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$\mathcal{I}$-Convergence of Sequences of p-Bochner, p-Dunford and p-Pettis Integrable Functions with Values in a Banach Space

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We study $\mathcal{I}$-convergence of sequences of functions. Mainly, we study Vitali type $\mathcal{I}$-convergence theorems for sequences of p-Bochner, p-Dunford and p-Pettis integrable functions with values in a Banach space. With the help of these theorems, we characterize p-Bochner relatively compact subsets of p-Bochner integrable functions and p-Pettis relatively $\mathcal{I}$-sequentially compact subsets of p-Dunford integrable functions. We study uniform $\mathcal{I}$-convergence of sequences of p-Bochner, p-Dunford and p-Pettis integrable functions, and weak uniform $\mathcal{I}$-convergence of sequences of p-Dunford and p-Pettis integrable functions, and discuss how they are related to the p-Bochner and p-Pettis $\mathcal{I}$-convergence. $\mathcal{I}$-convergence of compact mappings are studied with special emphasis to compact linear operators on normed linear spaces. We also study $\mathcal{I}$-exhaustiveness, $\mathcal{I}$-weak exhaustiveness and $\mathcal{I}$-α-convergence of sequences of metric space-valued functions defined on a metric space, and sequences of p-Dunford integrable functions are discussed in this perspective.

Keywords: $\mathcal{I}$-convergence in measure, p-Bochner $\mathcal{I}$-uniformly integrable, p-Pettis $\mathcal{I}$-uniformly integrable, p-Bochner δ-$\mathcal{I}$-Cauchy, p-Pettis δ-$\mathcal{I}$-Cauchy.

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1. Introduction

Sequences of functions and their convergence play an important role in analysis. It is a fundamental question whether different properties of the elements of a sequence of functions are inherited to the limit function. To meet this question, different modes of convergence emerged.

In 1951, Steinhaus [18] and Fast [11] introduce the concept of statistical convergence independently. Since then several generalisations and applications of this notion are made. The idea of statistical convergence first appears, under the name of almost convergence, in the first edition (Warsaw, 1935) of the monograph of Zygmund [19]. In 2000, Kostyrko et al. introduce the idea of $\mathcal{I}$-convergence based on the notion

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of the ideal $\mathcal{I}$ of subsets of the set $\mathbb{N}$ of positive integers [12]. The concept of $\mathcal{I}$-convergence is a generalisation of statistical convergence. There are several kinds of convergence that generalize the usual convergence for a sequence of points in a metric space. Among them, the notion of $\mathcal{I}$-convergence is quite interesting and useful in several settings.

Convergence theorems for integrals play an important role in integration theory. Among different types of convergence theorems for integrals, Viatli type convergence theorems are of great importance; as such, they are paid much attention by several authors. Even for functions with values in a Banach space, many authors discuss this type of convergence theorems. Musiał obtains such convergence theorems for Pettis integrable functions with values in a Banach space [16]. Balcerzak and Musiał study Vitali type $\mathcal{I}$-convergence theorems for Lebesgue and Pettis integrals [4].

The object of the present paper is to study Viatli type $\mathcal{I}$-convergence theorems for p-Bochner, p-Dunford and p-Pettis integrable functions with values in a Banach space for $p \in [1, \infty)$. The work of Balcerzak and Musiał [4] inspires and motivates us to study in this direction.

In [13], we make an extensive study on nets and sequences of p-Bochner, p-Dunford and p-Pettis integrable functions. We present in the present paper the sequence $\mathcal{I}$-versions of a number of results of the said paper.

At the outset of the paper, we give the definitions of some important terms frequently used in the paper, such as p-Bochner (resp. p-Dunford) $\mathcal{I}$-uniformly integrable, $\mathcal{I}$-Cauchy, δ-$\mathcal{I}$-Cauchy sequences of p-Bochner (resp. p-Dunford) integrable functions and obtain the interrelations among them. Then, we first present the Vitali type $\mathcal{I}$-convergence theorem for p-Bochner integrable functions. Depending upon this theorem, we obtain the same for p-Dunford and p-Pettis integrable functions. With the help of these theorems, we characterise p-Bochner relatively compact subsets of p-Bochner integrable functions and p-Pettis relatively $\mathcal{I}$-sequentially compact subsets of p-Dunford integrable functions.

Next, we consider uniform $\mathcal{I}$-convergence of compact mappings in metric spaces and show that uniform $\mathcal{I}$-limit of a sequence of compact mappings is compact. We apply the result to sequence of compact linear operators in Banach spaces and obtain a well known fundamental result of functional analysis. We study uniform $\mathcal{I}$-convergence in $L_p(\mu, X)$, $\mathcal{D}_p(\mu, X)$ and $\mathcal{P}_p(\mu, X)$, and weakly uniform $\mathcal{I}$-convergence of sequences in $\mathcal{D}_p(\mu, X)$ and $\mathcal{P}_p(\mu, X)$. It is shown that in $L_p(\mu, X)$, uniform $\mathcal{I}$-convergence implies p-Bochner $\mathcal{I}$-convergence. On the other hand in $\mathcal{D}_p(\mu, X)$, uniform $\mathcal{I}$-convergence implies p-Pettis $\mathcal{I}$-convergence and weak uniform $\mathcal{I}$-convergence implies p-scalarly strong $\mathcal{I}$-convergence.

We study $\mathcal{I}$-closedness and $\mathcal{I}$-completeness in different settings and show how the spaces $L_p(\mu, X)$, $\mathcal{D}_p(\mu, X)$ and $\mathcal{P}_p(\mu, X)$ behave in this respect.

At last, we study $\mathcal{I}$-exhaustiveness, $\mathcal{I}$-weak exhaustiveness and $\mathcal{I}$-α-convergence of sequences of metric space-valued functions defined on a metric space. Athanassiadou, Boccuto, Dimitriou and Papanastassiou introduce these notions for sequences of real-valued functions defined on a metric space [1]. We study here the same in respect of sequences in $\mathcal{D}_p(\mu, X)$.

2. Notations, definitions and preliminaries

Throughout the paper (Ω, Σ, μ) stands for a finite measure space and X, for a real Banach space with dual X^*, B_X and B_{X^*} denoting the closed unit ball of X and X^* respectively.

For definition and some properties of a μ-measurable function defined on a finite measure space with values in a Banach space, we refer to [10]. The collection of all equivalence classes of μ-measurable functions defined on Ω with values in X is denoted by $\mathcal{M}(\Omega, \Sigma, \mu, X)$, or in short, by $\mathcal{M}(\mu, X)$. If $X = \mathbb{R}$, then it is denoted, simply, as $\mathcal{M}(\mu)$.

A function $f \in X^\Omega$ is said to be *scalarly μ-measurable* if $x^*f \in \mathcal{M}(\mu)$ for each $x^* \in X^*$. The collection of all scalar equivalence classes of scalarly μ-measurable functions defined on Ω with values in X is denoted by $\mathcal{M}_s(\Omega, \Sigma, \mu, X)$, or in short, by $\mathcal{M}_s(\mu, X)$. Both $\mathcal{M}(\mu, X)$ and $\mathcal{M}_s(\mu, X)$ are real vector spaces and the former is a subspace of the latter.

For definitions and many important properties of Bochner, Dunford and Pettis integrable functions, we again refer to [10]. For $p \in [1, \infty]$, a function $f \in X^\Omega$ is said to be p-Bochner (resp. p-Dunford) integrable if it is μ-measurable (resp. scalarly μ-measurable) and $\|f\| \in L_p(\mu)$ (resp. $x^*f \in L_p(\mu)$ for each $x^* \in X^*$, or equivalently, for each $x^* \in B_{X^*}$), $L_p(\mu)$ being the set of all equivalence classes of p-Lebesgue integrable functions defined on Ω; it is said to be p-Pettis integrable if it is p-Dunford integrable and Pettis integrable. The set of all equivalence classes of p-Bochner integrable functions in X^Ω is denoted by $L_p(\Omega, \Sigma, \mu, X)$, or in short, by $L_p(\mu, X)$. If $X = \mathbb{R}$, then it reduces to $L_p(\mu)$. The set of all scalar equivalence classes of p-Dunford and p-Pettis integrable functions in X^Ω are denoted by $\mathcal{D}_p(\Omega, \Sigma, \mu, X)$ and $\mathcal{P}_p(\Omega, \Sigma, \mu, X)$, or in short, by $\mathcal{D}_p(\mu, X)$ and $\mathcal{P}_p(\mu, X)$ respectively. For a detailed discussion on p-Dunford and p-Pettis integrable functions with values in a Banach space, we refer to [7].

The *p-Bochner norm* of $f \in L_p(\mu, X)$, denoted as $\|f\|_p$, is defined to be the p-Lebesgue norm of $\|f\|$, under which $L_p(\mu, X)$ becomes a Banach space; the *p-Pettis norm* of $f \in \mathcal{D}_p(\mu, X)$, denoted as $\|f\|_{P,p}$, is defined as $\|f\|_{P,p} = \sup_{x^* \in B_{X^*}} \|x^*f\|_p$, under which $\mathcal{D}_p(\mu, X)$ becomes a normed linear space. It should be noted that $L_p(\mu, X) \subset \mathcal{P}_p(\mu, X) \subset \mathcal{D}_p(\mu, X)$ and for any $f \in L_p(\mu, X)$, $\|f\|_{P,p} \leq \|f\|_p$. Clearly p-Bochner norm in $L_p(\mu)$ is the p-Lebesgue norm. In fact, if $X = \mathbb{R}$, then the term 'Bochner' is replaced by 'Lebesgue'. If $p = 1$, we shall write simply Bochner, Dunford and Pettis instead of p-Bochner, p-Dunford and p-Pettis respectively.

It should also be noted that if $p \in [1, \infty]$ and $q \in [1, p]$, then $L_p(\mu, X) \subset L_q(\mu, X)$, and $\|f\|_q \leq \|f\|_p$ for all $f \in L_p(\mu, X)$, and $\mathcal{D}_p(\mu, X) \subset \mathcal{D}_q(\mu, X)$, $\mathcal{P}_p(\mu, X) \subset \mathcal{P}_q(\mu, X)$, and $\|f\|_{P,q} \leq \|f\|_{P,p}$ for all $f \in \mathcal{D}_p(\mu, X)$.

From now on, we assume, if not stated otherwise, that $p \in [1, \infty)$.

Let us recall the following definitions:

(a) A sequence of functions $\{f_n\}$ in $\mathcal{M}(\mu, X)$ is said to be *convergent in measure* to $f \in \mathcal{M}(\mu, X)$ (resp. *Cauchy in measure*) if for each number $\varepsilon > 0$, the sequence $\big\{\mu(\{\omega \in \Omega : \|f_n(\omega) - f(\omega)\| > \varepsilon\})\big\}$ $\big($resp. the double sequence $\big\{\mu(\{\omega \in \Omega : \|f_n(\omega) - f_m(\omega)\| > \varepsilon\})\big\}\big)$ is null in $\mathbb{R}$.

(b) A sequence of functions $\{f_n\}$ in $\mathcal{M}_s(\mu, X)$ is said to be *scalarly convergent in measure* to $f \in \mathcal{M}_s(\mu, X)$ (resp. scalarly Cauchy in measure) if for each $x^* \in X^*$, $\{x^* f_n\}$ is convergent in measure to $x^* f$ (resp. $\{x^* f_n\}$ is Cauchy in measure) [4, p. 2028].

(c) A sequence of functions $\{f_n\}$ in $\mathcal{M}_s(\mu, X)$ is said to be *scalarly equi-convergent in measure* to $f \in \mathcal{M}_s(\mu, X)$ (resp. scalarly equi-Cauchy in measure) if for each $\varepsilon > 0$, the sequence $\left\{ \sup_{x^* \in B_{X^*}} \mu(\{\omega \in \Omega : |x^*(f_n(\omega) - f(\omega))| > \varepsilon\})\right\}$ $\big($resp. the double sequence $\left\{ \sup_{x^* \in B_{X^*}} \mu(\{\omega \in \Omega : |x^*(f_n(\omega) - f_m(\omega))| > \varepsilon\})\right\}\big)$ is null in $\mathbb{R}$ [4, p. 2028].

Let us recall that a collection Φ, in $L_p(\mu)$, is said to be *p-uniformly integrable* if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\int_E |\phi(\omega)|^p d\mu < \varepsilon$ for all $\phi \in \Phi$ and for all $E \in \Sigma$ with $\mu(E) < \delta$.

The following definitions are sequence versions of [13, p. 52, Definition 3.22 & p. 53, Definition 3.28]:

A sequence $\{f_n\}$ in $L_p(\mu, X)$ (resp. in $\mathcal{D}_p(\mu, X)$) is said to be

(a) *p-Bochner uniformly integrable (resp. p-Pettis uniformly integrable)* if $\{\|f_n\|\}$ (resp. $\{x^* f_n : x^* \in B_{X^*}\}$) is p-uniformly integrable in $L_p(\mu)$,

(b) *p-Bochner (resp. p-Pettis) δ-Cauchy* if for each $\varepsilon > 0$, there exist an $N \in \mathbb{N}$ and a $\delta > 0$ such that $\displaystyle\int_E \|f_n - f_m\|^p d\mu < \varepsilon \left(\text{resp. } \sup_{x^* \in B_{X^*}} \int_E |x^*(f_n - f_m)|^p d\mu < \varepsilon\right)$ for all $n, m \in \mathbb{N}$ with $n, m \geq N$ and for all $E \in \Sigma$ with $\mu(E) < \delta$,

(c) *p-Bochner (resp. p-Pettis) convergent* to $f \in L_p(\mu, X)$ (resp. to $f \in \mathcal{D}_p(\mu, X)$) if $\{f_n\}$ is convergent to f in p-Bochner (resp. in p-Pettis) norm,

(d) *p-Bochner (resp. p-Pettis) Cauchy* if $\{f_n\}$ is Cauchy in p-Bochner (resp. in p-Pettis) norm.

Let us recall that a non-empty family $\mathcal{I} \subset 2^{\mathbb{N}}$ of subsets of $\mathbb{N}$, the set of all natural numbers, is said to be an *ideal* in $\mathbb{N}$ if (a) for each $A, B \in \mathcal{I}$, we have $A \cup B \in \mathcal{I}$, and (b) for each $A \in \mathcal{I}$ and for each $B \subset A$, we have $B \in \mathcal{I}$. An ideal $\mathcal{I}$ is called *non-trivial* if $\mathbb{N} \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is called *admissible* if and only if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Let $\mathcal{I}$ be a non-trivial ideal of subsets of $\mathbb{N}$ and let (Z, ρ) be a metric space.

