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Schrödinger-Poisson System Involving Potential Vanishing at Infinity and Unbounded Below

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This article concerns the following class of system

$$\begin{cases} -\Delta u + V(x)u + \ell(x)\phi u = f(u) + \lambda |u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = \ell(x)u^2 & \text{in } \mathbb{R}^3, \\ u, \phi \in D^{1,2}(\mathbb{R}^3),\ u, \phi \geq 0 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda \geq 0$ and $q \geq 2^* = 6$ is the critical Sobolev exponent in dimension 3, the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is superlinear and has subcritical growth, $V, \ell : \mathbb{R}^3 \to \mathbb{R}$ are measurable functions with $\ell \in L^2(\mathbb{R}^3)$, the potential V can change sign in $\mathbb{R}^3$ and vanish at infinity, that is, $V(x) \to 0$ as $|x| \to \infty$. Our approach is based on variational method combined with Benci-Fortunato's reduction argument [Topol. Methods Nonlinear Anal. 11 (1998) 283–293], Del Pino-Felmer's penalization technique [Calc. Var. Partial Diff. Equations 4 (1996) 121–137] and L^∞-estimate.

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1. Introduction

This article concerns the existence of nontrivial solution to the Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \ell(x)\phi u = f(u) + \lambda |u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = \ell(x)u^2 & \text{in } \mathbb{R}^3, \\ u, \phi \in D^{1,2}(\mathbb{R}^3),\ u, \phi \geq 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

where $\lambda \geq 0$ and $q \geq 2^* = 6$ is the critical Sobolev exponent in dimension 3. We assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous, superlinear and has subcritical growth and $V, \ell : \mathbb{R}^3 \to \mathbb{R}$ are measurable functions. Later, we will specify the assumptions on V, ℓ and f.

System (1), also known as the nonlinear Schrödinger-Maxwell problem, was proposed by Benci and Fortunato in [10] as a model describing solitary waves for the nonlinear

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stationary Schrödinger equations interacting with the electrostatic field. It has a strong physical meaning because it appears in quantum mechanics models (see e.g. [14, 18, 27]). The system (1) also appears in semiconductor theory, nonlinear optics and plasma physics, see [12, 29, 32]. For more details on the physical aspects, we refer the readers to [10, 12].

In recent years, there has been increasing attention to Schrödinger-Poisson system (1) or similar problem on obtaining existence of positive solutions, ground state solutions, multiple solutions and semiclassical states depending on the assumptions of the potentials V and ℓ. For recent papers, see [22, 23, 24, 41].

The largest part of the literature focusses on the study of problem (1) with $V(x) \geq V_0$ in $\mathbb{R}^3$ for some $V_0 > 0$. See, for instance, Ambrosetti [4], Ambrosetti-Ruiz [5], Kristály-Repovš [26], Ruiz [36] and references therein, for the case $V \equiv \ell \equiv 1$.

In [7], Azzollini and Pomponio studied (1) with $\ell \equiv 1$, $f(t) = |t|^{p-1}t$ and $\lambda = 0$. If V is a positive constant, they proved the existence of a ground state solution (u, ϕ) for $2 < p < 5$ and for V non-constant was considered $3 < p < 5$ with V possibly unbounded below. Existence and nonexistence results were proved when the nonlinearity exhibits a critical growth.

In [19], Cerami and Vaira considered a version of (1) with the potential V constant and ℓ vanishing at infinity. More precisely, they studied the problem

$$\begin{cases} -\Delta u + V(x)u + \ell(x)\phi u = g(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = \ell(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{2}$$

where $V \equiv 1$, $g(x,u) = a(x)|u|^{p-1}u$, $p \in (3,5)$, $a, \ell : \mathbb{R}^3 \to \mathbb{R}$ are nonnegative functions such that

$$\lim_{|x|\to\infty} a(x) = a_\infty > 0 \text{ and } \lim_{|x|\to\infty} \ell(x) = 0 \text{ with } 0 \leq \ell \in L^2(\mathbb{R}^3). \tag{3}$$

Assuming (3) and adequate hypotheses about the functions a and ℓ, they proved the existence of positive solution for (2).

In [8], Azzollini et al. studied (2) for the case $V \equiv 0$, $\ell \equiv q$ constant and the nonlinearity $g(x,u) = g(u)$ is of the type Berestycki and Lions [15], see the hypotheses $(g_1)-(g_4)$ in [8]. The case in which both V and a vanish at infinity, that is, $V(x) \to 0$ and $a(x) \to 0$ as $|x| \to \infty$, was studied in [30, 33]. Recently, planar Schrödinger-Poisson system (that is, $\mathbb{R}^3$ is replaced by $\mathbb{R}^2$ in (2)) was considered in [41], where $V \equiv 0$, $\ell \equiv 1$ and $a(x) = a(|x_1|, |x_2|)$ for all $x \in \mathbb{R}^2$ and $\liminf_{|x|\to\infty} a(x) \in (0, \infty)$. For recent works on planar Schrödinger-Poisson system, see [1, 17, 31] and references therein.

In [9], Batista and Furtado showed the existence of positive and nodal solutions to (2) with appropriate hypotheses on the functions V, a and ℓ, with possible sign-change in potential V. See also [21, 39, 40] for papers that consider sign-changing in the potential V.

Motivated by the above papers, in this article we study the problem (1) for a class of potential V which can change sign in $\mathbb{R}^3$ and vanish at infinity, that is, $V(x) \to 0$ as $|x| \to \infty$.

In what follows, we will denote by S the best Sobolev embedding constant of the injection $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, that is,

$$0 < S := \inf \left\{ |\nabla u|^2_{L^2(\mathbb{R}^3)} : u \in D^{1,2}(\mathbb{R}^3) \text{ and } |u|_{L^6(\mathbb{R}^3)} = 1 \right\} \tag{4}$$

and we also denote $\Omega := \{x \in \mathbb{R}^3 : V(x) < 0\}$. To state our result, we need the following assumptions on V and ℓ :

(V_1) $V \in L^{\frac{3}{2}}(\Omega)$ and $|V|_{L^{\frac{3}{2}}(\Omega)} < S$.

(V_2) $V(x) \leq V_\infty$ in $B_1(0)$, for some constant $V_\infty > 0$.

(V_3) There exist constants $\Lambda > 0$ and $R > 1$ such that

$$\frac{1}{R^4} \inf_{|x| \geq R} |x|^4 V(x) \geq \Lambda.$$

(ℓ) $\ell \in L^2(\mathbb{R}^3), \ell(x) \geq 0$ a.e. in $\mathbb{R}^3$ and $\ell \not\equiv 0$.

Remark 1.1. It is clear that if (V_3) occurs, then $\Omega \subset B_R(0)$. □

Related to the continuous function $f : \mathbb{R} \to \mathbb{R}$, we will assume the following hypotheses:

(f_1) $f(s) = 0$, for all $s \leq 0$ and $\limsup_{s \to 0^+} \dfrac{sf(s)}{s^6} < +\infty$;

(f_2) $\limsup_{s \to +\infty} \dfrac{sf(s)}{s^p} = 0$, for some $p \in (2, 6)$;

(f_3) there exists $\theta > 4$ such that $0 < \theta F(t) := \theta \int_0^t f(s)\, ds \leq tf(t)$, for all $t > 0$.

