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Optimization of Elliptic Type Differential Inclusions with Mixed Boundary Conditions

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This paper first deals with the Neumann problem for discrete, discrete-approximate problems and elliptic differential inclusions. Then using the obtained results for discrete inclusions in the form of Euler-Lagrange inclusions necessary and sufficient conditions for optimality are derived for the problems under consideration. Further, these problems are generalized to mixed problems, where the Neumann and Dirichlet conditions are satisfied separately in different parts of the set-valued mapping domain. As an example, a linear optimal control problem is considered, from which the Weierstrass-Pontryagin maximum condition follows. Thus, relying only on the previous auxiliary results, we can obtain the main results of this paper for mixed problems. In turn, the Neumann problem is generalized to the multidimensional case with a second order elliptic operator.

Keywords: Discrete-approximate, elliptic differential inclusion, Neumann, Dirichlet, mixed, necessary and sufficient conditions.

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1. Introduction

Many problems in economic dynamics, as well as classical problems on optimal control in vibrations, chemical, engineering, heat, diffusion processes, differential games, etc., can be described by ordinary and partial differential inclusions (DFIs). We refer the reader to the survey papers [1–4, 12, 13, 17, 19–25, 27–32, 36]. These intensively developing problems also include problems with elliptic differential equations/inclusions (see [8, 11, 26, 30] and references therein). In a closer article [26] we consider in detail the optimization of the Dirichlet problem for elliptic DFIs with another statement, where the right-hand side depend both on the searched function and on the first partial derivatives. This complicates the construction of the adjoint inclusion and "adjoint" boundary conditions, where it is additionally required that the resulting adjoint variables be equal to zero. In this respect, the reader can compare the results of Corollary 5.2 of the present paper to Theorem 7 [26]. The work [24] is devoted to the optimization of the first mixed initial-boundary value problem for hyperbolic DFIs with the Laplace operator. To this end, an

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auxiliary problem with a hyperbolic discrete inclusion is defined and, using LAMs, necessary and sufficient optimality conditions for hyperbolic discrete inclusions are proved. The article [25] discusses the problem of optimal control theory given by second-order sweeping processes with discrete and DFIs. Based on the discretization method in the form of Euler-Lagrange inclusions, optimality conditions for discrete approximate inclusions and transversality conditions are obtained. The establishment of adjoint Euler-Lagrange-type inclusions for DFIs is based on the presence of equivalence relations for LAM. The work [31] considers the Bolza problem for DFIs subject to general endpoint restrictions, where this serves a dual purpose. First, the finite difference method was developed and a discrete-approximate problem was constructed that ensures strong convergence of optimal solutions. Second, this direct method is used to obtain the necessary optimality conditions in the refined Euler-Lagrange form without the standard convexity assumptions. The paper [19] presents the necessary optimality conditions in optimal control problems, expressed as examples of the generalized Bolza problem, with the addition of the possibility of varying the fundamental planning interval.

Along with the optimization problems of DFIs, there are many interesting works [1, 2, 5–10, 14–16, 18, 33, 35] in the literature devoted to various qualitative problems of DFIs. The main goal of the paper [1] is to provide new explicit criteria for characterizing weak lower semicontinuous Lyapunov pairs or functions associated with first-order DFIs in Hilbert spaces. These inclusions are determined by the Lipschitz perturbation of a maximally monotone operator. The approach is based on advanced analysis of variations and generalized differentiation tools. The paper [6] studies three-point boundary value problems for perturbed second-order DFIs. The existence of solutions is proved under the condition that the given multivalued mapping is nonconvex. In [35], a measurable sweeping process with set-valued perturbation in a separable Hilbert space is considered. The retraction of the sweeping process is a function of bounded variation whose differential measure is absolutely continuous with respect to the positive Radon measure. In [8], using variational methods, sub-super solutions, and truncation methods, the existence of three nontrivial solutions of a quasilinear elliptic equation with the Neumann boundary condition was proved.

The sign information for each of these solutions is given: two of them have the opposite constant sign, and the third has an alternating sign. The article [9] is devoted to weakening the "classical" assumptions and presenting new arguments to prove the existence of DFIs solutions with proximal normal cones in Banach spaces. Basically, the concept of "directional prox-regularity" is defined and assumptions about the Banach spaces are given. In the article [2], the problem of finding a Lipschitz function in an open domain with given boundary values is studied, the gradient of which is required to satisfy some nonhomogeneous pointwise constraints a.e. in the domain. It is assumed that these constraints are given by a measurable set-valued mapping with convex, uniformly compact, and nonempty-interior values.

In [10], measurable viability results for the evolution DFI were obtained by approximating it with the help of Aumann's integral means. The feasibility of a pre-ordering is also studied by reducing it to the viability of its sections. The asymptotic behaviour of the solution of a DFI [18] provides a simple proof of a minmax theorem.

The novelty of the work lies in the fact that by constructing the corresponding discrete problems and approximate problems, based on the results of the equivalence of LAMs, sufficient optimality conditions for elliptic DFIs with mixed boundary conditions are systematically proved on a rectangular and then on an arbitrary bounded domain. To the best of our knowledge, such studies have not yet been carried out in the literature.

The rest of the present paper is organized as follows:

In Section 2, some definitions and concepts are introduced. Hamiltonian functions, Locally adjoint mappings (LAMs), convex upper approximations, locally tents are defined, which are the main tool for studying optimization problems described by discrete and DFIs. Then, optimization problems are formulated for elliptic discrete and differential DFIs with mixed boundary conditions both with Laplace operators and with a second-order multidimensional elliptic operators.

In Section 3 for problem with discrete inclusions we use one of the constructions of convex and nonsmooth analysis to get necessary and sufficient conditions of optimality in terms of Euler-Lagrange type inclusions. The main difficulty here lies in the calculation of dual cones to the cone of tangent directions arising in the equivalent convex programming problem.

In Section 4, we use difference approximations of partial derivatives to approximate the Neumann problem for elliptic DFIs and derive necessary and sufficient optimality conditions for a discrete approximate problem that requires LAM equivalence results [26] to pass to the latter.

In Section 5, based on the results obtained in Section 4, a sufficient condition for the optimality of the continuous elliptic Niemann problem is formulated. To do this, we pass to the formal limit as the discrete steps tend to zero. It turns out that the formulated conditions are sufficient optimality conditions. Thus, the discrete-approximate problem is a bridge between discrete and continuous Neumann-type problems.

In Section 6 we consider a multidimensional optimal control problem for elliptic DFIs with a second-order elliptic operator and Neumann boundary conditions. When proving the optimality conditions formulated in terms of the second-order adjoint operator, the familiar Green's formula is used.

In Section 7 we consider the extension of the Neumann type problems to a mixed problem with an arbitrary bounded set and its closed piecewise smooth curve, where the Dirichlet and Neumann boundary conditions hold separately along the two curves of the boundary curve. In fact, this generalization is the main result of this paper, and for the proof of the optimality condition in it, the Neumann problem for elliptic DFIs plays an important auxiliary role. As an example for linear optimal control problems, these results imply the WeierstrassPontryagin maximum condition [34].

2. Auxiliary concepts and problem statement

For the convenience of readers, we present the main information from the book [21]. Henceforth, $\langle u_1, u_2\rangle$ will denote the scalar product of the elements of n-dimensional Euclidian space; $u_1, u_2 \in \mathbb{R}^n$. A set-valued mapping $F : \mathbb{R}^{4n} \rightrightarrows \mathbb{R}^n$ is *convex*, if $gphF = \{(u_1, u_2, u_3, u_4, v)\colon = \{(u_1, u_2, u_3, u_4, v) : v \in F(u_1, u_2, u_3, u_4)\}$ is convex.

It is *convex-valued* if $F(u)$ is a convex set for each $u = (u_1, u_2, u_3, u_4) \in \operatorname{dom} F = \{u : F(u) \neq \varnothing\}$. F is *closed* if $gphF$ is a closed set in $\mathbb{R}^{5n}$.

The *Hamiltonian function* and the *argmaximum set* are defined as follows

$$H(u, v^*) = \sup_v \{\langle v, v^* \rangle : v \in F(u)\}, v^* \in \mathbb{R}^n,$$
$$F_{Arg}(u, v^*) \equiv F_A(u, v^*) = \{v \in F(u) : \langle v, v^* \rangle = H(u, v^*)\}.$$

The *cone of tangent directions* for a convex mapping F at a point $(\tilde{u}, \tilde{v}) \in gphF$ is defined by the formula

$$K_F(\tilde{u}, \tilde{v}) = \{(\bar{u}, \bar{v}) : \bar{u} = \lambda(u - \tilde{u}), \bar{v} = \lambda(v - \tilde{v}), \lambda > 0, \forall (u, v) \in \operatorname{gph} F\}.$$

Generally speaking, for an arbitrary set $A \subset \mathbb{R}^n$, for each tangent direction $\bar{u}$, there exists a function $\gamma(\lambda)$ such that $u + \lambda\bar{u} + \gamma(\lambda) \in A$ for sufficiently small $\lambda \geq 0$ and $\lambda^{-1}\gamma(\lambda) \to 0$, $\lambda \downarrow 0$. It follows that the cone of tangent directions involve directions for each of which there exists a function $\gamma(\lambda)$. But in order to predetermine properties of the set A this is not sufficient. Nevertheless, the following notion of a local tent allows us to predetermine mapping in A for the nearest tangent directions among themselves.