A sequence $\{z_n\}$ in Z is said to be *$\mathcal{I}$-convergent* to $z \in Z$, denoted as $z_n \to_{\mathcal{I}} z$, or, as $\mathcal{I}\text{-}\lim_n z_n = z$, if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : \rho(z_n, z) \geq \varepsilon\} \in \mathcal{I}$ [12, p. 671, Definition 3.1]. It is easy to note that a sequence $\{z_n\}$ in Z is $\mathcal{I}$-convergent to $z \in Z$ if and only if for each $\varepsilon > 0$, there exists an $A \in \mathcal{I}$ such that $\rho(z_n, z) < \varepsilon$ for all $n \in \mathbb{N} \setminus A$ [3, p. 4, Definition 2]. Also, it is well known that the limit of an $\mathcal{I}$-convergent sequence in a metric space is unique [12, p. 673, Proposition 3.1(i)].

It should be noted that usual convergence implies $\mathcal{I}$-convergence for any admissible ideal $\mathcal{I}$, while the converse is not true, in general. The converse is true if $\mathcal{I} = \mathcal{I}_{fin}$, the ideal of all finite subsets of $\mathbb{N}$. Thus the usual convergence in a metric space coincides with $\mathcal{I}_{fin}$-convergence. However, it can be easily shown that every $\mathcal{I}$-convergent sequence possesses a subsequence which is convergent to the same limit.

From now on, $\mathcal{I}$ stands for an admissible ideal on $\mathbb{N}$.

A double sequence $\{z_{mn}\}$ in a metric space (Z, ρ) is said to be *$\mathcal{I}$-convergent* to $z \in Z$ if for each $\varepsilon > 0$, there exists an $A \in \mathcal{I}$ such that $\rho(z_{mn}, z) < \varepsilon$ for all $m, n \in \mathbb{N} \setminus A$.

A sequence or a double sequence in a normed linear space is said to be *$\mathcal{I}$-null* if it is $\mathcal{I}$-convergent to 0.

A sequence $\{z_n\}$ in the metric space Z is said to be *$\mathcal{I}$-Cauchy* if for each $\varepsilon > 0$, there exists a $k \in \mathbb{N}$ such that the set $\{n \in \mathbb{N} : \rho(z_n, z_k) \geq \varepsilon\} \in \mathcal{I}$. It is Dems who introduces and studies $\mathcal{I}$-Cauchy sequences in a metric space [9, p. 124]. She establishes that a sequence $\{z_n\}$ in Z is $\mathcal{I}$-Cauchy if and only if for each $\varepsilon > 0$, there exists an $A \in \mathcal{I}$ such that $\rho(z_n, z_m) < \varepsilon$ for all $n, m \in \mathbb{N} \setminus A$ [9, p. 126, Proposition 4 ((1) $\Leftrightarrow$ (2))].

A subset S of Z is said to be *$\mathcal{I}$-complete* if every $\mathcal{I}$-Cauchy sequence in S is $\mathcal{I}$-convergent in S. It follows from [9, p. 124, Theorem 2] that a metric space is complete if and only if it is $\mathcal{I}$-complete.

A subset S of Z is said to be *$\mathcal{I}$-closed* in Z if the $\mathcal{I}$-limit of every $\mathcal{I}$-convergent sequence in S, is contained in S.

It is easy to note that a subset in a metric space is closed if and only if it is $\mathcal{I}$-closed.

3. Main results

We begin with the following definitions of some well-known terms in the context of ideals:

Definition 3.1. (a) A sequence of functions $\{f_n\}$ in $\mathcal{M}(\mu, X)$ is said to be *$\mathcal{I}$-convergent in measure* to $f \in \mathcal{M}(\mu, X)$ (resp. *$\mathcal{I}$-Cauchy in measure*) if for each $\varepsilon > 0$, the sequence $\big\{\mu(\{\omega \in \Omega : \|f_n(\omega) - f(\omega)\| > \varepsilon\})\big\}$ $\big($resp. the double sequence $\big\{\mu(\{\omega \in \Omega : \|f_n(\omega) - f_m(\omega)\| > \varepsilon\})\big\}\big)$ is $\mathcal{I}$-null in $\mathbb{R}$ [2, p. 723].

(b) A sequence of functions $\{f_n\}$ in $\mathcal{M}_s(\mu, X)$ is said to be *scalarly $\mathcal{I}$-convergent in measure* to $f \in \mathcal{M}_s(\mu, X)$ (resp. *scalarly $\mathcal{I}$-Cauchy in measure*) if for each $x^* \in X^*$, the sequence $\{x^* f_n\}$ is $\mathcal{I}$-convergent in measure to $x^* f$ (resp. $\{x^* f_n\}$ is $\mathcal{I}$-Cauchy in measure).

(c) A sequence of functions $\{f_n\}$ in $\mathcal{M}_s(\mu, X)$ is said to be *scalarly equi-$\mathcal{I}$-convergent in measure* to $f \in \mathcal{M}_s(\mu, X)$ (resp. *scalarly equi-$\mathcal{I}$-Cauchy in measure*) if for each $\varepsilon > 0$, the sequence $\big\{ \sup_{x^* \in B_{X^*}} \mu(\{\omega \in \Omega : |x^*(f_n(\omega) - f(\omega))| > \varepsilon\})\big\}$ $\big($resp. the double sequence $\big\{\sup_{x^* \in B_{X^*}} \mu(\{\omega \in \Omega : |x^*(f_n(\omega) - f_m(\omega))| > \varepsilon\})\big\}\big)$ is $\mathcal{I}$-null in $\mathbb{R}$ [4, p. 2035]. $\square$

It should be noted that the convergence in measure of a sequence in $\mathcal{M}(\mu, X)$ and the scalar equi-convergence in measure of a sequence in $\mathcal{M}_s(\mu, X)$ are each a metric convergence [13, p. 51, Lemma 3.20 (e) & Lemma 3.21 (e)]. Consequently, if a sequence in $\mathcal{M}(\mu, X)$ (resp. in $\mathcal{M}_s(\mu, X)$) is $\mathcal{I}$-convergent (resp. scalarly equi-$\mathcal{I}$-convergent) in measure, then it is $\mathcal{I}$-Cauchy (resp. scalarly equi-$\mathcal{I}$-Cauchy) in measure [9, p. 124, Proposition 1] and it has a subsequence which is convergent (resp. scalarly equi-convergent) in measure to the same limit.

Definition 3.2. A sequence $\{f_n\}$ in $L_p(\mu, X)$ (resp. in $\mathcal{D}_p(\mu, X)$) is said to be

(a) *p-Bochner $\mathcal{I}$-uniformly integrable (resp. p-Pettis $\mathcal{I}$-uniformly integrable)* if for each $\varepsilon > 0$, there exist an $A \in \mathcal{I}$ and a $\delta > 0$ such that $\int_E \|f_n\|^p d\mu < \varepsilon$ (resp. $\sup_{x^* \in B_{X^*}} \int_E |x^* f_n|^p d\mu < \varepsilon$) for all $n \in \mathbb{N} \setminus A$ and for all $E \in \Sigma$ with $\mu(E) < \delta$,

(b) *p-Bochner δ-$\mathcal{I}$-Cauchy (resp. p-Pettis δ-$\mathcal{I}$-Cauchy)* if for each $\varepsilon > 0$, there exist an $A \in \mathcal{I}$ and a number $\delta > 0$ such that $\int_E \|f_n - f_m\|^p d\mu < \varepsilon$ (respectively $\sup_{x^* \in B_{X^*}} \int_E |x^*(f_n - f_m)|^p d\mu < \varepsilon$) forall $n, m \in \mathbb{N} \setminus A$ and for all $E \in \Sigma$ with $\mu(E) < \delta$,

(c) *p-Bochner (resp. p-Pettis) $\mathcal{I}$-convergent* to $f \in L_p(\mu, X)$ (resp. to $f \in \mathcal{D}_p(\mu, X)$) if $\{f_n\}$ is $\mathcal{I}$-convergent to f in p-Bochner (resp. in p-Pettis) norm,

(d) *p-Bochner (resp. p-Pettis) $\mathcal{I}$-Cauchy* if $\{f_n\}$ is $\mathcal{I}$-Cauchy in p-Bochner (resp. in p-Pettis) norm. □

It is easy to note that

(a) a p-Bochner (resp. p-Pettis) uniformly integrable sequence of functions in the space $L_p(\mu, X)$ (resp. in $\mathcal{D}_p(\mu, X)$) is p-Bochner (resp. p-Pettis) $\mathcal{I}$-uniformly integrable,

(b) a p-Bochner (resp. p-Pettis) δ-Cauchy sequence of functions in $L_p(\mu, X)$ (resp. in $\mathcal{D}_p(\mu, X)$) is p-Bochner δ-$\mathcal{I}$-Cauchy (resp. p-Pettis δ-$\mathcal{I}$-Cauchy),

(c) a p-Bochner (resp. p-Pettis) convergent sequence in $L_p(\mu, X)$ (resp. in $\mathcal{D}_p(\mu, X)$) is p-Bochner (resp. p-Pettis) $\mathcal{I}$-convergent to the same limit,

(d) a p-Bochner (resp. p-Pettis) Cauchy sequence in $L_p(\mu, X)$ (resp. in $\mathcal{D}_p(\mu, X)$) is p-Bochner (resp. p-Pettis) $\mathcal{I}$-Cauchy.

If $\mathcal{I} = \mathcal{I}_{fin}$, then the converses of the above assertions are true.

From Chebyshev's lemma [13, p. 50, Corollary 3.18] follows the following lemma:

Lemma 3.3. (a) *Let $\{f_n\} \subset L_p(\mu, X)$ and let $f \in L_p(\mu, X)$. If $\{f_n\}$ is p-Bochner $\mathcal{I}$-convergent to f (resp. p-Bochner $\mathcal{I}$-Cauchy), then it is $\mathcal{I}$-convergent in measure to f (resp. $\mathcal{I}$-Cauchy in measure).*

(b) *Let $\{f_n\} \subset \mathcal{D}_p(\mu, X)$ and let $f \in \mathcal{D}_p(\mu, X)$. If $\{f_n\}$ is p-Pettis $\mathcal{I}$-convergent to f (resp. p-Pettis $\mathcal{I}$-Cauchy), then it is scalarly equi-$\mathcal{I}$-convergent in measure to f (resp. scalarly equi-$\mathcal{I}$-Cauchy in measure).*

Theorem 3.4. *The following statements hold good:*

(a) *A sequence $\{f_n\}$ in $L_p(\mu, X)$ is p-Bochner $\mathcal{I}$-uniformly integrable if and only if it is p-Bochner δ-$\mathcal{I}$-Cauchy.*

(b) *A sequence $\{f_n\}$ in $L_p(\mu, X)$ is p-Bochner $\mathcal{I}$-Cauchy if and only if it is $\mathcal{I}$-Cauchy in measure and is p-Bochner δ-$\mathcal{I}$-Cauchy.*

(c) *A sequence $\{f_n\}$ in $\mathcal{D}_p(\mu, X)$ is p-Pettis $\mathcal{I}$-uniformly integrable if and only if it is p-Pettis δ-$\mathcal{I}$-Cauchy and for each $\varepsilon > 0$, there exists an $A \in \mathcal{I}$ such that for each $n \in \mathbb{N} \setminus A$, there exists a $\delta > 0$ satisfying*

$$\sup_{x^* \in B_{X^*}} \int_E |x^* f_n|^p d\mu < \varepsilon$$

for all $E \in \Sigma$ with $\mu(E) < \delta$.

(d) *A sequence $\{f_n\}$ in $\mathcal{D}_p(\mu, X)$ is p-Pettis $\mathcal{I}$-Cauchy if and only if it is scalarly equi-$\mathcal{I}$-Cauchy in measure and is p-Pettis δ-$\mathcal{I}$-Cauchy.*

Proof. (a) Let $\{f_n\}$ be a sequence in $L_p(\mu, X)$. Let $\{f_n\}$ be p-Bochner $\mathcal{I}$-uniformly integrable. Let $\varepsilon > 0$. Then there exist an $A_1 \in \mathcal{I}$ and a $\delta_1' > 0$ such that

$$\int_E \|f_n\|^p d\mu < \frac{\varepsilon}{2^{p+1}}$$

for all $n \in \mathbb{N} \setminus A_1$ and for all $E \in \Sigma$ with $\mu(E) < \delta_1'$.

Now, for all $n, m \in \mathbb{N} \setminus A_1$ and for all $E \in \Sigma$ with $\mu(E) < \delta_1'$, we have

$$\int_E \|f_n - f_m\|^p d\mu \leq 2^p \left(\int_E \|f_n\|^p d\mu + \int_E \|f_m\|^p d\mu \right) < 2^p \left(\frac{\varepsilon}{2^{p+1}} + \frac{\varepsilon}{2^{p+1}} \right) = \varepsilon.$$

This shows that $\{f_n\}$ is p-Bochner δ-$\mathcal{I}$-Cauchy.

Conversely, let $\{f_n\}$ be p-Bochner δ-$\mathcal{I}$-Cauchy. Let $\varepsilon > 0$. Then there exist an $A_2 \in \mathcal{I}$ and a $\delta_1'' > 0$ such that

$$\int_E \|f_n - f_m\|^p d\mu < \frac{\varepsilon}{2^{p+1}}$$

for all $n, m \in \mathbb{N} \setminus A_2$ and for all $E \in \Sigma$ with $\mu(E) < \delta_1''$.

Let $m \in \mathbb{N} \setminus A_2$ be fixed. Since $\|f_m(\cdot)\| \in L_p(\mu)$, there exists a $\delta_1''' > 0$ such that

$$\int_E \|f_m\|^p d\mu < \frac{\varepsilon}{2^{p+1}}$$

for all $E \in \Sigma$ with $\mu(E) < \delta_1'''$.

Let $\delta_1 = \min\{\delta_1'', \delta_1'''\}$. Then $\delta_1 > 0$, and for all $n \in \mathbb{N} \setminus A_2$ and for all $E \in \Sigma$ with $\mu(E) < \delta_1$, we have

$$\int_E \|f_n\|^p d\mu \leq 2^p \left(\int_E \|f_n - f_m\|^p d\mu + \int_E \|f_m\|^p d\mu \right) < 2^p \left(\frac{\varepsilon}{2^{p+1}} + \frac{\varepsilon}{2^{p+1}} \right) = \varepsilon$$

which shows that $\{f_n\}$ is p-Bochner $\mathcal{I}$-uniformly integrable.

