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# Conditions for the Preservation of Motzkin Decomposability and the Pareto Bordered Property Under Addition

**Bogdan Daniel Moldovan, Cornel Pintea***
*Faculty of Mathematics and Computer Science, Babeş-Bolyai University, Cluj-Napoca, Romania*
*bogdan.moldovan1@ubbcluj.ro, cornel.pintea@ubbcluj.ro*

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We provide some sufficient conditions on pairs of Motzkin decompasable sets and Pareto bordered sets in order to get the Minkowski sum of their components Motzkin decomposable and Pareto bordered respectively. We also prove that minimal faces of a closed convex set are also extreme faces of the set and vice-versa. This result allows us to define the generalized Minkowski sets by using the extreme faces. A Minkowski type theorem is proved with extreme faces playing the role of the extreme points in the classical Minkowski Theorem. The special class of Pareto bordered sets, which is a subclass of that of generalized Minkowski sets, is also taken into account. Indeed, as mentioned above, we show that the Minkowski sum of some Pareto bordered sets with the same lineality remains Pareto bordered. Note that the class of generalized Minkowski sets is not closed with respect to the Minkowski sum. It is however worth to mention that the class of closed convex sets which are both Motzkin decomposable and generalized Minkowski (or shortly, $MdgM$ sets) is closed both with respect to Minkowski sum and Cartesian product [see J. E. Martínez-Legaz and C. Pintea: *Closed convex sets that are both Motzkin and generalized Minkowski sets*, J. Nonlinear Var. Analysis].

*Keywords:* Motzkin decomposable sets, Pareto bordered sets, generalized Minkowski sets.

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## 1. Introduction

In this paper we show that the Minkowski sum of some particular Pareto bordered sets [5, 8], with the same lineality, remain Pareto bordered. We also show that the Minkowski sum of two Motzkin decomposable sets [2], in some particular relative position, remains a Motzkin decomposable set. While the Pareto-like subset of a Motzkin decomposable set covers a compact part of its relative boundary, it covers the whole relative boundary in the case of a Pareto bordered set. The two classes are also somewhat extreme with respect to their Gaussian ranges too. Indeed, the Pareto bordered sets with nonempty interior and trivial lineality are characterized by the openness of their Gauss ranges and the Motzkin decomposable sets have closed Gauss range [4]. The openness of the Gauss ranges is a common feature for the Pareto bordered sets with trivial lineality as well as for the closed convex sets with compact proper faces, both with nonempty interiors. On the other hand, the Gauss range of the sum of two closed convex sets is the intersection of the Gauss

* Corresponding author.

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ranges of the two sets, whenever the Minkowski sum is a closed set. Therefore the sum of two Pareto bordered sets with nonempty interiors becomes a Pareto bordered set, once we know that their sum is a closed set. Similarly, the Minkowski sum of two Motzkin decomposable sets remains a Motzkin decomposable set, once we know that the Minkowski sum of their recession cones is a closed cone. In both cases the most challenging part is the closedness of the Minkowski sum of the two involved sets. However the closedness of the Minkowski sum of a Pareto bordered set with a generalized Minkowski set is rather obvious when the two sets have the same lineality and the orthogonal slice of the generalized Minkowski set is compact. We only consider here such pairs of generalized Minkowski sets. For two Motzkin decomposable sets we provide a sufficient condition which ensures the closedness of the Minkowski sum of their recession cones.

We also prove that minimal faces of a closed convex set, which are equally the lowest dimensional ones ([5, Corollary 2.4]), are also extreme faces of the set. This result allows us to define the generalized Minkowski sets, considered before in [5] via the minimal/lowest dimensional faces, by using the extreme faces this time. A Minkowski type theorem is proved for closed convex sets $C$ with compact orthogonal slice with extreme faces playing the role of the extreme points in the classical Minkowski Theorem.

The paper is organized on 5 sections as follows: In the second section we recall the definitions of the *recession* and *total normal* cones of a closed convex set. The later cone is used to define the *Gauss map* for whose range, we recall a few useful properties. The *Pareto like* set associated to a closed convex set along with the *Pareto bordered* sets are also defined here. The first couples of closed convex sets with Pareto bordered Minkowski sum are pointed out as well. The third section deals with Minkowski sums of Motzkin decomposable sets. As already mentioned before, the most challenging part for the Motzkin decomposability of such a Minkowski sum is the closedness of the Minkowski sum of such sets. It reduces to the closedness of the Minkowski sum of their recession cones. A sufficient condition for the later closedness relies on a result by Beutner ([1, Theorem 3.2]). In the fourth section we define the extreme faces of a closed convex set and prove a Minkowski type Theorem for closed convex sets with compact orthogonal slices. This result suggested us to define the generalized Minkowski sets by using the extreme faces. It turns out that the classes of minimal faces, lowest dimensional faces and that of extreme faces coincide. This shows that our current definition of generalized Minkowski agrees with the one given before in [5] by using minimal/lowest dimensional faces. In the last section we basically show that the sum of a Pareto bordered set and a $MdgM$ set is pareto bordered if the two sets have the same lineality.

## 2. Minkowski sums via Gauss range and lineality

### 2.1. The total normal cone of a closed convex set

**Definition 2.1.** A vector $x^* \in \mathbb{R}^n$ is said to be *normal* to a convex set $C \subseteq \mathbb{R}^n$ at a point $x \in C$ if $\langle c - x, x^* \rangle \leq 0$, for all $c \in C$. The set of vectors normal to $C$ at $x$ is a closed convex cone denoted by $N_C(x)$ and is called the *normal cone* to $C$ at $x$. $\square$

Note that $N_C(x) = \{0\}$ whenever $x \in \operatorname{int} C$. We extend the mapping $N_C : C \rightrightarrows \mathbb{R}^n$ to the whole space $\mathbb{R}^n$ taking $N_C(x) := \emptyset$ whenever $x \in \mathbb{R}^n \setminus C$. The mapping $N_C$ is multivalued at any point in $C \cap \operatorname{bd} C$. Denote by $N_C(\mathbb{R}^n)$ the range of $N_C$, i.e. the union of the normal cones to $C$ at all points of $C$, and call it the *total normal cone* of $C$. Note that $N_C(\mathbb{R}^n) = \{0\}$ if and only if $C$ is open.