Before introducing the notion of solution for (1), we will introduce adequate functional spaces to find a solution for (1). Note that $H^1(\mathbb{R}^3)$ is not the right space to study it because of hypotheses on V.

The natural functional spaces to find the solution of (1) are E and $D^{1,2}(\mathbb{R}^3)$, where

$$E := \left\{ u \in D^{1,2}(\mathbb{R}^3) : V(x)u^2 \in L^1(\mathbb{R}^3) \right\},$$

for more details on E, see Section 2.

By a solution of (1) we mean a pair $(u, \phi) \in E \times D^{1,2}(\mathbb{R}^3)$ such that $u, \phi \geq 0$ a.e in $\mathbb{R}^3$ and it holds that

$$\begin{cases} \displaystyle\int_{\mathbb{R}^3} \nabla u \nabla v + \int_{\mathbb{R}^3} V(x)uv + \int_{\mathbb{R}^3} \ell(x)\phi uv = \int_{\mathbb{R}^3} (f(u) + \lambda u^{q-1})v, \text{ for all } v \in E, \\ \displaystyle\int_{\mathbb{R}^3} \nabla \phi \nabla \varphi = \int_{\mathbb{R}^3} \ell(x)u^2\varphi, \text{ for all } \varphi \in D^{1,2}(\mathbb{R}^3). \end{cases}$$

Our main result can be stated as follows:

Theorem 1.2. *Suppose that* $(f_1), (f_2), (f_3), (V_1), (V_2), (V_3)$ *and* (ℓ) *hold. Then there are constants* $\lambda_0, \Lambda_0 > 0$*, such that for each* $\lambda \in [0, \lambda_0)$ *and* $\Lambda \geq \Lambda_0$*, the problem* (1) *admits a nontrivial solution* (u, ϕ)*. Moreover,* $u \in L^\infty(\mathbb{R}^3) \cap C^{1,\alpha}_{loc}(\mathbb{R}^3)$*, for some* $\alpha \in (0, 1)$.

The particular case of (1) with $\lambda = 0$ and $\ell \equiv 0$ (Schrödinger equation) was considered by Alves and Souto in [3], where the geometry (V_3) on V was considered for the first time in the literature. The authors proved the existence of a positive solution for the problem, where the nonlinearity f satisfies a subcritical growth condition. This geometric condition and some of its variations have been considered by several researchers, see for instance, [2, 16, 20, 34, 35, 37] and references therein.

Our research is based on the variational method. Inspired by [3, 16, 33, 41], we show the Theorem 1.2 by combining the famous *"Reduction argument"* of Benci and Fortunato [10], see also Benci et al. [13], with an adaptation of the celebrated *"Penalization method"* of del Pino and Felmer [25].

Below we list what we believe that are the main contributions of our paper.

(i) Regarding systems of the type Schrödinger-Poisson, our work completes [8, 19, 23, 30, 33, 34] where were considered potential vanishing at infinity and [9, 21, 39, 40] where the potential V can change sign.

(ii) Our work completes the papers [3, 35, 16, 33, 41] (where (V3) is considered), because the potential $V(x)$ may be negative in a domain of $\mathbb{R}^3$, see (V_1). In [3, 16] the set $\Omega =: \{x \in \Omega : V(x) < 0\}$ is empty. Since in our case we can have $\Omega \neq \emptyset$, in addition to the presence of two equations in (1), most of our estimates are different in several ways from those of the scalar problem in [3, 16].

(iii) We observe that, we do not assume that V is bounded from below. Thus, we introduce a more general class of potential V than the ones considered in [3, 16, 35, 33, 41]. A potential V that is unbounded below and vanishing at infinity $V : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ given by

$$V(x) = \begin{cases} -\dfrac{\epsilon}{|x|^\alpha}, & \text{if } |x| < R, \\ \left(\dfrac{R}{|x|}\right)^\beta \Lambda, & \text{if } |x| \geq R, \end{cases}$$

where $\epsilon > 0$ is small, $0 \leq \alpha < 2$ and $0 \leq \beta \leq 4$. It satisfies $V(x) \to 0$ as $|x| \to \infty$, the hypotheses $(V_2), (V_3)$ and (V_1) with $\Omega = B_1(0)$ because

$$\int_\Omega |V(x)|^{3/2} dx = \int_{B_1(0)} |V(x)|^{3/2} dx = w_3 \epsilon^{3/2} \int_0^s s^{2-3\alpha/2} ds = w_3 C_\alpha \epsilon^{3/2},$$

for some constant $C_\alpha > 0$ with w_3 denoting the volume of the unit ball in $\mathbb{R}^3$. But it does not belong to the potential class considered in [3, 35, 16, 33, 41].

A typical example of function f satisfying $(f_1) - (f_3)$ is given for $p \in (4, 6)$ by

$$f(s) = \begin{cases} 0, & \text{if } s \leq 0, \\ s^5, & \text{if } s \in (0,1), \\ s^{p-1}, & \text{if } s > 1. \end{cases}$$

The rest of the paper is organized as follows: In Section 2 we present a technical result about the space E. In Section 3 we show the existence of nontrivial solution for an auxiliary problem associated to (1). Section 4 is devoted to obtain a suitable L^∞-estimate of the solution of a auxiliary problem. Finally, in Section 5 we complete the proof of Theorem 1.2.

2. Preliminary results

In this section, we prove some preliminary results on the space E that will be useful in what follows. As mentioned in the introduction, the natural functional spaces to find the solution of (1) are $E \times D^{1,2}(\mathbb{R}^3)$, where

$$E := \left\{ u \in D^{1,2}(\mathbb{R}^3) : \int_\Omega V(x)|u|^2 < \infty \right\}$$

and $D^{1,2}(\mathbb{R}^3) := \{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$.

A fundamental tool in our analysis will be the following proposition.

Proposition 2.1. *Assume that condition* (V_1) *holds. Let us define, the function* $\|.\| : E \to \mathbb{R}$ *given by*

$$\|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)|u|^2),$$

then, $\|.\|$ *is a norm on* E *and* $E = (E, \|.\|)$ *is a Hilbert space. Moreover, the embedding* $E \hookrightarrow D^{1,2}(\mathbb{R}^3)$ *is continuous.*

Proof. We consider the bilinear form $(.\,,.) : E \times E \to \mathbb{R}$ defined by

$$(u, v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv). \tag{5}$$

Since by (V_1), $V(x) \geq 0$ in $\Omega^c := \mathbb{R}^3 \setminus \Omega$, we have by Hölder's inequality and the embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$,

$$\begin{aligned}\int_{\mathbb{R}^3} V(x)|u|^2 = \int_{\Omega^c} V(x)|u|^2 - \int_\Omega -V(x)|u|^2 &\geq -|V|_{L^{\frac{3}{2}}(\Omega)} \Big(\int_{\mathbb{R}^3} |u|^6 \Big)^{\frac{1}{3}} \\ &\geq -|V|_{L^{\frac{3}{2}}(\Omega)} S^{-1} \int_{\mathbb{R}^3} |\nabla u|^2, \end{aligned} \tag{6}$$

where S is given in (4). Then,

$$(u, u) \geq (1 - |V|_{L^{\frac{3}{2}}(\Omega)} S^{-1}) \int_{\mathbb{R}^3} |\nabla u|^2. \tag{7}$$

Hence, $(u, u) \geq 0$ for all $u \in E$ and $(u, u) = 0$ if and only if $u = 0$, see (V_1).

