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Convergence of the String-Averaging Algorithm for Common Fixed Point Problems with a Countable Family of Operators

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We apply a string-averaging algorithm for common fixed point problems with a countable family of quasi-nonexpansive operators in an arbitrary normed space and show its convergence. Our results are an extension of the recent results by T. Y. Kong , H. Pajoohesh and G. T. Herman obtained for operators which are projections on convex closed sets in a finite-dimensional Euclidean space.

Keywords: Convergence analysis, fixed point, inexact iterate, normed space, nonexpansive mapping.

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1. Introduction

The study of a fixed point problem is an important topic in nonlinear analysis [1, 3, 8, 10, 11, 12, 14, 15, 16, 17, 20, 21, 22, 24, 25, 26, 27, 28]. This problem finds various applications in mathematical analysis, optimization theory, and in engineering, medical and the natural sciences. A convex feasibility problem, which is a particular case of a common fixed problem, is of a great importance too [2, 5, 4, 7, 9, 13, 18, 23, 27, 28, 29]. In the study of common fixed point problems the goal is to find a common fixed point of a family of operators while for the feasibility problem the goal is to find a point which belongs to the intersection of a given family of sets. Usually in these studies the families of operators and sets are finite although there are a few interesting exceptions [19, 29].

Assume that C_i, $i = 1, \dots, m$, where $m \geq 2$ is a natural number, are closed and convex sets in a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and a complete norm $\| \cdot \|$ which is induced by the inner product. We consider the problem

$$\text{Find} \;\; z \in \cap_{i=1}^m C_i$$

under the assumption that $\cap_{i=1}^m C_i$ is nonempty. It is well-known [27] that for each $i \in \{1, \dots, m\}$ and each $x \in X$, there exists a unique element $P_{C_i}(x) \in C_i$ such that

$$\|x - P_{C_i}(x)\| = \inf\{\|x - y\| : \; y \in C_i\}.$$

For solving convex feasibility problem the following iterative method is used.

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Fix an integer $\bar{N} \geq 1$. Denote by $\mathcal{R}$ the set of all maps $r:\{1,2,\dots,\} \to \{1,\dots,m\}$ such that for every positive integer s,

$$\{1,\dots,m\} \subset \{r(s),\dots,r(s+\bar{N}-1)\}.$$

We associate with any map $r \in \mathcal{R}$ the following iterative algorithm:

Initialization: choose any starting point x_0 of the space X.

Iterative step: given a current iterate $x_k \in X$ calculate $x_{k+1} = P_{r(k+1)}(x_k)$.

It is known that iterates obtained by this method converge weakly to a solution of our feasibility problem. The same result is also guaranteed by the well-known Cimmino algorithm described below:

Initialization: choose any starting point x_0 of the space X.

Iterative step: given a current iterate $x_k \in X$ calculate $x_{k+1} = \sum_{i=1}^{m} m^{-1} P_i(x_k)$.

More recently, Y. Censor, T. Elfving, and G. T. Herman in [4] introduced string-averaging methods, which are in some sense a combination of the iterative algorithm and the Cimmino algorithm. In these dynamic string-averaging methods, which became very popular in the literature, a family of sets is divided into blocks and the algorithms operate in such a manner that all the blocks are processed in parallel.

Recently T. Y. Kong , H. Pajoohesh and G. T. Herman [19] studied the convex feasibility problem with a countable family of convex, closed sets in a finite-dimensional Euclidean space and obtained convergence results. In this paper we apply the string-averaging algorithm and extend their convergence results for common fixed point problems with a countable family of quasi-nonexpansive operators in an arbitrary normed space.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a normed space. For each $x \in X$ and each $r > 0$ set

$$B(x,r) = \{y \in X : \ \|x-y\| \leq r\}.$$

For each $x \in X$ and each nonempty set $D \subset X$, put

$$\rho(x,D) = \inf\{\|x-y\| : \ y \in D\}.$$

Denote by $\mathrm{Card}(E)$ the cardinality of a set E. Suppose that the sum over an empty set is zero. For each mapping $S : X \to X$, let $S^0(x) = x$ for all $x \in X$, $S^1 = S$ and $S^{i+1} = S \circ S^i$ for each integer $i \geq 0$ and define

$$\mathrm{Fix}(S) = \{z \in X : \ S(z) = z\}.$$

For each set $C \subset X$ denote by $cl(C)$ the closure of C in the norm topology.

Assume that $P_i : X \to X$, $i = 1, 2, \dots$ are continuous mappings such that for each $i \in \{1,2,\dots\}$, we have $\mathrm{Fix}(P_i) \neq \emptyset$ and that for each $i \in \{1,2,\dots\}$ each $z \in \mathrm{Fix}(P_i)$ and each $x \in X$,

$$\|z - P_i(x)\| \leq \|z - x\|. \tag{1}$$

In this paper we assume the following assumption:

(A) For each $M, \gamma > 0$ there exists $\delta > 0$ such that for each integer $i \geq 1$, each $x \in B(0, M)$ satisfying $\|x - P_i(x)\| \geq \gamma$ and each $z \in \mathrm{Fix}(P_i) \cap B(0, M)$ the following inequality holds: $\|z - x\| - \delta \geq \|z - P_i(x)\|$.

Note that (A) holds for many common fixed point problems [27, 28]. In particular, it holds when P_i, $i = 1, 2, \ldots$ are projections on convex, closed sets in a Hilbert space. It is not difficult to see that Assumption (A) holds when there exists a constant $\gamma > 0$ such that for each integer $i \geq 1$, each $x \in X$ and each for $z \in \mathrm{Fix}(P_i)$ we have

$$\|z - x\|^2 \geq \|z - P_i(x)\|^2 + \gamma \|x - P_i(x)\|^2.$$

Moreover if assumptions (A) holds, I is an identity operator in X, $\epsilon \in (0, 2^{-1})$ and $\{\gamma_i\}_{i=1}^{\infty} \subset [\epsilon, 1 - \epsilon]$, then (A) also holds for the family of mappings $\gamma_i P_i + (1 - \gamma_i) I$, $i = 1, 2, \ldots$.

$$\text{We assume that } F = \cap_{i=1}^{\infty} \mathrm{Fix}(P_i) \neq \emptyset \tag{2}$$

and that there exists a natural number s_0 such that

$$cl(P_{s_0}(X)) \cap B(0, r) \text{ is compact for every } r > 0. \tag{3}$$

In view of (3) any closed bounded subset of F is compact. Our goal is to find an approximate solution of the inclusion $z \in F$.

In the sequel we use the following auxiliary result.

Lemma 2.1. *Let $M, \epsilon > 0$. Then there exist $\delta > 0$ and a natural number q such that for each $x \in B(0, M)$ satisfying*

$$\rho(x, \{y \in X : \ \|P_i(y) - y\| \leq \delta\}) \leq \delta, \ i = 1, \ldots, q$$

the inequality $\rho(x, F) \leq \epsilon$ holds.