Recall that the total normal cone need not be convex. For an example of a closed convex set with non-convex total normal cone we refere the reader to [4]. However, the total normal cone, being the domain of a maximally monotone operator (namely, the subdifferential of the support function), is always nearly convex, that is, it is contained in the closure of one of its convex subsets. Recall also that a nonempty closed convex subset of $\mathbb{R}^n$ is compact if and only if its total normal cone is the whole of $\mathbb{R}^n$.

**Remark 2.2.** [4] If $C_1, C_2 \subseteq \mathbb{R}^n$ are two convex sets, then $C_1 + C_2$ is also a convex set and

$$N_{C_1+C_2}(\mathbb{R}^n) = N_{C_1}(\mathbb{R}^n) \cap N_{C_2}(\mathbb{R}^n). \qquad \square \tag{1}$$

The equality (1) is an immediate consequence of [7, Proposition 2.11.(ii)], which establishes that

$$N_{C_1+C_2}(c_1 + c_2) = N_{C_1}(c_1) \cap N_{C_2}(c_2)$$

for every $(c_1, c_2) \in C_1 \times C_2$. Since $C + 0^+(C) = C$ and $N_{0^+(C)}(\mathbb{R}^n) = N_{0^+(C)}(0)$, it follows that

$$N_C(\mathbb{R}^n) = N_{C+0^+(C)}(\mathbb{R}^n) = N_C(\mathbb{R}^n) \cap N_{0^+(C)}(\mathbb{R}^n) \subseteq N_{0^+(C)}(0). \tag{2}$$

## 2.2. The recession cone, the lineality and the Gauss range

Besides the normal cone, we will repeatedly use the recession cone

$$0^+(C) := \{d \in \mathbb{R}^n : d + C \subseteq C\}$$

of a closed convex set $C \subseteq \mathbb{R}^n$. Recall that $0^+C$ is a convex cone [9, Theorem 18.1] and $\operatorname{lin} C := 0^+C \cap (-0^+C)$ is a subspace of $\mathbb{R}^n$ called *the lineality* of $C$. In order to avoid crowded formulas we shall occasionally denote $\operatorname{lin} C$ simply by $L$. The obvious representation $C = C \cap (\operatorname{lin} C)^\perp + \operatorname{lin} C = C \cap L^\perp + L$ of $C$ (see [9, p. 65] or [11, Theorem 2.5.8, p. 78]) can be rewritten as

$$C = p_{L^\perp}^{-1}(p_{L^\perp}(C)) \tag{3}$$

where $p_X : \mathbb{R}^n \longrightarrow X$, $p_{X^\perp} : \mathbb{R}^n \longrightarrow X^\perp$ stand for the projections of $\mathbb{R}^n = X \oplus X^\perp$ on the linear subspace $X$ of $\mathbb{R}^n$ and on its orthogonal $X^\perp$ respectively. The intersection $C \cap (\operatorname{lin} C)^\perp = C \cap L^\perp$ will be occasionally called the *orthogonal slice* of $C$. In order to justify the representation (3) of $C$ we recall that

$$p_{Y^\perp}^{-1}(K) = K + Y; \tag{4}$$

$$\operatorname{lin} p_{Y^\perp}^{-1}(K) = p_{Y^\perp}^{-1}(\operatorname{lin} K) = \operatorname{lin} K + Y \tag{5}$$

$$\operatorname{lin} K_1 + \operatorname{lin} K_2 \subseteq \operatorname{lin} (K_1 + K_2), \tag{6}$$

where $Y \leq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ and $K, K_1, K_2$ are closed convex subsets of $Y^\perp$. While the relation (4) is rather obvious, the item (5) follows from [10, Theorem 5.26, p. 203] and (6) follows from [10, Corollary 5.25, p. 203].

The representation (3) of $C$ follows by combining the obvious equality

$$p_{L^\perp}(C) = C \cap L^\perp = C \cap (\text{lin } C)^\perp$$

with (5) and lin $[C \cap L^\perp] = \{0\}$ (see [11, Theorem 2.5.8, p. 78]). Closed convex sets of type $p_{L^\perp}^{-1}(K)$, where $K \subseteq L^\perp$ is a closed convex set, will be often used in the sections to come, but with $Y$ playing the role of $Y^\perp$ and vice-versa.

**Definition 2.3.** We call the set-valued mapping

$$G_C : \mathbb{R}^n \rightrightarrows S^{n-1},\ G_C(x) := N_C(x) \cap S^{n-1}$$

the *Gauss map* of $C$. □

Note that the *range* of $G_C$ is

$$G_C(\mathbb{R}^n) = N_C(\mathbb{R}^n) \cap S^{n-1}. \tag{7}$$

In view of (2), one has $N_C(\mathbb{R}^n) \cap 0^+(C) \subseteq N_{0^+(C)}(0) \cap 0^+(C) = \{0\}$, which shows, via (7), that $G_C(\mathbb{R}^n) \cap 0^+(C) = \emptyset$. Combining Remark 2.2 with Definition 2.3 it follows that

$$G_{C_1+C_2}(\mathbb{R}^n) = G_{C_1}(\mathbb{R}^n) \cap G_{C_2}(\mathbb{R}^n). \tag{8}$$

**Remark 2.4.** Let $C_1, C_2 \subseteq \mathbb{R}^n$ be closed convex sets such that $C_1 + C_2$ is closed. This is the case when one of the two sets is compact.

1. If the Gauss ranges $G_{C_1}(\mathbb{R}^n), G_{C_2}(\mathbb{R}^n)$ are both open subsets of $S^{n-1}$, then the Gauss range $G_{C_1+C_2}(\mathbb{R}^n)$ of $C_1 + C_2$ is an open subset of $S^{n-1}$ as well.
2. If the Gauss ranges $G_{C_1}(\mathbb{R}^n), G_{C_2}(\mathbb{R}^n)$ are both closed subsets of $S^{n-1}$, then the Gauss range $G_{C_1+C_2}(\mathbb{R}^n)$ of $C_1 + C_2$ is a closed subset of $S^{n-1}$ as well. □

A source of closed convex sets with open Gauss ranges is the collection of all compact convex sets along with the collection of all Pareto bordered sets which contain no lines and have nonempty interior. A common feature of the Pareto bordered sets which contain no lines and the closed convex sets with open Gauss range is the compactness of their proper faces (see [4] and [8]). On the other hand, a source of closed convex sets with closed Gauss ranges is the collection of all Motzkin decomposable sets (see [4]). Therefore each of these classes are, due to Remark 2.4, closed with respect to the Minkowski sum. However non of them is closed with respect to topological closedness, as the Minkowski sum of two Pareto bordered sets or of two Motzkin decomposable sets might not be closed. While the closedness of the sum of two Motzkin decomposable sets is approached in the next section through some sufficient conditions, we only provide in this section some couples of closed convex sets with Pareto bordered Minkowski sums. Let us also mention that some other examples of couples of Pareto bordered sets with Pareto bordered Minkowski sums are also provided in [8].