It is clear that $E \subset D^{1,2}(\mathbb{R}^3)$ and the embedding $E \hookrightarrow D^{1,2}(\mathbb{R}^3)$ is continuous, because of (7), with

$$\|u\|^2_{D^{1,2}(\mathbb{R}^3)} \leq C\|u\|^2, \text{ for all } u \in E, \text{ and } C := (1 - |V|_{L^{\frac{3}{2}}(\Omega)} S^{-1})^{-1}. \qquad \square$$

3. Auxiliary problems

Since for $q > 2^* := 6$ the functional associated with problem (1) is not well defined in $E \times D^{1,2}(\mathbb{R}^3)$ because we can not guarantee that $|u|^q \in L^1(\mathbb{R}^3)$ if $u \in E$, then to overcome this difficulty and apply the variational method to get the existence of weak of solution for (1), we will introduce a new nonlinearity. To this end, given $m \in \mathbb{N}$ (arbitrary), we define the function $g_{m,\lambda} : \mathbb{R} \to [0, \infty)$ by

$$g_{\lambda,m}(s) = \begin{cases} 0, & \text{if } s \leq 0, \\ f(s) + \lambda s^{q-1}, & \text{if } 0 \leq s \leq m, \\ f(s) + \lambda m^{q-p} s^{p-1}, & \text{if } s > m. \end{cases} \tag{8}$$

Using (f_1) and (f_2) we conclude that there exists a constant $C_0 \geq 1$, independent of m, such that $|f(s)| \leq C_0 s^{p-1}$ and $|f(s)| \leq C_0 s^5$, for all $s \in \mathbb{R}$. Hence, by (8) we obtain for all $s \in \mathbb{R}$,

$$|g_{\lambda,m}(s)| \leq C_0(1+\lambda m^{q-p})|s|^{p-1} \quad \text{and} \quad |g_{\lambda,m}(s)| \leq C_0(1+\lambda m^{q-p})|s|^5. \tag{9}$$

Moreover, for $G_{\lambda,m}(s) := \displaystyle\int_0^s g_{\lambda,m}(t)$ we have

$$G_{\lambda,m}(s) = \begin{cases} 0, & \text{if } s \leq 0, \\ F(s) + \lambda s^q, & \text{if } 0 \leq s \leq m, \\ F(s) + \dfrac{\lambda}{p} m^{q-p} s^p + \lambda\left(\dfrac{1}{q} - \dfrac{1}{p}\right) m^q, & \text{if } s > m \end{cases} \tag{10}$$

and for all $s \in \mathbb{R}$, we obtain

$$|G_{\lambda,m}(s)| \leq \frac{C_0}{p}(1+\lambda m^{q-p})|s|^p \quad \text{and} \quad |G_{\lambda,m}(s)| \leq \frac{C_0}{6}(1+\lambda m^{q-p})|s|^6. \tag{11}$$

Observe that by (11) the functional

$$\mathcal{I}_{\lambda,m}(u,\phi) := \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} \ell(x)\phi u^2 - \int_{\mathbb{R}^3} G_{\lambda,m}(u) - \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi|^2,$$

is well defined on $E \times D^{1,2}(\mathbb{R}^3)$. Moreover, $\mathcal{I}_{\lambda,m} \in C^1\left(E \times D^{1,2}(\mathbb{R}^3), \mathbb{R}\right)$ and critical points of $\mathcal{I}_{\lambda,m}$ are solutions of the modified problem

$$\begin{cases} -\Delta u + V(x)u + \ell(x)\phi u = g_{\lambda,m}(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = \ell(x)u^2 & \text{in } \mathbb{R}^3, \\ u \in E, \phi \in D^{1,2}(\mathbb{R}^3), \ u,\phi \geq 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{12}$$

It is clear that if $(u,\phi) \in E \times D^{1,2}(\mathbb{R}^3)$ is critical points of $\mathcal{I}_{\lambda,m}$ (i.e. (u,ϕ_u) is a solution of (12)) with $|u|_{L^\infty(\mathbb{R}^3)} \leq m$, then (u,ϕ) is solution of (1). However, the functional $\mathcal{I}_{\lambda,m}$ is strongly indefinite, which causes some difficulties to find its critical points.

Another difficulty is the lack of compactness in the embedding $E, D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ which produces some technical difficulties to prove the Palais-Smale condition and use variational methods. To overcome these difficulties and find solution of (1), inspired by [3, 13, 16], we will combine the *"Reduction argument"* described by Benci and Fortunato in [10], see also Benci et al. in [13], with an adaptation of the *"Penalization method "* of del Pino and Felmer [25].

3.1. The reduction argument and the truncated problem

For each $u \in E$, by Lax-Milgram's theorem, we deduce that there exists a unique $u \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla\phi_u \nabla v = \int_{\mathbb{R}^3} \ell(x)u^2 v, \text{ for all } v \in D^{1,2}(\mathbb{R}^3). \tag{13}$$

More precisely, the weak solution ϕ_u can be expressed as, see [6, 36],

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\ell(y)u^2(y)}{|x-y|}\,dy. \tag{14}$$

The next result gives the main properties of the nonlocal term ϕ_u. The proof follows standard arguments.

Lemma 3.1. *Assume that* (V_1) *and* (ℓ) *hold. The following assertions hold true:*

(i) *There exists a constant* $C > 0$ *such that* $\int_{\mathbb{R}^3} \ell(x)\phi_u u^2 \leq C\|u\|^4$ *for all* $u \in E$.

If $u_n \rightharpoonup u$ *in* E, *then:*

(ii) $\int_{\mathbb{R}^3} \ell(x)\phi_{u_n} u_n^2 \to \int_{\mathbb{R}^3} \ell(x)\phi_u u^2$,

(iii) $\int_{\mathbb{R}^3} |\nabla \phi_{u_n}|^2 \to \int_{\mathbb{R}^3} |\nabla \phi_u|^2$,

(iv) $\int_{\mathbb{R}^3} \ell(x)\phi_{u_n} u_n u \to \int_{\mathbb{R}^3} \ell(x)\phi_u u^2$.

