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On Set-Valued Derivations Modulo K

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Let Y be a real vector metric space and $K \subset Y$ be a closed convex cone with $K \cap (-K) = \{0\}$. We study properties of set-valued maps $F\colon \mathbb{R} \to 2^Y \setminus \{\emptyset\}$ which are additive modulo K, i.e. $F(x+y)+K = F(x)+F(y)+K$ for $x, y \in \mathbb{R}$, and satisfy condition $F(xy)+K = xF(y)+yF(x)+K$ for $x, y \in [0, \infty)$ (or $x, y \in \mathbb{R}$). Such maps are called set-valued derivations modulo K and generalize the well-known single-valued derivations of $\mathbb{R}$.

Keywords: K-additive set-valued map, (strong) set-valued K-derivation, K-lower boundedness, weak K-upper boundedness, K-continuity, null-finite set.

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1. Introduction

Let $(Q, +, \cdot)$ be a commutative ring and $(P, +, \cdot)$ be a subring of $(Q, +, \cdot)$. A function $f\colon P \to Q$ is called a *derivation* of P, if it satisfies the following two conditions:

$$f(x+y) = f(x) + f(y), \qquad x, y \in P,$$
$$f(xy) = xf(y) + yf(x), \qquad x, y \in P.$$

If $P = Q$ are both the ring of polynomials $\mathbb{R}[x]$, then the function $f\colon \mathbb{R}[x] \to \mathbb{R}[x]$ mapping every polynomial p into its classical derivative p' is clearly a derivation.

If $P = Q = \mathbb{R}$, then f is called a derivation of $\mathbb{R}$. It turned out that such derivations are equal to 0 on the set $\mathbb{Q}$ of all rational numbers and, consequently, $f = 0$ is the only one continuous derivation of $\mathbb{R}$ (see [15, Lemma 14.1.3 and Theorem 14.1.1]). However, non-trivial derivations of $\mathbb{R}$ also exist (they are discontinuus), because any non-zero real function which is defined on an algebraic base of $\mathbb{R}$ over $\mathbb{Q}$ can be uniquely extended to a derivation of $\mathbb{R}$ (see [15, Theorem 14.2.2]).

Using Minkowski's operations [1], in the paper [10] the notion of K-additive set-valued maps has been introduced, which generalizes the classical notion of additivity and refers to the notions of K-subadditivity (see [13], [14]) and K-midconvexity of set-valued maps (see [11], [12]).

Definition 1.1. Let X, Y be real vector spaces and $K \subset Y$ be a convex cone (i.e. $K + K \subset K$ and $tK \subset K$ for $t \geq 0$). Let $n(Y) := 2^Y \setminus \{\emptyset\}$. A set-valued map $F\colon X \to n(Y)$ is called K-*additive* (or *additive modulo* K), if

$$F(x+y) =_K F(x) + F(y), \qquad x, y \in X,$$

[1] $A + B := \{a + b\colon a \in A,\, b \in B\}$ and $tA := \{ta\colon a \in A\}$ for subsets A, B of a real vector space and $t \in \mathbb{R}$.

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where $=_K$ means the following equivalence relation in $n(Y)$:

$$\forall\, A, B \in n(Y) \quad A =_K B \iff A + K = B + K. \qquad \square \tag{1}$$

Let us remark (see [10, Lemma 9]) that for a real vector space Y and a convex cone $K \subset Y$ condition (1) is equivalent to

$$\forall\, A, B \in n(Y) \quad A =_K B \iff A \subset B + K \ \wedge\ B \subset A + K \tag{2}$$

and the relation $=_K$ satisfies for $A, B, C, D \in n(Y)$ the following two important properties:

(i) if $A =_K B$ and $C =_K D$, then $A + C =_K B + D$,

(ii) if $A =_K B$, then $tA =_K tB$ for every $t > 0$.

If $K = \{0\}$ the notion of K-additivity coincides with the definition of additivity of set-valued maps introduced by Henney [6]–[9] and continued by Nikodem [16].

Remark 1.2. If F is single-valued, then K-additivity of F is equivalent to the classical additivity of F provided $K \cap (-K) = \{0\}$.