(b) Let $\{f_n\}$ be a sequence in $L_p(\mu, X)$. Let $\{f_n\}$ be p-Bochner $\mathcal{I}$-Cauchy. That $\{f_n\}$ is $\mathcal{I}$-Cauchy in measure follows from Lemma 3.3(a) and that it is p-Bochner δ-$\mathcal{I}$-Cauchy follows from the inequality

$$\int_E \|f_n - f_m\|^p d\mu \leq \int_\Omega \|f_n - f_m\|^p d\mu$$

for all $E \in \Sigma$ and for all $n, m \in \mathbb{N}$.

Conversely, let $\{f_n\}$ be $\mathcal{I}$-Cauchy in measure and p-Bochner δ-$\mathcal{I}$-Cauchy. Let $\varepsilon > 0$. Then there exist a $B_1 \in \mathcal{I}$ and a $\delta_2 > 0$ such that

$$\int_E \|f_n - f_m\|^p d\mu < \frac{\varepsilon^p}{2}$$

for all $n, m \in \mathbb{N} \setminus B_1$ and for all $E \in \Sigma$ with $\mu(E) < \delta_2$.

Since $\{f_n\}$ is $\mathcal{I}$-Cauchy in measure, there exists, for the above $\delta_2 > 0$, a $B_2 \in \mathcal{I}$ such that for all $n, m \in \mathbb{N} \setminus B_2$

$$\mu\Big(\Big\{\omega \in \Omega : \|f_n(\omega) - f_m(\omega)\| > \frac{\varepsilon}{(2\mu(\Omega))^{\frac{1}{p}}}\Big\}\Big) < \delta_2.$$

Let $B = B_1 \cup B_2$. Then $B \in \mathcal{I}$, and for all $n, m \in \mathbb{N} \setminus B$, we have

$$\begin{aligned}
&\int_\Omega \|f_n - f_m\|^p d\mu \\
&\leq \int_{\|f_n - f_m\| > \frac{\varepsilon}{(2\mu(\Omega))^{\frac{1}{p}}}} \|f_n - f_m\|^p d\mu + \int_{\|f_n - f_m\| \leq \frac{\varepsilon}{(2\mu(\Omega))^{\frac{1}{p}}}} \|f_n - f_m\|^p d\mu \\
&< \frac{\varepsilon^p}{2} + \Big(\frac{\varepsilon}{(2\mu(\Omega))^{\frac{1}{p}}}\Big)^p \mu(\Omega) = \varepsilon^p.
\end{aligned}$$

Hence it follows that $\|f_n - f_m\|_p < \varepsilon$ for all $n, m \in \mathbb{N} \setminus B$ which implies that $\{f_n\}$ is p-Bochner $\mathcal{I}$-Cauchy.

(c) Let $\{f_n\}$ be a sequence in $\mathcal{D}_p(\mu, X)$. Let $\{f_n\}$ be p-Pettis $\mathcal{I}$-uniformly integrable. Let $\varepsilon > 0$. Then there exist a $C_1 \in \mathcal{I}$ and a $\delta_3' > 0$ such that

$$\sup_{x^* \in B_{X^*}} \int_E |x^* f_n|^p d\mu < \frac{\varepsilon}{2^{p+2}}$$

for all $n \in \mathbb{N} \setminus C_1$ and for all $E \in \Sigma$ with $\mu(E) < \delta_3'$. Now, for all $n, m \in \mathbb{N} \setminus C_1$, for all $E \in \Sigma$ with $\mu(E) < \delta_3'$ and for all $x^* \in B_{X^*}$, we have

$$\begin{aligned}
\int_E |x^*(f_n - f_m)|^p d\mu &\leq 2^p \Big(\sup_{x^* \in B_{X^*}} \int_E |x^* f_n|^p d\mu + \sup_{x^* \in B_{X^*}} \int_E |x^* f_m|^p d\mu\Big) \\
&< 2^p \left(\frac{\varepsilon}{2^{p+2}} + \frac{\varepsilon}{2^{p+2}}\right) = \frac{\varepsilon}{2}.
\end{aligned}$$

Therefore

$$\sup_{x^* \in B_{X^*}} \int_E |x^*(f_n - f_m)|^p d\mu \leq \frac{\varepsilon}{2} < \varepsilon$$

for all $n, m \in \mathbb{N} \setminus C_1$ and for all $E \in \Sigma$ with $\mu(E) < \delta_3'$. This shows that $\{f_n\}$ is p-Pettis δ-$\mathcal{I}$-Cauchy.

The second part follows trivially.

Conversely, let $\varepsilon > 0$. Since, by hypothesis, $\{f_n\}$ is p-Pettis δ-$\mathcal{I}$-Cauchy, there exist a $C_2 \in \mathcal{I}$ and a $\delta_3'' > 0$ such that

$$\sup_{x^* \in B_{X^*}} \int_E |x^*(f_n - f_m)|^p d\mu < \frac{\varepsilon}{2^{p+1}}$$

for all $n, m \in \mathbb{N} \setminus C_2$ and for all $E \in \Sigma$ with $\mu(E) < \delta_3''$. Again, by hypothesis, there exists a $C_3 \in \mathcal{I}$ such that for each $n \in \mathbb{N} \setminus C_3$, there exists a $\delta_3''' > 0$ satisfying

$$\sup_{x^* \in B_{X^*}} \int_E |x^* f_n|^p d\mu < \frac{\varepsilon}{2^{p+1}}$$

for all $E \in \Sigma$ with $\mu(E) < \delta_3'''$.

Let $C = C_2 \cup C_3$. Then $C \in \mathcal{I}$. We now fix $N \in \mathbb{N} \setminus C$. Then $N \in \mathbb{N} \setminus C_3$ and so there exists a $\delta_3^{iv} > 0$ such that

$$\sup_{x^* \in B_{X^*}} \int_E |x^* f_N|^p d\mu < \frac{\varepsilon}{2^{p+1}}$$

for all $E \in \Sigma$ with $\mu(E) < \delta_3^{iv}$. Let $\delta_3 = \min\{\delta_3'', \delta_3^{iv}\}$. Then $\delta_3 > 0$ and $\delta_3 \leq \delta_3^{iv}$. Hence

$$\sup_{x^* \in B_{X^*}} \int_E |x^* f_N|^p d\mu < \frac{\varepsilon}{2^{p+1}}$$

for all $E \in \Sigma$ with $\mu(E) < \delta_3$. Again, $N \in \mathbb{N} \setminus C_2$ and $\delta_3 \leq \delta_3''$. Hence

$$\sup_{x^* \in B_{X^*}} \int_E |x^*(f_n - f_N)|^p d\mu < \frac{\varepsilon}{2^{p+1}}$$

for all $n \in \mathbb{N} \setminus C_2$ and for all $E \in \Sigma$ with $\mu(E) < \delta_3$. Therefore, for all $n \in \mathbb{N} \setminus C$ and for all $E \in \Sigma$ with $\mu(E) < \delta_3$, we have

$$\begin{aligned}
\sup_{x^* \in B_{X^*}} \int_E |x^* f_n|^p d\mu &= \sup_{x^* \in B_{X^*}} \int_E |x^*(f_n - f_N + f_N)|^p d\mu \\
&\leq 2^p \Big(\sup_{x^* \in B_{X^*}} \int_E |x^*(f_n - f_N)|^p d\mu + \sup_{x^* \in B_{X^*}} \int_E |x^* f_N|^p d\mu \Big) \\
&< 2^p \left(\frac{\varepsilon}{2^{p+1}} + \frac{\varepsilon}{2^{p+1}} \right) = \varepsilon.
\end{aligned}$$

This shows that $\{f_n\}$ is p-Pettis $\mathcal{I}$-uniformly integrable.

(d) Let $\{f_n\}$ be a sequence in $\mathcal{D}_p(\mu, X)$. Let $\{f_n\}$ be p-Pettis $\mathcal{I}$-Cauchy.

That $\{f_n\}$ is scalarly equi-$\mathcal{I}$-Cauchy in measure follows from Lemma 3.3 (b), and that it is p-Pettis δ-$\mathcal{I}$-Cauchy follows from the fact that for any $x^* \in B_{X^*}$ and for any $E \in \Sigma$ and for all $n, m \in \mathbb{N}$ we have

$$\left(\int_E |x^*(f_n - f_m)|^p d\mu \right)^{\frac{1}{p}} \leq \sup_{x^* \in B_{X^*}} \left(\int_\Omega |x^*(f_n - f_m)|^p d\mu \right)^{\frac{1}{p}} = \|f_n - f_m\|_{P,p}.$$

Conversely, let $\{f_n\}$ be scalarly equi-$\mathcal{I}$-Cauchy in measure and p-Pettis δ-$\mathcal{I}$-Cauchy.

Let $\varepsilon > 0$. Then there exist a $D_1 \in \mathcal{I}$ and a $\delta_4 > 0$ such that

$$\sup_{x^* \in B_{X^*}} \int_E |x^*(f_n - f_m)|^p d\mu < \frac{\varepsilon^p}{4}$$

for all $n, m \in \mathbb{N} \setminus D_1$ and for all $E \in \Sigma$ with $\mu(E) < \delta_4$.

Again, since $\{f_n\}$ is scalarly equi-$\mathcal{I}$-Cauchy in measure, there exists, for above $\delta_4 > 0$, a $D_2 \in \mathcal{I}$ such that, for all $n, m \in \mathbb{N} \setminus D_2$,

$$\sup_{x^* \in B_{X^*}} \mu\Big(\Big\{ \omega \in \Omega : |x^*(f_n(\omega) - f_m(\omega))| > \frac{\varepsilon}{(4\mu(\Omega))^{\frac{1}{p}}} \Big\} \Big) < \delta_4.$$

Let $D = D_1 \cup D_2$. Then $D \in \mathcal{I}$, and for all $n, m \in \mathbb{N} \setminus D$, for all $E \in \Sigma$ with $\mu(E) < \delta_4$ and for all $x^* \in B_{X^*}$, we have

$$\begin{aligned}
&\int_\Omega |x^*(f_n - f_m)|^p d\mu \\
&\leq \int_{|x^*(f_n-f_m)|>\frac{\varepsilon}{(4\mu(\Omega))^{\frac{1}{p}}}} |x^*(f_n - f_m)|^p d\mu + \int_{|x^*(f_n-f_m)|\leq\frac{\varepsilon}{(4\mu(\Omega))^{\frac{1}{p}}}} |x^*(f_n - f_m)|^p d\mu \\
&< \frac{\varepsilon^p}{4} + \left(\frac{\varepsilon}{(4\mu(\Omega))^{\frac{1}{p}}}\right)^p \mu(\Omega) = \frac{\varepsilon^p}{2}.
\end{aligned}$$

This implies that $\quad \|f_n - f_m\|_{P,p} = \sup_{x^* \in B_{X^*}} \left(\int_\Omega |x^*(f_n - f_m)|^p d\mu\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{2^{\frac{1}{p}}} < \varepsilon$

for all $n, m \in \mathbb{N} \setminus D$, whence it follows that $\{f_n\}$ is p-Pettis $\mathcal{I}$-Cauchy. □

We now present the Vitali type $\mathcal{I}$-convergence theorem for sequences of p-Bochner integrable functions.

Theorem 3.5. *Let $\{f_n\}$ be a sequence in $L_p(\mu, X)$ and let $f \in \mathcal{M}(\mu, X)$. Then the following statements are equivalent:*

(a) *$f \in L_p(\mu, X)$ and $\{f_n\}$ is p-Bochner $\mathcal{I}$-convergent to f.*

(b) *$\{f_n\}$ is $\mathcal{I}$-convergent in measure to f and is p-Bochner $\mathcal{I}$-Cauchy.*

(c) *$\{f_n\}$ is $\mathcal{I}$-convergent in measure to f and is p-Bochner δ-$\mathcal{I}$-Cauchy.*

Proof. (a) $\Longrightarrow$ (b): That $\{f_n\}$ is $\mathcal{I}$-convergent in measure to f follows from Lemma 3.3 (a) and that it is p-Bochner $\mathcal{I}$-Cauchy follows from the fact that an $\mathcal{I}$-convergent sequence in a metric space is $\mathcal{I}$-Cauchy [9, p. 124, Proposition 1].

(b) $\Longrightarrow$ (a): Since $L_p(\mu, X)$ is complete, the sequence $\{f_n\}$ is p-Bochner $\mathcal{I}$-convergent in $L_p(\mu, X)$ [9, p. 124, Theorem 2(1)]. So, there exists a $g \in L_p(\mu, X)$ such that $\{f_n\}$ is p-Bochner $\mathcal{I}$-convergent to g. Hence, by Lemma 3.3 (a), it follows that $\{f_n\}$ is $\mathcal{I}$-convergent in measure to g. By uniqueness (up to equal μ-almost everywhere) of $\mathcal{I}$-limit of an $\mathcal{I}$-convergent sequence in measure, it follows that $f = g$ μ-a.e. which implies that $f \in L_p(\mu, X)$ and $\{f_n\}$ is p-Bochner $\mathcal{I}$-convergent to f.

(b) $\Longleftrightarrow$ (c): Follows from Theorem 3.4 (b) and the fact that $\{f_n\}$ is $\mathcal{I}$-Cauchy in measure whenever it is $\mathcal{I}$-convergent in measure. □

If we take $\mathcal{I} = \mathcal{I}_{fin}$ in Theorem 3.5, then we have the following Vitali type convergence theorem for p-Bochner integrable functions:

Corollary 3.6. *Let $\{f_n\}$ be a sequence in $L_p(\mu, X)$ and let $f \in \mathcal{M}(\mu, X)$. Then the following statements are equivalent:*

(a) *$f \in L_p(\mu, X)$ and $\{f_n\}$ is p-Bochner convergent to f.*

(b) *$\{f_n\}$ is convergent in measure to f and is p-Bochner Cauchy.*

(c) *$\{f_n\}$ is convergent in measure to f and is p-Bochner δ-Cauchy.*

A subset $\mathcal{S}$ of $\mathcal{M}(\mu, X)$ is said to be *relatively $\mathcal{I}$-sequentially compact in measure* if every sequence in S has a subsequence which is $\mathcal{I}$-convergent in measure.

A subset $\mathcal{S}$ of $L_p(\mu, X)$ is said to be *p-Bochner $\mathcal{I}$-precompact (resp. p-Bochner δ-$\mathcal{I}$-precompact)* if every sequence in $\mathcal{S}$ has a p-Bochner $\mathcal{I}$-Cauchy (resp. p-Bochner δ-$\mathcal{I}$-Cauchy) subsequence.