To overcome the lack of compactness in the embedding E, $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and for using an adaptation of penalization method combined with the reduction argument, we finally introduce two auxiliary functions. To this end, with $k = \frac{2\theta}{\theta - 4}$, consider the following functions $\widetilde{g}_{\lambda,m}, h_{\lambda,m} : \mathbb{R}^3 \times \mathbb{R} \to [0, \infty)$, given by

$$\widetilde{g}_{\lambda,m}(x,s) = \begin{cases} g_{\lambda,m}(s), & \text{if } \; k g_{\lambda,m}(s) \leq V(x)s, \\ \frac{V(x)}{k} s, & \text{if } \; k g_{\lambda,m}(s) > V(x)s \end{cases} \tag{15}$$

and $\widetilde{g}_{\lambda,m}(x,s) = 0$ for all $x \in \mathbb{R}^3$ and $s \leq 0$. Furthermore, we also define

$$h_{\lambda,m}(x,s) = \chi_{B_R(0)}(x) g_{\lambda,m}(s) + (1 - \chi_{B_R(0)}(x)) \widetilde{g}_{\lambda,m}(x,s), \tag{16}$$

where $\chi_{B_R(0)}$ is the characteristic function related to ball $B_R(0)$.
Now consider the auxiliary system of Schrödinger-Poisson equations

$$\begin{cases} -\Delta u + V(x)u + \ell(x)\phi u = h_{\lambda,m}(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = \ell(x)u^2 & \text{in } \mathbb{R}^3, \\ u \in E, \phi \in D^{1,2}(\mathbb{R}^3), \; u, \phi \geq 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{17}$$

We recall that the Lax-Milgram theorem implies that for all $u \in E$ there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$, given by the convolution $\phi_u = \frac{1}{4\pi}|x| * \ell(x)u^2$, such that $-\Delta \phi_u = \ell(x)u^2$. Substituting into (17) we find

$$\begin{cases} -\Delta u + V(x)u + \ell(x)\phi_u u = h_{\lambda,m}(x,u) & \text{in } \mathbb{R}^3, \\ u \in E, \; u \geq 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{18}$$

The Euler-Lagrange functional associated with the system (18) is $J_{\lambda,m} : E \to \mathbb{R}$, given by

$$J_{\lambda,m}(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) + \frac{1}{4}\int_{\mathbb{R}^3} \ell(x)\phi_u u^2 - \int_{\mathbb{R}^3} H_{\lambda,m}(x,u),$$

where $H_{\lambda,m}(x,s) = \int_0^s h_{\lambda,m}(x,t)$.

The functional $J_{\lambda,m}$ is of class $C^1(E,\mathbb{R})$ and

$$J'_{\lambda,m}(u)v = \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv) + \int_{\mathbb{R}^3} \ell(x)\phi_u uv - \int_{\mathbb{R}^3} h_{\lambda,m}(x,u)v, \text{ for all } u,v \in E.$$

In fact, it is sufficient to note that assuming that $(V_1), (\ell)$ hold, using Hardy-Littlewood-Sobolev inequality [28, Theorem 4.3] (i.e., if $0 < \beta < N$, $f \in L^q(\mathbb{R}^N)$, $g \in L^r(\mathbb{R}^N)$ with $\frac{1}{q} + \frac{1}{r} + \frac{\beta}{N} = 2$ and $1 < q, r < \infty$). Then,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)||g(x)|}{|x-y|^\beta} \leq C|f|_{L^q(\mathbb{R}^N)}|g|_{L^r(\mathbb{R}^N)},$$

for some contant $C = C(q,r,\beta,N)\big)$ and standard arguments, we can prove that the functional $J_1 : E \to \mathbb{R}$ given by

$$J_1(u) = \int_{\mathbb{R}^3} \ell(x)\phi_u u^2,$$

is of class $J_1 \in C^1(E,\mathbb{R})$ with

$$J_1'(u)v = 4\int_{\mathbb{R}^3} \ell(x)\phi_u uv, \text{ for all } u,v \in E.$$

Remark 3.2. (i) The functional associated with the system (17)

$$\mathcal{J}_{\lambda,m}(u,\phi) := \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} \ell(x)\phi u^2 - \int_{\mathbb{R}^3} H_{\lambda,m}(x,u) - \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi|^2, \tag{19}$$

is of class $C^1\left(E \times D^{1,2}(\mathbb{R}^3), \mathbb{R}\right)$ with

$$\frac{\partial \mathcal{J}_{\lambda,m}}{\partial u}(u,\phi)\xi = \int_{\mathbb{R}^3} (\nabla u \nabla \xi + V(x)u\xi) + \int_{\mathbb{R}^3} \ell(x)\phi u\xi - \int_{\mathbb{R}^3} h_{\lambda,m}(x,u)\xi, \text{ for all } \xi \in E,$$

and $$\frac{\partial \mathcal{J}_{\lambda,m}}{\partial \phi}(u,\phi)\eta = \frac{1}{2}\int_{\mathbb{R}^3} \ell(x)u^2\eta - \frac{1}{2}\int_{\mathbb{R}^3} \nabla\phi\nabla\eta, \text{ for all } \eta \in D^{1,2}(\mathbb{R}^3).$$

Hence, since $J_{\lambda,m}(u) = \mathcal{J}_{\lambda,m}(u,\phi_u)$ from a standard argument we conclude that the pair $(u,\phi_u) \in E \times D^{1,2}(\mathbb{R}^3)$ is a critical point of $\mathcal{J}_{\lambda,m}$ (i.e. (u,ϕ_u) is a solution of (17) if and only if u is a critical point of $J_{\lambda,m}$ and $\phi = \phi_u$.

(ii) From (i) we notice that problem (18) is strongly related to system (1), because if $(u,\phi_u) \in E \times D^{1,2}(\mathbb{R}^3)$ is a solution of (18) verifying $kg_{\lambda,m}(u) \leq V(x)u$ in $B_R^c(0)$ and $|u|_{L^\infty(\mathbb{R}^3)} \leq m$, then by the definitions of $h_{\lambda,m}$ and $g_{\lambda,m}$, we directly obtain $h_{\lambda,m}(x,u) = g_{\lambda,m}(u) = f(u) + \lambda u^{q-1}$; then u will also be a solution for (1). □

Now, note that using (8), (10) and a direct computation, we have

$$g_{\lambda,m}(s)s - \theta G_{\lambda,m}(s) = \begin{cases} 0, \text{ if } \ s \leq 0, \\ f(t)t - \theta F(t) + (\frac{q-\theta}{q})\lambda s^q, \text{ if } \ 0 \leq s \leq m, \\ f(t)t - \theta F(t) + \lambda m^{q-p}\frac{s^p}{p}(\frac{p-\theta}{p}) + \lambda m^q(\frac{q-p}{qp}), \text{ if } \ s > m. \end{cases} \tag{20}$$

Hence, the following properties can be easily proved:

Lemma 3.3. *The functions $h_{\lambda,m}$, given in* (16), *and its primitive $H_{\lambda,m}$ satisfy:*

(h_1) $\theta H_{\lambda,m}(x,s) \le h_{\lambda,m}(x,s)s$, *for all $x \in B_R(0)$ and $s \in \mathbb{R}$,*

(h_2) $2H_{\lambda,m}(x,s) \le h_{\lambda,m}(x,s)s \le \dfrac{V(x)}{k}s^2$, *for all $x \in \mathbb{R}^3 \setminus B_R(0)$ and $s \in \mathbb{R}$,*