In [3], [5] and [8] some special attention is paid to the closed convex sets $\emptyset \neq C \subset \mathbb{R}^n$ which satisfy the equality

$$M(C) = \text{rbd } C, \tag{9}$$

where rbd $C$ stands for the relative boundary of $C$ and $M(C)$ for the *Pareto-like set* of $C$, namely

$$M(C) = \{x \in C : (x - 0^+C) \cap C \subseteq x + 0^+C\} = \{x \in C : (x - 0^+C) \cap C = x + \text{lin } C\}.$$

Note that $M(C)$ is always contained in rbd $C$ and the closed convex sets satisfying (9) are therefore worth to be studied. Also the closed convex set $C$ is bounded if and only if its recession cone $0^+C$ consists of the zero vector alone [9, Theorem 8.4]. If the closed convex set $C$ is not bounded, note that its Pareto-like set $M(C)$ is contained in its relative boundary rbd C.

**Example 2.5.** The halfflats of $\mathbb{R}^n$ satisfy the equality (9). On the other hand the convex compact subsets of $\mathbb{R}^n$ do not satisfy the equality (9).

**Definition 2.6.** A closed convex set $C \subseteq \mathbb{R}^n$ with full Pareto-like relative boundary, i.e. which satisfy the equality $M(C) = \text{rbd } C$ and is not a halfflat, is said to be a *Pareto bordered set*.

Note that the Pareto bordered sets are unbounded, as the compact convex ones are obviously not Pareto bordered. By combining [4, Theorem 8, Corollary 11] with [8, Theorem 3.2], one can state the following

**Theorem 2.7.** *A closed convex set $C \subset \mathbb{R}^n$ with nonempty interior is Pareto bordered if and only if its Gauss range $G_C(\mathbb{R}^n)$ is a proper open subset of $S^{n-1}$.*

**Corollary 2.8.** *If $C_1, C_2 \subset \mathbb{R}^n$ are Pareto bordered with nonempty interiors such that $C_1 + C_2$ is closed and $C_1 + C_2 \subsetneq \mathbb{R}^n$, then $C_1 + C_2$ is a Pareto bordered set as well.*

**Corollary 2.9.** *If $C_1 \subset \mathbb{R}^n$ is a Pareto bordered set with nonempty interior and $C_2 \subset \mathbb{R}^n$ is a compact convex set, then $C_1 + C_2$ is a Pareto bordered set.*

We finally recall that a Pareto bordered set $C \subseteq \mathbb{R}^n$ which contains no lines is, according to [3, Proposition 4.8], a Minkowski set, i.e. the convex hull of its extreme points is the whole set $C$, or shortly $C = \text{conv ext } C$.

## 3. Minkowski sums of Motzkin decomposable sets

**Theorem 3.1.** ([1, Theorem 3.2]) *Let $X$ be a reflexive Banach space. Let $K$ and $L$ be closed convex cones of $X$. Suppose there exists a constant $d > 0$ such that $||x + y|| \geq d$, $\forall x \in K$, $\forall y \in L$, $||x|| = ||y|| = 1$. Then $K + L$ is closed.*

**Remark 3.2.** Let $K$ be a convex cone in $\mathbb{R}^n$. We first observe that $x^* \in N_K(0)$ if and only if $\langle k, x^* \rangle \leq 0$, for all $k \in K$. The above characterization of the vectors in $N_K(0)$ is, in fact, a characterization of the vectors in $N_K(\mathbb{R}^n)$ as well, which shows that $N_K(\mathbb{R}^n) = N_K(0)$. Indeed, if $x^* \in N_K(\mathbb{R}^n)$ if and only if $\langle k, x^* \rangle \leq 0$, for all $k \in K$. For the other implication we consider $x^* \in N_K(\mathbb{R}^n)$, i.e. $x^* \in N_K(x^0)$ for some $x^0 \in K$. Equivalently $\langle k - x^0, x^* \rangle \leq 0$. Since $K$ is a convex cone, it is closed under addition, i.e. $x^0 + k' \in K$ for all $k' \in K$. Taking $k = x^0 + k'$, we obtain $\langle k', x^* \rangle \leq 0$ for all $k' \in K$.

**Proposition 3.3.** *Let $K$ be a closed convex cone in $\mathbb{R}^n$ and $x^* \in G_K(\mathbb{R}^n)$. Then, $x^* \in \text{int}_{S^{n-1}} G_K(\mathbb{R}^n)$ if and only if $\langle k, x^* \rangle < 0$ for all $k \in S^{n-1} \cap K$.*

**Proof.** Assume that $\langle k, x^* \rangle < 0$ for all $k \in S^{n-1} \cap K$ and consider, for every $k \in K \cap S^{n-1}$, some open neighbourhoods $U_k, V_k$ of $k$ and of $x^*$ in $S^{n-1}$ respectively, such that $\langle u, v \rangle < 0$ for all $(u, v) \in U_k \times V_k$.