Let us recall that a subset $\mathcal{S}$ of $L_p(\mu, X)$ is said to be *p-Bochner uniformly integrable* if $\{\|f\| : f \in \mathcal{S}\}$ is p-uniformly integrable in $L_p(\mu)$.

Corollary 3.7. *Let $\mathcal{S} \subset L_p(\mu, X)$ be relatively $\mathcal{I}$-sequentially compact in measure. Then the following statements are equivalent:*

(a) *$\mathcal{S}$ is p-Bochner relatively compact.*

(b) *$\mathcal{S}$ is p-Bochner uniformly integrable.*

(c) *Every sequence in $\mathcal{S}$ is p-Bochner uniformly integrable.*

(d) *$\mathcal{S}$ is p-Bochner precompact.*

(e) *$\mathcal{S}$ is p-Bochner δ-precompact.*

(f) *Every sequence in $\mathcal{S}$ is p-Bochner $\mathcal{I}$-uniformly integrable.*

(g) *$\mathcal{S}$ is p-Bochner $\mathcal{I}$-precompact.*

(h) *$\mathcal{S}$ is p-Bochner δ-$\mathcal{I}$-precompact.*

Proof. (a) $\Longrightarrow$ (b): Let $\mathcal{S}$ be p-Bochner relatively compact. Then it is easy to see that the set $\{\|f\| : f \in \mathcal{S}\}$ is relatively compact and hence p-uniformly integrable in $L_p(\mu)$, whence it follows that $\mathcal{S}$ is p-Bochner uniformly integrable.

(a) $\Longrightarrow$ (d) & (b) $\Longrightarrow$ (c) $\Longrightarrow$ (f): Trivial.

(d) $\Longrightarrow$ (e): Follows from [13, p. 53, Theorem 3.29 (b)].

(d) $\Longrightarrow$ (g): Follows from the fact that every p-Bochner Cauchy sequence in $L_p(\mu, X)$ is p-Bochner $\mathcal{I}$-Cauchy.

(e) $\Longrightarrow$ (h): Follows from the fact that every p-Bochner δ-Cauchy sequence in $L_p(\mu, X)$ is p-Bochner δ-$\mathcal{I}$-Cauchy.

(f) $\Longrightarrow$ (h): Follows from Theorem 3.4 (a).

(g) $\Longrightarrow$ (h): Follows from Theorem 3.4 (b).

(h) $\Longrightarrow$ (a): Let $\{f_n\}$ be a sequence in $\mathcal{S}$. Then, by hypothesis, it has a subsequence $\{f_{n_k}\}$ which is $\mathcal{I}$-convergent in measure and hence, there is a subsequence $\{f_{n_{k_j}}\}$ of $\{f_{n_k}\}$ which is convergent in measure. Again, by hypothesis, $\{f_{n_{k_j}}\}$ has a subsequence $\{f_{n_{k_{j_l}}}\}$ which is p-Bochner δ-$\mathcal{I}$-Cauchy. Also, $\{f_{n_{k_{j_l}}}\}$ is convergent in measure and hence, is $\mathcal{I}$-convergent in measure. Then by Theorem 3.5 ((c) $\Longrightarrow$ (a)), it follows that $\{f_{n_{k_{j_l}}}\}$ is p-Bochner $\mathcal{I}$-convergent and hence, it has a subsequence which is p-Bochner convergent. Hence, it follows that $\mathcal{S}$ is p-Bochner relatively compact and (a) follows. □

Following [5, p. 247, Definition 3.3], we introduce the following definitions:

Definition 3.8. (a) Let $\{f_n\} \subset \mathcal{D}_p(\mu, X)$ and let $f \in \mathcal{D}_p(\mu, X)$. Then $\{f_n\}$ is said to be *p-scalarly strongly $\mathcal{I}$-convergent to f (resp. p-scalarly strongly $\mathcal{I}$-Cauchy)* if for each $x^* \in X^*$, $\{x^* f_n\}$ is p-Lebesgue $\mathcal{I}$-convergent to $x^* f$ (resp. p-Lebesgue $\mathcal{I}$-Cauchy).

(b) A sequence $\{f_n\} \subset \mathcal{D}_p(\mu, X)$ is said to be *p-scalarly $\mathcal{I}$-uniformly integrable (resp. p-scalarly δ-$\mathcal{I}$-Cauchy)* if for each $x^* \in X^*$, $\{x^* f_n\}$ is p-Lebesgue $\mathcal{I}$-uniformly integrable (resp. p-Lebesgue δ-$\mathcal{I}$-Cauchy). □

It is easy to verify that if $\{f_n\} \subset \mathcal{D}_p(\mu, X)$ is p-Pettis $\mathcal{I}$-convergent to f (resp. p-Pettis $\mathcal{I}$-Cauchy, resp. p-Pettis $\mathcal{I}$-uniformly integrable, resp. p-Pettis δ-$\mathcal{I}$-Cauchy), then it is p-scalarly strongly $\mathcal{I}$-convergent to f (resp. p-scalarly strongly $\mathcal{I}$-Cauchy, resp. p-scalarly $\mathcal{I}$-uniformly integrable, resp. p-scalarly δ-$\mathcal{I}$-Cauchy).

An application of Theorem 3.4 (a) to the sequence $\{x^* f_n\}$ for each $x^* \in X^*$ yields that a sequence $\{f_n\}$ is p-scalarly $\mathcal{I}$-uniformly integrable if and only if it is p-scalarly δ-$\mathcal{I}$-Cauchy.

An application of Theorem 3.5 to the sequence $\{x^* f_n\}$ for each $x^* \in X^*$ yields the following result:

Theorem 3.9. *Let $\{f_n\}$ be a sequence in $\mathcal{D}_p(\mu, X)$ and let $f \in \mathcal{M}_s(\mu, X)$. Then the following statements are equivalent:*

(a) *$f \in \mathcal{D}_p(\mu, X)$ and $\{f_n\}$ is p-scalarly strongly $\mathcal{I}$-convergent to f.*

(b) *$\{f_n\}$ is scalarly $\mathcal{I}$-convergent in measure to f and is p-scalarly strongly $\mathcal{I}$-Cauchy.*

(c) *$\{f_n\}$ is scalarly $\mathcal{I}$-convergent in measure to f and is p-scalarly δ-$\mathcal{I}$-Cauchy.*

For any $f \in X^\Omega$, we define the mapping $T_f : X^* \longrightarrow \mathbb{R}^\Omega$ as $T_f(x^*) = x^* f$ for $x^* \in X^*$. Clearly T_f is a weak*-pointwise continuous linear operator.

If $f \in \mathcal{M}_s(\mu, X)$, then $T_f \in (\mathcal{M}(\mu))^{X^*}$ and T_f is continuous with respect to weak*-topology of X^* and topology of pointwise convergence of $\mathcal{M}(\mu)$.

If $f \in \mathcal{D}_p(\mu, X)$, then $T_f \in L(X^*, L_p(\mu))$ with $\|T_f(x^*)\|_p = \|x^* f\|_p$ for each $x^* \in X^*$ and $\|T_f\| = \|f\|_{P,p}$ [14, p. 8, Corollary 13].

Let S be a non-empty set and let (Z, ρ) be a metric space.

A sequence of functions $\{f_n\}$ in Z^S is said to be *pointwise $\mathcal{I}$-convergent* to a function $f \in Z^S$ (resp. pointwise $\mathcal{I}$-Cauchy) on a subset T of S if for each $t \in T$, the sequence $\{f_n(t)\}$ is $\mathcal{I}$-convergent to $f(t)$ (resp. $\mathcal{I}$-Cauchy), i.e., if for each $t \in T$ and for each $\varepsilon > 0$, there exists an $A \in \mathcal{I}$ such that $\rho(f_n(t), f(t)) < \varepsilon$ for all $n \in \mathbb{N} \setminus A$ (resp. $\rho(f_n(t), f_m(t)) < \varepsilon$ for all $n, m \in \mathbb{N} \setminus A$).

A sequence of functions $\{f_n\}$ in Z^S is said to be *uniformly $\mathcal{I}$-convergent* to a function $f \in Z^S$ (resp. uniformly $\mathcal{I}$-Cauchy) on a subset T of S if for each $\varepsilon > 0$, there exists an $A \in \mathcal{I}$ such that $\rho(f_n(t), f(t)) < \varepsilon$ for all $n \in \mathbb{N} \setminus A$ and for all $t \in T$ [2, p. 717] (resp. $\rho(f_n(t), f_m(t)) < \varepsilon$ for all $n, m \in \mathbb{N} \setminus A$ and for all $t \in T$).

It is evident that a uniformly $\mathcal{I}$-Cauchy sequence of functions in Z^S is pointwise $\mathcal{I}$-Cauchy.

Balcerzak et al. pointed out that each uniformly $\mathcal{I}$-convergent sequence of functions $\{f_n\}$ with values in a metric space contains a subsequence which is uniformly convergent [2, p. 718, Remark 4], obviously to the same limit as the uniform $\mathcal{I}$-limit of $\{f_n\}$.

From these very definitions we derive the following two lemmas:

Lemma 3.10. *Let $\{f_n\} \subset \mathcal{M}_s(\mu, X)$, $f \in \mathcal{M}_s(\mu, X)$ and let $\mathcal{M}(\mu)$ be induced with the topology of convergence in measure. Then*

(a) *$\{f_n\}$ is scalarly $\mathcal{I}$-convergent in measure to f (resp. scalarly $\mathcal{I}$-Cauchy in measure) if and only if $\{T_{f_n}\}$ is pointwise $\mathcal{I}$-convergent to T_f (resp. $\{T_{f_n}\}$ is pointwise $\mathcal{I}$-Cauchy) in $(\mathcal{M}(\mu))^{X^*}$ on X^*.*

(b) *$\{f_n\}$ is scalarly equi-$\mathcal{I}$-convergent in measure to f (resp. scalarly equi-$\mathcal{I}$-Cauchy in measure) if and only if $\{T_{f_n}\}$ is uniformly $\mathcal{I}$-convergent to T_f (resp. $\{T_{f_n}\}$ is uniformly $\mathcal{I}$-Cauchy) in $(\mathcal{M}(\mu))^{X^*}$ on B_{X^*}.*

Lemma 3.11. *Let $\{f_n\}$ be a sequence in $\mathcal{D}_p(\mu, X)$ and let $f \in \mathcal{D}_p(\mu, X)$. Then*

(a) *$\{f_n\}$ is p-scalarly strongly $\mathcal{I}$-convergent to f (resp. p-scalarly strongly $\mathcal{I}$-Cauchy) if and only if $\{T_{f_n}\}$ is pointwise $\mathcal{I}$-convergent to T_f (resp. pointwise $\mathcal{I}$-Cauchy) in $(L_p(\mu))^{X^*}$ on X^*.*

(b) *$\{f_n\}$ is p-Pettis $\mathcal{I}$-convergent to f (resp. p-Pettis $\mathcal{I}$-Cauchy) if and only if $\{T_{f_n}\}$ is uniformly $\mathcal{I}$-convergent to T_f (resp. uniformly $\mathcal{I}$-Cauchy) in $(L_p(\mu))^{X^*}$ on B_{X^*}, i.e., if and only if $\{T_{f_n}\}$ is $\mathcal{I}$-convergent to T_f (resp. $\mathcal{I}$-Cauchy) in $L(X^*, L_p(\mu))$ in operator norm.*

The following result is the $\mathcal{I}$-version of [6, p. 287, Lemma 6.2.6]:

Lemma 3.12. *Let S be a non-empty set and let (Z, ρ) be a metric space. Let $\{g_n\}$ be a sequence in Z^S and let $g \in Z^S$. Then $\{g_n\}$ is uniformly $\mathcal{I}$-convergent to g on a subset T of S if and only if $\{g_n\}$ is pointwise $\mathcal{I}$-convergent to g on T and uniformly $\mathcal{I}$-Cauchy on T.*

Proof. The necessity part is trivial.

For the sufficiency part, let $\varepsilon > 0$. Then there exists an $A \in \mathcal{I}$ such that

$$\rho(g_n(s), g_m(s)) < \frac{\varepsilon}{2}$$

for all $n, m \in \mathbb{N} \setminus A$ and for all $s \in T$. Let $s \in T$ be arbitrarily fixed. Then $\{g_n(s)\}$ is $\mathcal{I}$-convergent to $g(s)$ and hence, there exists a $B_s \in \mathcal{I}$ such that

$$\rho(g_n(s), g(s)) < \frac{\varepsilon}{2}$$

for all $n \in \mathbb{N} \setminus B_s$. Let $m \in \mathbb{N} \setminus (A \cup B_s)$ be fixed. Then $m \in \mathbb{N} \setminus A$ and $m \in \mathbb{N} \setminus B_s$. Hence for all $n \in \mathbb{N} \setminus A$, we have

$$\rho(g_n(s), g(s)) \leq \rho(g_n(s), g_m(s)) + \rho(g_m(s), g(s)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $\rho(g_n(s), g(s)) < \varepsilon$ for all $s \in T$ and for all $n \in \mathbb{N} \setminus A$, whence the result follows. □

The following result follows from Lemma 3.10 and Lemma 3.12:

Lemma 3.13. *Let $\{f_n\} \subset \mathcal{M}_s(\mu, X)$ and let $f \in \mathcal{M}_s(\mu, X)$. Then $\{f_n\}$ is scalarly equi-$\mathcal{I}$-convergent in measure to f if and only if $\{f_n\}$ is scalarly $\mathcal{I}$-convergent in measure to f and is scalarly equi-$\mathcal{I}$-Cauchy in measure.*

Now, we are in a position to present the Vitali type $\mathcal{I}$-convergence theorem for sequences of p-Dunford integrable functions.