The cone $K_A(u)$ of the set A at a point $u \in A$ is called a *local tent* if for each point of relative interior $\bar{u} \in riK_A(u)$ there exists a convex cone $K \subseteq K_A(u)$ and a continuous mapping $\psi(\bar{u})$ defined in a neighbourhood of the origin such that

(a) $\bar{u} \in \operatorname{ri} K$, $\operatorname{Lin} K = \operatorname{Lin} K_A(u)$, where $\operatorname{Lin} K$ is the linear span of K,

(b) $\psi(\bar{u}) = \bar{u} + r(\bar{u})$, $\|\bar{u}\|^{-1} r(\bar{u}) \to 0$, as $\bar{u} \to 0$,

(c) $u + \psi(\bar{u}) \in A$, $\quad \bar{u} \in K \cap \mathrm{S}_\varepsilon(0)$ for some $\varepsilon > 0$, where $\mathrm{S}_\varepsilon(0)$ is the ball of radius ε and with centre of origin.

The local properties of differentiable functions can be described quite well using the notion of a derivative and the related notion of the gradient of a function. For convex functions, the concept of gradient is replaced by a subdifferential. In the case of a set-valued mapping, a similar role is played by the LAM, which is introduced in this paper.

Definition 2.1. Let $F : \mathbb{R}^{4n} \rightrightarrows \mathbb{R}^n$ be a convex mapping. Then the set-valued mapping $F^*(\cdot; (u, v)) : \mathbb{R}^n \rightrightarrows \mathbb{R}^{4n}$ defined by the formula

$$F^*(v^*; (u, v)) = \{u^* : (u^*, -v^*) \in K_F^*(u, v)\}$$

is called the *locally adjoint mapping* (LAM) to F at the point $(u, v) \in gphF$, where $K_F^*(u, v)$ is the dual to the cone $K_F(u, v)$. □

Note that the coderivative concept of Mordukhovich [32] is essentially different for nonconvex mappings, defined in terms of the underlying normal cone to their graphs. However, for the convex mappings the two notions are equivalent.

The function $h(\cdot, u)$ is said to be a *convex upper approximation* (CUA) of a function $g(\cdot) : \mathbb{R}^n \to \mathbb{R}^1 \cup \{+\infty\}, u \in \operatorname{dom} g = \{u : |g(u)| < +\infty\}$, if

(i) $h(\bar{u}, u) \geq \sup\limits_{r(\cdot)} \limsup\limits_{\lambda \downarrow 0} \frac{1}{\lambda}(g(u + \lambda\bar{u} + r(\lambda)) - g(u))$ for all $\bar{u} \neq 0, \lambda^{-1} r(\lambda) \to 0$,

(ii) $h(\cdot, u)$ is a convex closed positive-homogeneous function.

Then, the set $\partial h(0,u) = \{u^* \in \mathbb{R}^n : h(\bar{u},u) \geq \langle \bar{u}, u^* \rangle, \forall \bar{u} \in \mathbb{R}^n\}$ is called the *subdifferential* of g and is denoted by $\partial g(u)$, which for convex functions continuous at u coincides with the classical definition of a subdifferential. A function is called *proper* if it takes no value $-\infty$ and is not identically equal to $+\infty$.

The following definition of LAM is a useful generalization of Definition 2.1 to the nonconvex case where the Hamilton function is used.

Definition 2.2. For a nonconvex set-valued mapping $F : \mathbb{R}^{4n} \rightrightarrows \mathbb{R}^n$ the *locally adjoint mapping* (LAM) at a point $(\tilde{u},\tilde{v}) \in gph$ F is defined as follows

$$F^*(v^*, (\tilde{u},\tilde{v})) = \{u^* : H(u,v^*) - H(\tilde{u},v^*) \leq \langle u^*, u-\tilde{u} \rangle, \forall u \in \mathbb{R}^n\}, \ \tilde{v} \in F_A(\tilde{u}, v^*).$$

As can be seen from the definition of the Hamilton function $H(\cdot, v^*)$, due to the operation of taking the supremum, it is a concave function for a convex F. Therefore, by Theorem 2.1 [21, p .62] this definition of LAM is the same as Definition 2.1.

In the next section we consider the following optimization problem for elliptic discrete inclusions with "Neumann type" conditions, labelled (PD):

$$\text{(PD)} \qquad \text{minimize} \sum_{x=1,..,T-1,y=1,..,\Lambda-1} g(u_{x,y}, x, y), \tag{1}$$

$$u_{x+1,y} \in F(u_{x-1,y}, u_{x,y-1}, u_{x,y}, u_{x,y+1}, x, y), \tag{2}$$

$$\begin{gathered} u_{x,0} = u_{x,1} + \alpha_{0x}; u_{x,\Lambda} = u_{x,\Lambda-1} + \alpha_{\Lambda x}, \\ u_{0,y} = u_{1,y} + \beta_{0y}; u_{T,y} = u_{T-1,y} + \beta_{Ty}, \\ x = 1, \dots, T-1; \quad y = 1, \dots, \Lambda - 1, \end{gathered} \tag{3}$$

where $g_{x,y} : \mathbb{R}^n \to \mathbb{R}^1 \cup \{+\infty\}$, $F(\cdot, x, y) : \mathbb{R}^{4n} \rightrightarrows \mathbb{R}^n$ are set-valued mappings, $\alpha_{0x}, \alpha_{\Lambda x}, \beta_{0y}, \beta_{Ty}$ and T, Λ are fixed vectors and integers, respectively. We call it a *Neumann type problem for discrete inclusions of elliptic type*. Then a set of points $\{u_{x,y}\}_D = \{u_{x,y} : (x,y) \in D\}$ is called a *feasible solution* for the problem (1)–(3) if it satisfies the inclusions (2) and boundary conditions (3), where

$$D = \{(x,y) : x = 0, \dots, T; y = 0, \dots, \Lambda, \ (x,y) \neq (0,0), (0,\Lambda), (T,0), (T,\Lambda)\}.$$

For the problem (PD) in the non-convex case, the following condition is assumed below concerning the functions $g(\cdot, x, y)$ and the mappings $F(\cdot, x, y)$.

Conditions A: Assume that if $\{\tilde{u}_{x,y}\}_D$ is an optimal solution to the problem (PD), then the mappings $F(\cdot, x, y)$ are such that the cone of tangent directions $K_F(\tilde{u}_{x-1,y}, \tilde{u}_{x,y-1}, \tilde{u}_{x,y}, \tilde{u}_{x,y+1}, \tilde{u}_{x+1,y})$ is a local tent. Besides, let us assume that the functions $g(\cdot, x, y)$ admit CUA $h_{x,y}(\cdot, \tilde{u}_{x,y})$ continuous with respect to $\bar{u}$, whence it immediately follows that the subdifferential $\partial g(\tilde{u}_{x,y}, x, y) := \partial h_{x,y}(0, \tilde{u}_{x,y})$ is defined.

Conditions B: Let the problem (PD) be convex, that is, $F(\cdot, x, y)$ be a convex mapping, $g(\cdot, x, y)$ be a convex proper function. Then for some feasible solution $\{\hat{u}_{x,y}\}_D$ let $(\hat{u}_{x-1,y}, \hat{u}_{x,y-1}, \hat{u}_{x,y}, \hat{u}_{x,y+1}, \hat{u}_{x+1,y}) \in \text{rigphF}(\cdot, x, y)$, $\hat{u}_{x,y} \in \text{ridomg}(\cdot, x, y)$, $x = 1, \dots, T-1$; $y = 1, \dots, \Lambda - 1$.