Since $K \cap S^{n-1}$ is compact, its open covering $\{U_k\}_{k \in K \cap S^{n-1}}$ has a finite subcover $U_{k_1}, \ldots, U_{k_p}$. Since $\langle u, v \rangle < 0$ for all $(u, v) \in U \times V$, where

$$U = U_{k_1} \cup \cdots \cup U_{k_p} \text{ and } V = V_{k_1} \cap \cdots \cap V_{k_p},$$

we get that $\langle u, v \rangle < 0$ for all $(u, v) \in (K \cap S^{n-1}) \times V$, which shows, via Remark 3.2, that $V \subseteq G_K(\mathbb{R}^n)$, i.e. $x^* \in \operatorname{int} G_K(\mathbb{R}^n)$, as $x^* \in V \subseteq G_K(\mathbb{R}^n)$. Conversely, assume that $\langle k_0, x^* \rangle = 0$ for some $k_0 \in S^{n-1} \cap K$ and consider the function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ given by $\varphi(t) = \langle (\cos t)x^* + (\sin t)k_0, k_0 \rangle$ and observe that $\varphi(t) = \sin t$. Since $\varphi'(0) = 1$, it follows that $\varphi$ is strictly increasing in a neighbourhood of 0, namely there exists $\varepsilon > 0$ such that $\varphi(t) > \varphi(0) = 0$ for all $t \in (0, \varepsilon)$, i.e. $\langle (\cos t)x^* + (\sin t)k_0, k_0 \rangle > 0$ for all $t \in (0, \varepsilon)$. If $t_n \searrow 0$, then $S^{n-1} \ni x_n := (\cos t_n)x^* + (\sin t_n)k_0 \longrightarrow x^*$ and $\langle x_n, k_0 \rangle > 0$, for $n$ sufficiently large, that is $x_n \in S^{n-1} \setminus G_K(\mathbb{R}^n)$, for $n$ sufficiently large. Therefore $x^* \in \partial_{S^{n-1}} G_K(\mathbb{R}^n) \subseteq S^{n-1} \setminus \operatorname{int}_{S^{n-1}} G_K(\mathbb{R}^n)$, as $x^* \in G_K(\mathbb{R}^n)$ and it can be approximated by points of $S^{n-1} \setminus G_K(\mathbb{R}^n)$. □

**Theorem 3.4.** *Let $K, L \subseteq \mathbb{R}^n$ be closed convex cones. If the relation holds*

$$\operatorname{int}_{S^{n-1}} G_K(\mathbb{R}^n) \cap \operatorname{int}_{S^{n-1}} G_L(\mathbb{R}^n) \neq \emptyset,$$

*then there exists a constant $d > 0$ such that $||x + y|| \geq d$, for all $x \in K \cap S^{n-1}$ and all $y \in L \cap S^{n-1}$.*

**Proof.** Assume that $\operatorname{int}_{S^{n-1}} G_K(\mathbb{R}^n) \cap \operatorname{int}_{S^{n-1}} G_L(\mathbb{R}^n) \neq \emptyset$ and consider some $x^* \in \operatorname{int}_{S^{n-1}} G_K(\mathbb{R}^n) \cap \operatorname{int}_{S^{n-1}} G_L(\mathbb{R}^n) \neq \emptyset$, i.e. $\langle k, x^* \rangle < 0$ for all $k \in S^{n-1} \cap K$ and $\langle l, x^* \rangle < 0$ for all $l \in S^{n-1} \cap L$, due to Proposition 3.3. Therefore

$$\langle k + l, x^* \rangle < 0, \quad \text{for all } k \in S^{n-1} \cap K \text{ and all } l \in S^{n-1} \cap L$$

Since the function $S^{n-1} \times S^{n-1} \longrightarrow \mathbb{R},\ (u, v) \mapsto \langle u+v, x^* \rangle$ is continuous and negative on the compact subset $(S^{n-1} \cap K) \times (S^{n-1} \cap L)$ of $S^{n-1} \times S^{n-1}$, there exists $d > 0$ such that $\langle k + l, x^* \rangle \leq -d$ for all $k \in S^{n-1} \cap K$ and all $l \in S^{n-1} \cap L$. Therefore

$$d \leq \langle k + l, -x^* \rangle \leq \|k + l\| \cdot \| - x^*\| = \|k + l\|,$$

for all $k \in S^{n-1} \cap K$ and all $l \in S^{n-1} \cap L$. □

**Remark 3.5.** The converse of Theorem 3.4 is not true as shows the example of any proper subspace $Y \leq \mathbb{R}^n$ and its orthogonal complement $Y^\perp \leq \mathbb{R}^n$, which is also proper. Indeed $\operatorname{int}_{S^{n-1}} G_Y(\mathbb{R}^n) = \emptyset = \operatorname{int}_{S^{n-1}} G_{Y^\perp}(\mathbb{R}^n)$, as

$$\begin{aligned}
\dim\ G_Y(\mathbb{R}^n) &= \dim N_Y(\mathbb{R}^n) \cap S^{n-1} \\
&= \dim(Y^\perp \cap S^{n-1}) = \dim Y^\perp - 1 \leq n - 2 < \dim(S^{n-1}) \\
\dim\ G_{Y^\perp}(\mathbb{R}^n) &= \dim N_{Y^\perp}(\mathbb{R}^n) \cap S^{n-1} \\
&= \dim(Y \cap S^{n-1}) = \dim Y - 1 \leq n - 2 < \dim(S^{n-1}).
\end{aligned}$$

On the other hand the intersection of the compact symmetric sets $Y \cap S^{n-1}$ and $Y^\perp \cap S^{n-1}$ is empty as $Y \cap Y^\perp = \{0\}$. Since the function

$$S^{n-1} \times S^{n-1} \longrightarrow \mathbb{R},\ (u, v) \mapsto \|u + v\|$$

is continuous and its restriction to

$$Y \cap S^{n-1}] \times [-(Y^\perp \cap S^{n-1})] = [Y \cap S^{n-1}] \times [Y^\perp \cap S^{n-1}]$$

is positive, it follows that there exists $d > 0$ such that

$$\|u + v\| \geq d, \ \forall (u, v) \in [Y \cap S^{n-1}] \times [Y^{\perp} \cap S^{n-1}].$$

We conjecture however that the converse of Theorem 3.4 holds true under the additional requirement on the closed convex cones $K, L \subseteq \mathbb{R}^n$ and the stronger condition $\|u + v\| \geq d$ for some $d > 0$ and all $u, v \in (K \cup L) \cap S^{n-1}$. A positive answer to this conjecture is implied, for such cones, by the following, still conjectured, relation $\emptyset \neq \mathrm{int}_{S^{n-1}} \, G_{\mathrm{conv}(K\cup L)} \, (\mathbb{R}^n)$, as the inclusion

$$\mathrm{int}_{S^{n-1}} \, G_{\mathrm{conv}(K\cup L)} \, (\mathbb{R}^n) \subseteq \mathrm{int}_{S^{n-1}} \, G_K \, (\mathbb{R}^n) \cap \mathrm{int}_{S^{n-1}} \, G_L \, (\mathbb{R}^n)$$

holds. Indeed, the obvious inclusions $K, L \subseteq \mathrm{conv}(K \cup L)$ imply that

$$N_{\mathrm{conv}(K\cup L)}(0) \subseteq N_K(0), \ N_L(0), \ \text{i.e. } N_{\mathrm{conv}(K\cup L)}(0) \subseteq N_K(0) \cap N_L(0).$$