Theorem 3.14. *Let $\{f_n\}$ be a sequence in $\mathcal{D}_p(\mu, X)$ and let $f \in \mathcal{M}_s(\mu, X)$. Then the following statements are equivalent:*

(a) *$f \in \mathcal{D}_p(\mu, X)$ and $\{f_n\}$ is p-Pettis $\mathcal{I}$-convergent to f.*

(b) *$f \in \mathcal{D}_p(\mu, X)$, $\{f_n\}$ is scalarly equi-$\mathcal{I}$-convergent in measure to f and $\{f_n - f\}$ is p-Pettis $\mathcal{I}$-uniformly integrable.*

(c) *$\{f_n\}$ is scalarly equi-$\mathcal{I}$-convergent in measure to f and is p-Pettis δ-$\mathcal{I}$-Cauchy.*

(d) *$\{f_n\}$ is scalarly equi-$\mathcal{I}$-convergent in measure to f and is p-Pettis $\mathcal{I}$-Cauchy.*

(e) *$\{f_n\}$ is scalarly $\mathcal{I}$-convergent in measure to f and is p-Pettis $\mathcal{I}$-Cauchy.*

(f) *$f \in \mathcal{D}_p(\mu, X)$, $\{f_n\}$ is p-scalarly strongly $\mathcal{I}$-convergent to f and is p-Pettis $\mathcal{I}$-Cauchy.*

Proof. (a) $\Longrightarrow$ (b): Let $f \in \mathcal{D}_p(\mu, X)$ and let $\{f_n\}$ be p-Pettis $\mathcal{I}$-convergent to f. That $\{f_n\}$ is scalarly equi-$\mathcal{I}$-convergent in measure to f follows from Lemma 3.3 (b) and that $\{f_n - f\}$ is is p-Pettis $\mathcal{I}$-uniformly integrable follows from the inequality

$$\sup_{x^* \in B_{X^*}} \int_E |x^*(f_n - f)|^p d\mu \leq \sup_{x^* \in B_{X^*}} \int_\Omega |x^*(f_n - f)|^p d\mu \leq \|f_n - f\|_{P,p}^p$$

for all $E \in \Sigma$.

(b) $\Longrightarrow$ (c): It follows from hypothesis and Theorem 3.4 (c) that $\{f_n - f\}$ is p-Pettis δ-$\mathcal{I}$-Cauchy whence it follows easily that $\{f_n\}$ is p-Pettis δ-$\mathcal{I}$-Cauchy.

(c) $\Longrightarrow$ (d): Follows from Lemma 3.13 and Theorem 3.4 (d).

(d) $\Longrightarrow$ (e): Follows from Lemma 3.13.

(e) $\Longrightarrow$ (f): From hypothesis, it follows that $\{f_n\}$ is scalarly $\mathcal{I}$-convergent in measure to f and is p-scalarly strongly $\mathcal{I}$-Cauchy. Hence, the result follows from Theorem 3.9 ((b) $\Longrightarrow$ (a)).

(f) $\Longrightarrow$ (a): From Lemma 3.11 (a), it follows that $\{T_{f_n}\}$ is pointwise $\mathcal{I}$-convergent to T_f in $(L_p(\mu))^{X^*}$ on X^* and hence on B_{X^*}. Again, from Lemma 3.11 (b), it follows that $\{T_{f_n}\}$ is uniformly $\mathcal{I}$-Cauchy on B_{X^*}. Hence, by Lemma 3.12, $\{T_{f_n}\}$ is uniformly $\mathcal{I}$-convergent to T_f on B_{X^*}. Hence the result follows from Lemma 3.11(b). □

A subset $\mathcal{S}$ of $\mathcal{D}_p(\mu, X)$ is said to be *p-Pettis $\mathcal{I}$-precompact (resp. p-Pettis δ-$\mathcal{I}$-precompact)* if every sequence in $\mathcal{S}$ has a p-Pettis $\mathcal{I}$-Cauchy (resp. p-Pettis δ-$\mathcal{I}$-Cauchy) subsequence.

By $\mathcal{I}$-closure (resp. scalar equi-$\mathcal{I}$-closure) in measure of a subset of $\mathcal{M}(\mu, X)$ (resp. of $\mathcal{M}_s(\mu, X)$), we mean its $\mathcal{I}$-closure with respect to the topology of convergence (resp. the topology of scalar equi-convergence) in measure.

Corollary 3.15. *The $\mathcal{I}$-closure (resp. scalar equi-$\mathcal{I}$-closure) in measure of a p-Bochner (resp. p-Pettis) δ-$\mathcal{I}$-precompact subset of $L_p(\mu, X)$ (resp. of $\mathcal{D}_p(\mu, X)$) coincides with its p-Bochner (resp. p-Pettis) $\mathcal{I}$-closure.*

Proof. Let $\mathcal{S}$ be a p-Bochner (resp. p-Pettis) δ-$\mathcal{I}$-precompact subset of $L_p(\mu, X)$ (resp. $\mathcal{D}_p(\mu, X)$) and let $\{f_n\}$ be a sequence in $\mathcal{S}$ which is $\mathcal{I}$-convergent (resp. scalarly equi-$\mathcal{I}$-convergent) in measure to $f \in \mathcal{M}(\mu, X)$ (resp. to $f \in \mathcal{M}_s(\mu, X)$).

Then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$, convergent (resp. scalarly equi-convergent) in measure to f. By hypothesis, $\{f_{n_k}\}$ has a subsequence $\{f_{n_{k_j}}\}$ which is p-Bochner (resp. p-Pettis) δ-$\mathcal{I}$-Cauchy. Since $\{f_{n_k}\}$ is convergent (resp. scalarly equi-convergent) in measure to f, $\{f_{n_{k_j}}\}$ is so. Hence, it is $\mathcal{I}$-convergent (resp. scalarly equi-$\mathcal{I}$-convergent) in measure to f. An application of Theorem 3.5 ((c) $\Longrightarrow$ (a)) (resp. Theorem 3.14 ((c) $\Longrightarrow$ (a))) implies that $f \in L_p(\mu, X)$ (resp. $f \in \mathcal{D}_p(\mu, X)$) and $\{f_{n_{k_j}}\}$ is p-Bochner (resp. p-Pettis) $\mathcal{I}$-convergent to f. Hence, it follows that $\mathcal{I}$-closure (resp. scalar equi-$\mathcal{I}$-closure) in measure of $\mathcal{S}$ is contained in its p-Bochner (resp. p-Pettis) $\mathcal{I}$-closure. On the other hand, it follows from Lemma 3.3 that the p-Bochner (resp. p-Pettis) $\mathcal{I}$-closure of any subset of $L_p(\mu, X)$ (resp. $\mathcal{D}_p(\mu, X)$) is contained in its $\mathcal{I}$-closure (resp. scalar equi-$\mathcal{I}$-closure) in measure. □

Let us recall that a subset S of a metric space (X, d) is said to be relatively $\mathcal{I}$-sequentially compact if every sequence in S has an $\mathcal{I}$-convergent subsequence.

The following lemma follows from the fact that every $\mathcal{I}$-convergent sequence in a metric space has a convergent subsequence:

Lemma 3.16. *A subset of a metric space is relatively $\mathcal{I}$-sequentially compact if and only if it is relatively compact.*

A subset $\mathcal{F}$ of $\mathcal{M}_s(\mu, X)$ is said to be *relatively scalarly equi-sequentially (resp. relatively scalarly equi-$\mathcal{I}$-sequentially) compact in measure* if every sequence in $\mathcal{F}$ has a subsequence which is scalarly equi-convergent (resp. scalarly equi-$\mathcal{I}$-convergent) in measure in $\mathcal{M}_s(\mu, X)$.

Corollary 3.17. *Let $\mathcal{S} \subset \mathcal{D}_p(\mu, X)$. Then the following statements are equivalent:*

(a) *$\mathcal{S}$ is p-Pettis relatively compact.*

(b) *$\mathcal{S}$ is p-Pettis relatively $\mathcal{I}$-sequentially compact.*

(c) *$\mathcal{S}$ is relatively scalarly equi-$\mathcal{I}$-sequentially compact in measure and is p-Pettis $\mathcal{I}$-precompact.*

(d) *$\mathcal{S}$ is relatively scalarly equi-$\mathcal{I}$-sequentially compact in measure and is p-Pettis δ-$\mathcal{I}$-precompact.*

Proof. (a) $\Longleftrightarrow$ (b): Follows from Lemma 3.16.

(b) $\Longrightarrow$ (c): Follows from Theorem 3.14 ((a) $\Longrightarrow$ (d)).

(c) $\Longrightarrow$ (d): Follows from Theorem 3.4 (d).

(d) $\Longrightarrow$ (b): Let $\{f_n\}$ be a sequence in $\mathcal{S}$. Since $\mathcal{S}$ is relatively scalarly equi-$\mathcal{I}$-sequentially compact in measure and since scalar equi-convergence in measure is a metric convergence, it follows from Lemma 3.16 that $\mathcal{S}$ is relatively scalarly equi-sequentially compact in measure. Therefore, $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ which is scalarly equi-convergent in measure to some $f \in \mathcal{M}_s(\mu, X)$. Again, by hypothesis, there exists a subsequence $\{f_{n_{k_j}}\}$ of $\{f_{n_k}\}$ which is p-Pettis δ-$\mathcal{I}$-Cauchy. Also, $\{f_{n_{k_j}}\}$ is scalarly equi-convergent in measure to f and hence, is scalarly equi-$\mathcal{I}$-convergent in measure to f. So, it follows from Theorem 3.14 ((c) $\Longrightarrow$ (a)) that $f \in \mathcal{D}_p(\mu, X)$ and $\{f_{n_{k_j}}\}$ is p-Pettis $\mathcal{I}$-convergent to f. Hence, (b) follows. □

The following result follows by some routine arguments:

Lemma 3.18. *Let $\{f_n\}$ be a sequence in $\mathcal{D}_p(\mu, X)$ and let $f \in \mathcal{D}_p(\mu, X)$. Then any two of the following statements imply the other:*

(a) *$\{f_n\}$ is p-Pettis $\mathcal{I}$-uniformly integrable.*

(b) *$\{f_n - f\}$ is p-Pettis $\mathcal{I}$-uniformly integrable.*

(c) *$\{x^*f : x^* \in B_{X^*}\}$ is p-uniformly integrable.*

Theorem 3.19. *Let $\{f_n\}$ be a sequence in $\mathcal{D}_p(\mu, X)$ and let $f \in \mathcal{M}_s(\mu, X)$. Then the following statements are equivalent:*

(a) *$f \in \mathcal{D}_p(\mu, X)$, $\{f_n\}$ is p-Pettis $\mathcal{I}$-convergent to f and is p-Pettis $\mathcal{I}$-uniformly integrable.*

(b) *$f \in \mathcal{D}_p(\mu, X)$, $\{f_n\}$ is p-Pettis $\mathcal{I}$-convergent to f and for each $\varepsilon > 0$, there exists an $A \in \mathcal{I}$ such that for each $n \in \mathbb{N} \setminus A$, there exists a $\delta > 0$ satisfying*

$$\sup_{x^* \in B_{X^*}} \int_E |x^* f_n|^p d\mu < \varepsilon \quad \textit{for all } E \in \Sigma \textit{ with } \mu(E) < \delta.$$

(c) *$\{f_n\}$ is scalarly equi-$\mathcal{I}$-convergent in measure to f and is p-Pettis $\mathcal{I}$-uniformly integrable.*

(d) *$f \in \mathcal{D}_p(\mu, X)$, $\{f_n\}$ is p-Pettis $\mathcal{I}$-convergent to f and $\{x^*f : x^* \in B_{X^*}\}$ is p-uniformly integrable.*

Proof. (a) $\Longrightarrow$ (b): Follows from Theorem 3.4 (c).

(b) $\Longrightarrow$ (c): Follows from Theorem 3.14 ((a) $\Longrightarrow$ (c)) and Theorem 3.4 (c).

(c) $\Longrightarrow$ (d): First and second part follow from Theorem 3.4 (c) and Theorem 3.14 ((c) $\Longrightarrow$ (a)), and the third part follows from Theorem 3.14 ((a) $\Longrightarrow$ (b)) and Lemma 3.18 ((a) & (b) $\Longrightarrow$ (c)).

(d) $\Longrightarrow$ (a): Follows from Theorem 3.14 ((a) $\Longrightarrow$ (b)) and Lemma 3.18 ((b) & (c) $\Longrightarrow$ (a)). □

A sequence $\{y_n\}$ in a normed linear space Y is said to be *weakly $\mathcal{I}$-convergent* to an element $y \in Y$ if for each $y^* \in Y^*$, the dual of Y, the sequence $\{y^*(y_n)\}$ is $\mathcal{I}$-convergent to $y^*(y)$ in $\mathbb{R}$. In this case, y is said to be a weak $\mathcal{I}$-limit of $\{y_n\}$ in Y, denoted as weak $\mathcal{I}\text{-}\lim y_n$, and $\{y_n\}$ is said to be weakly $\mathcal{I}$-convergent in Y.

It can be easily verified that weak $\mathcal{I}$-limit of a weakly $\mathcal{I}$-convergent sequence in a normed linear space is unique.

Also, it is easy to note that a norm $\mathcal{I}$-convergent sequence in a normed linear space is weakly $\mathcal{I}$-convergent to the same limit.

Theorem 3.20. *Let $\{f_n\} \subset \mathcal{P}(\mu, X)$ and let $f \in \mathcal{D}(\mu, X)$. Let us consider the following statements:*

(a) *For each $E \in \Sigma$, $\{(Pettis) \int_E f_n d\mu\}$ is weakly $\mathcal{I}$-convergent in X, and for each $E \in \Sigma$ and for each $x^* \in X^*$, $\{\int_E x^* f_n d\mu\}$ is $\mathcal{I}$-convergent to $\int_E x^* f d\mu$ in $\mathbb{R}$.*

(b) *$f \in \mathcal{P}(\mu, X)$, and for each $E \in \Sigma$, $\{(Pettis) \int_E f_n d\mu\}$ is weakly $\mathcal{I}$-convergent to $(Pettis) \int_E f d\mu$ in X.*

(c) *For each $E \in \Sigma$, $\{(Pettis) \int_E f_n d\mu\}$ is norm $\mathcal{I}$-convergent in X, and for each $E \in \Sigma$ and for each $x^* \in X^*$, $\{\int_E x^* f_n d\mu\}$ is $\mathcal{I}$-convergent to $\int_E x^* f d\mu$ in $\mathbb{R}$.*

(d) *$f \in \mathcal{P}(\mu, X)$, and for each $E \in \Sigma$, $\{(Pettis) \int_E f_n d\mu\}$ is norm $\mathcal{I}$-convergent to $(Pettis) \int_E f d\mu$ in X.*

Then (a) $\Longleftrightarrow$ (b) $\Longleftarrow$ (c) $\Longleftrightarrow$ (d).