Proof. Assume the contrary. Then for each integer $q \geq 1$ there exists

$$x_q \in B(0, M) \tag{4}$$

and for each $i \in \{1, \ldots, q\}$ there exists

$$y_{q,i} \in B(x_q, q^{-1}) \tag{5}$$

such that
$$\|y_{q,i} - P_i(y_{q,i})\| \leq q^{-1}, \tag{6}$$

and
$$\rho(x_q, F) > \epsilon. \tag{7}$$

In view of (1), (2), (4) and (5), the sequence $\{P_{s_0}(y_{q,1})\}_{q=1}^{\infty}$ is bounded. By (3) it has a convergent subsequence. Together with (5) and (6) this implies that $\{x_q\}_{q=1}^{\infty}$ has a convergent subsequence too. We may assume without loss of generality that $\{x_q\}_{q=1}^{\infty}$ converges. Set

$$x_* = \lim_{q \to \infty} x_q.$$

By (5) and (6), for each integer $i \geq 1$, we have $\lim_{q \to \infty} y_{q,i} = x_*$ and $\lim_{q \to \infty} P_i(y_{q,i}) = x_*$.

Since P_i, $i = 1, 2, \ldots$ are continuous we conclude that

$$P_i(x_*) = x_*, \ i = 1, 2, \ldots \quad \text{and} \quad x_* \in F.$$

On the other hand by (7), we have $\rho(x_*, F) \geq \epsilon$.

The contradiction we have reached proves Lemma 2.1. □

By an *index vector*, we a mean a vector $t = (t_1, \dots, t_p)$ such that t_i is a natural number for all $i = 1, \dots, p$. For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q,\ P[t] = P_{t_q} \cdots P_{t_1}. \tag{8}$$

It is not difficult to see that for each index vector t,

$$P[t](x) = x \ \text{ for all } x \in F, \tag{9}$$

$$\|P[t](x) - P[t](y)\| \leq \|x - P[t](y)\| \leq \|x - y\| \tag{10}$$

for every $x \in F$ and every $x \in X$.

Denote by Ω the set of all index vectors and by $\mathcal{M}$ the collection of all functions $w : \Omega \to [0, \infty)$ such that

$$\mathrm{supp}(w) := \{t \in \Omega :\ w(t) > 0\} \text{ is finite} \tag{11}$$

and

$$\sum_{t \in \mathrm{supp}(w)} w(t) = \sum_{t \in \Omega} w(t) = 1. \tag{12}$$

Let $w \in \mathcal{M}$. Define

$$P_w(x) = \sum_{t \in \mathrm{supp}(w)} w(t) P[t](x) = \sum_{t \in \Omega} w(t) P[t](x),\ x \in X. \tag{13}$$

It is easy to see that

$$P_w(x) = x \text{ for all } x \in F, \tag{14}$$

and

$$\|P_w(x) - P_w(y)\| = \|x - P_w(y)\| \leq \|x - y\| \tag{15}$$

for each $x \in F$ and each $y \in X$.

The dynamic string-averaging method with variable strings and variable weights can now be described by the following algorithm.

Initialization: select an arbitrary point $x_0 \in X$.

Iterative step: given a current iteration vector x_k pick $w_k \in \mathcal{M}$ and calculate the next iteration vector x_{k+1} by $x_{k+1} \in P_{w_k}(x_k)$.

As a matter of fact in this paper we study an inexact version of this algorithm. In order to meet this goal we need the following definitions (see [27, 28]).

Let $\delta \geq 0$, $x \in X$ and let $t = (t_1, \dots, t_{p(t)})$ be an index vector. Define

$$A_0(x, t, \delta) = \left\{ (y, \lambda) \in X \times R^1 : \begin{array}{l} \text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that} \\ y_0 = x \text{ and for all } i = 1, \dots, p(t), \\ \|y_i - P_{t_i}(y_{i-1})\| \leq \delta,\ y = y_{p(t)}, \\ \lambda = \max\{\|y_i - y_{i-1}\| :\ i = 1, \dots, p(t)\}. \end{array} \right\} \tag{16}$$

Let $\delta \geq 0$, $x \in X$ and let $w \in \mathcal{M}$. Define

$$A(x,w,\delta)=\left\{(y,\lambda)\in X\times R^1:\ \begin{array}{l}\text{there exist } (y_t,\lambda_t)\in A_0(x,t,\delta),\ t\in\mathrm{supp}(w)\\ \text{such that } \|y-\sum_{t\in\mathrm{supp}(w)} w(t)y_t\|\le\delta,\\ \lambda=\max\{\lambda_t:\ t\in\mathrm{supp}(w)\}.\end{array}\right\} \quad (17)$$

Let $D\subset\Omega$ be nonempty and i,s be natural numbers. Set

$$D(i,s)=\left\{(t=(t_1,\dots,t_{p(t)})\in D:\ \begin{array}{l}\text{there is } j\in\{1,\dots,\min\{s,p(t)\}\}\\ \text{for which } t_j=i\end{array}\right\}. \quad (18)$$

There are only a few results on common fixed point problems with infinitely many operators. It should be mentioned that the string-averaging algorithm for common fixed point problems with a countable family of operators was also studied in [6] for firmly nonexpansive operators which are a subclass of mappings satisfying assumption (A). Our algorithm differs from the algorithm of [6] where for each iterate it was assumed that the whole family of operators is available and the next iterate was calculated by using the sum of a countable set of operators. Moreover we take into account the influence of computational errors. Note that if in our algorithm for each integer $k\ge 0$ we choose $w_k\in\mathcal{M}$ such that $\mathrm{supp}(w_k)$ is a singleton, we obtain the iterative method for solving common fixed point problem while if each $t\in\mathrm{supp}(w_k)$ has a length 1 we obtain the Cimmino algorithm.

Our paper contains results (Theorems 3.1, 4.1, 5.1 and 6.1)) which are proved in Section 3–6, respectively. In Theorem 3.1 we show the convergence of iterates to a common fixed point under the presence of summable computational errors. Theorem 4.1 shows that our iterates converge to a small neighborhood of F under the presence of sufficiently small nonsummable computational errors. Theorem 5.1 is proved in Section 5 while Theorem 6 is proved in Section 6. They are extensions of Theorems 3.1 and 4.1 respectively. Note that in Theorems 3.1 and 4.1 computational errors occur only at the final stage of the iterative step while in Theorems 5.1 and 6.1 they are taken into account at all the intermediate calculations.