In Sections 5–7 we study different mixed optimization problems for partial elliptic type DFIs. In Section 5 we will consider the following problem with a Neumann boundary condition:

$$\text{(PC)} \qquad \text{minimize } J[u(x,y)] := \iint\limits_{Q} g(u(x,y),x,y)dxdy, \tag{4}$$

$$\Delta u(x,y) \in F(u(x,y),x,y), \ (x,y) \in Q = (0,T) \times (0,\Lambda), \tag{5}$$

$$\frac{\partial u(x,0)}{\partial v_1} = \alpha_0(x), \ \frac{\partial u(x,\Lambda)}{\partial v_2} = \alpha_\Lambda(x), \ x \in [0,T],$$

$$\frac{\partial u(0,y)}{\partial v_3} = \beta_0(y), \ \frac{\partial u(T,y)}{\partial v_4} = \beta_T(y), \ y \in [0,\Lambda], \tag{6}$$

$$\alpha_0(T) = \beta_T(0), \ \alpha_0(0) = \beta_0(0), \ \alpha_L(0) = \beta_0(L), \ \alpha_\Lambda(T) = \beta_T(L).$$

Here Δ is a Laplace's operator $\Delta := \partial^2/\partial x^2 + \partial^2/\partial y^2$, $F(\cdot,x,y) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ set-valued mapping for all (x,y) in the rectangle region $Q \subset \mathbb{R}^2$, $g(\cdot,x,y) : \mathbb{R}^n \to \mathbb{R}^1$ is a continuous proper convex function and the functions α_0, α_Λ, β_0, β_T are continuous, $v_1 = (0,-1)$, $v_2 = (0,1)$, $v_3 = (-1,0)$, $v_4 = (1,0)$. In (6) the conditions $\alpha_0(T) = \beta_T(0)$, $\alpha_0(0) = \beta_0(0)$, $\alpha_\Lambda(0) = \beta_0(\Lambda)$, $\alpha_\Lambda(T) = \beta_T(\Lambda)$ ensure the required continuity on the boundary of the rectangle $\overline{Q} = [0,T] \times [0,\Lambda]$. Recall that, without loss of generality, formula $\partial u(x,y)/\partial \bar{v} = u_x(x,y)\bar{v}_1 + u_y(x,y)\bar{v}_2$ is valid for a normal vector $\bar{v} = (\bar{v}_1, \bar{v}_2)$ and boundary conditions (6) in terms of directional derivatives can be formulated in terms of partial derivatives with respect to x and y as follows:

$$u_y(x,0) = \alpha_0(x), \ u_y(x,\Lambda) = \alpha_\Lambda(x), \ u_x(0,y) = \beta_0(y), \ u_x(T,y) = \beta_T(y). \tag{7}$$

We label this continuous problem (PC) and call it the Neumann problem for elliptic DFIs. The problem is to find a classical solution $\tilde{u}(x)$ of the boundary value problem (5), (7) that minimizes $J[u(\cdot)]$. Section 7 studies a problem with mixed Neumann and Dirichlet boundary conditions $u(x,0) = \alpha_0(x)$, $u(0,y) = \beta_0(y)$, $u_y(x,\Lambda) = \alpha_\Lambda(x)$, $u_x(T,y) = \beta_T(y)$ in a rectangular region where it seems interesting to the design of a "adjoint" boundary condition. Then this problem applies to the cases with arbitrary bounded set $G \subset \mathbb{R}^2$ with closed piecewisesmooth curve. Thus, we consider the extension of the problem (PC) to a mixed problem with an arbitrary bounded set $G \subset \mathbb{R}^2$ instead of a rectangular Q and its closed piecewise smooth curve $\Gamma = \overline{\Gamma}_1 \cup \Gamma_2$, where the Dirichlet and Neumann boundary conditions hold separately along the curves $\overline{\Gamma}_1$ and Γ_2 of the curve Γ. In fact, this generalization is the main result of this article, and for the proof of the optimality condition in it, the problem (PC) plays an important auxiliary role.

The subject of the research in Section 6 in the following multidimensional optimal control problem (PM) for elliptic DFIs:

$$\text{(PM)} \qquad \text{minimize } J[u(\cdot)] := \int\limits_{G} g(u(x),x)dx, \tag{8}$$

$$Lu(x) \in F(u(x),x), \quad x \in G, \tag{9}$$

$$\frac{\partial u(x)}{\partial v} = \alpha(x), \quad x \in S, \tag{10}$$

where $F(\cdot,x) : \mathbb{R}^1 \rightrightarrows \mathbb{R}^1$ is a convex closed set-valued mapping for all $x = (x_1, ..., x_n)$ in the bounded set $G \subset \mathbb{R}^n$ with a closed piecewise-smooth surface S, and where

$g : \mathbb{R}^1 \times G \to \mathbb{R}^1$ is a continuous and convex on u function, α is continuous and $dx = dx_1 dx_2 \dots dx_n$, L is a second order elliptic operator:

$$Lu := \sum_{i,j=1}^{n} \left(a_{ij}(x) u_{x_j}\right)_{x_i} + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x)u, a_{ij}(x) \in C^1(\overline{G}),$$
$$b_i(x) \in C^1(\overline{G}), c(x) \in C(\overline{G}),$$

where $\|a_{ij}(x)\|$ is a positively definite matrix, $C(\overline{G})$ and $C^1(\overline{G})$ are the spaces of continuous and continuously differentiable functions in G, respectively. Let $C^2(G)$ be the space of functions $u(x)$ having continuous second-order derivatives $u_{x_i x_j}$, $i, j = 1, \dots, n$. Then a function $u(x)$ in $C^2(G) \cap C(\overline{G})$, that satisfies the inclusion (9) in G and the Neumann boundary condition (10) we call a classical solution of the problem (PM).

3. Optimality conditions for the Neumann type elliptic discrete inclusions

At first, we consider the convex problem (PD).

Theorem 3.1. *Let $F(\cdot, x, y)$ be a convex set-valued mapping, $g(\cdot, x, y)$ be a convex proper function continuous at points of some feasible solution $\{\hat{u}_{x,y}\}_D$. Then for the optimality of the solution $\{\tilde{u}_{x,y}\}_D$ of the problem (PD) it is necessary that there exist a scalar number $\alpha \in \{0, 1\}$ and vectors $\left\{\psi^*_{x,y}, \eta^*_{x,y}, \xi^*_{x,y}, u^*_{x,y}\right\}$, not all equal to zero, such that:*

(i) $\left(\psi^*_{x,y}, \xi^*_{x,y}, u^*_{x-1,y}, \eta^*_{x,y}\right) \in F^*\left(u^*_{x,y}; \left(\tilde{u}_{x-1,y}, \tilde{u}_{x,y-1}, \tilde{u}_{x,y}, \tilde{u}_{x,y+1}, \tilde{u}_{x+1,y}\right), x, y\right)$
$+\{0\} \times \{0\} \times \left\{\psi^*_{x+1,y} + \xi^*_{x,y+1} + \eta^*_{x,y-1} - \alpha \partial g\left(\tilde{u}_{x,y}, x, y\right)\right\} \times \{0\}$,

(ii) $u^*_{0,y} + \psi^*_{1,y} = 0$; $\eta^*_{x,0} - \xi^*_{x,1} = 0$; $\xi^*_{x,\Lambda} - \eta^*_{x,\Lambda-1} = 0$; $\psi^*_{T,y} + u^*_{T-1,y} = 0$.

In addition, under the Conditions B, *the conditions* (i) *and* (ii) *are also sufficient for the optimality of $\{\tilde{u}_{x,y}\}_D$.*

Proof. We use results of convex programming. Let $w = (u_0, u_1, \dots, u_T)$, which has dimension $m = 2n(\Lambda - 1) + n(T - 1)(\Lambda + 1)$, where for $x_1 = 1, \dots, T-1$, $u_x = (u_{x,0}, u_{x,1}, \dots, u_{x,\Lambda}) \in \mathbb{R}^{n(\Lambda+1)}$ is an $n(\Lambda + 1)$-dimensional vector and further $u_0 = (u_{0,1}, \dots, u_{0,\Lambda-1}) \in \mathbb{R}^{n(\Lambda-1)}$, $u_T = (u_{T,1}, \dots, u_{T,\Lambda-1}) \in \mathbb{R}^{n(\Lambda-1)}$. Let us now define the following convex sets in the space $\mathbb{R}^m$:

$$M_{x,y} = \left\{ w = (u_0, u_1, \dots, u_T) : \begin{array}{l} (u_{x-1,y}, u_{x,y-1}, u_{x,y}, u_{x,y+1}, u_{x+1,y}) \in \operatorname{gph} F(\cdot, x, y\}, \\ x = 1, \dots, T-1;\ y = 1, \dots, \Lambda - 1 \end{array} \right\}$$

$$P_{\alpha_0} = \left\{ w = (u_0, \dots, u_T) : u_{x,0} = u_{x,1} + \alpha_{0x},\ x = 1, \dots, T-1 \right\},$$
$$P_{\beta_0} = \left\{ w = (u_0, \dots, u_T) : u_{0,y} = u_{1,y} + \beta_{0y},\ y = 1, \dots, \Lambda - 1 \right\},$$
$$P_{\alpha_\Lambda} = \left\{ w = (u_0, \dots, u_T) : u_{x,\Lambda} = u_{x,\Lambda-1} + \alpha_{\Lambda x},\ x = 1, \dots, T-1 \right\},$$
$$P_{\beta_T} = \left\{ w = (u_0, \dots, u_T) : u_{T,y} = u_{T-1,y} + \beta_{Ty},\ y = 1, \dots, \Lambda - 1 \right\}.$$

Then it is clear that the given problem (PD) is equivalent to the following convex minimization problem

$$\text{minimize}\, g(w) \equiv \sum\nolimits_{\substack{x=1,.,T-1;\\ y=1,.,\Lambda-1}} g\left(u_{x,y}, x, y\right),$$

$$w \in N = \left[\bigcap\nolimits_{\substack{x=1,\ldots,T-1\\ y=1,\ldots,\Lambda-1}} M_{x,y}\right] \cap P_{\alpha_0} \cap P_{\beta_0} \cap P_{\alpha_\Lambda} \cap P_{\beta_T}. \tag{11}$$