Proof. (a) $\Longrightarrow$ (b): Let $E \in \Sigma$. Then, by hypothesis, $\{(Pettis) \int_E f_n d\mu\}$ is weakly $\mathcal{I}$-convergent in X. Let $x_E \in X$ be the weak $\mathcal{I}$-limit of $\{(Pettis) \int_E f_n d\mu\}$. Then, for each $x^* \in X^*$, $\{\int_E x^* f_n d\mu\}$ is $\mathcal{I}$-convergent to $x^*(x_E)$ in $\mathbb{R}$. So, it follows from hypothesis and uniqueness of $\mathcal{I}$-limit in $\mathbb{R}$ that $\int_E x^* f d\mu = x^*(x_E)$ for each $x^* \in X^*$ which implies that $f \in \mathcal{P}(\mu, X)$ and $x_E = (Pettis) \int_E f d\mu$. Obviously, $\{(Pettis) \int_E f_n d\mu\}$ is weakly $\mathcal{I}$-convergent to $(Pettis) \int_E f d\mu$.

(d) $\Longrightarrow$ (b) $\Longrightarrow$ (a) & (d) $\Longrightarrow$ (c) Trivial.

(c) $\Longrightarrow$ (d) Follows from (a) $\Longrightarrow$ (b) and the fact that a norm $\mathcal{I}$-convergent sequence in a normed linear space is weakly $\mathcal{I}$-convergent to the same limit. □

Corollary 3.21. *Let $\{f_n\}$ be a sequence in $\mathcal{P}_p(\mu, X)$ and let $f \in \mathcal{D}_p(\mu, X)$. If $\{f_n\}$ is r-Pettis $\mathcal{I}$-convergent to f for some $r \in [1, p]$, then $f \in \mathcal{P}_p(\mu, X)$.*

Proof. From hypothesis, it follows that $\{f_n\}$ is Pettis $\mathcal{I}$-convergent to f and hence, it is Pettis $\mathcal{I}$-Cauchy. Therefore, for each $E \in \Sigma$, $\{(Pettis) \int_E f_n d\mu\}$ is norm $\mathcal{I}$-Cauchy in X [15, p. 325] and hence, it is norm $\mathcal{I}$-convergent in X. From hypothesis, it also follows that $\{f_n\}$ is r-scalarly strongly $\mathcal{I}$-convergent to f and hence, it is scalarly strongly $\mathcal{I}$-convergent to f. Therefore, for each $E \in \Sigma$ and for each $x^* \in X^*$, $\{\int_E x^* f_n d\mu\}$ is $\mathcal{I}$-convergent to $\int_E x^* f d\mu$. Hence, it follows from Theorem 3.20 ((c) $\Longrightarrow$ (d)) that $f \in \mathcal{P}(\mu, X)$ and consequently, $f \in \mathcal{P}_p(\mu, X)$. □

The following result follows from Corollary 3.21 and the fact that a subset of a metric space is closed if and only if it is $\mathcal{I}$-closed:

Corollary 3.22. *$\mathcal{P}_p(\mu, X)$ is r-Pettis $\mathcal{I}$-closed and hence, r-Pettis closed in $\mathcal{D}_p(\mu, X)$ for all $r \in [1, p]$.*

A sequence $\{y_n\}$ in a normed linear space Y is said to be *weakly $\mathcal{I}$-Cauchy* if for each $y^* \in Y^*$, the dual of Y, the sequence $\{y^*(y_n)\}$ is $\mathcal{I}$-Cauchy in $\mathbb{R}$.

A normed linear space is said to be *weakly $\mathcal{I}$-sequentially complete* if every weakly $\mathcal{I}$-Cauchy sequence in it is weakly $\mathcal{I}$-convergent in it.

Corollary 3.23. *Let $\{f_n\}$ be a sequence in $\mathcal{P}(\mu, X)$ and let $f \in \mathcal{D}(\mu, X)$ such that for each $E \in \Sigma$ and for each $x^* \in X^*$, $\{\int_E x^* f_n d\mu\}$ is $\mathcal{I}$-convergent to $\int_E x^* f d\mu$ in $\mathbb{R}$. If X is weakly $\mathcal{I}$-sequentially complete, then $f \in \mathcal{P}(\mu, X)$.*

Proof. From the hypothesis, it follows that for each $E \in \Sigma$, $\{(Pettis) \int_E f_n d\mu\}$ is weakly $\mathcal{I}$-Cauchy in X. Since X is weakly $\mathcal{I}$-sequentially complete, $\{(Pettis) \int_E f_n d\mu\}$ is weakly $\mathcal{I}$-convergent in X for each $E \in \Sigma$. Hence, the result follows from Theorem 3.20 ((a) $\Longrightarrow$ (b)). □

Corollary 3.24. *If X is weakly $\mathcal{I}$-sequentially complete, then $\mathcal{P}_p(\mu, X)$ is r-scalarly strongly $\mathcal{I}$-closed in $\mathcal{D}_p(\mu, X)$ for all $r \in [1, p]$.*

Proof. Let $\{f_n\}$ be a sequence in $\mathcal{P}_p(\mu, X)$ which is r-scalarly strongly $\mathcal{I}$-convergent to some $f \in \mathcal{D}_p(\mu, X)$. Then it is scalarly strongly $\mathcal{I}$-convergent to f.

Therefore, for each $E \in \Sigma$ and for each $x^* \in X^*$, $\{\int_E x^* f_n d\mu\}$ is $\mathcal{I}$-convergent to $\int_E x^* f d\mu$ in $\mathbb{R}$. Now, an application of Corollary 3.23 yields that $f \in \mathcal{P}_p(\mu, X)$ and the result follows. □

The following result is the Pettis version of Theorem 3.14:

Theorem 3.25. *Let $\{f_n\}$ be a sequence in $\mathcal{P}_p(\mu, X)$ and let $f \in \mathcal{M}_s(\mu, X)$. Then the following statements are equivalent:*

(a) *$f \in \mathcal{P}_p(\mu, X)$ and $\{f_n\}$ is p-Pettis $\mathcal{I}$-convergent to f.*

(b) *$f \in \mathcal{P}_p(\mu, X)$, $\{f_n\}$ is scalarly equi-$\mathcal{I}$-convergent in measure to f and $\{f_n - f\}$ is p-Pettis $\mathcal{I}$-uniformly integrable.*

(c) *$\{f_n\}$ is scalarly equi-$\mathcal{I}$-convergent in measure to f and is p-Pettis δ-$\mathcal{I}$-Cauchy.*

(d) *$\{f_n\}$ is scalarly equi-$\mathcal{I}$-convergent in measure to f and is p-Pettis $\mathcal{I}$-Cauchy.*

(e) *$\{f_n\}$ is scalarly $\mathcal{I}$-convergent in measure to f and is p-Pettis $\mathcal{I}$-Cauchy.*

(f) *$f \in \mathcal{P}_p(\mu, X)$, $\{f_n\}$ is p-scalarly strongly $\mathcal{I}$-convergent to f and is p-Pettis $\mathcal{I}$-Cauchy.*

Proof. (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (e) & (f)$\Longrightarrow$ (a): Follow from Theorem 3.14 and the fact that $\mathcal{P}_p(\mu, X) \subset \mathcal{D}_p(\mu, X)$.
(e) $\Longrightarrow$ (f): It follows from Theorem 3.14 ((e) $\Longrightarrow$ (f)) that $f \in \mathcal{D}_p(\mu, X)$, $\{f_n\}$ is p-scalarly strongly $\mathcal{I}$-convergent to f and is p-Pettis $\mathcal{I}$-Cauchy. Again, by Theorem 3.14 ((f) $\Longrightarrow$ (a)), it follows that $\{f_n\}$ is p-Pettis $\mathcal{I}$-convergent to f. Hence, by Corollary 3.21, $f \in \mathcal{P}_p(\mu, X)$. □

Let (S, d) and (Z, ρ) be two metric spaces. Let us recall that a mapping $g \in Z^S$ is said to be compact if for every bounded subset B of S, $g(B) \subset K$ for some compact subset K of Z [8, p. 101, Definition].

The following result is the $\mathcal{I}$-version of [8, p. 102, Theorem 8.2]:

Theorem 3.26. *Let (S, d) be a metric space and let (Z, ρ) be a complete metric space. Let $\{g_n\} \subset Z^S$ be a sequence of compact mappings and let $g \in Z^S$. If $\{g_n\}$ is uniformly $\mathcal{I}$-convergent to g on S, then g is compact.*

Proof. If $\{g_n\}$ is uniformly $\mathcal{I}$-convergent to g on S, then it has a subsequence which is uniformly convergent to g on S [2, p. 718, Remark 4]. Hence the result follows from [8, p. 102, Theorem 8.2]. □

Let Y and Z be two normed linear spaces. Then it is known that a linear operator $T \in L(Y, Z)$ is a compact linear operator if and only if $T(B_Y) \subset K$ for some compact subset K of Z. The set of all compact linear operators in $L(Y, Z)$ is denoted by $\mathcal{K}(Y, Z)$. It is evident that a bounded linear operator T belongs to $\mathcal{K}(Y, Z)$ if and only if T is a compact mapping on B_Y into Z.

As a particular case of Theorem 3.26, we have the following result which is the $\mathcal{I}$-version of a fundamental result on compact linear operators on a normed linear space into a Banach space:

Corollary 3.27. *Let Y be a normed linear space and let Z be a Banach space. Let $\{T_n\}$ be a sequence in $\mathcal{K}(Y, Z)$ and let $T \in L(Y, Z)$. If $\{T_n\}$ is $\mathcal{I}$-convergent to T in operator norm, then $T \in \mathcal{K}(Y, Z)$.*

From Lemma 3.11 (b) and Corollary 3.27 follows the following result:

Corollary 3.28. *Let $\{f_n\}$ be a sequence in $\mathcal{D}_p(\mu, X)$ and let $f \in \mathcal{D}_p(\mu, X)$. Let T_{f_n} be compact for each n. If $\{f_n\}$ is p-Pettis $\mathcal{I}$-convergent to f, then T_f is compact.*

It is known that each $f \in \mathcal{P}(\mu, X)$ defines a countably additive μ-continuous vector measure $\nu_f : \Sigma \to X$, called the *induced vector measure* of f.

A function $f \in X^\Omega$ is said to be *strongly p-Pettis integrable* if $f \in \mathcal{P}_p(\mu, X)$ and if the range of its induced vector measure ν_f is relatively compact in X.

Let us denote, by $\mathcal{K}_p(\mu, X)$, the set of all strongly p-Pettis integrable functions in X^Ω. Evidently, $\mathcal{K}_p(\mu, X)$ is a linear subspace of $\mathcal{P}_p(\mu, X)$.

Corollary 3.29. *$\mathcal{K}_p(\mu, X)$ is r-Pettis closed in $\mathcal{D}_p(\mu, X)$ and hence, in $\mathcal{P}_p(\mu, X)$, for all $r \in [1, p]$.*

Proof. Let $r \in [1, p]$. Let $f \in \mathcal{D}_p(\mu, X)$ and let $\{f_n\}$ be a sequence in $\mathcal{K}_p(\mu, X)$ which is r-Pettis $\mathcal{I}$-convergent to f. Then Corollary 3.21 implies that $f \in \mathcal{P}_p(\mu, X)$.

Since $\{f_n\} \subset \mathcal{K}_p(\mu, X)$, it follows from [14, p. 8, Corollary 10 ((a) $\Longleftrightarrow$ (b))] that $i_p \circ T_{f_n}$ is compact for each n, where $i_p : L_p(\mu) \to L_1(\mu)$ is the inclusion map. Now, it is easy to note that

$$\|i_p \circ T_{f_n} - i_p \circ T_f\| = \|f_n - f\|_P \leq \|f_n - f\|_{P,r}$$

for each n. Hence, $\{i_p \circ T_{f_n}\}$ is $\mathcal{I}$-convergent to $i_p \circ T_f$ in operator norm as $\{f_n\}$ is r-Pettis $\mathcal{I}$-convergent to f. So, it follows from Corollary 3.27 that $i_p \circ T_f$ is compact which implies by [14, p. 8, Corollary 10 ((a) $\Longleftrightarrow$ (b))] that $f \in \mathcal{K}_p(\mu, X)$. This shows that $\mathcal{K}_p(\mu, X)$ is r-Pettis $\mathcal{I}$-closed in $\mathcal{D}_p(\mu, X)$ and hence, in $\mathcal{P}_p(\mu, X)$. Hence the result follows from the fact that a subset in a metric space is closed if and only if it is $\mathcal{I}$-closed. □

The following lemma follows trivially from the very definition of a uniformly $\mathcal{I}$-Cauchy sequence of functions:

Lemma 3.30. *Let $\{f_n\} \subset X^\Omega$ be uniformly $\mathcal{I}$-Cauchy.*

(a) *If $\{f_n\} \subset \mathcal{M}(\mu, X)$, then it is $\mathcal{I}$-Cauchy in measure.*

(b) *If $\{f_n\} \subset L_p(\mu, X)$, then it is p-Bochner $\mathcal{I}$-Cauchy.*

(c) *If $\{f_n\} \subset \mathcal{M}_s(\mu, X)$, then it is scalarly equi-$\mathcal{I}$-Cauchy in measure.*

(d) *If $\{f_n\} \subset \mathcal{D}_p(\mu, X)$, then it is p-Pettis $\mathcal{I}$-Cauchy.*

Theorem 3.31. *Let $\{f_n\} \subset X^\Omega$ and let $f \in X^\Omega$. Let $\{f_n\}$ be uniformly $\mathcal{I}$-convergent to f.*

(a) *If $\{f_n\} \subset \mathcal{M}(\mu, X)$, then $f \in \mathcal{M}(\mu, X)$ and $\{f_n\}$ is $\mathcal{I}$-convergent in measure to f.*

(b) *If $\{f_n\} \subset L_p(\mu, X)$, then $f \in L_p(\mu, X)$ and $\{f_n\}$ is p-Bochner $\mathcal{I}$-convergent to f.*

Proof. (a) It follows from hypothesis that $\{f_n\}$ contains a subsequence which is uniformly convergent to f [2, p. 718, Remark 4] which implies that $f \in \mathcal{M}(\mu, X)$.

That $\{f_n\}$ is $\mathcal{I}$-convergent in measure to f can be proved in a similar manner as done in [2, p. 723, Remark 14] for sequences of real-valued functions.