3. The first result

Theorem 3.1. *Assume that* $M>0$, $B(0,M)\cap F\neq\emptyset$, (19)

$\{\Delta_i\}_{i=0}^\infty\subset[0,\infty)$ *satisfies*

$$\sum_{i=0}^\infty \Delta_i<\infty, \quad (20)$$

there exists a sequence of natural numbers $\{d_s\}_{s=1}^\infty$ *and for each integer* $s\ge 1$, *there exists* $\{a_{k,s}\}_{k=1}^\infty\subset[0,\infty)$ *such that*

$$\sum_{k=1}^\infty a_{k,s}=\infty. \quad (21)$$

Let $\epsilon\in(0,1]$. *Then there exists a natural number* n *such that for each* $\{w_i\}_{i=0}^\infty\subset\mathcal{M}$ *satisfying for each integer* $s\ge 1$, *each* $p\in\{1,\dots,s\}$ *and each integer* $i\ge 0$,

$$\sum\left\{w_i(t):\ t\in\mathrm{supp}(w_i)(p,d_s)\right\}\ge a_{i,s}, \quad (22)$$

each

$$x_0\in B(0,M) \quad (23)$$

and each $\{x_i\}_{i=1}^{\infty} \subset X$ *satisfying for each natural number* i,

$$\|x_{i+1} - P_{w_i}(x_i)\| \leq \Delta_i \tag{24}$$

there exists $z \in F$ *such that* $\|z - x_i\| \leq \epsilon$ *for all integers* $i \geq n$ *and that the sequence* $\{x_i\}_{i=0}^{\infty}$ *converges to a point of* F.

Proof. In view of (19), there exists $z \in B(0, M) \cap F$. (25)

Lemma 2.1 implies that there exist $\delta \in (0, \epsilon/2)$ and an integer $q \geq 1$ such that the following property holds:

(i) for each $x \in B(0, 3M + \sum_{j=0}^{\infty} \Delta_j)$ satisfying

$$\rho(x, \{y \in X : \|y - P_i(y)\| \leq \delta\}) \leq \delta, \; i = 1, \dots, q$$

the inequality $\rho(x, F) < \epsilon/4$ holds. Choose

$$\delta_0 \in (0, \delta d_q^{-1}). \tag{26}$$

Assumption (A) implies that there exists

$$\gamma \in (0, \delta_0) \tag{27}$$

such that the following property holds:

(ii) for each $i \geq 1$, each $x \in B(0, 3M + \sum_{j=0}^{\infty} \Delta_j)$ satisfying $\|x - P_i(x)\| \geq \delta_0$ and each $z \in \mathrm{Fix}(P_i) \cap B(0, M)$ the inequality

$$\|z - x\| - \gamma \geq \|z - P_i(x)\|$$

holds. By (20), there exists an integer $n_1 > 2$ such that

$$\sum_{i=n_1}^{\infty} \Delta_i < \delta_0/4. \tag{28}$$

In view of (20) and (21), there exists an integer $n > n_1 + 2$ such that

$$\sum_{i=n_1}^{n-1} a_{i,q} > 4M\gamma^{-1} + 4\gamma^{-1} \sum_{j=0}^{\infty} \Delta_j. \tag{29}$$

Assume that $\{w_i\}_{i=0}^{\infty} \subset \mathcal{M}$, for each integer $s \geq 1$, each $p \in \{1, \dots, s\}$ and each integer $i \geq 0$, (22) holds, $\{x_i\}_{i=0}^{\infty} \subset X$ satisfies (23) and (24) for each natural number i. By (23) and (25),

$$\|x_0 - z\| \leq 2M. \tag{30}$$

Let $i \geq 0$ be an integer. It follows from (15), (24), (25) and (30) that

$$\|x_{i+1} - z\| \leq \|x_{i+1} - P_{w_i}(x_i)\| + \|P_{w_i}(x_i) - z\| \leq \Delta_i + \|x_i - z\|,$$

$$\|x_i - z\| \leq \|x_0 - z\| + \sum_{j=0}^{\infty} \Delta_j \leq 2M + \sum_{j=0}^{\infty} \Delta_j, \tag{31}$$

$$\|x_i\| \leq 3M + \sum_{j=0}^{\infty} \Delta_j. \tag{32}$$

We show that there exists $\tau \in \{n_1, \dots, n-1\}$ such that $\rho(x_\tau, F) < \epsilon/4$.
Assume the contrary. Then

$$\rho(x_i, F) \geq \epsilon/4, \ i \in \{n_1, \dots, n-1\}. \tag{33}$$

Property (i), (32) and (33) imply that for every $i \in \{n_1, \dots, n-1\}$,

$$\max\{\rho(x_i, \{y \in X : \|y - P_k(y)\| \leq \delta\}) : \ k = 1, \dots, q\} > \delta. \tag{34}$$

Let $t \in \{n_1, \dots, n-1\}$. In view of (34), there exists

$$k_0 \in \{1, \dots, q\} \tag{35}$$

such that
$$\rho(x_i, \{y \in X : \ \|y - P_{k_0}(y)\| \leq \delta\}) > \delta. \tag{36}$$

Assume that
$$t \in \mathrm{supp}(w_i)(k_0, d_q). \tag{37}$$

Set
$$y_0 = x_i, \ y_j = P_{t_j}(y_{j-1}) \text{ for all } j = 1, \dots, p(t), \ y_{p(t)} = P[t](x_i). \tag{38}$$

By (18) and (37), there exists
$$j_0 \in \{1, \dots, \min\{p(t), d_q\}\} \tag{39}$$

such that
$$t_{j_0} = k_0. \tag{40}$$

We show that there exists $j \in \{1, \dots, j_0\}$ such that $\|y_j - y_{j-1}\| > \delta_0$.

Assume the contrary. Then
$$\|y_j - y_{j-1}\| \leq \delta_0, \ j = 1, \dots, j_0. \tag{41}$$

By (38)–(41) we have

$$\delta_0 \geq \|y_{j_0} - y_{j_0-1}\| = \|P_{t_{j_0}}(y_{j_0-1}) - y_{j_0-1}\| = \|P_{k_0}(y_{j_0-1}) - y_{j_0-1}\|. \tag{42}$$

By (27), (38), (39) and (41) we conclude

$$\|x_i - y_{j_0-1}\| = \|y_0 - y_{j_0-1}\| \leq \delta_0 j_0 \leq \delta_q d_q \leq \delta. \tag{43}$$

Equation (42) and (43) imply that

$$\|P_{k_0}(y_{j_0-1}) - y_{j_0-1}\| \leq \delta_0, \ \|x_i - y_{j_0-1}\| \leq \delta.$$

This contradicts (36). The contradiction we have reached shows that there exists $j_1 \in \{1, \dots, j_0\}$ such that

$$\|P_{t_{j_1}}(y_{j_1-1}) - y_{j_1-1}\| = \|y_{j_1} - y_{j_1-1}\| > \delta_0. \tag{44}$$

By (10), (25), (31) and (38), for $j = 1, \dots, p(t)$,

$$\begin{aligned} \|z - y_j\| &\leq \|z - P_{t_j}(y_{j-1})\| \leq \|z - y_{j-1}\| \leq \|z - y_0\| = \|z - x_i\| \\ &\leq 2M + \sum_{m=0}^{\infty} \Delta_m, \end{aligned} \tag{45}$$

and
$$\|y_j\| \leq 3M + \sum_{m=0}^{\infty} \Delta_m, \ j = 0, \dots (t). \tag{46}$$