Indeed, if F is K-additive, then, using (2) for every $x, y \in X$ we get

$$F(x+y) \in F(x) + F(y) + K,$$
$$F(x) + F(y) \in F(x+y) + K.$$

The first inclusion implies $F(x+y) - F(x) - F(y) \in K$, and thus

$$-F(x+y) + F(x) + F(y) \in -K, \qquad x, y \in X,$$

and the second one implies

$$-F(x+y) + F(x) + F(y) \in K, \qquad x, y \in X.$$

In this way we obtain

$$-F(x+y) + F(x) + F(y) \in K \cap (-K) = \{0\}, \qquad x, y \in X.$$

Hence F is additive.

On the other hand, if F is additive, then $F(x+y) = F(x) + F(y)$ for $x, y \in X$ and hence, $F(x+y) + K = F(x) + F(y) + K$ for $x, y \in X$, which means that F is K-additive. This completes the proof. $\square$

Now, denote by $\mathcal{CC}(Y)$ the family of all nonempty compact convex sets in a real vector metric space Y. In the whole paper we will use the fact that

$$(s+t)A = sA + tA, \qquad r(A+B) = rA + rB$$

for every $A, B \in \mathcal{CC}(Y)$, $r \in \mathbb{R}$ and $s, t \geq 0$ (see e.g. [16, Lemma 1.1]).

2. Two definitions of K-derivations of $\mathbb{R}$

Before we introduce a notion of a set-valued derivation modulo K, let us observe that the definition of a derivation of $\mathbb{R}$ is equivalent to the following one.

Definition 2.1. The function $f : \mathbb{R} \to \mathbb{R}$ is a *derivation* of $\mathbb{R}$, if

$$f(x+y) = f(x) + f(y), \qquad x, y \in \mathbb{R}, \tag{3}$$

$$f(xy) = xf(y) + yf(x), \qquad x, y \in [0, \infty). \quad \square \tag{4}$$

Indeed, if f satisfies (3) and (4), since every additive function is odd, we obtain:

(i) if $x \geq 0$ and $y \leq 0$, then

$$f(xy) = -f(x \cdot (-y)) = -[xf(-y) - yf(x)] = xf(y) + yf(x),$$

(ii) if $x \leq 0$ and $y \leq 0$, then

$$f(xy) = f(-x \cdot (-y)) = -xf(-y) - yf(-x) = xf(y) + yf(x).$$

Consequently, $$f(xy) = xf(y) + yf(x), \qquad x, y \in \mathbb{R}.$$

That is why we can define set-valued K-derivations of $\mathbb{R}$ in two ways.

Definition 2.2. Let Y be a real vector metric space, K be a convex cone in Y such that $K \cap (-K) = \{0\}$ and $F\colon \mathbb{R} \to \mathcal{CC}(Y)$. The set-valued map F is called:

(i) a *K-derivation*, if $$F(x+y) =_K F(x) + F(y), \qquad x, y \in \mathbb{R}, \tag{5}$$
and $$F(xy) =_K xF(y) + yF(x), \qquad x, y \in [0, \infty), \tag{6}$$

(ii) a *strong K-derivation*, if F satisfies (5) and

$$F(xy) =_K xF(y) + yF(x), \qquad x, y \in \mathbb{R}. \qquad \Box \tag{7}$$

Clearly, every strong K-derivation is a K-derivation, but the converse implication does not hold, which shows the next example.

Example 2.3. Let $K = [0, \infty)$ and $F\colon \mathbb{R} \to \mathcal{CC}(\mathbb{R})$ be given by $F(x) = [0, |x|]$ for every $x \in \mathbb{R}$. It is clear that such a map is a K-derivation, because

$$F(x+y) + K = [0, \infty) = F(x) + F(y) + K, \qquad x, y \in \mathbb{R},$$

and $$F(xy) + K = [0, \infty) = xF(y) + yF(x) + K, \qquad x, y \geq 0.$$

However, F is not a strong K-derivation, because

$$F(xy) + K = [0, \infty) \neq [-2xy, \infty) = xF(y) + yF(x) + K, \qquad x, y < 0. \qquad \Box$$

The next theorem gives us a full characterization of strong K-derivations.