Therefore $\mathrm{int} N_{\mathrm{conv}(K\cup L)}(\mathbb{R}^n) = \mathrm{int} N_{\mathrm{conv}(K\cup L)}(0)$ is contained both in $\mathrm{int} N_K(0) = \mathrm{int} N_K(\mathbb{R}^n)$ and in $\mathrm{int} N_L(0) = \mathrm{int} N_L(\mathbb{R}^n)$. We used here the fact that the convex hull of the union of two cones with a common apex $\mathbf{s}$ is still a cone with apex $\mathbf{s}$ (see [10, Theorem 4.8]). $\square$

**Definition 3.6.** A closed convex set $C \subseteq \mathbb{R}^n$ is said to be Motzkin decomposable [2] if there exists a compact convex set $K$ and a closed convex cone $D$ such that

$$C = K + D. \tag{10}$$

A Motzkin decomposable closed convex set and its recession cone have the same total normal cone and the same Gauss ranges therefore. Indeed we have:

$$N_C(\mathbb{R}^n) = N_{K+D}(\mathbb{R}^n) = N_K(\mathbb{R}^n) \cap N_D(\mathbb{R}^n) = \mathbb{R}^n \cap N_D(\mathbb{R}^n) = N_D(\mathbb{R}^n).$$

Therefore $G_C(\mathbb{R}^n) = S^{n-1} \cap N_C(\mathbb{R}^n) = S^{n-1} \cap N_D(\mathbb{R}^n) = G_D(\mathbb{R}^n)$.

**Theorem 3.7.** *Let $M_1, M_2 \subseteq \mathbb{R}^n$ be Motzkin decomposable sets. If*

$$\mathrm{int}_{S^{n-1}} \, G_{M_1} \, (\mathbb{R}^n) \cap \mathrm{int}_{S^{n-1}} \, G_{M_2} \, (\mathbb{R}^n) \neq \emptyset,$$

*then $M_1 + M_2$ is a Motzkin decomposable closed convex set.*

**Proof.** The statement follows by considering Motzkin decompositions of $M_1$ and $M_2$, i.e.

$$M_1 = K_1 + O^+(C_1), \ M_2 = K_2 + O^+(C_2),$$

where $K_1$ and $K_2$ are compact convex components of $M_1$ and $M_2$ respectively. Since

$$M_1 + M_2 = K_1 + K_2 + O^+(C_1) + O^+(C_2)$$

and $K_1 + K_2$ is obviously convex and compact, we conclude that $M_1 + M_2$ is closed, as $O^+(C_1) + O^+(C_2)$ is closed due to [1, Theorem 3.2] and Theorem 3.4. Indeed,

$$\begin{aligned} \mathrm{int}_{S^{n-1}} \, G_{O^+(C_1)+O^+(C_2)} \, (\mathbb{R}^n) &= \mathrm{int}_{S^{n-1}} \, G_{O^+(C_1)} \, (\mathbb{R}^n) \cap \mathrm{int}_{S^{n-1}} \, G_{O^+(C_2)} \, (\mathbb{R}^n) \\ &= \mathrm{int}_{S^{n-1}} \, G_{M_1} \, (\mathbb{R}^n) \cap \mathrm{int}_{S^{n-1}} \, G_{M_2} \, (\mathbb{R}^n) \neq \emptyset. \end{aligned}$$

Now the closedness of $M_1 + M_2 = K_1 + K_2 + O^+(C_1) + O^+(C_2)$ relies on the compactness of $K_1 + K_2$ and on the closedness of $O^+(C_1) + O^+(C_2)$. In fact $M_1 + M_2$ is Motzkin decomposable and $O^+(M_1 + M_2) = O^+(M_1) + O^+(M_2)$. $\square$

**Remark 3.8.** In dimension two, Theorem 3.7 can be equally proved via [5, Corollary 23] which states that a closed convex set $M \subseteq \mathbb{R}^2$ is Motzkin decomposable if and only if its Gauss range $G_M(\mathbb{R}^2)$ is closed. Therefore $G_{M_1+M_2}(\mathbb{R}^2) = G_{M_1}(\mathbb{R}^2) \cap G_{M_2}(\mathbb{R}^2)$ is closed as $G_{M_1}(\mathbb{R}^2)$ and $G_{M_2}(\mathbb{R}^2)$ are both closed.

## 4. Extreme faces and generalized Minkowski sets

We showed in [5, Corollary 2.4] that the collection $M(C)$ of all minimal faces of a closed convex set $C \subseteq \mathbb{R}^n$ is the same with the collection of lowest dimensional faces of $C$. In this subsection we will show that $M(C)$ further coincides with the collection of all extreme faces of $C$.

**Definition 4.1.** A face $F$ of the closed convex set $C$ is said to be an *extreme face* of $C$ if $(1-t)u+tv \in F$ for some $u, v \in C$ and $t \in (0,1)$ implies that $u - v \in$ lin $C$.

Denote by $EF(C)$ the collection of all extreme faces of $C$. Note that $EF(C) = \text{ext } C$ whenever $C$ contains no lines, i.e. lin $C = \{0\}$. We shall denote the convex hull of $\cup\{\mathcal{F} \mid F \in EF(C)\}$ by conv $EF(C)$.