Property (ii), (25), (38), (44) and (46) imply that

$$\|z - y_{j_1}\| \leq \|z - y_{j_1-1}\| - \gamma. \tag{47}$$

In view of (45),

$$\|z - x_i\| - \gamma \geq \|z - y_{j_1-1}\| - \gamma \geq \|z - y_{j_1}\| \geq \|z - y_{p(t)}\| = \|z - P[t](x_i)\|. \tag{48}$$

Thus for each integer t satisfying (37),

$$\|z - P[t](x_t)\| \leq \|z - x_i\| - \gamma. \tag{49}$$

By the convexity of the norm, (11)–(13), (15), (22), (25), (35), (37) and (38), we get

$$\begin{aligned}
\|z - P_{w_i}(x_i)\| = \|z - \sum_{t\in \mathrm{supp}(w_i)} P[t](x_i)\| &\leq \sum_{t\in \mathrm{supp}(w_i)} w_i(t)\|z - P[t](x_i)\| \\
&\leq \sum\{w_i(t)\|z - P[t](x_i)\| : \ t \in \mathrm{supp}(w_i)(k_0, d_q)\} \\
&\quad + \sum\{w_i(t)\|z - P[t](x_i)\| : \ t \in \mathrm{supp}(w_i) \setminus \mathrm{supp}(w_i)(k_0, d_q)\} \\
&\leq \sum\{w_i(t)(\|z - x_i\| - \gamma) : \ t \in \mathrm{supp}(w_i)(k_0, d_q)\} \\
&\quad + \sum\{w_i(t)\|z - x_i\| : \ t \in \mathrm{supp}(w_i) \setminus \mathrm{supp}(w_i)(k_0, d_q)\} \\
&\leq (\|z - x_i\| - \gamma) \sum\{w_i(t) : \ t \in \mathrm{supp}(w_i)(k_0, d_q)\} \\
&\quad + \|z - x_i\| \sum\{w_i(t) : \ t \in \mathrm{supp}(w_i) \setminus \mathrm{supp}(w_i)(k_0, d_q)\} \\
&\leq \|z - x_i\| - \gamma a_{i,q}.
\end{aligned} \tag{50}$$

thus by (24) and (50), for each $i \in \{n_1, \dots, n-1\}$,

$$\|z - x_{i+1}\| \leq \|z - P_{w_i}(x_i)\| + \|P_{w_i}(x_i) - x_{i+1}\| \leq \|z - x_i\| - \gamma a_{i,q} + \Delta_i. \tag{51}$$

By (31) and (51),

$$\begin{aligned}
2M + \sum_{i=0}^{\infty} \Delta_i &\geq \|z - x_{n_1}\| \geq \|z - x_{n_1}\| - \|z - x_n\| \\
&= \sum_{i=n_1}^{n-1} (\|z - x_i\| - \|z - x_{i+1}\|) \geq \sum_{t=n_1}^{n-1} \gamma a_{i,q} - \sum_{j=n_1}^{\infty} \Delta_j
\end{aligned}$$

and

$$\sum_{i=n_1}^{n-1} a_{i,q} \leq \gamma^{-1}(2M + 3\sum_{i=0}^{\infty} \Delta_i).$$

This contradicts (29). The contradiction we have reached proves that there exists $\tau \in \{n_1, \dots, n-1\}$ such that $\rho(x_\tau, F) < \epsilon/4$. This implies that there exists

$$z_0 \in F \tag{52}$$

such that

$$\|x_\tau - z_0\| < \epsilon/4. \tag{53}$$

By (15), (24), (28), (52) and (53), for each integer $t \geq \tau$, we obtain

$$\|z_0 - x_{i+1}\| \leq \|z_0 - P_{w_i}(x_i)\| + \|P_{w_i}(x_i) - x_{i+1}\| \leq \|z_0 - x_i\| + \Delta_i$$

and
$$\|x_i - z_0\| \leq \|x_\tau - z_0\| + \sum_{j=\tau}^{\infty} \Delta_j < \epsilon.$$

Since ϵ is any number from $(0, 1]$ we conclude that the sequence $\{x_i\}_{i=0}^{\infty}$ converges and its limit belongs to F. Thus, Theorem 3.1 is proved. □

4. The second result

Theorem 4.1. *Assume that*

$$M > 1,\ \epsilon \in (0, 1),\ F \subset B(0, M - 1) \tag{54}$$

and that for each integer $s \geq 1$, d_s, b_s are natural numbers and $a_s > 0$. Then there exist $\delta \in (0, \epsilon)$ and a natural number n such that for each $\{w_i\}_{i=0}^{\infty} \subset \mathcal{M}$ satisfying for each integer $p \geq 1$ and each integer $i \geq b_p$,

$$\sum\{w_i(t) :\ t \in supp(w_i)(p, d_p)\} \geq a_p, \tag{55}$$

each $x_0 \in B(0, M)$ and each $\{x_i\}_{i=1}^{\infty} \subset X$ satisfying

$$\|x_{i+1} - P_{w_i}(x_i)\| \leq \delta,\ i = 0, 1, \ldots \tag{56}$$

the inequality $\rho(x_i, F) \leq \epsilon$ holds for all integers $i \geq n$.

Proof. Let $\{\Delta_i\}_{i=0}^{\infty} \subset (0, \infty)$ satisfy $\sum_{i=0}^{\infty} \Delta_i < \infty$.

Theorem 3.1 implies that there exists a natural number n such that the following property holds:

(i) for each $\{v_i\}_{i=0}^{\infty} \subset \mathcal{M}$ satisfying for each integer $p \geq 1$ and each integer $i \geq b_p$,

$$\sum\{v_i(t) :\ t \in \mathrm{supp}(v_i)(p, d_p)\} \geq a_p$$

and each $\{y_i\}_{i=0}^{\infty} \subset X$ satisfying $y_0 \in B(0, M)$, and for each integer $i \geq 0$ with $\|y_{i+1} - P_{v_i}(y_i)\| \leq \Delta_i$ there exists $z \in F$ such that $\|z - x_i\| \leq \epsilon/4$ for all integers $i \geq n$.