Now we can apply to problem (11) the results of convex programming [21], for which, in turn, it is necessary to calculate the dual cones

$$K^*_{M_{x,y}}(w),\ K^*_{P_{\alpha_0}}(w),\ K^*_{P_{\beta_0}}(w),\ K^*_{P_{\alpha_\Lambda}}(w),\ K^*_{P_{\beta_T}}(w),\ w \in N.$$

Therefore, we calculate the corresponding cones of tangent directions. So

$$\begin{aligned} K_{M_{x,y}}(w) = \{\bar{w} = (\bar{u}_0, \ldots, \bar{u}_T) : (\bar{u}_{x-1,y}, \bar{u}_{x,y-1}, \bar{u}_{x,y}, \bar{u}_{x,y+1}, \bar{u}_{x+1,y}) \\ \in K_F\left(u_{x-1,y}, u_{x,y-1}, u_{x,y}, u_{x,y+1}, u_{x+1,y}\right)\}, \end{aligned} \tag{12}$$

where the components $\bar{u}_{i,j}$ of the vector $\bar{w}$ are arbitrary. Then considering (12) we can easily calculate the dual cone $K^*{}_{M_{X,y}}(w)$ to the cone $K_{M_{x,y}}(w)$ as follows

$$\begin{aligned} K^*_{M_{x,y}}(w) = \big\{w^* = (u_0^*, \ldots, u_T^*) : \big(u^*_{x-1,y}, u^*_{x,y-1}, u^*_{x,y}, u^*_{x,y+1}, u^*_{x+1,y}\big) \\ \in K^*_F\left(u_{x-1,y}, u_{x,y-1}, u_{x,y}, u_{x,y+1}, u_{x+1,y}\right),\ u^*_{i,j} = 0,\ (i,j) \neq (x-1,y), \\ (x, y-1), (x,y), (x,y+1), (x+1,y)\},\ x = 1, \ldots, T-1;\ y = 1, \ldots, \Lambda - 1, \end{aligned}$$

where we took into account that the components $\bar{u}_{i,j}, (i,j) \neq (x-1,y)$, $(x,y-1)$, (x,y), $(x,y+1)$, $(x+1,y)$ of the vector $\bar{w}$ are arbitrary. Now, to calculate the dual cones to the cone of tangent directions of sets $P_{\alpha_0}, P_{\beta_0}, P_{\alpha_\Lambda}, P_{\beta_T}$, we must apply Proposition 3.1 [23]. Let's calculate, for example $K^*_{P_{\alpha_0}}(w)$; on this Proposition 3.1, it is easy to show that

$$K^*_{P_{\alpha_0}}(w) = \big\{w^* = (u_0^*, \ldots, u_T^*) : u^*_{x,y} = 0, y \neq 0, 1, x = 1, \ldots, T-1\big\}. \tag{13}$$

By analogy it can be seen that

$$\begin{aligned} K^*_{P_{\beta_0}}(w) &= \big\{w^* = (u_0^*, \ldots, u_T^*) : u^*_{x,y} = 0, x \neq 0, 1, y = 1, \ldots, \Lambda - 1\big\}, \\ K^*_{P_{\alpha_\Lambda}}(w) &= \big\{w^* = (u_0^*, \ldots, u_T^*) : u^*_{x,y} = 0, y \neq \Lambda - 1, \Lambda, x = 1, \ldots, T-1\big\}, \\ K^*_{P_{\beta_T}}(w) &= \big\{w^* = (u_0^*, \ldots, u_T^*) : u^*_{x,y} = 0, x \neq T-1, T, y = 1, \ldots, \Lambda - 1\big\}. \end{aligned} \tag{14}$$

Further we conclude from the condition of the theorem that $g(w)$ is continuous at the point $w^0 = (\hat{u}_0, \ldots, \hat{u}_T)$ and $\tilde{w} = (\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_T)$ is a solution to the convex problem (11). Then by Theorem 3.4 [21, p. 99] there exist vectors

$$\begin{gathered} w^*(x,y) \in K^*_{M_{x,y}}(\tilde{w}), \tilde{w}^* \in K^*_{P_{\alpha_0}}(\tilde{w}), \hat{w}^* \in K^*_{P_{\beta_0}}(\tilde{w}), \\ w^{\Lambda^*} \in K^*_{P_{\alpha_\Lambda}}(\tilde{w}), w^{T^*} \in K^*_{P_{\beta_T}}(\tilde{w}), w^{0^*} \in \partial_w g(\tilde{w}) \end{gathered}$$

and scalar α equal to zero or one, such that

$$\sum_{\substack{x=1,\ldots,T-1;\\ y=1,\ldots,\Lambda-1}} w^*(x,y) + \tilde{w}^* + \hat{w}^* + w^{\Lambda^*} + w^{T^*} = \alpha w^{0^*}, \tag{15}$$

while α and the indicated vectors are not equal to zero at the same time, moreover, for $\alpha = 1$, these conditions are sufficient condition for optimality.

Now, using relations (13), (14), we write (15) in component-wise form

$$u^*_{x,y}(x+1,y)+u^*_{x,y}(x,y+1)+u^*_{x,y}(x,y)+u^*_{x,y}(x,y-1)+u^*_{x,y}(x-1,y)=\alpha u^{0^*}_{x,y},$$
$$u^*_{1,y}(0,y)-u^*_{0,y}(1,y)=0, u^*_{x,1}(x,0)-u^*_{x,0}(x,1)=0,$$
$$u^*_{x,\Lambda-1}(x,\Lambda)-u^*_{x,\Lambda}(x,\Lambda-1)=0,\ u^*_{T-1,y}(T,y)-u^*_{T,y}(T-1,y)=0,$$
$$x=1,\dots,T-1,\ y=1,\dots,\Lambda-1. \tag{16}$$

By LAM, see Definition 2.1, and the formula for $K^*{}_{M_{x,y}}(\tilde{w})$, we obtain

$$\begin{gathered}\left(u^*_{x-1,y}(x,y),u^*_{x,y-1}(x,y),u^*_{x,y}(x,y),u^*_{x,y+1}(x,y)\right)\\ \in F^*\left(-u^*_{x+1,y}(x,y),\left(\tilde{u}_{x-1,y},\tilde{u}_{x,y-1},\tilde{u}_{x,y},\tilde{u}_{x,y+1},\tilde{u}_{x+1,y}\right),x,y\right),\\ x=1,\dots,T-1,\ y=1,\dots,\Lambda-1.\end{gathered} \tag{17}$$

Hence, after introducing the new notations $u^*_{x-1,y}(x,y)=\psi^*_{x,y}$, $u^*_{x,y-1}(x,y)=\xi^*_{x,y}$, $u^*_{x,y+1}(x,y)=\eta^*_{x,y}$, $-u^*_{x+1,y}(x,y)=u^*_{x,y}$ from (16) and (17) we find that the necessary condition of the theorem is satisfied. On the other hand, it follows from Conditions B that the representation (15) is satisfied with the parameter $\alpha=1$ for the point $w^{0^*}\in\partial_w g(\tilde{w})\cap K^*_N(\tilde{w})$. Therefore, conditions (i) and (ii) are sufficient for the optimality of $\{\tilde{u}_{x,y}\}_D$. This completes the proof of the theorem □

Remark 3.2. If in a convex problem (PD) the functions and set-valued mappings are polyhedral, i.e., the graph of a set-valued mapping F is the algebraic sum of some polytope and a cone, then Conditions B is redundant. This follows from the fact that in the proof of the above-mentioned Theorem 3.4 [21, p.99], by virtue of Lemma 1.22 [21, p. 23], on the calculation of the dual cone to the intersection of a finite number of polyhedral cones, the execution of which guarantees that $\alpha=1$.

Theorem 3.3. *Let Conditions* A *be satisfied for a nonconvex problem* (P_D). *Then for the optimality of* $\{\tilde{u}_{x,y}\}_D$ *in this nonconvex problem it is necessary that there exist a number* $\alpha\in\{0,1\}$ *and vectors* $\left\{\psi^*_{x,y},\eta^*_{x,y},\xi^*_{x,y},u^*_{x,y}\right\}$ *not all equal to zero satisfying conditions* (i), (ii) *of Theorem* 3.1.

