F_0}u$, where P_{F_0} is the metric projection of E onto F_0.

Theorem 4.5. *Suppose that* p, C, E, c, D, F, L, P_D^F, P_C^E, F_0, $J_{p,E}$, $J_{p,F}$, L^*, $\{r_n\}$, $\{\lambda_n\}$, $\{\alpha_n\}$, a_1, a_2, b_1, b_2 *and* P_{F_0} *are the same as Theorem 4.4. Let* $u \in E$ *and* $\{x_n\} \subset E$ *be a sequence generated by*

$$\begin{array}{rcl} x_1 & \in & E \\ y_n & = & J_{p,E}^{-1}(J_{p,E}x_n - r_n L^* J_{p,F}(Lx_n - P_D^F Lx_n)) \\ x_{n+1} & = & J_{p,E}^{-1}(J_{p.E}y_n - \lambda_n J_{p.E}(y_n - P_C^E y_n) - \alpha_n J_{p,E}(y_n - u)) \end{array}$$

for every $n \in \boldsymbol{N}$. *Then,* $\{x_n\}$ *converges strongly to* $P_{F_0}u$.

Next, we obtain the following results for the split common null point problem (monotone inclusion problem) from Examples 3.1 and 3.5.

Theorem 4.6. *Let* $p > 1$, C *be a nonempty, closed and convex subset of a* p*-uniformly convex and smooth Banach space* E *with a constant* $c > 0$ *in Theorem 2.1,* D *be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space* F *and* L *be a bounded linear operator of* E *into* F *such that* $L \neq 0$ *and* $L(C) \subset D$. *Let* $\beta_1 > 0$, $\beta_2 > 0$, $B_1 : D \to F^*$ *be a* β_1*-inverse strongly monotone operator and* $B_2 : C \to E^*$ *a* β_2*-inverse strongly monotone operator with* $F_0 = B_2^{-1}0 \cap L^{-1}B_1^{-1}0 \neq \emptyset$. *Let* $u \in E$ *and* $\{x_n\} \subset C$ *be a sequence generated by*

$$\begin{array}{rcl} x_1 & \in & C \\ y_n & = & \Pi_C^p J_{p,E}^{-1}(J_{p,E}x_n - r_n L^* J_{p,F} J_{2,F}^{-1} B_1 L x_n - \alpha_n J_{p,E}(x_n - u)) \\ x_{n+1} & = & \Pi_C^p J_{p,E}^{-1}(J_{p.E}y_n - \lambda_n J_{p.E} J_{2,E}^{-1} B_2 y_n) \end{array}$$

for every $n \in \boldsymbol{N}$, *where* Π_C^p *is the generalized projection of* E *onto* C, $J_{p,E}$ *and* $J_{p,F}$ *are the generalized duality mappings of* E *and* F, *respectively,* $J_{2,E}$ *and* $J_{2,F}$ *are the normalized duality mappings of* E *and* F, *respectively,* L^* *is the adjoint operator of* L,

$$a_1 \leq r_n \leq b_1 < \frac{c}{\|L\|^p}\left(\frac{\beta_1 p}{p-1}\right)^{p-1} \ (\forall n \in \boldsymbol{N}),$$

$$a_2 \leq \lambda_n \leq b_2 < c\left(\frac{\beta_2 p}{p-1}\right)^{p-1} \ (\forall n \in \boldsymbol{N})$$

for some $a_1, a_2, b_1, b_2 > 0$ *with* $a_1 \leq b_1$ *and* $a_2 \leq b_2$ *and* $\{\alpha_n\} \subset (0, 1]$ *with* $\alpha_n \to 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Then,* $\{x_n\}$ *converges strongly to* $P_{F_0}u$, *where* P_{F_0} *is the metric projection of* E *onto* F_0.