Theorem 2.4. *Let Y be a real vector metric space, K be a closed convex cone in Y such that $K \cap (-K) = \{0\}$ and $F\colon \mathbb{R} \to \mathcal{CC}(Y)$. Then, F is a strong K-derivation if and only if it is a single-valued derivation.*

Proof. First assume that F is a strong K-derivation. Since F is K-additive, by [10, Theorem 12], $F(0) =_K \{0\}$. Moreover, using (7), we obtain

$$-2xF(-x) = -xF(-x) - xF(-x) =_K F((-x)^2) = F(x^2) =_K 2xF(x)$$

for $x \in \mathbb{R}$, so, dividing both sides by $-2x > 0$, we get

$$F(-x) =_K -F(x), \qquad x < 0. \tag{8}$$

Thus $$\{0\} =_K F(0) =_K F(x) + F(-x) =_K F(x) - F(x), \qquad x < 0, \tag{9}$$

which means that $F(x)$ is a singleton for every $x < 0$.

Indeed, if $F(x)$ were not be a singleton for some $x < 0$, then there would exist $y, z \in F(x)$, $y \neq z$, and hence, by (9), $y - z, z - y \in K$, which would contradict $K \cap (-K) = \{0\}$.

Hence, in view of (8), we get $F(x) =_K -F(-x)$, $x > 0$, where $F(-x)$ is a singleton. Consequently, $F(x)$ are also singletons for every $x > 0$.

In this way we proved that F is single-valued. We have to prove yet that F is a derivation. In view of Remark 1.2, F is additive. Moreover, if F satisfies (7), then, by (2), for every $x, y \in \mathbb{R}$ we get

$$F(xy) \in xF(y) + yF(x) + K \quad \text{and} \quad xF(y) + yF(x) \in F(xy) + K.$$

The first inclusion implies

$$-F(xy) + xF(y) + yF(x) \in -K, \qquad x, y \in \mathbb{R},$$

and the second one implies

$$-F(xy) + xF(y) + yF(x) \in K, \qquad x, y \in \mathbb{R}.$$

Consequently, $-F(xy) + xF(y) + yF(x) \in K \cap (-K) = \{0\}$, $x, y \in \mathbb{R}$.

Hence F is a derivation. The converse implication is obvious. □

In the above theorem we proved that considerations on strong K-derivations of $\mathbb{R}$ narrow to single-valued derivations of $\mathbb{R}$, which properties are known (see [15, Chapter XIV]). That is why from now on we would like to focus on the case of K-derivations of $\mathbb{R}$.

3. Properties of K-derivations of $\mathbb{R}$

Lemma 3.1. *Let Y be a real vector metric space and K be a closed convex cone in Y such that $K \cap (-K) = \{0\}$. If $F\colon \mathbb{R} \to \mathcal{CC}(Y)$ is a K-derivation, then the following conditions hold:*

$$F(0) =_K \{0\}, \quad F(1) =_K \{0\}, \quad F(-1) =_K \{0\}, \tag{10}$$

$$F(tx) =_K tF(x), \qquad t \in \mathbb{Q}_+, \ x \in \mathbb{R}, \tag{11}$$

$$F(-x) =_K x^2 F\left(\tfrac{1}{x}\right), \qquad x \in \mathbb{R} \setminus \{0\}, \tag{12}$$

$$F(x) =_K \{0\}, \qquad x \in \mathbb{Q}. \tag{13}$$

Proof. Since F is K-additive, in view of [10, Theorem 12], $F(0) =_K \{0\}$. Now, using an analogous method, we prove that $F(1) =_K \{0\}$.