Proof. In this case, Theorem 3.25 [21, p. 134] is used instead of Theorem 3.4 [21, p. 99], where Conditions A again ensure that the relation (15) is satisfied for the nonconvex problem (PD), from which the necessary condition follows, as in Theorem 3.1. □

4. Optimization of a discrete-approximate problem to a continuous problem (PC) with the Neumann boundary condition

For the construction of a discrete-approximate problem for the problem (PC) we choose the steps δ and h on the x and y axes, respectively, using the grid functions $u_{x,y}=u_{\delta h}(x,y)$ on a uniform grid on Q. Denoting $A_1u=\partial^2u/\partial x^2, A_2u=\partial^2u/\partial y^2$ we approximate the Laplace operator $\Delta u=A_1u+A_2u$ using difference operators

$$B_1u(x,y)=\frac{u(x+\delta,y)-2u(x,y)+u(x-\delta,y)}{\delta^2},$$
$$B_2u(x,y)=\frac{u(x,y+h)-2u(x,y)+u(x,y-h)}{h^2}.$$

Then, in accordance with (PC), we obtain the following approximating Neumann boundary value problem:

$$\text{(PA)} \qquad \text{minimize } J_{\delta h}[u(x,y)] := \sum_{(x,y)\in\overline{\Omega}_{\delta h}} \delta h g(u(x,y),x,y),$$
$$B_1 u(x,y) + B_2 u(x,y) \in F(u(x,y),x,y),\ (x,y)\in\overline{\Omega}_{\delta h},$$
$$u(0,y) = u(\delta,y) + \delta\beta_{0x_2};\ u(T,y) = u(T-\delta,y) + \delta\beta_{Ty};$$
$$u(x,0) = u(x,h) + h\alpha_{0x};\ u(x,\Lambda) = u(x,\Lambda-h,) + h\alpha_{\Lambda x},\ (x,y)\in\ell_{\delta h},$$

where $\overline{\Omega}_{\delta h} = \{(x,y) : x = 0,\delta,...,T;\ y = 0,h,...,\Lambda,\ (x,y)\neq(0,0),(0,\Lambda),(T,0),(T,\Lambda)\}$ and $\ell_{\delta h}$ is the set of its boundary points.

To reduce problem (PA) to a problem of the form (PD), we introduce a new mapping $R(\cdot,x,y) : \mathbb{R}^{4n} \rightrightarrows \mathbb{R}^n$

$$R(u_1,u_2,u_3,u_4,x,y) = 2(1+\sigma)u_3 - u_1 - \sigma(u_4+u_2) + \delta^2 F(u_3,x,y),\sigma = \delta^2/h^2 \quad (18)$$

and we rewrite problem (PA) as follows:

$$\text{minimize } J_{\delta h}[u(x,y)], \quad (19)$$
$$u(x+\delta,y) \in R(u(x-\delta,y),u(x,y-h),u(x,y),u(x,y+h),x,y),(x,y)\in\overline{\Omega}_{\delta h},$$
$$u(0,y) = u(\delta,y) + \delta\beta_{0x_2};\ u(T,y) = u(T-\delta,y) + \delta\beta_{Ty}; \quad (20)$$
$$u(x,0) = u(x,h) + h\alpha_{0x};\ u(x,\Lambda) = u(x,\Lambda-h,) + h\alpha_{\Lambda x},(x,y)\in\ell_{\delta h}.$$

Following Theorem 3.1, for problem (19), (20) to be optimal at $\{\tilde{u}(x,y)\}_{\bar{\Omega}_{\delta h}}$, it is necessary that there are vectors $\{u^*(x,y),\psi^*(x,y),\xi^*(x,y),\eta^*(x,y)\}$ and numbers $\alpha = \alpha_{\delta h}\in\{0,1\}$ not all zero such that

$$\begin{gathered}(\psi^*(x,y),\xi^*(x,y),u^*(x-\delta,y),\eta^*(x,y))\\ \in R^*\left(u^*(x,y);(\tilde{u}(x-\delta,y),\tilde{u}(x,y-h),\tilde{u}(x,y),\tilde{u}(x,y+h),\tilde{u}(x+\delta,y)),x,y\right) \quad (21)\\ +\{0\}\times\{0\}\times\{\psi^*(x+\delta,y)+\xi^*(x,y+h)+\eta^*(x,y-h)-\lambda\partial g(\tilde{u}(x,y),x,y)\}\\ \times\{0\};\ u^*(0,y)+\psi^*(\delta,y)=0;\eta^*(x,0)-\xi^*(x,h)=0;\xi^*(x,\Lambda)-\eta^*(x,\Lambda-h)\\ =0;\ \psi^*(T,y)+u^*(T-\delta,y)=0;\ x=\delta,\dots,T-\delta;\ y=h,\dots,\Lambda-h.\end{gathered}$$

The main problem in (21) is to express LAM R^* in terms of F^*. Just the equivalence Theorem 3 [26] makes it possible to express R^* in terms of F^* and the above condition (21) has the form

$$\begin{gathered}\frac{u^*(x-\delta,y)-\psi^*(x+\delta,y)-\xi^*(x,y+h)-\eta^*(x,y-h)-2(1+\sigma)u^*(x,y)}{\delta^2}\\ \in F^*\left(u^*(x,y);(\tilde{u}(x,y),B_1\tilde{u}(x,y)+B_2\tilde{u}(x,y),x,y\right)-\alpha\partial g(\tilde{u}(x,y),x,y)\ , \quad (22)\\ u^*(x,y)=-\psi^*(x,y),\quad \xi^*(x,y)=\eta^*(x,y)=-\sigma u^*(x,y),\\ \sigma=\delta^2/h^2,\quad x=\delta,\dots,T-\delta;\quad y=h,\dots,\Lambda-h, \quad (23)\\ u^*(0,y)+\psi^*(\delta,y)=0,\eta^*(x,0)-\xi^*(x,h)=0,\quad \xi^*(x,\Lambda)-\eta^*(x,\Lambda-h)=0,\\ \psi^*(T,y)+u^*(T-\delta,y)=0,x=\delta,\dots,T-\delta;y=h,\dots,\Lambda-h.\end{gathered}$$

It is easy to check that the left-hand side of the discrete-approximation inclusion (22) has the following form, which is useful for what follows

$$\frac{1}{\delta^2}\left[u^*(x-\delta,y)+u^*(x+\delta,y)+\sigma\left(u^*(x,y+h)+u^*(x,y-h)\right)-2(1+\sigma)u^*(x,y)\right]$$
$$=\frac{u^*(x+\delta,y)-2u^*(x,y)+u^*(x-\delta,y)}{\delta^2}+\frac{u^*(x,y+h)-2u^*(x,y)+u^*(x,y-h)}{h^2}. \tag{24}$$

At the same time, from the boundary conditions (23) we get

$$u^*(0,y)-u^*(\delta,y)=0;\ u^*(x,h)-u^*(x,0)=0;\ u^*(x,\Lambda-h)-u^*(x,\Lambda)=0;$$
$$u^*(T-\delta,y)-u^*(T,y)=0;\ x=\delta,\dots,T-\delta;\ y=h,\dots,\Lambda-h. \tag{25}$$

Taking into account the relations (22), (24), and (25) divided by δ and h, respectively, we can formulate the following result for the problem (PA).

Theorem 4.1. *Let $g(\cdot,x,y)$ be a convex proper function and Condition* B *is satisfied. Then, for the optimality of the solution $\{\tilde{u}(x,y)\}_{\bar{\Omega}_{\delta h}}$ in the convex problem* (PA)*, it is necessary and sufficient that there exist functions $\{u^*(x,y)\}_{\bar{\Omega}_{\delta h}}$, such that*

(i) $B_1u^*(x,y)+B_2u^*(x,y)$
$\in F^*\left(u^*(x,y);(\tilde{u}(x,y),B_1\tilde{u}(x,y)+B_2\tilde{u}(x,y)),x,y\right)-\partial g(\tilde{u}(x,y),x,y),$

(ii) $\dfrac{u^*(\delta,y)-u^*(0,y)}{\delta}=0,\ \dfrac{u^*(x,h)-u^*(x,0)}{h}=0,\ \dfrac{u^*(x,\Lambda)-u^*(x,\Lambda-h)}{h}=0,$
$\dfrac{u^*(T,y)-u^*(T-\delta,y)}{\delta}=0,\ x=\delta,\dots,T-\delta,\ y=h,\dots,\Lambda-h.$

Proof. It remains only to show that under the Conditions B we have $\alpha=1$. Indeed, this follows from the fact that in the proof, as we have seen, the essential is Theorem 3.4 [21, p. 99], in turn, in the proof of which, by virtue of Theorem 1.30 [21, p. 47], the dual cone to the intersection of a finite number of cones with their relative interior is simply the algebraic sum of the dual cones, which provides $\alpha=1$. □

Remark 4.2. As in Theorem 3.1 the conditions (i), (ii) of the Theorem 4.1 are necessary for optimality in the nonconvex case of the problem (PA) under the Conditions A.