Set
$$\delta_0 = \min\{\Delta_i :\ i = 0, \ldots, n\}, \tag{57}$$

and
$$\delta = \min\{\epsilon, \delta_0\}(4n)^{-1}. \tag{58}$$

Assume that $\{x_i\}_{i=0}^{\infty} \subset X$, $\{w_i\}_{i=0}^{\infty} \subset \mathcal{M}$, for each integer $p \geq 1$, each integer $i \geq b_p$, (55) and (56) hold and $\|x_0\| \leq M$. Equations (56)–(58) and property (i) applied to

$$v_i = w_i,\ i = 0, 1, \ldots,\ y_i = x_i,\ i = 0, \ldots, n,$$

$$y_{i+1} = P_{w_i}(y_i) \text{ for each integer } i \geq n$$

imply that there exists $z_1 \in F$ such that

$$\|x_n - z_1\| = \|y_n - z_1\| \leq \epsilon/4.$$

We show by induction that for each integer $p \geq 1$ there exists $z_p \in F$ such that

$$\|x_{pn} - z_p\| \leq \epsilon/4.$$

Assume that $p \geq 1$ is an integer and that for each integer $i \in \{1, \dots, p\}$ there exists

$$z_i \in F \tag{59}$$

such that

$$\|x_{ni} - z_i\| \leq \epsilon/4. \tag{60}$$

By (54), (59) and (60), we have $\|x_{pn}\| \leq M$. Set

$$v_i = w_{i+pn},\ i = 0, 1, \dots,\ y_i = x_{i+pn},\ i = 0, \dots, n,$$

and

$$y_{i+1} = P_{v_i}(y_i) \text{ for each integer } i \geq n.$$

By our definition and property (i), there exists $z_{p+1} \in F$ such that

$$\|x_{(p+1)n} - z_{p+1}\| = \|y_n - z_{p+1}\| \leq \epsilon/4.$$

Thus for each integer $p \geq 1$ there exists

$$z_p \in F \tag{61}$$

and

$$\|x_{pn} - z_p\| \leq \epsilon/4. \tag{62}$$

Let $p \geq 1$ be an integer. For each $i \in \{pn, \dots, (p+1)n - 1\}$ by (15), (56)–(58), (61) and (62), we conclude

$$\|x_{i+1} - z_p\| \leq \|x_{i+1} - P_{w_i}(x_i)\| + \|P_{w_i}(x_i) - z_p\| \leq \delta + \|x_i - z_p\|$$

and

$$\|x_i - z_p\| \leq \|x_{pn} - z_p\| + \delta n < \epsilon.$$

Theorem 4.1 is proved. □

5. The third result

Theorem 5.1. *Assume that* $M > 0$,

$$B(0, M) \cap F \neq \emptyset, \tag{63}$$

$\{\Delta_i\}_{i=0}^{\infty} \subset [0, \infty)$ *satisfies*

$$\sum_{i=0}^{\infty} \Delta_i < \infty, \tag{64}$$

$\bar{q}$ *is a natural number and for each integer* $s \geq 1$, *there exists* $\{a_{k,s}\}_{k=1}^{\infty} \subset [0, \infty)$

such that

$$\sum_{k=1}^{\infty} a_{k,s} = \infty. \tag{65}$$

Let $\epsilon \in (0, 1)$. *Then there exist a natural number* n *such that for each* $\{w_i\}_{i=0}^{\infty} \subset \mathcal{M}$ *satisfying for each integer* $i \geq 1$, *each* $t \in \mathrm{supp}(w_i)$,

$$p(t) \leq \bar{q} \tag{66}$$

and for each integer $s \geq 1$, *each* $p \in \{1, \dots, s\}$ *and each integer* $i \geq 0$,

$$\sum \{w_i(t) :\ t \in \mathrm{supp}(w_i)(p, \bar{q})\} \geq a_{i,s}, \tag{67}$$

each $\{x_i\}_{i=0}^{\infty} \subset X$ *and each* $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ *satisfying*

$$x_0 \in B(0, M) \tag{68}$$

$$\text{and for each natural number } i, \ (x_i, \lambda_i) \in A(x_{i-1}, w_{i-1}, \Delta_{i-1}) \tag{69}$$

there exists $z \in F$ *such that* $\|z - x_i\| \le \epsilon$ *for all integers* $i \ge n$ *and the sequence* $\{x_i\}_{i=0}^{\infty}$ *converges to a point of* F.

Proof. In view of (63), there exists

$$z \in B(0, M) \cap F. \tag{70}$$

Lemma 2.1 implies that there exist $\delta \in (0, \epsilon/4)$ and an integer $q \ge 1$ such that the following property holds:

(i) for each $x \in B(0, 3M + (\bar{q} + 1)\sum_{j=0}^{\infty} \Delta_j)$ satisfying

$$\rho(x, \{y \in X : \ \|y - P_i(y)\| \le \delta\}) \le \delta, \ i = 1, \dots, q$$

the inequality $\rho(x, F) < \epsilon/4$ holds.

By (A) choose
$$\delta_0 \in (0, \delta(\bar{q} + 1)^{-1}) \tag{71}$$
and
$$\gamma \in (0, \delta_0) \tag{72}$$

such that the following property holds:

(ii) for each integer $i \ge 1$, each $x \in B(0, 3M + (\bar{q} + 1)\sum_{j=0}^{\infty} \Delta_j)$ satisfying

$$\|x - P_i(x)\| \ge \delta_0/4$$

and each $z \in \mathrm{Fix}(P_i) \cap B(0, M)$ the inequality $\|z - x\| - \gamma \ge \|z - P_i(x)\|$ holds.

By (61), there exists an integer $n_1 > 2$ such that

$$\sum_{i=n_1}^{\infty} \Delta_i < \bar{q}^{-1}\gamma/4. \tag{73}$$

In view of (65), there exists an integer $n > n_1 + 2$ such that

$$\sum_{i=n_1}^{n-1} a_{i,q} > 4M\gamma^{-1} + 4\gamma^{-1}(\bar{q} + 1)\sum_{j=0}^{\infty} \Delta_j + 2\bar{q} + 2. \tag{74}$$

Assume that $\{w_i\}_{i=0}^{\infty} \subset \mathcal{M}$ satisfies (66) and (67), for each integer $s \ge 1$, each $p \in \{1, \dots, s\}$ and each integer $i \ge 0$, $\{x_i\}_{i=0}^{\infty} \subset X$ and each $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfy (68) and (69) for each natural number i. By (68) and (70) we obtain

$$\|x_0 - z\| \le 2M. \tag{75}$$

Let $i \ge 0$ be an integer. It follows from (69) that

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, w_i, \Delta_i). \tag{76}$$

By (17) and (76), there exist

$$(y_t^{(i)}, \lambda_t^{(i)}) \in A_0(x_i, t, \Delta_i),\ t \in \mathrm{supp}(w_i) \tag{77}$$

such that

$$\|x_{i+1} - \sum_{t \in \mathrm{supp}(w_i)} w_i(t) y_t^{(i)}\| \leq \Delta_i, \tag{78}$$

$$\lambda_{i+1} = \max\{\lambda_t^{(i)} :\ t \in \mathrm{supp}(w_i)\}. \tag{79}$$