(b) Since $L_p(\mu, X) \subset \mathcal{M}(\mu, X)$, it follows from part (a) that $f \in \mathcal{M}(\mu, X)$ and $\{f_n\}$ is $\mathcal{I}$-convergent in measure to f. It follows from Lemma 3.12 and Lemma 3.30(b) that $\{f_n\}$ is p-Bochner $\mathcal{I}$-Cauchy. Hence, the result follows from Theorem 3.5 ((b) $\implies$ (a)). □

Let S be a non-empty set and let (Z, ρ) be a metric space. A subset U of Z^S is said to be pointwise (resp. uniformly) $\mathcal{I}$-sequentially closed if the pointwise (resp. the uniform) $\mathcal{I}$-limit of every pointwise (resp. uniformly) $\mathcal{I}$-convergent sequence in U, is contained in U.

An immediate consequence of Theorem 3.31 is the following result:

Corollary 3.32. *Each of $\mathcal{M}(\mu, X)$ and $L_p(\mu, X)$ is uniformly $\mathcal{I}$-sequentially closed in X^Ω, and hence, $L_p(\mu, X)$ is uniformly $\mathcal{I}$-sequentially closed in $\mathcal{M}(\mu, X)$.*

Let S be a non-empty set. A sequence of functions $\{f_n\}$ in X^S is said to be weakly pointwise (resp. weakly uniformly) $\mathcal{I}$-convergent to an element $f \in X^S$ on a subset T of S if for each $x^* \in X^*$, the sequence $\{x^* f_n\}$ is pointwise (resp. uniformly) $\mathcal{I}$-convergent to $x^* f$ on T.

It is evident that a pointwise (resp. uniformly) $\mathcal{I}$-convergent sequence of functions in X^S is weakly pointwise (resp. weakly uniformly) $\mathcal{I}$-convergent to the same limit.

Corollary 3.33. *Let $\{f_n\} \subset X^\Omega$ and let $f \in X^\Omega$. Let $\{f_n\}$ be weakly uniformly $\mathcal{I}$-convergent to f on Ω.*

(a) *If $\{f_n\} \subset \mathcal{M}_s(\mu, X)$, then $f \in \mathcal{M}_s(\mu, X)$ and $\{f_n\}$ is scalarly $\mathcal{I}$-convergent in measure to f.*

(b) *If $\{f_n\} \subset \mathcal{D}_p(\mu, X)$, then $f \in \mathcal{D}_p(\mu, X)$ and $\{f_n\}$ is p-scalarly strongly $\mathcal{I}$-convergent to f.*

(c) *If $\{f_n\} \subset \mathcal{P}_p(\mu, X)$ and if X is weakly $\mathcal{I}$-sequentially complete, then $f \in \mathcal{P}_p(\mu, X)$ and $\{f_n\}$ is p-scalarly strongly $\mathcal{I}$-convergent to f.*

Proof. (a) Follows by an application of Theorem 3.31 (a) to the sequence $\{x^* f_n\}$ in $\mathcal{M}(\mu)$ for each $x^* \in X^*$.

(b) Follows by an application of Theorem 3.31 (b) to the sequence $\{x^* f_n\}$ in $L_p(\mu)$ for each $x^* \in X^*$.

(c) Part (b) implies $f \in \mathcal{D}_p(\mu, X)$ and that $\{f_n\}$ is p-scalarly strongly $\mathcal{I}$-convergent to f. Hence, it follows from Corollary 3.24 that $f \in \mathcal{P}_p(\mu, X)$. □

Let S be a non-empty set. A subset U of X^S is said to be weakly uniformly $\mathcal{I}$-sequentially closed if the weak uniform $\mathcal{I}$-limit of every weakly uniformly $\mathcal{I}$-convergent sequence in U, is contained in U.

The following result immediately follows from Corollary 3.33:

Corollary 3.34. (a) *Each of $\mathcal{M}_s(\mu, X)$ and $\mathcal{D}_p(\mu, X)$ is weakly uniformly $\mathcal{I}$-sequentially closed in X^Ω and hence, $\mathcal{D}_p(\mu, X)$ is weakly uniformly $\mathcal{I}$-sequentially closed in $\mathcal{M}_s(\mu, X)$.*

(b) *If X is weakly $\mathcal{I}$-sequentially complete, then $\mathcal{P}_p(\mu, X)$ is weakly uniformly $\mathcal{I}$-sequentially closed in X^Ω and hence, in $\mathcal{D}_p(\mu, X)$.*

Theorem 3.35. *Let $\{f_n\} \subset X^\Omega$ and let $f \in X^\Omega$. Let $\{f_n\}$ be uniformly $\mathcal{I}$-convergent to f.*

(a) *If $\{f_n\} \subset \mathcal{M}_s(\mu, X)$, then $f \in \mathcal{M}_s(\mu, X)$ and $\{f_n\}$ is scalarly equi-$\mathcal{I}$-convergent in measure to f.*

(b) *If* $\{f_n\} \subset \mathcal{D}_p(\mu, X)$, *then* $f \in \mathcal{D}_p(\mu, X)$ *and* $\{f_n\}$ *is p-Pettis* $\mathcal{I}$*-convergent to* f.

(c) *If* $\{f_n\} \subset \mathcal{P}_p(\mu, X)$, *then* $f \in \mathcal{P}_p(\mu, X)$ *and* $\{f_n\}$ *is p-Pettis* $\mathcal{I}$*-convergent to* f.

Proof. (a) Since a uniformly $\mathcal{I}$-convergent sequence of functions in X^Ω is weakly uniformly $\mathcal{I}$-convergent to the same limit, Corollary 3.33(a) implies $f \in \mathcal{M}_s(\mu, X)$ and that $\{f_n\}$ is scalarly $\mathcal{I}$-convergent in measure to f. Also, it follows from Lemma 3.12 and Lemma 3.30 (c) that $\{f_n\}$ is scalarly equi-$\mathcal{I}$-Cauchy in measure. Hence, Lemma 3.13 implies that $\{f_n\}$ is scalarly equi-$\mathcal{I}$-convergent in measure to f.

(b) Follows from part (a) that $f \in \mathcal{M}_s(\mu, X)$ and $\{f_n\}$ is scalarly equi-$\mathcal{I}$-convergent in measure to f. Also, it follows from Lemma 3.12 and Lemma 3.30 (d) that $\{f_n\}$ is p-Pettis $\mathcal{I}$-Cauchy. Hence, the result follows from Theorem 3.14 ((d) $\Longrightarrow$ (a)).

(c) Follows from part (b) and Corollary 3.21. □

The following result immediately follows from Theorem 3.35:

Corollary 3.36. *Each of* $\mathcal{M}_s(\mu, X)$, $\mathcal{D}_p(\mu, X)$, $\mathcal{P}_p(\mu, X)$ *is uniformly* $\mathcal{I}$*-sequentially closed in* X^Ω *and hence, each of* $\mathcal{D}_p(\mu, X)$, $\mathcal{P}_p(\mu, X)$ *is uniformly* $\mathcal{I}$*-sequentially closed in* $\mathcal{M}_s(\mu, X)$, *and* $\mathcal{P}_p(\mu, X)$ *is uniformly* $\mathcal{I}$*-sequentially closed in* $\mathcal{D}_p(\mu, X)$.

Let S be a non-empty set and let (Z, ρ) be a metric space. A subset U of Z^S is said to be pointwise (resp. uniformly) $\mathcal{I}$-sequentially complete if every pointwise (resp. uniformly) $\mathcal{I}$-Cauchy sequence in U is pointwise (resp. uniformly) $\mathcal{I}$-convergent on S to an element of U.

Lemma 3.37. *Let* S *be a non-empty set and let* (Z, ρ) *be a metric space. Then the following statements are equivalent:*

(a) Z *is* $\mathcal{I}$*-complete.*

(b) Z^S *is pointwise* $\mathcal{I}$*-sequentially complete.*

(c) Z^S *is uniformly* $\mathcal{I}$*-sequentially complete.*

Proof. (a) $\Longrightarrow$ (b): Let Z be $\mathcal{I}$-complete. Let $\{f_n\}$ be a pointwise $\mathcal{I}$-Cauchy sequence in Z^S. Let $s \in S$. Then $\{f_n(s)\}$ is an $\mathcal{I}$-Cauchy sequence in Z. Hence, by hypothesis, it is $\mathcal{I}$-convergent to an element of Z. Let $f(s) = \mathcal{I}\text{-}\lim_n f_n(s)$. This is done for each $s \in S$. Clearly, f is well-defined as the $\mathcal{I}$-limit of an $\mathcal{I}$-convergent sequence in a metric space is unique. Hence, $f \in Z^S$ and $\{f_n\}$ is pointwise $\mathcal{I}$-convergent to f on S, and the result follows.

(b) $\Longrightarrow$ (c): Let Z^S be pointwise $\mathcal{I}$-sequentially complete. Let $\{f_n\}$ be a uniformly $\mathcal{I}$-Cauchy sequence in Z^S. Then it is pointwise $\mathcal{I}$-Cauchy in Z^S. Hence, by hypothesis, it is pointwise $\mathcal{I}$-convergent to some $f \in Z^S$. An application of Lemma 3.12 yields that $\{f_n\}$ is uniformly $\mathcal{I}$-convergent to f on S, and the result follows.

(c) $\Longrightarrow$ (a): Let Z^S be uniformly $\mathcal{I}$-sequentially complete. Let $\{z_n\}$ be an $\mathcal{I}$-Cauchy sequence in Z. Let us define, for each n, the mapping $f_n \in Z^S$ by $f_n(s) = z_n$ for all $s \in S$. Then it follows easily that $\{f_n\}$ is uniformly $\mathcal{I}$-Cauchy in Z^S. Hence, by hypothesis, $\{f_n\}$ is uniformly $\mathcal{I}$-convergent to some $f \in Z^S$. Then $\{z_n\}$ is $\mathcal{I}$-convergent to $f(s) \in Z$ for all $s \in S$. From uniqueness of $\mathcal{I}$-limit in a metric space, it follows that f is a constant mapping. Let $f(s) = z \in Z$ for all $s \in S$. Then $\{z_n\}$ is $\mathcal{I}$-convergent to z, and the result follows. □

Let S be a non-empty set. A sequence of functions $\{f_n\}$ in X^S is said to be *weakly pointwise (resp. weakly uniformly)* $\mathcal{I}$*-Cauchy* on a subset T of S if for each $x^* \in X^*$,

$\{x^* f_n\}$ is pointwise (resp. uniformly) $\mathcal{I}$-Cauchy on T. It is evident that a weakly uniformly $\mathcal{I}$-Cauchy sequence is weakly pointwise $\mathcal{I}$-Cauchy.

A subset U of X^S is called *weakly pointwise (resp. weakly uniformly) $\mathcal{I}$-sequentially complete* if every weakly pointwise (resp. weakly uniformly) $\mathcal{I}$-Cauchy sequence in U is weakly pointwise (resp. weakly uniformly) $\mathcal{I}$-convergent on S to an element of U.

Lemma 3.38. *For a non-empty set S the following statements are equivalent:*

(a) *X is weakly $\mathcal{I}$-sequentially complete.*

(b) *X^S is weakly pointwise $\mathcal{I}$-sequentially complete.*

(c) *X^S is weakly uniformly $\mathcal{I}$-sequentially complete.*

Proof. (a) $\implies$ (b): Let X be weakly $\mathcal{I}$-sequentially complete. Let $\{f_n\}$ be a weakly pointwise $\mathcal{I}$-Cauchy sequence in X^S. Let $s \in S$. Then $\{f_n(s)\}$ is a weakly $\mathcal{I}$-Cauchy sequence in X. Hence, by hypothesis, it is weakly $\mathcal{I}$-convergent to an element of X. Let $f(s) = \text{weak } \mathcal{I}\text{-}\lim_n f_n(s)$. This is done for each $s \in S$. Clearly, f is well-defined as the weak $\mathcal{I}$-limit of a weakly $\mathcal{I}$-convergent sequence in a normed linear space is unique. Hence, $f \in X^S$ and $\{f_n\}$ is weakly pointwise $\mathcal{I}$-convergent to f on S, and the result follows.

(b) $\implies$ (c): Let X^S be weakly pointwise $\mathcal{I}$-sequentially complete. Let $\{f_n\}$ be a weakly uniformly $\mathcal{I}$-Cauchy sequence in X^S. Then it is weakly pointwise $\mathcal{I}$-Cauchy in X^S. Hence, by hypothesis, it is weakly pointwise $\mathcal{I}$-convergent to some $f \in X^S$. Therefore, for each $x^* \in X^*$, $\{x^* f_n\}$ is uniformly $\mathcal{I}$-Cauchy and pointwise $\mathcal{I}$-convergent to $x^* f$ in $\mathbb{R}^S$. An application of Lemma 3.12 to $\{x^* f_n\}$, for each $x^* \in X^*$, yields that $\{x^* f_n\}$ is uniformly $\mathcal{I}$-convergent to $x^* f$ on S which implies that $\{f_n\}$ is weakly uniformly $\mathcal{I}$-convergent to f on S, and the result follows.

(c) $\implies$ (a): Let X^S be weakly uniformly $\mathcal{I}$-sequentially complete. Let $\{x_n\}$ be a weakly $\mathcal{I}$-Cauchy sequence in X. Let us define, for each n, the mapping $f_n \in X^S$ by $f_n(s) = x_n$ for all $s \in S$. Then it follows easily that $\{f_n\}$ is weakly uniformly $\mathcal{I}$-Cauchy in X^S. Hence, by hypothesis, $\{f_n\}$ is weakly uniformly $\mathcal{I}$-convergent to some $f \in X^S$. Then $\{x_n\}$ is weakly $\mathcal{I}$-convergent to $f(s)$ for all $s \in S$. From uniqueness of weak $\mathcal{I}$-limit in a normed linear space, it follows that f is a constant mapping. Let $f(s) = x \in X$ for all $s \in S$. Then $\{x_n\}$ is weakly $\mathcal{I}$-convergent to x, and the result follows. □

The following result follows by a routine argument along with an application of Lemma 3.37 ((a) $\implies$ (b) & (a) $\implies$ (c)):

Lemma 3.39. *Let S be a non-empty set and let (Z, ρ) be a complete metric space. Then the following statements hold good:*

(a) *Every $\mathcal{I}$-closed subset of Z is $\mathcal{I}$-complete.*

(b) *Every pointwise (resp. uniformly) $\mathcal{I}$-sequentially closed subset of Z^S is pointwise (resp. uniformly) $\mathcal{I}$-sequentially complete.*

Lemma 3.40. *Let S be a non-empty set. Then*

(a) *every pointwise $\mathcal{I}$-sequentially closed subset of X^S is pointwise $\mathcal{I}$-sequentially complete,*

(b) *if X is weakly $\mathcal{I}$-sequentially complete, then every weakly pointwise (resp. weakly uniformly) $\mathcal{I}$-sequentially closed subset of X^S is weakly pointwise (resp. weakly uniformly) $\mathcal{I}$-sequentially complete.*

Proof. (a) Follows from Lemma 3.39 (b), as X is complete.