5. Sufficient conditions of optimality for elliptic DFIs with Neumann boundary conditions

In this section, using Theorems 4.1, we formulate a sufficient condition for the optimality of a continuous problem (PC). To do this, we pass to the formal limit as $\delta,h\to 0$ in condition (i) and (ii) of Theorem 4.1, as a result of which we obtain

(a) $\Delta u^*(x,y)\in F^*(u^*(x,y),(\tilde{u}(x,y),\Delta\tilde{u}(x,y)),x,y)-\partial g(\tilde{u}(x,y),x,y),\ (x,y)\in Q,$

(b) $\dfrac{\partial u^*(x,y)}{\partial v}=0,\ (x,y)\in\partial Q,$

where ∂Q is the boundary of the region Q. We add here the following condition (c), which ensures non-emptiness of LAM for each fixed $(x,y)\in Q$ (Theorem 2.1 [21, p. 62]):

(c) $\Delta\tilde{u}(x,y)\in F\left(\tilde{u}(x,y),u^*(x,y),x,y\right).$

Theorem 5.1. *Let $g(\cdot,x,y)$ be a continuous proper convex function and $F(\cdot,x,y)$ be a convex set-valued mapping. Then, for a solution $\tilde{u}(x)$ to be optimal, it suffices that there exists a classical solution $u^*(x,y)$ such that conditions* (a)–(c) *hold.*

Proof. From condition (a) by Theorem 2.1 [21, p. 62] we get

$$\Delta u^*(x,y) \in \partial_u \left[H\left(\tilde{u}(x,y), u^*(x,y), x, y\right) - g(\tilde{u}(x,y),x,y)\right], \ (x,y)\in Q$$

or

$$H\left(u(x,y), u^*(x,y), x, y\right) - H\left(\tilde{u}(x,y), u^*(x,y), x, y\right)$$
$$-g(u(x,y),x,y) + g(\tilde{u}(x,y),x,y) \leq \left\langle \Delta u^*(x,y), u(x,y)-\tilde{u}(x,y)\right\rangle.$$

Then, taking into account condition (c) of the theorem and the definition of the Hamilton function, we obtain

$$g(u(x,y),x,y) - g(\tilde{u}(x,y),x,y)$$
$$\geq \left\langle \Delta(u(x,y)-\tilde{u}(x,y)), u^*(x,y)\right\rangle - \left\langle u(x,y)-\tilde{u}(x,y), \Delta u^*(x,y)\right\rangle. \tag{26}$$

On the other hand, it is easy to verify the validity of the following relations

$$\left\langle u^*(x,y), \frac{\partial^2(u(x,y)-\tilde{u}(x,y))}{\partial x^2}\right\rangle - \left\langle \frac{\partial^2 u^*(x,y)}{\partial x^2}, u(x,y)-\tilde{u}(x,y)\right\rangle$$
$$= \frac{\partial}{\partial x}\left\langle u^*(x,y), \frac{\partial(u(x,y)-\tilde{u}(x,y))}{\partial x}\right\rangle - \frac{\partial}{\partial x}\left\langle \frac{\partial u^*(x,y)}{\partial x}, u(x,y)-\tilde{u}(x,y)\right\rangle, \tag{27}$$

$$\left\langle u^*(x,y), \frac{\partial^2(u(x,y)-\tilde{u}(x,y))}{\partial y^2}\right\rangle - \left\langle \frac{\partial^2 u^*(x,y)}{\partial y^2}, u(x,y)-\tilde{u}(x,y)\right\rangle$$
$$= \frac{\partial}{\partial y}\left\langle u^*(x,y), \frac{\partial(u(x,y)-\tilde{u}(x,y))}{\partial y}\right\rangle - \frac{\partial}{\partial y}\left\langle \frac{\partial u^*(x,y)}{\partial y}, u(x,y)-\tilde{u}(x,y)\right\rangle. \tag{28}$$

Therefore, from (26)–(28), we have

$$g(u(x,y),x,y) - g(\tilde{u}(x,y),x,y)$$
$$\geq \frac{\partial}{\partial x}\left\langle u^*(x,y), \frac{\partial(u(x,y)-\tilde{u}(x,y))}{\partial x}\right\rangle - \frac{\partial}{\partial x}\left\langle \frac{\partial u^*(x,y)}{\partial x}, u(x,y)-\tilde{u}(x,y)\right\rangle$$
$$+ \frac{\partial}{\partial y}\left\langle u^*(x,y), \frac{\partial(u(x,y)-\tilde{u}(x,y))}{\partial y}\right\rangle - \frac{\partial}{\partial y}\left\langle \frac{\partial u^*(x,y)}{\partial y}, u(x,y)-\tilde{u}(x,y)\right\rangle. \tag{29}$$

As a result of the integrating inequality (29) obtain

$$\iint_Q [g(u(x,y),x,y) - g(\tilde{u}(x,y),x,y)]dxdy$$
$$\geq \int_0^\Lambda \left[\left\langle u^*(T,y), \frac{\partial(u(T,y)-\tilde{u}(T,y))}{\partial x}\right\rangle - \left\langle \frac{\partial u^*(T,y)}{\partial x}, u(T,y)-\tilde{u}(T,y)\right\rangle\right] dy$$
$$- \int_0^\Lambda \left[\left\langle u^*(0,y), \frac{\partial(u(0,y)-\tilde{u}(0,y))}{\partial x}\right\rangle - \left\langle \frac{\partial u^*(0,y)}{\partial x}, u(0,y)-\tilde{u}(0,y)\right\rangle\right] dy$$
$$+ \int_0^T \left[\left\langle u^*(x,\Lambda), \frac{\partial(u(x,\Lambda)-\tilde{u}(x,\Lambda))}{\partial y}\right\rangle - \left\langle \frac{\partial u^*(x,\Lambda)}{\partial y}, u(x,\Lambda)-\tilde{u}(x,\Lambda)\right\rangle\right] dx$$
$$- \int_0^T \left[\left\langle u^*(x,0), \frac{\partial(u(x,0)-\tilde{u}(x,0))}{\partial y}\right\rangle - \left\langle \frac{\partial u^*(x,0)}{\partial y}, u(x,0)-\tilde{u}(x,0)\right\rangle dx. \tag{30}$$

Now we recall that $u(x,t), \tilde{u}(x,t)$ are feasible solutions and satisfy the Neumann boundary conditions

$$u_y(x,0) = \tilde{u}_y(x,0) = \alpha_0(x), \ u_y(x,\Lambda) = \tilde{u}_y(x,\Lambda) = \alpha_\Lambda(x),$$
$$u_x(0,y) = \tilde{u}_x(0,y) = \beta_0(y), \ u_x(T,y) = \tilde{u}_x(T,y) = \beta_T(y).$$

Hence from (30) it follows that

$$\iint\limits_Q [g(u(x,y),x,y) - g(\tilde{u}(x,y),x,y)]dxdy$$
$$\geq \int_0^\Lambda \left\langle \frac{\partial u^*(0,y)}{\partial x}, u(0,y) - \tilde{u}(0,y) \right\rangle dy - \int_0^L \left\langle \frac{\partial u^*(T,y)}{\partial x}, u(T,y) - \tilde{u}(T,y) \right\rangle dy$$
$$+ \int_0^T \left[\left\langle \frac{\partial u^*(x,0)}{\partial y}, u(x,0) - \tilde{u}(x,0) \right\rangle \right] dx - \int_0^T \left\langle \frac{\partial u^*(x,\Lambda)}{\partial y}, u(x,\Lambda) - \tilde{u}(x,\Lambda) \right\rangle dx.$$

But, on the other hand, according to the "adjoint" boundary conditions (b) of the theorem $u^*_x(0,y) = u^*_x(T,y) = u^*_y(x,0) = u^*_y(x,\Lambda) = 0$, from the last inequality we obtain that

$$\iint\limits_Q g(u(x,y),x,y)dxdy \ \geq \iint\limits_Q g(\tilde{u}(x,y),x,y)dxdy$$

for all feasible solutions $u(x,y)$. The proof of the theorem is completed. □

Let us now consider a generalization of the Neumann problem (4)-(6) to the case of an arbitrary bounded $G \subset \mathbb{R}^2$, instead of the rectangle Q :

$$\text{minimize } J[u(x,y)] := \iint\limits_G g(u(x,y),x,y)dxdy,$$
$$\Delta u(x,y) \in F(u(x,y),x,y), (x,y) \in G \subset \mathbb{R}^2, \tag{31}$$
$$\frac{\partial u(x,y)}{\partial v} = \alpha(x,y), (x,y) \in S,$$

where Δ is Laplace's operator as before, $F(\cdot,x,y) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping, $G \subset \mathbb{R}^2$ is bounded region, which has a closed piecewise-smooth simple curve S as its boundary, $g(\cdot,x,y) : \mathbb{R}^n \to \mathbb{R}^1$ is a continuous convex function, $\alpha(x,y)$ is continuous.