By (16) and (77), for each $t \in \mathrm{supp}(w_i)$, there is a finite sequence $\{y_j^{(i,t)}\}_{j=0}^{p(t)} \subset X$

such that

$$y_0^{(i,t)} = x_i, \tag{80}$$

and for all $j = 1, \dots, p(t)$,

$$\|y_j^{(i,t)} - P_{t_j}(y_{j-1}^{(i,t)})\| \leq \Delta_i, \tag{81}$$

$$y_{p(t)}^{(i,t)} = y_t^{(i)}, \tag{82}$$

$$\lambda_t^{(i)} = \max\{\|y_j^{(i,t)} - y_{i-1}^{(i,t)}\| :\ i = 1, \dots, p(t)\}. \tag{83}$$

Let $t \in \mathrm{supp}(w_i),\ j \in \{0, \dots, p(t) - 1\}$. By (1), (70) and (87),

$$\|z - y_j^{(i,t)}\| \leq \|z - P_{t_j}(y_{j-1}^{(i,t)})\| + \|P_{t_j}(y_{j-1}^{(i,t)}) - y_j^{(i,t)}\| \leq \|z - y_{j-1}^{(i,t)}\| + \Delta_i. \tag{84}$$

Equations (66), (80) and (88) imply that for each $j = 1, \dots, p(t)$,

$$\|z - y_j^{(i,t)}\| \leq \|z - y_0^{(i,t)}\| + j\Delta_i \leq \|z - x_i\| + \bar{q}\Delta_i. \tag{85}$$

By the convexity of the norm, (78), (82) and (85), we obtain

$$\begin{aligned}\|z - x_{i+1}\| &\leq \|z - \sum_{t \in \mathrm{supp}(w_i)} w_i(t) y_t^{(i)}\| + \|\sum_{t \in \mathrm{supp}(w_i)} w_i(t) y_t^{(i)} - x_{i+1}\| \\ &\leq \sum_{t \in \mathrm{supp}(w_i)} w_i(t) \|z - y_t^{(i)})\| + \Delta_i \leq \|z - x_i\| + (\bar{q} + 1)\Delta_i. \end{aligned} \tag{86}$$

By (70), (75), (82), (85) and (86), for each $t \in \mathrm{supp}(w_i)$, each $j \in \{0, \dots, p(t)\}$, each $i = 0, 1, \dots,$

$$\|z - x_i\| \leq \|z - x_0\| + (\bar{q} + 1) \sum_{k=0}^{\infty} \Delta_k \leq 2M + (\bar{q} + 1) \sum_{k=0}^{\infty} \Delta_k, \tag{87}$$

$$\|x_k\| \leq 3M + (\bar{q} + 1) \sum_{k=0}^{\infty} \Delta_k, \tag{88}$$

$$\|z - y_t^{(i)}\| \leq \|z - x_0\| + (\bar{q} + 1) \sum_{k=0}^{\infty} \Delta_k \leq 2M + (\bar{q} + 1) \sum_{k=0}^{\infty} \Delta_k, \tag{89}$$

$$\|y_t^{(i)}\| \leq 3M + (\bar{q} + 1) \sum_{k=0}^{\infty} \Delta_k, \tag{90}$$

$$\|z - y_j^{(i,t)}\| \leq \|z - x_0\| + (\bar{q} + 1) \sum_{k=0}^{\infty} \Delta_k \leq 2M + (\bar{q} + 1) \sum_{k=0}^{\infty} \Delta_k, \tag{91}$$

$$\|y_j^{(i,t)}\| \leq 3M + (\bar{q} + 1) \sum_{k=0}^{\infty} \Delta_k. \tag{92}$$

We show that there exists $\tau \in \{n_1, \dots, n\}$ such that $\rho(x_\tau, F) < \epsilon/4$.

Assume the contrary. Then $$\rho(x_i, F) \geq \epsilon/4, \ i \in \{n_1, \dots, n-1\}. \tag{93}$$

Let $i \in \{n_1, \dots, n-1\}$. Property (i), (88) and (93) imply that

$$\max\{\rho(x_i, \{y \in X : \|y - P_k(y)\| \leq \delta\}) : \ k = 1, \dots, q\} > \delta.$$

This implies that there exists $$k_0 \in \{1, \dots, q\} \tag{94}$$

such that $$\rho(x_i, \{y \in X : \ \|y - P_{k_0}(y)\| \leq \delta\}) > \delta. \tag{95}$$

Assume that $$t \in \mathrm{supp}(w_i)(k_0, \bar{q}). \tag{96}$$

By (18), (66) and (96), there exists $$j_0 \in \{1, \dots, p(t)\} \tag{97}$$

$(p(t) \leq \bar{q})$ such that $$t_{j_0} = k_0. \tag{98}$$

We show that there exists $j \in \{1, \dots, j_0\}$ such that $\|y_j^{(i,t)} - y_{j-1}^{(i,t)}\| > \delta_0$.

Assume the contrary. Then $$\|y_j^{(i,t)} - y_{j-1}^{(i,t)}\| \leq \delta_0, \ j = 1, \dots, j_0. \tag{99}$$

By (81) and (99),

$$\begin{aligned}\delta_0 &\geq \|y_{j_0}^{(i,t)} - y_{j_0-1}^{(i,t)}\| \geq \|P_{t_{j_0}}(y_{j_0-1}^{(i,t)}) - y_{j_0-1}^{(i,t)}\| - \|y_{j_0}^{(i,t)} - P_{t_{j_0}}(y_{j_0-1}^{(i,t)})\| \\ &\geq \|P_{t_{j_0}}(y_{j_0-1}^{(i,t)}) - y_{j_0-1}^{(i,t)}\| - \Delta_i. \end{aligned} \tag{100}$$

In view of (71), (72), (98) and (100),

$$\|P_{k_0}(y_{j_0-1}^{(i,t)}) - y_{j_0-1}^{(i,t)}\| = \|P_{t_{j_0}}(y_{j_0-1}^{(i,t)}) - y_{j_0-1}^{(i,t)}\| \leq \delta_0 + \Delta_i \leq \delta. \tag{101}$$

By (66), (71), (80) and (99),

$$\|x_i - y_{j_0-1}^{(i,t)}\| = \|y_0^{(i,t)} - y_{j_0-1}^{(i,t)}\| \leq \delta_0 j_0 \leq \delta_0 \bar{q} \leq \delta. \tag{102}$$

Together with (101) and (102) this contradicts (95). The contradiction we have reached shows that there exists

$$j_1 \in \{1, \dots, j_0\} \tag{103}$$

such that $$\|y_{j_1}^{(i,t)} - y_{j_1-1}^{(i,t)}\| > \delta_0. \tag{104}$$

By (73), (81) and (104),

$$\|P_{t_{j_1}}(y_{j_1-1}^{(i,t)}) - y_{j_1-1}^{(i,t)}\| \geq \|y_{j_1}^{(i,t)} - y_{j_1-1}^{(i,t)}\| - \|P_{t_{j_1}}(y_{j_1-1}^{(i,t)}) - y_{j_1}^{(i,t)}\| > \delta_0 - \Delta_i > \delta_0/2. \tag{105}$$