By (6), $$2F(1) = F(1) + F(1) =_K F(1 \cdot 1) = F(1),$$

and hence, using induction, we get

$$F(1) =_K \frac{1}{2}F(1) =_K \frac{1}{4}F(1) =_K \ldots =_K \frac{1}{2^n}F(1) \quad \text{for every } n \in \mathbb{N},$$

and hence $$F(1) + K = \frac{1}{2^n}F(1) + K \quad \text{for } n \in \mathbb{N}. \tag{14}$$

First, we show that $F(1) + K \subset K$. So, take any $y \in F(1)$ and $k \in K$. By (14), we can find sequences $(y_n)_{n\in\mathbb{N}} \subset F(1)$ and $(k_n)_{n\in\mathbb{N}} \subset K$ such that

$$y + k = \frac{y_n}{2^n} + k_n.$$

But $F(1)$ is compact, so we can assume that $(y_n)_{n\in\mathbb{N}}$ is convergent. Thus

$$k_n = y + k - \frac{y_n}{2^n} \to y + k$$

and whence $y + k \in \operatorname{cl} K = K$.

Next, we prove that $K \subset F(1) + K$. To this end fix $k \in K$. By (14), for fixed $y \in F(1)$ the sequence $\left\{\frac{y}{2^n} + k\right\}_{n\in\mathbb{N}}$ is contained in $F(1) + K$ and, moreover, it converges to k. Hence $k \in \operatorname{cl}(F(1) + K) = F(1) + K$.

In this way we showed that $F(1) + K = K$, which means that $F(1) =_K \{0\}$. Then, in view of (5),

$$\{0\} =_K F(0) = F(1-1) =_K F(1) + F(-1) =_K \{0\} + F(-1) = F(-1).$$

Since F is K-additive, condition (11) is an immediate consequence of [10, Theorem 18] [2]. According to (6),

$$\{0\} =_K F(1) = F\left(x \cdot \tfrac{1}{x}\right) =_K xF\left(\tfrac{1}{x}\right) + \tfrac{1}{x}F(x), \qquad x > 0.$$

Hence, multiplying both sides by x,

$$\{0\} =_K x^2F\left(\tfrac{1}{x}\right) + F(x), \qquad x > 0,$$

and thus

$$F(-x) =_K x^2F\left(\tfrac{1}{x}\right) + F(x) + F(-x) =_K x^2F\left(\tfrac{1}{x}\right) + F(0) =_K x^2F\left(\tfrac{1}{x}\right)$$

for $x > 0$. Moreover,

$$\{0\} =_K F(1) =_K F\left(-x \cdot \left(-\tfrac{1}{x}\right)\right) =_K -xF\left(-\tfrac{1}{x}\right) - \tfrac{1}{x}F(-x), \qquad x < 0.$$

Thus, multiplying both sides by $-x$,

$$x^2F\left(-\tfrac{1}{x}\right) + F(-x) =_K \{0\}, \qquad x < 0,$$

and, consequently,

$$\begin{aligned} F(-x) &=_K x^2F(0) + F(-x) =_K x^2\left(F\left(\frac{1}{x}\right) + F\left(-\frac{1}{x}\right)\right) + F(-x) \\ &= x^2F\left(\frac{1}{x}\right) + \{0\} =_K x^2F\left(\frac{1}{x}\right) \end{aligned}$$

for $x < 0$, which means that (12) holds.

Finally, observe that (10) and (11) imply

$$F(x) =_K xF(1) =_K \{0\}, \qquad x \in \mathbb{Q}_+.$$

[2] Let us only mention that in [10, Theorem 18] we assume additionally that F is not single-valued. But this assumption is necessary only to obtain that $\{t \in \mathbb{R}\colon F(tx) =_K tF(x) \text{ for } x \in \mathbb{R}\} \subset [0,\infty)$, that is why we can omit it.

Hence, according to (5),

$$F(x) =_K F(x) + \{0\} =_K F(x) + F(-x) =_K F(0) =_K \{0\}, \qquad x \in \mathbb{Q}_-.$$

This completes the proof of (13). □

Now, let us recall necessary notions of boundedness and continuity of set-valued maps with respect to K.