Theorem 4.7. *Suppose that* p, C, E, c, D, F, L, β_1, β_2, B_1, B_2, F_0, Π_C^p, $J_{p,E}$, $J_{p,F}$, $J_{2,E}$, $J_{2,F}$, L^*, $\{r_n\}$, $\{\lambda_n\}$, $\{\alpha_n\}$, a_1, a_2, b_1, b_2 *and* P_{F_0} *are the same as Theorem 4.6. Let* $u \in E$ *and* $\{x_n\} \subset C$ *be a sequence generated by*

$$\begin{array}{rcl} x_1 & \in & C \\ y_n & = & \Pi_C^p J_{p,E}^{-1}(J_{p,E}x_n - r_n L^* J_{p,F} J_{2,F}^{-1} B_1 L x_n) \\ x_{n+1} & = & \Pi_C^p J_{p,E}^{-1}(J_{p.E}y_n - \lambda_n J_{p.E} J_{2,E}^{-1} B_2 y_n - \alpha_n J_{p,E}(y_n - u)) \end{array}$$

for every $n \in \boldsymbol{N}$. *Then,* $\{x_n\}$ *converges strongly to* $P_{F_0}u$.

Theorem 4.8. *Let $p > 1$, E be a p-uniformly convex and smooth Banach space with a constant $c > 0$ in Theorem 2.1, F be a smooth, strictly convex and reflexive Banach space and L be a bounded linear operator of E into F with $L \neq 0$. Let $B_1 \subset F \times F^*$ and $B_2 \subset E \times E^*$ be maximal monotone operators with $F_0 = B_2^{-1}0 \cap L^{-1}B_1^{-1}0 \neq \emptyset$. Let $u \in E$ and $\{x_n\} \subset E$ be a sequence generated by*

$$\begin{aligned} x_1 &\in E \\ y_n &= J_{p,E}^{-1}(J_{p,E}x_n - r_n L^* J_{p,F}(Lx_n - J_r^1 Lx_n) - \alpha_n J_{p,E}(x_n - u)) \\ x_{n+1} &= J_{p,E}^{-1}(J_{p.E}y_n - \lambda_n J_{p.E}(y_n - J_r^2 y_n)) \end{aligned}$$

for every $n \in \boldsymbol{N}$, where $r > 0$, J_r^1 and J_r^2 are the metric resolvents of B_1 and B_2, respectively, $J_{p,E}$ and $J_{p,F}$ are the generalized duality mappings of E and F, respectively, L^* is the adjoint operator of L,

$$a_1 \leq r_n \leq b_1 < \frac{c}{\|L\|^p}\Big(\frac{p}{p-1}\Big)^{p-1} \ (\forall n \in \boldsymbol{N}),$$

$$a_2 \leq \lambda_n \leq b_2 < c\Big(\frac{p}{p-1}\Big)^{p-1} \ (\forall n \in \boldsymbol{N})$$

for some $a_1, a_2, b_1, b_2 > 0$ with $a_1 \leq b_1$ and $a_2 \leq b_2$ and $\{\alpha_n\} \subset (0,1]$ with $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $P_{F_0}u$, where P_{F_0} is the metric projection of E onto F_0.

Theorem 4.9. *Assume that p, E, c, F, L, B_1, B_2, F_0, r, J_r^1, J_r^2, $J_{p,E}$, $J_{p,F}$, L^*, $\{r_n\}$, $\{\lambda_n\}$, $\{\alpha_n\}$, a_1, a_2, b_1, b_2 and P_{F_0} are the same as Theorem 4.8. Let $u \in E$ and $\{x_n\} \subset E$ be a sequence generated by*

$$\begin{aligned} x_1 &\in E \\ y_n &= J_{p,E}^{-1}(J_{p,E}x_n - r_n L^* J_{p,F}(Lx_n - J_r^1 Lx_n)) \\ x_{n+1} &= J_{p,E}^{-1}(J_{p.E}y_n - \lambda_n J_{p.E}(y_n - J_r^2 y_n) - \alpha_n J_{p,E}(y_n - u)) \end{aligned}$$

for every $n \in \boldsymbol{N}$. Then, $\{x_n\}$ converges strongly to $P_{F_0}u$.

Remark 4.10. From Remark 2.8 and Theorems 4.8 and 4.9, we have the results for the split equilibrium problem. The equilibrium problem contains as special cases, for example, optimization problem, problem of Nash equilibria, complementarity problem, fixed point problem and variational inequalities problem, see [9] for details.

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