(h_3) $h_{\lambda,m}(x,s) \le \widetilde{g}_{\lambda,m}(x,s) \le g_{\lambda,m}(s)$, *for all $x \in \mathbb{R}^3$ and $s \in \mathbb{R}$,*

(h_4) $h_{\lambda,m}(x,s) = g_{\lambda,m}(s)$, *for all $x \in B_R(0)$.*

Lemma 3.4. *Assume that condition $(f_1), (f_2), (f_3), (V_1), (\ell)$ and $\Omega \subset B_R(0)$ hold. Let (u_n) be a $(PS)_c$ sequence for $J_{\lambda,m}$. Then (u_n) is bounded in E.*

Proof. From (h_1), we have

$$J_{\lambda,m}(u_n) - \frac{1}{\theta}J'_{\lambda,m}(u_n)u_n \ge \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\int_{\mathbb{R}^3}(|\nabla u_n|^2 + V(x)u_n^2) + \Big(\frac{1}{4} - \frac{1}{\theta}\Big)\int_{\mathbb{R}^3}\ell(x)\phi_{u_n}u_n^2$$
$$+ \frac{1}{\theta}\int_{B_R(0)^c}\Big[h_{\lambda,m}(x,u_n)u_n - \theta H_{\lambda,m}(x,u_n)\Big].$$

Using $\Omega \subset B_R(0)$, (h_2) and $\theta > 4$, we get

$$\frac{1}{\theta}\int_{B_R(0)^c}\Big[h_{\lambda,m}(x,u_n)u_n - H_{\lambda,m}(x,u_n)\Big] \ge \Big(\frac{2-\theta}{\theta}\Big)\frac{1}{2k}\int_{B_R(0)^c}V(x)u_n^2$$
$$\ge \Big(\frac{2-\theta}{\theta}\Big)\frac{1}{2k}\int_{\Omega^c}V(x)u_n^2 = \Big(\frac{2-\theta}{\theta}\Big)\frac{1}{2k}\int_{\mathbb{R}^3}V(x)u_n^2 + \Big(\frac{\theta-2}{\theta}\Big)\frac{1}{2k}\int_{\Omega}V(x)u_n^2.$$

Furthermore, similarly as in (6),

$$\int_{\Omega}V(x)u_n^2 \ge -|V|_{L^{\frac{3}{2}}(\Omega)}S^{-1}\int_{\mathbb{R}^3}|\nabla u_n|^2,$$

hence, we obtain

$$J_{\lambda,m}(u_n) - \frac{1}{\theta}J'_{\lambda,m}(u_n)u_n \ge \Big(1 - \frac{|V|_{L^{\frac{3}{2}}(\Omega)}S^{-1}}{k}\Big)\frac{\theta-2}{2\theta}\int_{\mathbb{R}^3}|\nabla u_n|^2$$
$$+ \Big(1 - \frac{1}{k}\Big)\frac{\theta-2}{2\theta}\int_{\mathbb{R}^3}V(x)u_n^2 + \Big(\frac{\theta-4}{4\theta}\Big)\int_{\mathbb{R}^3}\ell(x)\phi_{u_n}u_n^2$$
$$\ge \Big(1 - \frac{1}{k}\Big)\frac{\theta-2}{2\theta}\int_{\mathbb{R}^3}(|\nabla u_n|^2 + V(x)u_n^2) + \Big(\frac{\theta-4}{4\theta}\Big)\int_{\mathbb{R}^3}\ell(x)\phi_{u_n}u_n^2$$
$$= \Big(1 - \frac{1}{k}\Big)\frac{\theta-2}{2\theta}\|u_n\|^2 + \Big(\frac{\theta-4}{4\theta}\Big)\int_{\mathbb{R}^3}\ell(x)\phi_{u_n}u_n^2,$$

where we use $|V|_{L^{\frac{3}{2}}(\Omega)}S^{-1} < 1$ and $\|u_n\| = (u_n,u_n)^{1/2}$, see (5).

Hence, it follows that given $\epsilon > 0$ there exists $n_0 \in N$ such that

$$c + 1 + \epsilon\|u_n\| + o_n(1) \ge \Big(1 - \frac{1}{k}\Big)\frac{\theta-2}{2\theta}\|u_n\|^2, \quad \text{for all} \quad n \ge n_0,$$

which implies that (u_n) is bounded in E. $\square$

Lemma 3.5. *Assume that condition $(f_1), (f_2), (f_3), (V_1), (\ell)$ and $\Omega \subset B_R(0)$ hold. Let (u_n) be a $(PS)_c$ sequence for $J_{\lambda,m}$. Then, for each $\epsilon > 0$, there exists $r = r(\epsilon) > 0$, such that*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3\setminus B_r(0)} [|\nabla u_n|^2 + V(x)|u_n|^2] + \int_{\mathbb{R}^3\setminus B_r(0)} \ell(x)\phi_{u_n} u_n^2 < \epsilon. \tag{21}$$

Proof. Let $\eta_r \in C^\infty(\mathbb{R}^3, [0,1])$, such that $\eta_r(x) = 0$ in $B_r(0)$, $\eta_r(x) = 1$ in $B_{2r}(0)^c$ and $|\nabla \eta_r(x)| \le C/r$ in $\mathbb{R}^3$, where $C > 0$ is a constant independent on R.

By Lemma 3.4, the sequence (u_n) is bounded in E, so $(u_n\eta_r)$ is also bounded in E. Hence, $J'_{\lambda,m}(u_n)(u_n\eta_r) = o_n(1)$ that is

$$\begin{aligned} &\int_{\mathbb{R}^3} \eta_r[|\nabla u_n|^2 + V(x)|u_n|^2] + \int_{\mathbb{R}^3} \ell(x)\phi_{u_n} u_n^2\eta_r \\ &= \int_{\mathbb{R}^3} h_{\lambda,m}(x,u_n)\eta_r u_n - \int_{\mathbb{R}^3} u_n \nabla\eta_r \nabla u_n + o_n(1). \end{aligned}$$

Let $r > R$ that will be fixed later. By (V_1), we deduce that the set $\Omega \subset B_r(0)$. Then, by (h_2) we get

$$\int_{\mathbb{R}^3} h_{\lambda,m}(x,u_n)\eta_r u_n \le \frac{1}{k}\int_{\mathbb{R}^3} \eta_r V(x)|u_n|^2.$$

Using this, the definition of η_r and Hölder's inequality, we obtain

$$I_{n,r} \le \frac{C}{r}\Big(1 - \frac{1}{k}\Big)^{-1} |\nabla u_n|_{L^2(\mathbb{R}^3)} \left(\int_{r\le|x|\le 2r} |u_n|^2\right)^{\frac{1}{2}} + o_n(1),$$

where we define

$$I_{n,r} := \int_{\mathbb{R}^3} \eta_r[|\nabla u_n|^2 + V(x)|u_n|^2] + \int_{\mathbb{R}^3} \ell(x)\phi_{u_n} u_n^2\eta_r.$$