Definition 3.2. Let X, Y be real topological vector spaces and K be a convex cone in Y. A set-valued map $F\colon X \to n(Y)$ is called:

- *weakly K-upper bounded* on a set $A \subset X$, if there exists a set $B \in \mathcal{B}(Y)$ such that $F(x) \cap (B - K) \neq \emptyset$ for all $x \in A$;
- *K-lower bounded* on a set $A \subset X$, if there exists a set $B \in \mathcal{B}(Y)$ such that $F(x) \subset B + K$ for all $x \in A$;
- *K-continuous at $x_0 \in X$*, if for every neighborhood $W \subset Y$ of 0 there is a neighborhood U of x_0 such that, for all $x \in U$,
$$F(x) \subset F(x_0) + W + K \quad \text{and} \quad F(x_0) \subset F(x) + W + K;$$
- *K-continuous on X*, if it is K-continuous at every point of X. □

Remark 3.3. Clearly, in the case when $K = \{0\}$, weak K-upper boundedness and K-lower boundedness mean weak boundedness and boundedness of F, respectively, and K-continuity means continuity with respect to the Hausdorff topology on $n(Y)$.

Let us also recall the notion of null-finite sets introduced in [1] and [2] (see also the notion of a shift-compact set in [3]).

Definition 3.4. The set A in a topological group X is called *null-finite*, if there is a sequence $(x_n)_{n\in\mathbb{N}}$ convergent to 0 in X such that the set $\{n \in \mathbb{N}\colon x + x_n \in A\}$ is finite for every $x \in X$. □

There is a connection between null-finite sets and Haar-null as well as Haar-meager sets.

The notion of a Haar-null set and a Haar-meager set have been introduced by Christensen [4] and Darji [5], respectively. According to [1, Theorem 4.3 and Proposition 5.1], the subset B of a complete metric group X is Haar-null (Haar-meager), if there exists a continuous function $f\colon 2^\omega \to X$ such that $f^{-1}(B + x)$ is a meager set (a set of Haar measure zero, respectively) for each $x \in X$.

Theorem 3.5. ([2, Theorem 5.1], [2, Theorem 6.1]) *If X is a complete metric group, then every Borel null-finite set $B \subset X$ is:*

- *Haar-meager and Haar-null,*
- *(consequently) meager and, if X is additionally locally compact, of Haar measure zero.*

Now, we can obtain the following important result.

Theorem 3.6. *Let Y be a real vector metric space and K be a closed convex cone in Y such that $K \cap (-K) = \{0\}$. If $F\colon \mathbb{R} \to \mathcal{CC}(Y)$ is a K-derivation which is weakly K-upper bounded or K-lower bounded on a non-null finite set, then $F(x) =_K \{0\}$ for every $x \in \mathbb{R}$.*

Proof. By [10, Theorem 13] every K-additive set-valued map which is weakly K-upper bounded or K-lower bounded on a non-null-finite set is K-continuous. Hence, according to [10, Theorem 15],

$$F(tx) =_K tF(x), \qquad x \in \mathbb{R}, \;\; t \geq 0.$$

Thus, by (10),
$$F(x) =_K xF(1) =_K \{0\}, \qquad x \geq 0,$$

and, consequently,
$$F(-x) =_K \{0\}, \qquad x \leq 0.$$

Hence, $F(x) =_K F(x) + \{0\} =_K F(x) + F(-x) =_K F(0) =_K \{0\}, \;\; x \leq 0,$

which completes the proof. □

At the end of this section let us give the easiest method of constructing examples of non-trivial K-derivations, i.e. K-derivations which are not equal to $\{0\}$ modulo K.

Lemma 3.7. *Let Y be a real vector metric space, K be a convex cone in Y such that $K \cap (-K) = \{0\}$, $z_0 \in K \setminus \{0\}$, $m, M\colon \mathbb{R} \to \mathbb{R}$ satisfy $M(x) \geq m(x)$ for $x \in \mathbb{R}$, and the set-valued map $F\colon \mathbb{R} \to \mathcal{CC}(Y)$ be given by*

$$F(x) := [m(x), M(x)]z_0, \qquad x \in \mathbb{R}.$$

Then, F is a K-derivation of $\mathbb{R}$ if and only if m is a derivation of $\mathbb{R}$.