**Theorem 4.2.** *A face $F$ of the closed convex set $C$ is extreme if and only if $F \cap \text{ext } [C \cap L^\perp] \neq \emptyset$ and $F = x + L$ for some $x \in F \cap \text{ext } [C \cap L^\perp]$, where $L = \text{lin } C$.*

**Proof.** Indeed, if $F = x + L$ for some $x \in F \cap \text{ext } [C \cap L^\perp]$ and $(1-t)u + tv \in F$, $u, v \in C$ implies $(1-t)u + tv = x + z$ for some $z \in L = \text{lin } C$. Therefore we have:

$$\begin{aligned}
(1-t)u + tv = x + z &\Leftrightarrow (1-t)(u-z) + t(v-z) = x \\
&\Leftrightarrow (1-t)p_L(u-z) + (1-t)p_{L^\perp}(u-z) + tp_L(v-z) + tp_{L^\perp}(v-z) = x \\
&\Leftrightarrow (1-t)p_L(u-z) + tp_L(v-z) = x - (1-t)p_{L^\perp}(u-z) - tp_{L^\perp}(v-z)
\end{aligned}$$

Since

$$(1-t)p_L(u-z) + tp_L(v-z) \in L \text{ and } x - (1-t)p_{L^\perp}(u-z) - tp_{L^\perp}(v-z) \in L^\perp,$$

as $x \in C \cap L^\perp$, we conclude that $(1-t)p_{L^\perp}(u-z) + tp_{L^\perp}(v-z) = x \in \text{ext } [C \cap L^\perp]$, which implies that

$$\begin{aligned}
p_{L^\perp}(u-z) = p_{L^\perp}(v-z) &\Leftrightarrow u - z - p_L(u-z) = v - z - p_L(v-z) \\
&\Leftrightarrow u - v = p_L(u-z) - p_L(v-z) \in L
\end{aligned}$$

as $u - z, v - z \in C + \text{lin } C \subseteq C$ and

$$\begin{aligned}
p_{L^\perp}(u-z) &= u - z - p_L(u-z) \in C + L \subseteq C, \\
p_{L^\perp}(v-z) &= v - z - p_L(v-z) \in C + L \subseteq C.
\end{aligned}$$

Conversely, since $F$ is a face of $C$ it follows, via [5, Proposition 2.6], that lin $F =$ lin $C = L$, namely $F + L = F$. For $x \in F$ we have

$$x = p_L(x) + p_{L^\perp}(x) \Leftrightarrow x - p_L(x) = p_{L^\perp}(x).$$

But since $p_{L^\perp}(x) = x - p_L(x) \in F + L = F$ and $p_{L^\perp}(x) \in L^\perp$, it follows that

$$p_{L^\perp}(x) \in F \cap L^\perp \subset C \cap L^\perp.$$

We will next show that $p_{L^\perp}(x) \in \text{ext}(C \cap L^\perp)$. In this respect we assume that $p_{L^\perp}(x) = (1-t)u + tv, t \in [0,1]$ and $u, v \in C \cap L^\perp$. Since $p_{L^\perp}(x) \in F$, $u, v \in C$ and $F$ is an extreme face of $C$, we deduce that $u - v \in L$. Taking into account that $u, v \in C \cap L^\perp \Rightarrow u - v \in L^\perp$ this implies that $u - v \in L \cap L^\perp = \{0\}$ and therefore $u = v$. Thus $p_{L^\perp}(x) \in \text{ext}(C \cap L^\perp)$.

In what follows we will show that $F = p_{L^\perp}(x) + L$. Indeed, we have $p_{L^\perp}(x) + L \subseteq F + L = F$. To show the opposite inclusion, we consider $f \in F$. Since $p_{L^\perp}(x) \in F$ and $F$ is convex, it follows that $\frac{1}{2}(f + p_{L^\perp}(x) \in F$. Taking into account that $F$ is an extreme face and $F \subset C$ we conclude that $f - p_{L^\perp}(x) \in L$ and therefore $f \in p_{L^\perp}(x) + L$ and the proof of the inclusion $F \subseteq p_{L^\perp}(x) + L$ is now complete. $\square$

**Theorem 4.3.** (Generalized Minkowski Theorem) *If $C \subset \mathbb{R}^n$ is a closed convex set such that $C \cap (\text{lin } C)^\perp$ is compact, then*

$$C = \text{conv } EF(C).$$

**Proof.** Since $C \cap (\text{lin } C)^\perp$ is compact, it follows, via the Minkowski Theorem, that $C \cap (\text{lin } C)^\perp = \text{conv ext } C \cap (\text{lin } C)^\perp$. Since $C = C \cap (\text{lin } C)^\perp + \text{lin } C$ [9, p. 65], it follows that

$$\begin{aligned} C &= C \cap (\text{lin } C)^\perp + \text{lin } C \\ &= \text{conv ext } C \cap (\text{lin } C)^\perp + \text{conv lin } C \\ &= \text{conv } [\text{ext } C \cap (\text{lin } C)^\perp + \text{lin } C] \\ &= \text{conv } \cup \left\{x + \text{lin } C \mid x \in \text{ext } C \cap (\text{lin } C)^\perp\right\} \\ &= \text{conv } \cup \{\mathcal{F} \mid F \in EF(C)\}. \end{aligned}$$

and the proof is complete. $\square$

**Definition 4.4.** A closed convex set $C \subset \mathbb{R}^n$ is said to be a *generalized Minkowski set* if $C = \text{conv } EF(C)$.

**Remark 4.5.** Let $C \subseteq \mathbb{R}^n$ be a closed convex set and $F$ be a face of $C$. The following statements are equivalent:

1. $F$ is an extreme face of $C$;
2. $F$ is a minimal face of $C$;
3. $F$ is a lowest dimensional face $C$;

Indeed, the equivalence (1)$\Longleftrightarrow$(3) follows by combining Theorem 4.2 with [5, Theorem 2.2] and the equivalence (1)$\Longleftrightarrow$(3) follows by combining Theorem 4.2 with [5, Theorem 2.2] and [5, Corollary 2.2]. $\square$

Once we know that the collection $EF(C)$ of extreme faces of the closed convex set $C \subseteq \mathbb{R}^n$ coincides with the collection $M(C)$ of minimal faces of $C$, we observe that our current definition of *generalized Minkowski sets* coincides with the one we proposed in [5, Definition 4.1]. While Theorem 4.3 is a good motivation for our current definition of generalized Minkowski sets, a more general version is [5, Theorem 4.3] with the set $M(C)$ playing the role of $EF(C)$. Indeed, we have the following result:

**Theorem 4.6.** ([5, Theorem 4.3]) *Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set and $U \subseteq \mathbb{R}^n$ be a supplementary subspace to* lin $C$. *Then, $C$ is a generalized Minkowski set if and only if $C \cap U$ is a Minkowski set.*

In particular, $C$ is a generalized Minkowski set if and only if the orthogonal slice $C \cap (\text{lin } C)^{\perp}$ is a Minkowski set.