By (7), we deduce that

$$I_{n,r} \le \frac{C}{r}\Big(1 - \frac{1}{k}\Big)^{-1} \frac{\|u_n\|}{(1 - |V|_{L^{\frac{3}{2}}(\Omega)} S^{-1})^{1/2}} \Big(\int_{r\le|x|\le 2r} |u_n|^2\Big)^{\frac{1}{2}} + o_n(1).$$

Since (u_n) is bounded in E then, up to subsequence, $u_n \to u$ in $L^2_{loc}(\mathbb{R}^3)$ and there exists a constant $M > 0$ such that

$$\limsup_{n\to\infty} I_{n,r} \le \frac{C}{r}\Big(1 - \frac{1}{k}\Big)^{-1} \frac{M}{(1 - |V|_{L^{\frac{3}{2}}(\Omega)} S^{-1})^{1/2}} \Big(\int_{r\le|x|\le 2r} |u|^2\Big)^{\frac{1}{2}}. \tag{22}$$

Applying Hölder's inequality, we get

$$\Big(\int_{r\le|x|\le 2r} |u|^2\Big)^{\frac{1}{2}} \le 2r\sqrt[3]{w_3}\Big(\int_{r\le|x|\le 2r} |u|^6\Big)^{\frac{1}{6}}, \tag{23}$$

where w_3 is the volume of the unit ball in $\mathbb{R}^3$.

Finally, since $u \in L^6(\mathbb{R}^3)$ by choosing $r > R$ such that

$$2\left(1-\frac{1}{k}\right)^{-1}\sqrt[3]{w_3}\,C\,\frac{M}{(1-|V|_{L^{\frac{3}{2}}(\Omega)}S^{-1})^{1/2}}\left(\int_{r\le|x|\le 2r}|u|^6\right)^{\frac{1}{6}}<\epsilon. \tag{24}$$

Combining (22), (23) and (24), we obtain (21). □

Proposition 3.6. *Assume that condition* $(f_1), (f_2), (f_3), (V_1), (\ell)$ *and* $\Omega \subset B_R(0)$ *hold. The functional* $J_{\lambda,m}$ *verifies the* $(PS)_c$ *condition in* E.

Proof. We claim that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3} h_{\lambda,m}(x,u_n)u_n = \int_{\mathbb{R}^3} h_{\lambda,m}(x,u)u. \tag{25}$$

Indeed, from the Lemma (21), we have

$$\limsup_{n\to\infty}\left|\int_{B^c_{2r}(0)} h_{\lambda,m}(x,u_n)u_n\right| = o_r(1). \tag{26}$$

Since, by the Sobolev embedding theorem, $u_n \to u$ in $L^p(B_{2r}(0))$ and

$$|h_{\lambda,m}(x,s)| \le g_{\lambda,m}(s) \le c_0(1+\lambda m^{q-p})s^{p-1},$$

see (9), using the dominated convergence theorem, we have

$$\int_{B_{2r}(0)} h_{\lambda,m}(x,u_n)u_n \to \int_{B_{2r}(0)} h_{\lambda,m}(x,u)u. \tag{27}$$

As consequence of (26) and (27), it follows that the claim (25) is true.

Similarly, we can also prove that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3} h_{\lambda,m}(x,u_n)u = \int_{\mathbb{R}^3} h_{\lambda,m}(x,u)u. \tag{28}$$

Now, since $u_n \rightharpoonup u$ in E, by Lemma 3.1,

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\ell(x)\phi_{u_n}u_n u = \int_{\mathbb{R}^3}\ell(x)\phi_u u^2. \tag{29}$$

By (28), (29), $u_n \rightharpoonup u$ in E and $J'_{\lambda,m}(u_n)u = o_n(1)$, we get

$$\int_{\mathbb{R}^3}[|\nabla u|^2 + V(x)|u|^2] + \int_{\mathbb{R}^3}\ell(x)\phi_u u^2 = \int_{\mathbb{R}^3} h_{\lambda,m}(x,u)u.$$

From this equality, $J'_{\lambda,m}(u_n)u_n = o_n(1)$ and (25), we obtain

$$\begin{aligned}\int_{\mathbb{R}^3}[|\nabla u_n|^2 + V(x)|u_n|^2] + \int_{\mathbb{R}^3}\ell(x)\phi_{u_n}u_n^2 &= \int_{\mathbb{R}^3} h_{\lambda,m}(x,u_n)u_n + o_n(1)\\ = \int_{\mathbb{R}^3}[|\nabla u|^2 + V(x)|u|^2] + \int_{\mathbb{R}^3}\ell(x)\phi_u u^2 + o_n(1).\end{aligned}$$

Thus, using once more (29) we conclude that

$$\int_{\mathbb{R}^3}[|\nabla u_n|^2+V(x)|u_n|^2]\to\int_{\mathbb{R}^3}[|\nabla u|^2+V(x)|u|^2].$$

then, since E is a Hilbert space, we have $u_n\to u$ in E. □

We now turn our attention to show that $J_{\lambda,m}$ has the geometric conditions of the mountain pass theorem.

Lemma 3.7. *Assume that condition $(f_1),(f_2),(f_3),(V_1),(\ell)$ and $\Omega\subset B_R(0)$ hold. Then, for each $R>1$ we have:*

(i) *There exist $\rho,\alpha>0$, such that $J_{\lambda,m}(u)\geq\alpha$ for all $u\in E$, $\|u\|=\rho$.*

(ii) *There exists $e\in C_0^\infty(B_1(0))$, independent of R, such that $J_{\lambda,m}(e)<0$ and $\|e\|>\rho$.*

Proof. Recall that $|H_{\lambda,m}(x,s)|\leq G_{\lambda,m}(s)\leq\frac{c_0}{6}(1+\lambda m^{q-p})s^6$ for all $s\in\mathbb{R}$ and so

$$J_{\lambda,m}(u)\geq\frac{1}{2}\|u\|^2-\frac{C_0(1+\lambda m^{q-p})}{6}\int_{\mathbb{R}^3}|u|^6,$$

hence, $$J_{\lambda,m}(u)\geq\|u\|^2\Big(\frac{1}{2}-\frac{C_0(1+\lambda m^{q-p}S^{-1})}{6}\|u\|^4\Big),\text{ for all } u\in E,$$

which implies that the assertion (i) holds true for $\|u\|=\rho$ sufficiently small and

$$\alpha=\rho^2\left(\frac{1}{2}-\frac{C_0(1+\lambda m^{q-p}S^{-1})}{6}\rho^4\right).$$

Note that $J_{\lambda,m}(0)=0$. Let $\psi\in C_0^\infty(B_1(0))\setminus\{0\}$, by (10) and (16) we have $H_{\lambda,m}(x,t\psi)=G_{\lambda,m}(t\psi)\geq F(t\psi)$. Hence, from (f_3) there exist $C_1,C_2>0$ such that for all $t\geq0$,

$$H_{\lambda,m}(x,t\psi)\geq C_1(t\psi)^\theta-C_2\text{ in } B_1(0). \tag{30}$$

This combined with (i) of Lemma 3.1 imply that

$$\begin{aligned}J_{\lambda,m}(t\psi)&\leq\frac{t^2}{2}\|\psi\|^2+\int_{\mathbb{R}^3}\ell(x)\phi_{t\psi}(t\psi)^2-C_1t^\theta\int_{B_1(0)}|\psi|^\theta+C_2|B_1(0)|\\&\leq\frac{t^2}{2}\|\psi\|^2+Ct^4\|\psi\|^2-C_1t^\theta\int_{B_1(0)}|\psi|^\theta+C_2|B_1(0)|.\end{aligned}$$