Proof. First assume that m is a derivation of $\mathbb{R}$. In view of [10, Lemma 5], since m is additive, F is K-additive. Moreover, for $x, y \geq 0$,

$$\begin{aligned} F(xy) &= [m(xy), M(xy)]z_0 = [xm(y) + ym(x), M(xy)]z_0 \\ &\subset [xm(y) + ym(x), \infty)z_0 \\ &= x[m(y), M(y)]z_0 + y[m(x), M(x)]z_0 + [0, \infty)z_0 \\ &\subset xF(y) + yF(x) + K, \end{aligned}$$

and

$$\begin{aligned} xF(y) + yF(x) &= x[m(y), M(y)]z_0 + y[m(x), M(x)]z_0 \\ &\subset [xm(y) + ym(x), \infty)z_0 = [m(xy), \infty)z_0 \\ &= [m(xy), M(xy)]z_0 + [0, \infty)z_0 \subset F(xy) + K. \end{aligned}$$

Hence, $F(xy) =_K xF(y) + yF(x)$ for $x, y \geq 0$, which means that F is a K-derivation of $\mathbb{R}$.

Next, assume that F is a K-derivation of $\mathbb{R}$. According to [10, Lemma 5], K-additivity of F implies additivity of m. Moreover, since F satisfies (6), for $x, y \geq 0$ we get

$$\begin{aligned} m(xy)z_0 \in F(xy) &\subset xF(y) + yF(x) + K \\ &= [xm(y) + ym(x), xM(y) + yM(x)]z_0 + K \end{aligned}$$

and

$$\begin{aligned} (xm(y) + ym(x))z_0 \in xF(y) + yF(x) &\subset F(xy) + K \\ &= [m(xy), M(xy)]z_0 + K. \end{aligned}$$

Hence, for every $x, y \geq 0$, there are

$$\alpha \in [xm(y) + ym(x), xM(y) + yM(x)], \qquad \beta \in [m(xy), M(xy)] \tag{15}$$

such that $\qquad (m(xy) - \alpha)z_0 \in K, \qquad (xm(y) + ym(x) - \beta)z_0 \in K.$

But $z_0 \in K \setminus \{0\}$ and $K \cap (-K) = \{0\}$, so

$$m(xy) - \alpha \geq 0, \qquad xm(y) + ym(x) - \beta \geq 0$$

and, in view of (15), we get

$$m(xy) \geq \alpha \geq xm(y) + ym(x), \qquad xm(y) + ym(x) \geq \beta \geq m(xy)$$

for $x, y \geq 0$, which proves that m satisfies (4). Consequently, m is a derivation. □

4. When K-additive maps are K-derivations

Now, we would like to give a full characterization of K-derivations of $\mathbb{R}$. Our result generalizes [15, Theorem 14.3.1].

Theorem 4.1. *Let Y be a real vector metric space, K be a closed convex cone in Y such that $K \cap (-K) = \{0\}$ and $F\colon \mathbb{R} \to \mathcal{CC}(Y)$. Then F is a K-derivation if and only if F is a K-additive set-valued map satisfying*

$$F(-x) =_K x^2 F\left(\tfrac{1}{x}\right), \qquad x \in \mathbb{R} \setminus \{0\}. \tag{16}$$

Proof. If F is a K-derivation, then it is K-additive and, according to Lemma 3.1, F satisfies (16).

Now, assume that F is K-additive and (16) holds. We have to show that (6) holds. In view of (16) and K-additivity of F, for $x \in \mathbb{R} \setminus \{-1, 0, 1\}$ we get

$$\begin{aligned}
F(x) + \frac{1}{x^2}F(x) &=_K F(x) + F\left(-\frac{1}{x}\right) =_K F\left(x - \frac{1}{x}\right) \\
&=_K F\left(-\frac{1-x^2}{x}\right) =_K \left(\frac{1-x^2}{x}\right)^2 F\left(\frac{x}{1-x^2}\right) \\
&=_K \left(\frac{1-x^2}{x}\right)^2 F\left(\frac{1}{x^2-1} - \frac{1}{x-1}\right) \\
&=_K \left(\frac{1-x^2}{x}\right)^2 F\left(\frac{1}{x^2-1}\right) + \left(\frac{1-x^2}{x}\right)^2 F\left(-\frac{1}{x-1}\right) \\
&=_K \frac{1}{x^2}\left(x^2-1\right)^2 F\left(\frac{1}{x^2-1}\right) + \left(\frac{1+x}{x}\right)^2 (1-x)^2 F\left(\frac{1}{1-x}\right)
\end{aligned}$$

and hence, $\qquad F(x) + \frac{1}{x^2}F(x) =_K \frac{1}{x^2}F\left(1-x^2\right) + \left(\frac{x+1}{x}\right)^2 F(x-1). \qquad (17)$