**Remark 4.7.** A Minkowski set $C \subset \mathbb{R}^n$ is a generalized Minkowski set. On the other hand, the class of generalized Minkowski set is significantly much larger that the class of Minkowski sets. For example the cylinder

$$C = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 \leq 1\}$$

is a generalized Minkowski set but not a Minkowski set. □

Indeed, while $C$ has no extreme points, its collection of extreme faces is

$$EF(C) = \{(x, y, 0) + \langle(0, 0, 1)\rangle \;:\; x^2 + y^2 = 1\}$$

and $C = \text{conv } EF(C)$.

**Remark 4.8.** If $C \subset \mathbb{R}^n$ is a Minkowski set such that dim $C < n$, then the set $C + (\text{aff}(C) - \text{aff}(C))^{\perp}$ is a generalized Minkowski set, but not a Minkowski set. Indeed $C + (\text{aff}(C) - \text{aff}(C))^{\perp}$ has no extreme points and its extreme faces are

$$\{x + (\text{aff}(C) - \text{aff}(C))^{\perp} \mid x \in \text{ext}(C)\},$$

as well as

$$\begin{aligned} C + (\text{aff}(C) - \text{aff}(C))^{\perp} &= \text{conv ext} C + \text{conv } (\text{aff}(C) - \text{aff}(C))^{\perp} \\ &= \text{conv } (\text{ext} C + (\text{aff}(C) - \text{aff}(C))^{\perp}) \\ &= \text{conv } \cup \{\mathcal{F} \mid \mathcal{F} \in EF(C + (\text{aff}(C) - \text{aff}(C))^{\perp})\} \\ &= \text{conv } EF(C + (\text{aff}(C) - \text{aff}(C))^{\perp}). \end{aligned}$$

Note that $C + (\text{aff}(C) - \text{aff}(C))^{\perp} = p_Y^{-1}(p_Y(C))$, where $Y = \text{aff}(C) - \text{aff}(C)$. □

Closed convex sets of type $p_Y^{-1}(K)$, where $Y$ is a linear subspace of $\mathbb{R}^n$ and $K \subseteq Y$ is a closed convex set, will be often considered in what follows.

**Proposition 4.9.** *Let $Y \leq \mathbb{R}^n$ be a subspace of $\mathbb{R}^n$ and $K \subseteq Y$ is a closed convex set. Then the following statements hold:*

1. $p_Y^{-1}(K) \cap (\text{lin } p_Y^{-1}(K))^{\perp} = K \cap (\text{lin } K)^{\perp}$.
2. $EF(p_Y^{-1}(K)) = p_Y^{-1}(\text{ext } K \cap (\text{lin } K)^{\perp})$.
3. $p_Y^{-1}(K)$ *is generalized Minkowski if and only if $K$ is generalized Minkowski.*

**Proof.** (1) Indeed, by using the equation (5), i.e. lin $p_Y^{-1}(K) = \text{lin } K + Y^\perp$ and

$$(\text{lin } K + Y^\perp)^\perp = (\text{lin } K)^\perp \cap (Y^\perp)^\perp = (\text{lin } K)^\perp \cap Y$$

we obtain the following equalities

$$p_Y^{-1}(K) \cap (\text{lin } p_Y^{-1}(K))^\perp = p_Y^{-1}(K) \cap Y \cap (\text{lin } K)^\perp = K \cap (\text{lin } K)^\perp.$$

(2) Indeed, according to Theorem 4.2, a face $F$ of the closed convex set $p_Y^{-1}(K)$ is extreme if and only if $F \cap \text{ext } [p_Y^{-1}(K) \cap (\text{lin } p_Y^{-1}(K))^\perp] \neq \emptyset$ and $F = x + \text{lin } p_Y^{-1}(K)$ for some $x \in F \cap \text{ext } [p_Y^{-1}(K) \cap (\text{lin } p_Y^{-1}(K))^\perp]$. By using (1), the above statement reeds as follows: A face $F$ of the closed convex set $p_Y^{-1}(K)$ is extreme if and only if $F \cap \text{ext } K \cap (\text{lin } K)^\perp \neq \emptyset$ and

$$F = x + \text{lin } p_Y^{-1}(K) = x + Y = p_Y^{-1}(x)$$

for some $x \in F \cap \text{ext } K \cap (\text{lin } K)^\perp$ and the equality is now completely justified.
(3) By combining (1) with [5, Theorem 4.3], the closed convex set $p_Y^{-1}(K)$ is generalized Minkowski if and only if $p_Y^{-1}(K) \cap (\text{lin } p_Y^{-1}(K))^\perp = K \cap (\text{lin } K)^\perp$ is Minkowski. By using [5, Theorem 4.3] once again, we obtain that $K$ is generalized Minkowski if and only if $K \cap (\text{lin } K)^\perp$ is Minkowski. □

**Remark 4.10.** An important class of generalized Minkowski sets is the one having the Pareto bordered quality, as [5, Theorem 5.6], shows. Note however that the classs of generalized Minkowki sets is significantly larger than the one of Pareto bordered sets, as [3, Example 4.9] shows. In fact we will be only studying, in the next section, the Pareto bordered quality of the Minkowski sum of two Pareto bordered sets. □

## 5. Minkowski sums of Pareto bordered sets

In this section we investigate the generalized Minkowski quality of the Minkowski sum of two generalized Minkowski sets. We will treat here the Minkowski sum of two generalized Minkowski sets $C_1, C_2 \subseteq \mathbb{R}^n$ with the same lineality $X := \text{lin } C_1 = \text{lin } C_2$.

For such a couple we obviously have

$$\begin{aligned} C_1 + C_2 &= C_1 \cap (\text{lin } C_1)^\perp + \text{lin } C_1 + C_2 \cap (\text{lin } C_2)^\perp + \text{lin } C_2 \\ &= C_1 \cap X^\perp + C_2 \cap X^\perp + X \\ &= (C_1 + C_2) \cap X^\perp + X, \end{aligned}$$

as $C_1 \cap X^\perp + C_2 \cap X^\perp = (C_1 + C_2) \cap X^\perp$. Indeed, $C_1 \cap X^\perp \subseteq C_1$ and $C_2 \cap X^\perp \subseteq C_2$ as well as $C_1 \cap X^\perp \subseteq X^\perp$ and $C_2 \cap X^\perp \subseteq X^\perp$. Therefore $C_1 \cap X^\perp + C_2 \cap X^\perp \subseteq C_1 + C_2$ and $C_1 \cap X^\perp + C_2 \cap X^\perp \subseteq X^\perp + X^\perp = X^\perp$, namely $C_1 \cap X^\perp + C_2 \cap X^\perp \subseteq (C_1 + C_2) \cap X^\perp$.