Since $\theta>4$, we obtain $J_{\lambda,m}(t_0\psi)<0$ for some $t_0>0$, which proves (ii). □

Now note that for every $u\in H_0^1(B_1(0))$ the function $\widetilde{u}:\mathbb{R}^3\to\mathbb{R}$ given by $\widetilde{u}=u$ in $B_1(0)$ and $\widetilde{u}=0$ in $\mathbb{R}^3\setminus B_1(0)$ belongs to E, hence the functional $\overline{J}:H_0^1(B_1(0))\to\mathbb{R}$ given by

$$\overline{J}(u)=\frac{1}{2}\int_{B_1(0)}(|\nabla u|^2+V_\infty u^2)+\frac{1}{4}\int_{B_1(0)}\phi_{\widetilde{u}}u^2-\int_{B_1(0)}F(u),$$

is well defined, where $V(x)\leq V_\infty$ in $B_1(0)$ for some $V_\infty\geq0$ $\big($which is possible because of $(V_2)\big)$ and $\phi_{\widetilde{u}}\in D^{1,2}(\mathbb{R}^3)$ is a unique solution (via Stampacchia theorem) of the equation

$$-\Delta\phi=\widetilde{u}^2\quad\text{in }\ \mathbb{R}^3. \tag{31}$$

Let $\overline{\Gamma} = \{\gamma \in C([0,1], B_1(0)) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}$

and $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}$.

Since $J_{\lambda,m}(\widetilde{u}) \le \overline{J}(u)$ for all $u \in H_0^1(B_1(0))$ and $\overline{\Gamma} \subset \Gamma$, we conclude that

$$c_V \le \overline{c}, \quad \text{for all} \quad R > 1, \tag{32}$$

where $$c_V = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_{\lambda,m}(\gamma(t)) \quad \text{and} \quad \overline{c} = \inf_{\gamma\in\overline{\Gamma}}\max_{t\in[0,1]} \overline{J}(\gamma(t)).$$

Theorem 3.8. *Assume that the conditions* $(f_1), (f_2), (f_3), (V_1), (\ell)$ *and* $\Omega \subset B_R(0)$ *hold. Then, for each* $R > 1$, *problem* (18) *has a nonnegative weak solution* $u = u(\lambda, m)$ *such that*

$$\|u\|^2 \le 2k\overline{c}, \quad \textit{for all} \quad R > 1, \tag{33}$$

where $\overline{c}$ *is given in* (32).

Proof. By Lemma 3.7, Proposition 3.6 and the Mountain pass theorem, there exists $u = u(\lambda, m) \in E$ such that $J_{\lambda,m}(u) = c$ and $J'_{\lambda,m}(u) = 0$. Hence,

$$\int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv) + \int_{\mathbb{R}^3} \ell(x)\phi_u uv = \int_{\mathbb{R}^3} h_{\lambda,m}(x,u)v, \quad \text{for all } v \in E. \tag{34}$$

Moreover, using the same arguments considered in the proof of Lemma 3.4,

$$c_V = J_{\lambda,m}(u) - \frac{1}{\theta} J'_{\lambda,m}(u)u, \quad k = 2\theta/(\theta - 2)$$

and (32), it follows that

$$\overline{c} \ge \left(1 - \frac{1}{k}\right)\frac{\theta-2}{2\theta}\|u\|^2 + \left(\frac{\theta-4}{4\theta}\right)\int_{\mathbb{R}^3} \ell(x)\phi_u u^2 \ge \frac{1}{2k}\|u\|^2,$$

which proves (33).

Noting that $\ell(x)$ and ϕ_u are nonnegative, testing (34) with $u^- := \min\{u, 0\}$ and recalling that $u = u^+ + u^-$ where $u^+ := \max\{u, 0\}$, we have

$$\|u^-\|^2 + \int_{\{u<0\}} \ell(x)\phi_u(u^-)^2 = \int_{\mathbb{R}^3} h_{\lambda,m}(x,u)u^- = 0,$$

hence, $\|u^-\|^2 = 0$ and so $u \ge 0$. □

4. L^∞-estimate for solution of the auxiliary problem (18)

In the next result we obtain $L^\infty(\mathbb{R}^3)$-estimate for the solution of (18), via Moser iteration method, which is very important in our arguments to find solution of (1).

Proposition 4.1. *Let* $u_{\lambda,m} \in E$ *be the solution of* (18) *found in Theorem* 3.8. *Then there exists a constant* $M_0 > 0$ *that depends only on* θ, p, q, C_0 *and* $|V|_{L^{\frac{3}{2}}(\Omega)}$, *such that*

$$|u_{\lambda,m}|_{L^\infty(\mathbb{R}^3)} \le M_0(1 + \lambda m^{q-p})^{\frac{1}{6-p}} |u_{\lambda,m}|_{L^6(\mathbb{R}^3)}, \quad \textit{for all} \ \ R > 1, \tag{35}$$

where C_0 *is the constant in* (9).

Proof. By Theorem 3.8 we know that $-\Delta u + V(x)u + \ell(x)\phi_u u = h_{\lambda,m}(x,u)$ in $\mathbb{R}^3$. Moreover, $\phi_u u \geq 0$ in $\mathbb{R}^3$ and $V(x)u \geq 0$ in $\mathbb{R}^3 \setminus \Omega$. Then,

$$-\Delta u \leq b(x,u) \text{ in } \mathbb{R}^3,$$

where $b(x,u) := h_{\lambda,m}(x,u) - \chi_\Omega V(x)u$ and χ_Ω denote the characteristic function of the set Ω. Define the function $a : \mathbb{R}^3 \to \mathbb{R}$ by

$$a(x) := \frac{|b(x,u)|}{1+|u|}.$$

Note that by Lemma 3.3 and (9), we have $|a(x)| \leq C_0(1+\lambda m^{q-p})|u|^4 + \chi(x)|V(x)|$ in $\mathbb{R}^3$. Hence,

$$\begin{aligned}\int_{\mathbb{R}^3} |a(x)|^{\frac{3}{2}} &= \int_\Omega |a(x)|^{\frac{3}{2}} + \int_{\mathbb{R}^3\setminus\Omega} |a(x)|^{\frac{3}{2}} \\ &\leq 2[C_0(1+\lambda m^{q-p})]^{\frac{3}{2}}\Big(\int_\Omega |V(x)|^{\frac{3}{2}} + \int_{\mathbb{R}^3} |u|^6\Big). \qquad (36)\end{aligned}$$

In particular, it follows that

$$-\Delta u \leq a(x)(1+|u|) \text{ in } \mathbb{R}^3 \text{ and } a(x) \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