In view of [10, Theorem 12], K-additivity implies

$$F(0) =_K \{0\}. \tag{18}$$

Then, setting $x = 1$ in (16), we get $F(-1) =_K F(1)$, which means that

$$\{0\} =_K F(0) =_K F(1) + F(-1) =_K F(1) + F(1) =_K 2F(1),$$

so,
$$F(-1) =_K F(1) =_K \{0\}. \tag{19}$$

Thus, by K-additivity,

$$F(t-1) =_K F(t) + F(-1) =_K F(t) + \{0\} = F(t), \qquad t \in \mathbb{R},$$
$$F(1-t) =_K F(1) + F(-t) =_K \{0\} + F(-t) = F(-t), \qquad t \in \mathbb{R},$$

and hence, by (17),

$$F(x) + \frac{1}{x^2} F(x) \ =_K \frac{1}{x^2} F\left(-x^2\right) + \left(\frac{x+1}{x}\right)^2 F(x), \qquad x \in \mathbb{R} \setminus \{-1, 0, 1\}.$$

Multiplying both sides by x^2, we obtain

$$\begin{aligned} (x^2+1)\, F(x) \ &=_K F\left(-x^2\right) + (x^2+2x+1)\, F(x) \\ &= (x^2+1)\, F(x) + 2xF(x) + F\left(-x^2\right) \end{aligned}$$

for $x \in (0, \infty) \setminus \{1\}$, and thus,

$$\begin{aligned} \{0\} \ &=_K (x^2+1)\, F(0) =_K (x^2+1)\, F(x) + (x^2+1)\, F(-x) \\ &=_K (x^2+1)\, F(x) + F\left(-x^2\right) + 2xF(x) + (x^2+1)\, F(-x) \\ &=_K (x^2+1)\, F(0) + F\left(-x^2\right) + 2xF(x) =_K F\left(-x^2\right) + 2xF(x) \end{aligned}$$

for $x \in (0, \infty) \setminus \{1\}$. Consequently,

$$\begin{aligned} F\left(x^2\right) \ &=_K F\left(x^2\right) + \{0\} =_K F\left(x^2\right) + F\left(-x^2\right) + 2xF(x) \\ &=_K F(0) + 2xF(x) =_K 2xF(x) \end{aligned}$$

for $x \in (0, \infty) \setminus \{1\}$ and, according to (18) and (19),

$$F(x^2) =_K 2xF(x), \qquad x \geq 0. \tag{20}$$

If $x, y > 0$, then, according to (20),

$$\begin{aligned} 2xF(y) + 2yF(x) &=_K 2xF(0) + 2xF(y) + 2yF(x) + 2yF(0) \\ &=_K 2xF(x) + 2xF(-x) + 2xF(y) + 2yF(x) + 2yF(y) + 2yF(-y) \\ &=_K 2xF(x+y) + 2yF(x+y) + 2xF(-x) + 2yF(-y) \\ &=_K 2(x+y)F(x+y) + 2xF(-x) + 2yF(-y) \\ &=_K F\left((x+y)^2\right) + 2xF(-x) + 2yF(-y) \\ &=_K F\left(x^2\right) + F(2xy) + F\left(y^2\right) + 2xF(-x) + 2yF(-y) \\ &=_K 2xF(x) + 2F(xy) + 2yF(y) + 2xF(-x) + 2yF(-y) \\ &=_K 2F(xy) + 2xF(0) + 2yF(0) =_K 2F(xy). \end{aligned}$$

Moreover, if $x = 0$ (or $y = 0$), then, according to (18),

$$F(0 \cdot y) = F(0) =_K \{0\} = \{0\} + y\{0\} =_K 0 \cdot F(y) + yF(0),$$

so (6) holds, which completes the proof. $\square$

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