For the oppposite inclusion, we consider $c_1 \in C_1$ and $c_2 \in C_2$ such that $c_1 + c_2 \in X^\perp$, i.e. $p_{X^\perp}(c_1 + c_2) = c_1 + c_2 \Leftrightarrow c_1 + c_2 = p_{X^\perp}(c_1) + p_{X^\perp}(c_2) \in C_1 \cap X^\perp + C_2 \cap X^\perp$.

On the other hand the representation

$$C_1 + C_2 = (C_1 + C_2) \cap [\text{lin } (C_1 + C_2)]^\perp + \text{lin } (C_1 + C_2), \tag{11}$$

might be quite different from $C_1 + C_2 = (C_1 + C_2) \cap X^\perp + X$ unless

$$\text{lin } (C_1 + C_2) = \text{lin } C_1 = \text{lin } C_2 = X. \tag{12}$$

Note that the representation (11) assures the generalized Minkowski quality of the sum $C_1+C_2$ once the Minkowski quality of $(C_1+C_2)\cap[\text{lin }(C_1+C_2)]^{\perp}$ is clear ([5, Theorem 4.3]). Although the requirement (12) looks rather restrictive, there are some quite large classes of couples $C_1, C_2$ satisfying it. Among them we consider the couple

$$C_1 = p_Y^{-1}(K_1),\ C_2 = p_Y^{-1}(K_2),$$

where $Y \leq \mathbb{R}^n$ is a subspace and $K_1, K_2 \subseteq Y$ are two closed convex sets. Under suitable sufficient conditions on $K_1$ and $K_2$, the requirement (12) is fulfilled.

**Remark 5.1.** If $Y \leq \mathbb{R}^n$ is a subspace and $K_1, K_2 \subseteq Y$ are two closed convex sets, then

$$p_Y^{-1}(K_1) + p_Y^{-1}(K_2) = p_Y^{-1}(K_1 + K_2);$$

$$\text{lin }(p_Y^{-1}(K_1) + p_Y^{-1}(K_2)) = \text{lin }(K_1 + K_2) + Y. \qquad \square$$

Indeed, the following the equalities hold

$$\begin{aligned} p_Y^{-1}(K_1) + p_Y^{-1}(K_2) &= K_1 + Y + K_2 + Y = K_1 + K_2 + Y = p_Y^{-1}(K_1 + K_2) \\ \text{lin }(p_Y^{-1}(K_1) + p_Y^{-1}(K_2)) &= \text{lin }(p_Y^{-1})(K_1 + K_2) = p_Y^{-1}(\text{lin }(K_1 + K_2)) \\ &= \text{lin }(K_1 + K_2) + Y. \end{aligned}$$

If $\text{lin }(K_1+K_2) = 0$, then $\text{lin }(p_Y^{-1}(K_1) + p_Y^{-1}(K_2)) = Y$. On the other hand $\text{lin}K_1 + \text{lin}K_2 \subseteq \text{lin}(K_1+K_2) = 0$, implies $\text{lin}K_1 = \text{lin}K_2 = 0$ for such closed convex sets. In what follows we will provide some sufficient conditions on the closed convex sets $K_1$ and $K_2$ in order to get for $K_1+K_2$ the Minkowski quality and therefore its trivial liniality along with the generalized Minkowski quality of $p_Y^{-1}(K_1) + p_Y^{-1}(K_2)$.

**Remark 5.2.** Let $Y \leq \mathbb{R}^n$ be a subspace of $\mathbb{R}^n$ and $K \subseteq Y^{\perp}$ be a closed convex set. Then $p_Y^{-1}(K)$ is Pareto bordered if and only if $K$ is Pareto bordered. Indeed, we proved in [8] that a closed convex subset $C$ is Pareto bordered if and only if $C \cap (\text{lin } C)^{\perp}$ is Pareto bordered. Therefore the Pareto bordered quality of $p_Y^{-1}(K)$ reduces to the Pareto bordered quality of

$$p_Y^{-1}(K) \cap (\text{lin } p_Y^{-1}(K))^{\perp} = K \cap (\text{lin } K)^{\perp}$$

and therefore to the Pareto bordered quality of $K$.

**Theorem 5.3.** *Let $K_1 \subseteq \mathbb{R}^n$ be a Pareto bordered set which contains no lines and $K_2 \subseteq \mathbb{R}^n$ be a compact set. Then $p_Y^{-1}(p_Y(K_1)) + p_Y^{-1}(p_Y(K_2))$ is a Pareto bordered set, where $Y$ stands for the linear subspace* $\text{aff}(K_1) - \text{aff}(K_1)$.

**Proof.** The Pareto bordered quality of

$$p_Y^{-1}(p_Y(K_1)) + p_Y^{-1}(p_Y(K_2)) = p_Y^{-1}(p_Y(K_1) + p_Y(K_2))$$

reduces to the Pareto bordered quality of $p_Y(K_1) + p_Y(K_2)$, due to Remark 5.2. The Pareto bordered quality of $K_1$ is preserved through the projection $p_Y$, as the restriction

$$\text{aff}(K_1) \longrightarrow Y,\ u \mapsto p_Y(u) \tag{13}$$

is an affine isomorphism and a homeomorphism as well.

Moreover the obvious fact $\text{int}_{\text{aff}(K_1)} K_1 \neq \emptyset$, combined with the quality of the restriction (13) to be a homeomorphism, shows that $\text{int}_Y p_Y(K_1) \neq \emptyset$. Since $p_Y(K_1)$ is Pareto bordered and $p_Y(K_2)$ is convex and compact, it follows, according to Corollary 2.9, that their Minkowski sum $p_Y(K_1) + p_Y(K_2)$ is Pareto bordered. $\square$

**Corollary 5.4.** *Let* $Y \leq \mathbb{R}^n$ *is a subspace such that* $\dim Y = n-2 \geq 1$ *and* $K_1, K_2 \subseteq Y$ *are two convex sets such that* $K_1$ *is Minkowski, and* $K_2$ *is compact, then* $p_Y^{-1}(K_1) + p_Y^{-1}(K_2)$ *is generalized Minkowski.*

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