Corollary 5.2. *For the optimality of the solution $\tilde{u}(x,y)$ among all feasible solutions in convex problem* (31) *it suffices that there exists a classical solution $u^*(x,y)$ such that conditions* (a)–(c) *of Theorem* 5.1 *are satisfied, rewritten in terms of G and S.*

Proof. Recall that the integrated inequality (26) over the set G has the form

$$\iint\limits_G [g(u(x,y),x,y) - g(\tilde{u}(x,y),x,y)]dxdy$$
$$\geq \iint\limits_G \left[\langle \Delta(u(x,y) - \tilde{u}(x,y)), u^*(x,y)\rangle - \langle u(x,y) - \tilde{u}(x,y), \Delta u^*(x,y)\rangle \right] dxdy.$$

To calculate the integral of the right-hand side of this inequality in this general case, we use the familiar Green's theorem. By applying this theorem, we have

$$\iint_G [\langle \Delta(u(x,y)-\tilde{u}(x,y)), u^*(x,y)\rangle - \langle u(x,y)-\tilde{u}(x,y), \Delta u^*(x,y)\rangle]\, dxdy \tag{32}$$
$$= \int_S \left[\left\langle \frac{\partial(u(x,y)-\tilde{u}(x,y))}{\partial v}, u^*(x,y)\right\rangle - \left\langle u(x,y)-\tilde{u}(x,y), \frac{\partial u^*(x,y)}{\partial v}\right\rangle\right] ds,$$

where ds is a symbolic arc length element and v is the unit outward normal vector to the curve S. By the condition of the theorem we have $\partial u^*(x,y)/\partial v = 0$, $(x,y) \in S$ and then from the relation (32) we obtain

$$\iint_G [\langle \Delta(u(x,y)-\tilde{u}(x,y)), u^*(x,y)\rangle - \langle u(x,y)-\tilde{u}(x,y), \Delta u^*(x,y)\rangle]\, dxdy$$
$$= \int_S \left\langle \frac{\partial(u(x,y)-\tilde{u}(x,y))}{\partial v}, u^*(x,y)\right\rangle ds. \tag{33}$$

On the other hand, since $u(x,y)$ and $\tilde{u}(x,y)$ are feasible solutions, we conclude that $\partial u(x,y)/\partial n, = \partial\tilde{u}(x,y)/\partial n = \alpha(x,y)$ for $(x,y) \in S$ and the right side of equality (33) is equal to zero, which means that

$$\iint_G g(u(x,y),x,y)dxdy \geq \iint_G g(\tilde{u}(x,y),x,y)dxdy$$

for arbitrarily feasible solutions $u(x,y), (x,y) \in G$. The theorem is proved. □

Corollary 5.3. *If $F(\cdot,x,y)$ is a closed mapping, then the conditions* (a), (c) *of Corollary* 5.2 *can be rewritten as follows*

(i) $\Delta u^*(x,y) \in \partial_u H(\tilde{u}(x,y), u^*(x,y), x, y) - \partial g(\tilde{u}(x,y), x, y)$,

(ii) $\Delta\tilde{u}(x,y) \in \partial_{v^*} H(\tilde{u}(x,y), u^*(x,y), x, y)$.

Proof. Indeed, by Theorem 2.1 [21, p. 62] and Theorem 1.31 [21, p. 47] the following relations are correct:

$$F^*(v^*, (u,v), x, y) = \partial_u H(u, v^*, x, y), v \in F(u, v^*, x, y),$$
$$\partial_{v^*} H(u, v^*, x, y) = F(u, v^*, x, y).$$

Hence conditions (i), (ii) of corollary and (a), (c) of Corollary 5.2 are equivalent. □

6. Multidimensional optimal control problem for elliptic DFIs with Neumann boundary conditions

Here we investigate the posed problem (PM) in Section 2 with elliptic operator L.

Theorem 6.1. *Suppose that $g(\cdot,x)$ is a continuous proper convex function and that $F(\cdot,x) : \mathbb{R}^1 \rightrightarrows \mathbb{R}^1$ is a convex closed mapping. Then for optimality of a solution $\tilde{u}(x)$ in the problem (PM) it is sufficient that there exists a classical solution $u^*(x)$ of the following problem:*

(i) $L^*u^*(x) \in F^*\left(u^*(x);(\tilde{u}(x), L\tilde{u}(x)), x\right) - \partial g((\tilde{u}(x), x),\ x \in G,$

(ii) $L\tilde{u}(x) \in F\left(\tilde{u}(x), u^*(x), x\right), \dfrac{\partial u^*(x)}{\partial v} = 0,\ x \in S,$

where L^* is the operator adjoint to L:

$$L^*u^*(x) := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u^*(x)}{\partial x_j}\right) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(b_i(x)u^*(x)\right) + c(x)u^*(x).$$

Proof. On the basis of condition (i) of the theorem we have

$$\begin{aligned} &H\left(u(x), u^*(x), x\right) - H\left(\tilde{u}(x), u^*(x), x\right) \\ &\leq g(u(x), x) - g(\tilde{u}(x), x) + L^*u^*(x) \cdot (u(x) - \tilde{u}(x)), \end{aligned}$$

where due to (ii) $H\left(\tilde{u}(x), u^*(x), x\right) = u^*(x) \cdot L\tilde{u}(x)$ and

$$\begin{aligned} &\int_G [g(u(x), x) - g(\tilde{u}(x), x)]dx \\ &\geq \int_G \left[u^*(x)L(u(x) - \tilde{u}(x))dx - (u(x) - \tilde{u}(x))L^*u^*(x)\right] dx. \end{aligned} \tag{34}$$

Then using the Green's formula for a multidimensional case with second order elliptic operator we derive that

$$\begin{aligned} &\int_G \left[u^*(x)L(u(x) - \tilde{u}(x)) - (u(x) - \tilde{u}(x))L^*u^*(x)\right] dx \\ &= \int_S \left[u^*(x)\frac{\partial(u(x) - \tilde{u}(x))}{\partial v} - (u(x) - \tilde{u}(x))\frac{\partial u^*(x)}{\partial v} + M(u(x) - \tilde{u}(x))u^*(x)\right] ds, \end{aligned}$$

where $M = \sum_{i=1}^{i=n} v_i b_i$, $\partial/\partial v = \sum_{i,j=1}^{n} a_{ij} v_j \partial/\partial x_i$ is the operator of differentiation in the direction of the so-called co-normal

$$v = \left(\sum_{j=1}^{j=n} a_{1j}v_j, \ldots, \sum_{j=1}^{j=n} a_{nj}v_j\right)$$

of the operator L. Using the boundary conditions $\partial u(x)/\partial v = \partial\tilde{u}(x)/\partial v = \alpha(x)$, $x \in S$, and the Neumann condition (ii) $\partial u^*(x)/\partial v = 0$, $x \in S$, it follows that the right hand side of the inequality (34) is equal to zero, which means that $J[u(x)] \geq J[\tilde{u}(x)]$ for all feasible solutions in problem(36). The theorem is proved. □

Remark 6.2. The problem for Neumann type elliptic DFIs (PM) can be extended to problem with vector-valued elliptic operators Lu acting on vector-valued functions $u = u\left(x_1, \ldots, x_n\right)$. In this case, in the proof of Theorem 6.1, in all calculations, the ordinary products of functions are replaced by inner products.

Corollary 6.3. *For the optimality of $\tilde{u}(x)$ to the nonconvex problem* (PM) *it is sufficient that there exists a classical solution $u^*(x)$ that satisfies the conditions:*

(i) $L^*u^*(x) + u^*(x) \in F^*\left(u^*(x), (\tilde{u}(x), L\tilde{u}(x)), x\right),\ u^*(x) = 0,\ x \in S,$

(ii) $u^*(x) \cdot L\tilde{u}(x) = H\left(\tilde{u}(x), u^*(x), x\right),$

(iii) $g(u, x) - g(\tilde{u}(x), x) \geq u^*(x)(u - \tilde{u}(x)) \quad \forall u.$

Proof. It follows from the condition (i) of theorem and definition of LAM that for any feasible solution

$$H\left(u(x,y),u^*(x,y),x,y\right)-H\left(\tilde{u}(x,y),u^*(x,y),x,y\right)$$
$$\leq\left(L^*u^*(x)+u^*(x)\right)\left(u(x)-\tilde{u}(x)\right),(x,y)\in G,$$

whence the condition (ii) gives

$$u^*(x)L(u(x)-\tilde{u}(x))\leq\left(L^*u^*(x)+u^*(x)\right)\left(u(x)-\tilde{u}(x)\right). \tag{35}$$

Then from the condition (iii) and from inequality (35) we obtain

$$g(u(x),x)-g(\tilde{u}(x),x)\geq u^*(x)L(u(x)-\tilde{u}(x))$$
$$-(u(x)-\tilde{u}(x))L^*u^*(x),x\in G. \tag{36}$$

Integrating inequality (36) the further proof of the corollary is similar to the proof of Theorem 6.1. □

7. Optimization of elliptic DFIs with mixed Dirichlet and Neumann conditions

In this section, using the results of Sections 5 and 6 consider problems with mixed boundary conditions; first consider the following problem in rectangular region with mixed Dirichlet and Neumann type boundary conditions:

$$\text{minimize } J[u(x,y)]:=\iint\limits_Q g(u(x,y),x,y)dxdy,$$
$$\Delta u(x,y)\in F(u(x,y),x,y),(x,y)\in Q=[0,T]\times[0,\Lambda],F(\cdot x,y):\mathbb{R}^n\rightrightarrows\mathbb{R}^n,$$
$$u(x,0)=\alpha_0(x),\ u(0,y)=\beta_0(y),\ \frac{\partial u(x,\Lambda)}{\partial y}=\alpha_\Lambda(x),\ \frac{\partial u(T,y)}{\partial x}=\beta_T(y), \tag{37}$$
$$\alpha_0(T)=\beta_T(0),\ \alpha_0(0)=\beta_0(0),\ \alpha_\Lambda(0)=\beta_0(\Lambda),\ \alpha_\Lambda(T)=\beta_T(\Lambda),$$

where for definiteness on the left and lower sides of the rectangle Q Dirichlet type conditions are satisfied, and on the upper and right sides of the rectangle Neumann type conditions are satisfied.