Property (ii) and (70), (92), (105) imply that

$$\|z - P_{t_{j_1}}(y_{j_1-1}^{(i,t)})\| \leq \|z - y_{j_1-1}^{(i,t)})\| - \gamma. \tag{106}$$

By (81), (85) and (106),

$$\begin{aligned}\|z - y_{j_1}^{(i,t)}\| &\leq \|z - P_{t_{j_1}}(y_{j_1-1}^{(i,t)})\| + \|P_{t_{j_1}}(y_{j_1-1}^{(i,t)}) - y_{j_1-1}^{(i,t)})\| \\ &\leq \|z - y_{j_1-1}^{(i,t)})\| - \gamma + \Delta_i \leq \|z - x_i\| + \Delta_i(\bar{q} - 1) + \Delta_i - \gamma. \end{aligned} \tag{107}$$

It follows from (73), (82), (84) and (107) that

$$\|z - y_t^{(i)}\| \le \|z - y_{j_1}^{(i,t)}\| + \bar{q}\gamma_i \le \|z - x_i\| + 2\bar{q}\Delta_i - \gamma \le \|z - x_i\| - \gamma/2$$

and
$$\|z - y_t^{(i)}\| \le \|z - x_i\| - \gamma/2 \tag{108}$$

for each $t \in \mathrm{supp}(w_i)(k_0, \bar{q})$. By the convexity of the norm, (11), (12), (78), (85), (94), (108), we get

$$\begin{aligned}
\|z - x_{i+1}\| &\le \|z - \sum_{t \in \mathrm{supp}(w_i)} w_i(t) y_t^{(i)}\| + \|\sum_{t \in \mathrm{supp}(w_i)} w_i(t) y_t^{(i)} - x_{i+1}\| \\
&\le \Delta_i + \sum_{t \in \mathrm{supp}(w_i)} w_i(t)\|z - y_t^{(i)}\| \\
&= \Delta_i + \sum\{w_i(t)\|z - y_t^{(i)}\| :\ t \in \mathrm{supp}(w_i)(k_0, \bar{q})\} \\
&\qquad + \sum\{w_i(t)\|z - y_t^{(i)})\| :\ t \in \mathrm{supp}(w_i) \setminus \mathrm{supp}(w_i)(k_0, \bar{q})\} \\
&\le \Delta_i + \sum\{w_i(t)(\|z - x_i\| - \gamma/2) :\ t \in \mathrm{supp}(w_i)(k_0, \bar{q})\} \\
&\qquad + \sum\{w_i(t)(\|z - x_i\| + \bar{q}\Delta_i) :\ t \in \mathrm{supp}(w_i) \setminus \mathrm{supp}(w_i)(k_0, \bar{q})\} \\
&\le \Delta_i + \|z - x_i\| + \bar{q}\Delta_i - (\gamma/2)\sum\{w_i(t) :\ t \in \mathrm{supp}(w_i)(k_0, \bar{q})\} \\
&\le \|z - x_i\| + (\bar{q} + 1)\Delta_i - 2^{-1}\gamma a_{i,q}.
\end{aligned}$$

Thus for each $i \in \{n_1, \dots, n-1\}$,

$$\|z - x_{i+1}\| \le \|z - x_i\| + (\bar{q} + 1)\Delta_i - 2^{-1}\gamma a_{i,q}. \tag{109}$$

By (73), (87), (95) and (109),

$$\begin{aligned}
2M + (\bar{q} + 1) + \sum_{i=0}^{\infty} \Delta_i &\ge \|z - x_{n_1}\| \ge \|z - x_{n_1}\| - \|z - x_n\| \\
&= \sum_{i=n_1}^{n-1} (\|z - x_i\| - \|z - x_{i+1}\|) \ge 2^{-1}\gamma \sum_{i=n_1}^{n-1} \gamma a_{i,q} - (\bar{q} + 1)\sum_{i=n_1}^{\infty} \Delta_i \\
&\ge 2^{-1}\gamma \sum_{i=n_1}^{n-1} a_{i,q} - (\bar{q} + 1)\gamma
\end{aligned}$$

and
$$\sum_{i=n_1}^{n-1} a_q \le 2\gamma^{-1}(2M + (\bar{q} + 1)\sum_{j=0}^{\infty} \Delta_j) + 2\bar{q} + 2.$$

This contradicts (74). The contradiction we have reached proves that there exists

$$\tau \in \{n_1, \dots, n-1\} \tag{110}$$

such that $\rho(x_\tau, F) < \epsilon/4$. This implies that there exists

$$z_0 \in F \tag{111}$$

such that
$$\|x_\tau - z_0\| < \epsilon/4. \tag{112}$$

Let $i \geq \tau$ be an integer, $t \in \mathrm{supp}(w_i)$, $j \in \{1, \dots, p(t)\}$. It follows from (1), (66), (80), (84) and (111) that

$$\|z_0 - y_j^{(i,t)}\| \leq \|z_0 - P_{t_j}(y_{j-1}^{(i,t)})\| + \|P_{t_j}(y_{j-1}^{(i,t)}) - y_j^{(i,t)}\| \leq \|z - y_{j-1}^{(i,t)}\| + \Delta_i$$

and

$$\|z_0 - y_j^{(i,t)}\| \leq \|z_0 - y_0^{(i,t)}\| + j\Delta_i \leq \|z_0 - x_i\| + \bar{q}\Delta_i.$$

Together with the convexity of the norm and (78) this implies that

$$\begin{aligned}\|z_0 - x_{i+1}\| &\leq \|z_0 - \sum_{t\in \mathrm{supp}(w_i)} w_i(t)y_t^{(i)}\| + \|\sum_{t\in \mathrm{supp}(w_i)} w_i(t)y_t^{(i)} - x_{i+1}\| \\ &\leq \Delta_i + \sum_{t\in \mathrm{supp}(w_i)} w_i(t)\|z_0 - y_t^{(i)}\| \leq \|z_0 - x_i\| + \bar{q}\Delta_i + \Delta_i.\end{aligned}$$

Together with (73), (98) and (110) this implies that for each integer $i \geq \tau$,

$$\|z_0 - x_i\| \leq \|z_0 - x_\tau\| + (\bar{q}+1)\sum_{j=n_1}^{\infty} \Delta_j < \epsilon.$$

Since ϵ is any element of $(0,1]$ we conclude that the sequence $\{x_i\}_{i=0}^{\infty}$ converge and its limit belongs to F. Theorem 5.1 is proved. □

6. The fourth result

Theorem 6.1. *Assume that* $M > 1$, $F \subset B(0, M-1)$, (113)

$\bar{q}$ is a natural number, for each natural number s, $a_s > 0$ and b_s is a natural number. Let $\epsilon \in (0,1)$. Then there are $\delta \in (0,\epsilon)$ and a natural n such that for each $\{w_i\}_{i=0}^{\infty} \subset \mathcal{M}$ satisfying for each integer $i \geq 1$, each $t \in \mathrm{supp}(w_i)$,

$$p(t) \leq \bar{q}, \tag{114}$$

for each integer $p \geq 1$ and each integer $i \geq b_p$,

$$\sum\{w_i(t) :\ t \in \mathrm{supp}(w_i)(p, \bar{q})\} \geq a_p, \tag{115}$$

each $\{x_i\}_{i=0}^{\infty} \subset X$, each $\{\lambda_i\}_{i=1}^{\infty} \subset [0,\infty)$ satisfying

$$x_0 \in B(0, M) \tag{116}$$

for each natural number i, $(x_i, \lambda_i) \in A(x_{i-1}, w_{i-1}, \delta)$ (117)

the inequality $\rho(x_i, F) < \epsilon$ holds for all integers $i \geq n$.