(b) Follows by a routine argument along with an application of Lemma 3.38 ((a) $\Longrightarrow$ (b)) (resp. Lemma 3.38 ((a) $\Longrightarrow$ (c))). $\square$

Theorem 3.41. (a) *Each of* $\mathcal{M}(\mu, X)$, $L_p(\mu, X)$, $\mathcal{M}_s(\mu, X)$, $\mathcal{D}_p(\mu, X)$ *and* $\mathcal{P}_p(\mu, X)$ *is uniformly* $\mathcal{I}$*-sequentially complete.*

(b) *If* X *is weakly* $\mathcal{I}$*-sequentially complete, then each of* $\mathcal{M}_s(\mu, X)$, $\mathcal{D}_p(\mu, X)$ *and* $\mathcal{P}_p(\mu, X)$ *is weakly uniformly* $\mathcal{I}$*-sequentially complete.*

Proof. (a) Since X is complete, the result follows from Corollary 3.32, Corollary 3.36 and Lemma 3.39 (b).

(b) Follows from Corollary 3.34 and Lemma 3.40 (b). $\square$

Following Athanassiadou, Boccuto, Dimitriou and Papanastassiou [1, p. 398–399, Definition 2.3], we define here $\mathcal{I}$-exhaustiveness, $\mathcal{I}$-weak exhaustiveness and $\mathcal{I}$-α-convergence for sequences of functions defined on a metric space with values also in a metric space:

Let (Y, d), (Z, ρ) be two metric spaces and let $\{g_n\}$ be a sequence in Z^Y.

(a) $\{g_n\}$ is said to be $\mathcal{I}$-*exhaustive* at a point $y_0 \in Y$ if for any $\varepsilon > 0$, there exist a $\delta > 0$ and an $A \in \mathcal{I}$ such that $\rho(g_n(y), g_n(y_0)) < \varepsilon$ for all $y \in B_\delta(y_0)$ and for all $n \in \mathbb{N} \setminus A$.

(b) $\{g_n\}$ is said to be $\mathcal{I}$-*weakly exhaustive* at a point $y_0 \in Y$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $y \in B_\delta(y_0)$, there exists an $A_y \in \mathcal{I}$ such that $\rho(g_n(y), g_n(y_0)) < \varepsilon$ for all $n \in \mathbb{N} \setminus A_y$.

(c) $\{g_n\}$ is said to be $\mathcal{I}$-α-*convergent* to a function $g \in Z^Y$ at a point $y_0 \in Y$ if for each sequence $\{y_n\}$ in Y, $\mathcal{I}$-converging to y_0, the sequence $\{g_n(y_n)\}$ is $\mathcal{I}$-convergent to $g(y_0)$ in Z.

The sequence $\{g_n\}$ is said to be $\mathcal{I}$-*exhaustive (resp.* $\mathcal{I}$*-weakly exhaustive)* on Y if it is $\mathcal{I}$-exhaustive (resp. $\mathcal{I}$-weakly exhaustive) at each point of Y, and it is said to be $\mathcal{I}$-α-*convergent* to $g \in Z^Y$ on Y if it is $\mathcal{I}$-α-convergent to g at each point of Y.

It is evident that if a sequence in Z^Y is $\mathcal{I}$-exhaustive at a point, then it is weakly $\mathcal{I}$-exhaustive at that point.

Theorem 3.42. *Let* (Y, d) *and* (Z, ρ) *be two metric spaces and let* $y_0 \in Y$*. Let* $\{g_n\} \subset Z^Y$ *be uniformly* $\mathcal{I}$*-convergent to* $g \in Z^Y$ *in some neighbourhood of* y_0 *and let* g *be continuous at* y_0*. Then* $\{g_n\}$ *is* $\mathcal{I}$*-exhaustive at* y_0*, and is* $\mathcal{I}$*-*α*-convergent to* g *at* y_0*.*

Proof. There exists, by hypothesis, a $\delta_1 > 0$ such that $\{g_n\}$ is uniformly $\mathcal{I}$-convergent to g on $B_{\delta_1}(y_0)$.

Now, let $\varepsilon > 0$. Then there exists an $A \in \mathcal{I}$ such that

$$\rho(g_n(y), g(y)) < \frac{\varepsilon}{3} \quad \text{for all } y \in B_{\delta_1}(y_0) \text{ and for all } n \in \mathbb{N} \setminus A.$$

Since g is continuous at y_0, there exists a $\delta_2 > 0$ such that

$$\rho(g(y), g(y_0)) < \frac{\varepsilon}{3} \quad \text{for all } y \in B_{\delta_2}(y_0).$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$, and for all $y \in B_\delta(y_0)$ and for all $n \in \mathbb{N} \setminus A$, we have

$$\rho(g_n(y), g_n(y_0)) \leq \rho(g_n(y), g(y)) + \rho(g(y), g(y_0)) + \rho(g(y_0), g_n(y_0)) < \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon$$

which implies that $\{g_n\}$ is $\mathcal{I}$-exhaustive at y_0.

Now, for the second part, let $\{y_n\}$ be a sequence in Y, $\mathcal{I}$-converging to y_0. Then, there exists a $B \in \mathcal{I}$ such that $y_n \in B_\delta(y_0)$ for all $n \in \mathbb{N} \setminus B$.

Let $C = A \cup B$. Then $C \in \mathcal{I}$ and for all $n \in \mathbb{N} \setminus C$, we have

$$\rho(g_n(y_n), g(y_0)) \leq \rho(g_n(y_n), g(y_n)) + \rho(g(y_n), g(y_0)) < \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} < \varepsilon$$

which implies that $\{g_n(y_n)\}$ is $\mathcal{I}$-convergent to $g(y_0)$ in Z.

Hence it follows that $\{g_n\}$ is $\mathcal{I}$-α-convergent to g at y_0 and the proof is complete. □

Corollary 3.43. *Let (Y, d) and (Z, ρ) be two metric spaces. Let $\{g_n\} \subset Z^Y$ be uniformly $\mathcal{I}$-convergent to $g \in Z^Y$ on Y and let g be continuous on Y. Then $\{g_n\}$ is $\mathcal{I}$-exhaustive on Y, and is $\mathcal{I}$-α-convergent to g on Y.*

Let (Y, d) and (Z, ρ) be two metric spaces. A sequence of functions $\{g_n\}$ in Z^Y is said to be equi-$\mathcal{I}$-Lipschitz of order $k > 0$ if there exist an $M > 0$ and an $A \in \mathcal{I}$ such that

$$\rho(g_n(u), g_n(v)) \leq M(d(u, v))^k$$

for all $u, v \in Y$ and for all $n \in \mathbb{N} \setminus A$.

Theorem 3.44. *Let (Y, d) and (Z, ρ) be two metric spaces. If $\{g_n\} \subset Z^Y$ is equi-$\mathcal{I}$-Lipschitz of order $k > 0$, then it is $\mathcal{I}$-exhaustive on Y.*

Proof. By hypothesis, there exist an $M > 0$ and an $A \in \mathcal{I}$ such that

$$\rho(g_n(u), g_n(v)) \leq M(d(u, v))^k$$

for all $u, v \in Y$ and for all $n \in \mathbb{N} \setminus A$.

Let $y_0 \in Y$ and let $\varepsilon > 0$. Let $\delta = \left(\frac{\varepsilon}{2M}\right)^{\frac{1}{k}}$. Then $\delta > 0$, and for all $y \in B_\delta(y_0)$ and for all $n \in \mathbb{N} \setminus A$, we have $\rho(g_n(y), g_n(y_0)) \leq M(d(y, y_0))^k < \varepsilon$ which implies that $\{g_n\}$ is $\mathcal{I}$-exhaustive at y_0 and the result follows. □

Let us recall that a sequence $\{y_n\}$ in a normed linear space is said to be $\mathcal{I}$-bounded if there exists an $M > 0$ such that $\{n \in \mathbb{N} : \|y_n\| > M\} \in \mathcal{I}$ [17, Definition 4.1]. It is easy to see that the sequence $\{y_n\}$ is $\mathcal{I}$-bounded if and only if there exist an $M > 0$ and an $A \in \mathcal{I}$ such that $\|y_n\| \leq M$ for all $n \in \mathbb{N} \setminus A$.

It is easy to note that an $\mathcal{I}$-convergent sequence in a normed linear space is $\mathcal{I}$-bounded though an $\mathcal{I}$-convergent sequence in a metric space is not necessarily bounded.

Corollary 3.45. *Let Y and Z be two normed linear spaces and let $\{T_n\} \subset L(Y, Z)$. If $\{T_n\}$ is $\mathcal{I}$-bounded, then it is equi-$\mathcal{I}$-Lipschitz of order 1 and hence, is $\mathcal{I}$-exhaustive on Y.*

Proof. That $\{T_n\}$ is equi-$\mathcal{I}$-Lipschitz of order 1 can be easily verified. Hence, the second part follows from Theorem 3.44. □

Lemma 3.46. *Let Y and Z be two normed linear spaces. Let $\{T_n\} \subset L(Y,Z)$ and let $T \in L(Y,Z)$. If $\{T_n\}$ is pointwise $\mathcal{I}$-convergent to T on Y and is $\mathcal{I}$-α-convergent to T at a point $y_0 \in Y$, then $\{T_n\}$ is $\mathcal{I}$-α-convergent to T on Y.*

Proof. Let $y \in Y$ and let $\{y_n\}$ be a sequence in Y such that $\{y_n\}$ is $\mathcal{I}$-convergent to y. Since $\{T_n\}$ is pointwise $\mathcal{I}$-convergent to T on Y, it follows that $\{T_n(y)\}$ and $\{T_n(y_0)\}$ are $\mathcal{I}$-convergent to $T(y)$ and $T(y_0)$ respectively. Now, the sequence $\{y_n - y + y_0\}$ is $\mathcal{I}$-convergent to y_0. Since $\{T_n\}$ is $\mathcal{I}$-α-convergent to T at y_0, it follows that $\{T_n(y_n - y + y_0)\}$ is $\mathcal{I}$-convergent to $T(y_0)$, i.e., $\{T_n(y_n) - T_n(y) + T_n(y_0)\}$ is $\mathcal{I}$-convergent to $T(y_0)$. Hence, from above, it follows that $\{T_n(y_n)\}$ is $\mathcal{I}$-convergent to $T(y)$. This shows that $\{T_n\}$ is $\mathcal{I}$-α-convergent to T at any point $y \in Y$. Hence, the result follows. □

Theorem 3.47. *Let $\{f_n\} \subset \mathcal{D}_p(\mu, X)$ and let $f \in \mathcal{D}_p(\mu, X)$. Then the following statements hold good:*

(a) *If $\{f_n\}$ is p-Pettis $\mathcal{I}$-bounded, then $\{T_{f_n}\}$ is $\mathcal{I}$-exhaustive on X^*.*

(b) *If $\{f_n\}$ is p-scalarly strongly $\mathcal{I}$-convergent to f, then $\{T_{f_n}\}$ is $\mathcal{I}$-weakly exhaustive on X^*.*

(c) *If $\{f_n\}$ is p-Pettis $\mathcal{I}$-bounded and p-scalarly strongly $\mathcal{I}$-convergent to f, then $\{T_{f_n}\}$ is $\mathcal{I}$-exhaustive on X^* and is $\mathcal{I}$-α-convergent to T_f on X^*.*

(d) *If $\{f_n\}$ is p-Pettis $\mathcal{I}$-convergent to f, then $\{T_{f_n}\}$ is $\mathcal{I}$-exhaustive on X^* and is $\mathcal{I}$-α-convergent to T_f on X^*.*

Proof. (a) If $\{f_n\}$ is p-Pettis $\mathcal{I}$-bounded, then $\{T_{f_n}\}$ is $\mathcal{I}$-bounded in $L(X^*, L_p(\mu))$ as $\|T_{f_n}\| = \|f_n\|_{P,p}$ for each n [14, p. 8, Corollary 13 (b)]. Hence the result follows from Corollary 3.45.

(b) If $\{f_n\}$ is p-scalarly strongly $\mathcal{I}$-convergent to f, then by Lemma 3.11 (a), $\{T_{f_n}\}$ is pointwise $\mathcal{I}$-convergent to T_f in $(L_p(\mu))^{X^*}$ on X^*. Also, T_f is continuous. Now, following the line of proof of [1, p. 400, Proposition 2.7 ((ii) $\Longrightarrow$ (i))] for sequences of real-valued functions defined on a metric space, it can be proved that $\{T_{f_n}\}$ is $\mathcal{I}$-weakly exhaustive on X^*.

(c) That $\{T_{f_n}\}$ is $\mathcal{I}$-exhaustive on X^* follows from part (a). Also, it follows from Lemma 3.11 (a) that $\{T_{f_n}\}$ is pointwise $\mathcal{I}$-convergent to T_f in $(L_p(\mu))^{X^*}$ on X^*. Then $\{T_{f_n}\}$ can be proved to be $\mathcal{I}$-α-convergent to T_f on X^* as done by Athanassiadou et al. for sequences of real-valued functions defined on a metric space [1, p. 399, Proposition 2.4 ((iii) $\Longrightarrow$ (i))].

(d) If $\{f_n\}$ is p-Pettis $\mathcal{I}$-convergent to f, then it is p-Pettis $\mathcal{I}$-bounded as well as p-scalarly strongly $\mathcal{I}$-convergent to f. Hence, the result follows from part (c). □

Corollary 3.48. *Let $\{f_n\} \subset \mathcal{D}_p(\mu, X)$ and let $f \in X^\Omega$.*

(a) *If $\{f_n\}$ is uniformly $\mathcal{I}$-convergent to f on Ω, then $f \in \mathcal{D}_p(\mu, X)$ and $\{T_{f_n}\}$ is $\mathcal{I}$-exhaustive on X^* and is $\mathcal{I}$-α-convergent to T_f on X^*.*

(b) *If $\{f_n\}$ is weakly uniformly $\mathcal{I}$-convergent to f on Ω, then $f \in \mathcal{D}_p(\mu, X)$ and $\{T_{f_n}\}$ is $\mathcal{I}$-weakly exhaustive on X^*.*

Proof. (a) Follows from Theorem 3.35 (b) and Theorem 3.47 (d).

(b) Follows from Corollary 3.33 (b) and Theorem 3.47 (b). □

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