Theorem 7.1. *Let $g(\cdot,x,y)$ be a continuous proper convex function and $F(\cdot,x,y)$ be a convex set-valued mapping. Then for the optimality of the solution $\tilde{u}(x,y)$ in the mixed problem* (37) *it is sufficient that there exists a classical solution $u^*(x,y)$ such that the conditions* (a), (c) *of Theorem* 5.1 *hold with following adjoint boundary conditions:*

$$\frac{\partial u^*(x,\Lambda)}{\partial y}=\frac{\partial u^*(T,y)}{\partial x}=0,\ u^*(x,0)=u^*(0,y)=0.$$

Proof. We recall that the inequality (30) is obtained under conditions (a), (c) of Theorem 5.1 and without condition (b) of this theorem. Now for the proof of the theorem it is enough to show that, with the initial-boundary condition of the problem (26) and the adjoint initial boundary condition of the theorem, the right hand side of the inequality (30) is zero.

Indeed, substituting $u_y^*(x,\Lambda) = u_x^*(T,y) = 0$ and $u^*(x,0) = u^*(0,y) = 0$ into the relations (30) we have

$$\iint\limits_Q [g(u(x,y),x,y) - g(\tilde{u}(x,y),x,y)]dxdy \tag{38}$$
$$\geq \int_0^\Lambda \left\langle u^*(T,y), \frac{\partial u(T,y)}{\partial x} - \frac{\partial \tilde{u}(T,y)}{\partial x} \right\rangle dy + \int_0^\Lambda \left\langle \frac{\partial u^*(0,y)}{\partial x}, u(0,y) - \tilde{u}(0,y) \right\rangle dy$$
$$+ \int_0^T \left\langle u^*(x,\Lambda), \frac{\partial u(x,\Lambda)}{\partial y} - \frac{\partial \tilde{u}(x,\Lambda)}{\partial y} \right\rangle dx + \int_0^T \left\langle \frac{\partial u^*(x,0)}{\partial y}, u(x,0) - \tilde{u}(x,0) \right\rangle dx.$$

In turn, remember that both $u(x,y), \tilde{u}(x,y)$ are feasible solutions and the following relations are satisfied:

$$\begin{gathered} u(x,0) = \tilde{u}(x,0) = \alpha_0(x), u(0,y) = \tilde{u}(0,y) = \beta_0(y), \\ u_y(x,\Lambda) = \tilde{u}_y(x,\Lambda) = \alpha_\Lambda(x), u_x(T,y) = \tilde{u}_x(T,y) = \beta_T(y). \end{gathered} \tag{39}$$

Then taking into account (39) in the inequality (38) immediately we get the needed result, that is

$$\iint\limits_Q [g(u(x,y),x,y) - g(\tilde{u}(x,y),x,y)]dxdy \geq 0$$

and $\tilde{u}(x,y)$ is optimal. □

Consider now generalization of the mixed problem (37) to the case an arbitrary bounded set $G \subset \mathbb{R}^2$ with closed piecewise-smooth curve $\Gamma = \overline{\Gamma}_1 \cup \Gamma_2$

$$\text{minimize} \quad J[u(x,y)] := \iint\limits_G g(u(x,y),x,y)dxdy,$$
$$\Delta u(x,y) \in F(u(x,y),x,y), \ (x,y) \in G, \ F(\cdot,x,y) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n, \tag{40}$$
$$u(x,y) = \beta(x,y), \ (x,y) \in \overline{\Gamma}_1, \ \frac{\partial u(x,y)}{\partial v} = \alpha(x,y), \ (x,y) \in \Gamma_2 = \Gamma \backslash \overline{\Gamma}_1,$$

where the Dirichlet and Neumann boundary conditions are satisfied separately along the curves $\overline{\Gamma}_1$ and Γ_2 of the curve Γ.

Theorem 7.2. *Assume that the conditions of Theorem* 5.1 *are satisfied. Then for the optimality of the solution $\tilde{u}(x,y)$ among all feasible solutions in mixed convex problem* (40) *it is sufficient that there exists a classical solution $u^*(x,y)$ such that the conditions* (a), (c) *of Theorem* 5.1 *hold with the following conditions:*

$$u^*(x,y) = 0, \ (x,y) \in \overline{\Gamma}_1, \ \frac{\partial u^*(x,y)}{\partial v} = 0, \ (x,y) \in \Gamma_2.$$

Proof. We will use the relation (32) of Corollary 5.2 By this relation

$$\iint\limits_G [g(u(x,y),x,y) - g(\tilde{u}(x,y),x,y)]dxdy \tag{41}$$
$$\geq \int\limits_\Gamma \left[\left\langle \frac{\partial (u(x,y) - \tilde{u}(x,y))}{\partial v}, u^*(x,y) \right\rangle - \left\langle u(x,y) - \tilde{u}(x,y), \frac{\partial u^*(x,y)}{\partial v} \right\rangle \right] ds.$$

Since $\Gamma = \overline{\Gamma}_1 \cup \Gamma_2$ from (41) we deduce that

$$\begin{aligned}
&\iint\limits_G [g(u(x,y),x,y) - g(\tilde{u}(x,y),x,y)]dxdy \qquad (42)\\
&\geq \int\limits_{\overline{\Gamma}_1} \left[\left\langle \frac{\partial(u(x,y)-\tilde{u}(x,y))}{\partial v}, u^*(x,y)\right\rangle - \left\langle u(x,y)-\tilde{u}(x,y), \frac{\partial u^*(x,y)}{\partial v}\right\rangle\right] ds\\
&\quad + \int\limits_{\Gamma_2} \left[\left\langle \frac{\partial(u(x,y)-\tilde{u}(x,y))}{\partial v}, u^*(x,y)\right\rangle - \left\langle u(x,y)-\tilde{u}(x,y), \frac{\partial u^*(x,y)}{\partial v}\right\rangle\right] ds.
\end{aligned}$$

But in consequence of the conditions of the theorem we have

$$\begin{aligned}
&u(x,y) = \tilde{u}(x,y) = \beta(x,y),\ u^*(x,y) = 0,\ (x,y) \in \overline{\Gamma}_1,\\
&\frac{\partial u(x,y)}{\partial v} = \frac{\partial \tilde{u}(x,y)}{\partial v} = \alpha(x,y),\ \frac{\partial u^*(x,y)}{\partial v} = 0,\ (x,y) \in \Gamma_2.
\end{aligned} \qquad (43)$$

Hence, taking (43) in (42) implies the needed result $J[u(x,y)] \geq J[\widetilde{u}(x,y)]$ for each feasible solution. □

Let us consider the following linear optimal control problem with mixed boundary condition

$$\text{minimize} \quad J[u(x,y)] = \iint\limits_G g(u(x,y),x,y)dxdy,$$

$$\Delta u(x,y) = Au(x,y) + Bw(x,y),\ w(x,y) \in U,\ (x,y) \in G, \qquad (44)$$

$$u(x,y) = \beta(x,y),\ (x,y) \in \overline{\Gamma}_1;\ \frac{\partial u(x,y)}{\partial v} = \alpha(x,y),\ (x,y) \in \Gamma_2 = \Gamma \backslash \overline{\Gamma}_1,$$

where A and B are $n \times n$ and $n \times r$ matrices, $G \subset \mathbb{R}^2$ is a bounded set with closed piecewise-smooth curve $\Gamma = \overline{\Gamma}_1 \cup \Gamma_2, U \subset \mathbb{R}^r$ is a convex compact and $g(\cdot, x, y)$ is continuously differentiable function. In this case $F(u) = Au + BU$. Since

$$F^*(v^*, (u,v)) = \begin{cases} A^*v^*, & -B^*v^* \in K_U^*(w),\\ \varnothing, & -B^*v^* \notin K_U^*(w), \end{cases}$$

where $v = Au + Bw$ and $K_U^*(w), w \in U$ is the dual cone, using Theorem 7.2 we get the relations

$$\begin{aligned}
&\Delta u^*(x,y) = A^*u^*(x,y) - g'(\tilde{u}(x,y),x,y),\ (x,y) \in G,\\
&u^*(x,y) = 0,\ (x,y) \in \overline{\Gamma}_1,\ \frac{\partial u^*(x,y)}{\partial v} = 0,\ (x,y) \in \Gamma_2,\\
&\langle B\tilde{w}(x,y), u^*(x,y)\rangle = \max_{w \in U} \langle Bw, u^*(x,y)\rangle.
\end{aligned} \qquad (45)$$

The obtained result can be formulated as follows.

Theorem 7.3. *A solution $\tilde{u}(x)$ corresponding to the control function $\tilde{w}(x,y)$ is optimal in linear control problem* (44) *if there exists a solution $u^*(x,y)$ that satisfies conditions* (45).

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