Proof. Let $\{\Delta_i\}_{i=0}^{\infty} \subset (0,\infty)$ satisfy $\sum_{i=0}^{\infty} \Delta_i < \infty$.

Theorem 5.1 implies that there exists a natural number n such that the following property holds:

(i) for each $\{v_i\}_{i=0}^{\infty} \subset \mathcal{M}$ satisfying for each integer $i \geq 1$ and each $t \in \mathrm{supp}(v_i)$, $p(t) \leq \bar{q}$, for each integer $p \geq 1$, each integer $i \geq b_p$,

$$\sum \{v_i(t) : \ t \in \mathrm{supp}(v_i)(p, \bar{q})\} \geq a_p,$$

for each $\{y_i\}_{i=0}^{\infty} \subset X$ and each $\{\mu_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfying $y_0 \in B(0, M)$, for each natural number i, $(y_i, \mu_i) \in A(y_{i-1}, v_{i-1}, \Delta_{i-1})$ the inequality $\rho(y_i, F) < \epsilon/4$ holds for each integer $i \geq n$. Set

$$\delta_0 = \min\{\Delta_i : \ i = 0, \dots, n\}, \tag{118}$$

$$\delta = \min\{\epsilon, \delta_0\}(4n)^{-1}(\bar{q} + 1)^{-1}. \tag{119}$$

Assume that $\{w_i\}_{i=0}^{\infty} \subset \mathcal{M}$, (114) holds for each integer $i \geq 1$ and each $t \in \mathrm{supp}(w_i)$, for each integer $p \geq 1$ and each integer $i \geq b_p$ (115) holds, the sequences $\{x_i\}_{i=0}^{\infty} \subset X$, $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfy (116), (117) for each integer $i \geq 1$. By equations (118), (119) and property (i) applied to

$$v_i = w_i \ (i = 0, 1, \dots), \ y_i = x_i \ (i = 0, \dots, n), \ \mu_i = \lambda_i \ (i = 1, \dots, n),$$
$$(y_{i+1}, \mu_{i+1}) \in A(y, w_i, \delta) \ (i = n, n + 1, \dots)$$

the equations $\rho(x_n, F) = \rho(y_n, F) < \epsilon/4$ and $\|x_n\| \leq M$ hold. Assume that $p \geq 1$ is an integer and that for each integer $i \in \{1, \dots, p\}$

$$\rho(x_{in}, F) < \epsilon/4. \tag{120}$$

By (119) and (120),

$$\|x_{pn}\| \leq M. \tag{121}$$

Set $v_i = w_{i+pn} \ (i=0, 1, \dots), \ y_i = x_{i+pn} \ (i=0, \dots, n), \ \mu_i = \lambda_{i+pn} \ (i=1, \dots, n),$

$$(y_{i+1}, \mu_{i+1}) \in A(y_i, v_i, 0) \ \text{ for each integer } i \geq n.$$

Property (i) and (118), (119) and (121) imply that $\rho(x_{(p+1)n}, F) = \rho(y_n, F) < \epsilon/4$. Therefore for each integer integer $p \geq 1$, we have $\rho(x_{pn}, F) < \epsilon/4$ and there exists

$$z_p \in F \ \text{ such that } \ \|x_{pn} - z_p\| < \epsilon/4. \tag{122}$$

Let $p \geq 1$ be an integer, $i \in \{0, \dots, n-1\}$. By (16), (17), for each $t \in \mathrm{supp}(w_{i+pn})$ there exist $(y_t, \lambda_t) \in A_0(x_{i+pn}, t, \delta)$ such that

$$\|x_{i+pn+1} - \sum_{t \in \mathrm{supp}(w_i + pn)} w_{i+pn}(t) y_t\| \leq \delta, \tag{123}$$

$$\lambda_{i+1+p} = \max\{\lambda_t : \ t \in \mathrm{supp}(w_{i+pn})\}$$

and for each $t \in \mathrm{supp}(w_i + pn)$, there is a finite sequence $\{y_j^{(t)}\}_{j=0}^{p(t)} \subset X$ such that $y_0^{(t)} = x_{i+pn}$, and for all $j = 1, \dots, p(t)$, $\|y_j^{(t)} - P_{t_j}(y_{j-1}^{(t)})\| \leq \delta$ and $y_{p(t)}^{(t)} = y_t$. By the relation above, (1) and (122), for each $t \in \mathrm{supp}(w_{p+in})$, $j \in \{1, \dots, p(t)\}$,

$$\|z_p - y_j^{(t)}\| \leq \|z_p - P_{t_j}(y_{j-1}^{(i)})\| + \|P_{t_j}(y_{j-1}^{(t)} - y_j^{(t)}\| \leq \|z_p - y_{j-1}^{(t)}\| + \delta,$$
$$\|z_p - y_j^{(t)}\| \leq \|z_p - y_0^{(t)}\| + j\delta \leq \|z_p - x_{pn+i}\| + \bar{q}\delta,$$
$$\|z_p - y_t\| \leq \|z_p - x_{pn+i}\| + \bar{q}\delta, \ t \in \mathrm{supp}(w_{p+in}). \tag{124}$$

By (123) and (124) and convexity of the norm,

$$\begin{aligned}\|z_p - x_{pn+i+1}\| &\le \|z_p - \sum_{t\in\mathrm{supp}(w_{p+in})} w_{p+in}(t)y_t\| + \|\sum_{t\in\mathrm{supp}(w_{p+in})} w_{p+in}(t)y_t - x_{i+1}\| \\ &\le \sum_{t\in\mathrm{supp}(w_{p+in})} w_{p+in}(t)\|z_p - y_t\| + \delta \le \|z_p - x_{pn+i}\| + \bar{q}\delta + \delta \\ &= \|z_p - x_{pn+i}\| + (\bar{q}+1)\,\delta.\end{aligned}$$

Together with (122) and (123) for $i = 0, \ldots, n$,

$$\|z_p - x_{pn+i}\| \le \|z_{pn} - x_{pn}\| + n(\bar{q}+1)\,\delta < \epsilon.$$

Theorem 6.1 is proved. □

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