So, using the Trudinger-Brezis-Kato iteration argument, found in Struwe's book [38, Lemma B3, p. 270], we conclude that $u \in L^r(\mathbb{R}^3)$ for all $r > 1$. Moreover, since $|a|_{L^{\frac{3}{2}}(\mathbb{R}^3)}$ does not depend on $R > 1$ (see (33) and (36)) the norm $|u|_{L^r(\mathbb{R}^3)}$ also does not depend on $R > 1$. Finally, fixing r large enough, the bootstrap arguments implies that there exist $M_0 > 0$, independent of $R > 1$, such that

$$|u|_{L^\infty(\mathbb{R}^3)} \leq M_0(1+\lambda m^{q-p})^\sigma. \qquad \square$$

5. Proof of Theorem 1.2

In this section, in addition to conditions (V_1) and (V_2) we will assume that (V_3) occurs to show the solution $u = u_{\lambda,m}$ of the auxiliary problem (18), given in Theorem 3.8, produces a solution (u,ϕ) to the original problem (1). For this purpose, it is sufficient to prove that the following inequalities hold, see Remark 3.2,

$$|u_{\lambda,m_0}|_{L^\infty(\mathbb{R}^3)} \leq m_0 \text{ and } kg_{\lambda,m}(u) \leq V(x)u \text{ in } B_R^c(0),$$

for all $\lambda \in [0,\lambda_0)$, for some $\lambda_0 > 0$ and $m_0 \in \mathbb{N}$.

Note that as a consequence of Theorem 3.8 and Proposition 4.1, there exist $\lambda_0 > 0$ and $m_0 \in \mathbb{N}$ such that

$$|u_{\lambda,m_0}|_{L^\infty(\mathbb{R}^3)} \leq m_0, \text{ for all } \lambda \in [0,\lambda_0). \qquad (37)$$

In fact, by (33) and (35), there exist constants $M_0, \sigma > 0$ (depending only on $\theta, p, q, C_0, |V|_{L^{\frac{3}{2}}(\Omega)}$) such that $|u_{\lambda,m_0}|_{L^\infty(\mathbb{R}^3)} \leq M_0(1+\lambda m_0^{q-p})^{\frac{1}{6-p}}$. Thus by fixing $m_0 > M_0$ and choosing $\lambda_0 > 0$ sufficiently small such that $M_0(1+\lambda m_0^{q-p})^\sigma \leq m_0$ we obtain (37).

Lemma 5.1. *Assume the same hypotheses of Theorem* 3.8. *For each* $\lambda \in (0, \lambda_0]$ *let* u_{λ,m_0} *be the solution for the auxiliary problem* (18) *with* $m = m_0$, *such that* $J_{\lambda,m_0}(u_{\lambda,m_0}) = c_{\lambda,m_0}$. *Then,*

$$u_{\lambda,m_0}(x) \leq \frac{R}{|x|} m_0, \text{ for all } |x| \geq R.$$

Proof. For each $\lambda \in (0, \lambda_0]$ consider the harmonic function $v : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}$ given by

$$v(x) = \frac{R}{|x|} m_0 \text{ for } x \neq 0,$$

from (37) the function

$$w(x) := \begin{cases} (u_{\lambda,m_0} - v)^+(x), & \text{if } |x| \geq R, \\ 0, & \text{if } |x| \leq R, \end{cases}$$

belongs to $D^{1,2}(\mathbb{R}^3)$ and as $0 \leq w \leq u_{\lambda,m_0}$ we have $w \in E$. Taking w as a test function in (34) and using that $\phi_{\lambda,m_0}, u_{\lambda,m_0}, w \geq 0$ in $\mathbb{R}^3$, $\Delta v = 0$ in $\mathbb{R}^3 \setminus B_R(0)$, it follows that

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla w|^2 &= \int_{\mathbb{R}^3} \nabla(u_{\lambda,m} - v) \nabla w \\ &= \int_{\mathbb{R}^3 \backslash B_R(0)} [h_{\lambda,m_0}(x,u) - V(x) u_{\lambda,m_0} - \ell(x) \phi_{u_{\lambda,m_0}} u_{\lambda,m_0}] w \\ &\leq \int_{\mathbb{R}^3 \backslash B_R(0)} [h_{\lambda,m_0}(x,u) - V(x) u_{\lambda,m_0}] w \\ &\leq \left(\frac{1}{k} - 1\right) \int_{\mathbb{R}^3 \backslash B_R(0)} V(x) u_{\lambda,m_0} w \leq 0. \end{aligned}$$

hence, $w \equiv 0$ in $\mathbb{R}^3$. Therefore, $u_{\lambda,m_0}(x) \leq \frac{R}{|x|} m_0$ for all $|x| \geq R$. □

Finalizing Proof of Theorem 1.2: Let $u_{\lambda,m}$ be the solution of the problem (18) obtained by Theorem 3.8 and λ_0, m_0 given in (37). Since $|u_{\lambda,m}|_{L^\infty(\mathbb{R}^3)} \leq m_0$, by (9) and Lemma 5.1, we have

$$\begin{aligned} \frac{g_{\lambda,m_0}(u_{\lambda,m_0})}{u_{\lambda,m_0}} &\leq C_0(1 + \lambda_0 m_0^{q-p}) |u_{\lambda,m_0}|^4 \\ &\leq C_0(1 + \lambda_0 m_0^{q-p}) \frac{R^4}{|x|^4} m_0^4 \text{ in } B_R^c(0), \end{aligned}$$

then by (V_2), it follows that

$$\frac{k g_{\lambda,m_0}(u_{\lambda,m_0})}{u_{\lambda,m_0}} \leq k C_0(1 + \lambda_0 m_0^{q-p}) \, m_0^4 \, \frac{V(x)}{\Lambda} \text{ in } B_R^c(0).$$

Thus, choosing $\Lambda \geq \Lambda_0 = k C_0(1 + \lambda_0 m_0^{q-p}) m_0^4$ and $\lambda \in (0, \lambda_0)$, we have

$$k g_{\lambda,m_0}(u_{\lambda,m_0}(x)) \leq V(x) u_{\lambda,m_0} \text{ in } B_R^c(0). \tag{38}$$

Using once more $|u_{\lambda,m_0}|_{L^\infty(\mathbb{R}^3)} \leq m_0$ combined with (8), (15), (16) and (38), we get

$$h_{\lambda,m_0}(u_{\lambda,m_0}) = \widetilde{g}_{\lambda,m_0}(x, u_{\lambda,m_0}) = f(u_{\lambda,m_0}) + \lambda |u_{\lambda,m_0}|^{q-2} u_{\lambda,m_0} \text{ in } B_R(0)^c. \qquad (39)$$

Therefore, for all $v \in E$ we have

$$\int_{\mathbb{R}^3} \big(\nabla u_{\lambda,m_0} \nabla v + V(x) u_{\lambda,m_0} v + \ell(x) \phi_{u_{\lambda,m_0}} u_{\lambda,m_0} v\big) = \int_{\mathbb{R}^3} \big(f(u_{\lambda,m_0}) + (u_{\lambda,m_0})^{q-1}\big) v,$$

which combined with Remark 3.2 concludes the proof.

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