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Existence of Optimal and ε-Optimal Controls for the Stochastic Cahn-Hilliard Navier-Stokes System

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We study in this article a stochastic version of a coupled of the Cahn-Hilliard Navier-Stokes model in two dimensional bounded domain. The model consists of the Navier-Stokes equations for the velocity, coupled with a Cahn-Hilliard model for the order (phase) parameter. We prove the existence of optimal and ϵ-optimal controls for the stochastic Cahn-Hilliard Navier-Stokes model.

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1. Introduction

Optimal control problems for fluid flows have been a topic of interest for experimenters and designers for many decades. Mathematicians and computational scientists have lately shown a strong interest in them as well because of the beautiful mathematics behind them. It is well known that the Navier-Stokes equations describe the motion of a fluid. When we consider the evolution of an incompressible isothermal mixture of two immiscible fluids, then phase separation will occur. This phenomenon is described by a diffuse interface model, which is also known as Cahn-Hilliard Navier-Stokes system (see [19], [33], [34] for more details). Optimal control problems for the stochastic Navier-Stokes equations have been studied extensively for both the case of distributed control. Motivated by the study of the stochastic optimal control problems for single fluid flows and its application in various engineering fields, atmospheric sciences, hydrodynamics etc, we are interested to study

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the stochastic control problem for the binary mixture described by the stochastic Cahn-Hilliard Navier-Stokes system. Optimal control problems associated with the stochastic Cahn-Hilliard Navier-Stokes system have many applications where one is interested in influencing the phase separation process, such as in the separation of binary alloys, the formation of polymeric membranes, etc.

In this paper, we consider the evolutionary system for Cahn-Hilliard Navier-Stokes perturbed by a multiplicative noise. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a filtered probability space satisfying the usual hypotheses i.e., $\{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous filtration such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null subsets of $(\Omega, \mathcal{F})$. We assume that the domain D of the fluid is a bounded domain in $\mathbb{R}^2$. We have the system

$$\begin{cases} d\mathrm{u} + (-\nu\Delta\mathrm{u} + (\mathrm{u}\cdot\nabla)\mathrm{u} + \nabla\mathrm{p} - \mathcal{K}\mu\nabla\phi - \Phi(t,\mathrm{u}))dt = \mathrm{g}_2(t,\mathrm{u})\,dW, \\ \mathrm{div}\ \mathrm{u} = 0, \\ \frac{\partial\phi}{\partial t} + \mathrm{u}\cdot\nabla\phi = \beta\Delta\mu, \\ \mu = -\delta\Delta\phi + \alpha\mathrm{f}(\phi), \end{cases} \tag{1.1}$$

in $\Omega \times D \times (0,T)$, subject to the following boundary and initial conditions

$$\begin{cases} \partial_n\phi = \partial_n\Delta\phi = 0, \quad \text{on}\,\partial D\times(0,T), \\ \mathrm{u} = 0, \quad \text{on} \quad \partial D\times(0,T), \\ (\mathrm{u},\phi)(0) = (\mathrm{u}_0,\phi_0) \quad \text{in} \quad D, \end{cases} \tag{1.2}$$

where n is the outward normal to ∂D.

Note that $(1.2)_1$ implies that $\partial_n\mu = 0 \quad \text{on} \quad \partial D\times(0,T)$, and ensure the following mass conservation law:

$$\int_{\mathrm{D}} \phi(t,x)dx = \int_D \phi_0(x)dx, \forall t\in[0,T]. \tag{1.3}$$

In (1.1) and (1.2), $T > 0$ fixed, the unknown functions are the velocity $\mathrm{u} = (\mathrm{u}_1, \mathrm{u}_2)$ of the fluid, its pressure p and the order (phase) parameter ϕ, ∇ and Δ denote the gradient and Laplacian operators. The function Φ depends on the velocity field and acts as a feedback control of the system. The term $\mathrm{g}_2(t,\mathrm{u})dW$ represents random external forces depending eventually on u, where W is a real-valued Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The quantity μ is the chemical potential of the binary mixture which is given by the variational derivative of the following free energy functional

$$\mathcal{F}(\phi) = \int_{\mathrm{D}} (\frac{\delta}{2}\left|\nabla\phi\right|^2 + \alpha\mathrm{F}(\phi))dx,$$

where $\mathrm{F}(r) = \int_0^r f(\zeta)d\zeta$ is the suitable double-well potential. The constants ν,β and $\mathcal{K}$ are positive and correspond to the kinematic viscosity of the fluid, mobility constant and capillarity (stress) coefficient, respectively. Here, δ and α are two positive parameters describing the interactions between the two phases. In particular, δ is related to the thickness of the interface separating the two fluids.

A typical example of potential F is that of logarithmic type. However, this potential is often replaced by a polynomial approximation of the type $\mathrm{F}(r) = y_1 r^4 - y_2 r^2$, y_1, y_2 being positive constants.

As noted in [34], $(1.1)_1$ can be replaced by

$$du + (\nu\Delta \mathrm{u} + (\mathrm{u}.\nabla)\mathrm{u} + \nabla\widetilde{\mathrm{p}} + \mathcal{K}\mathrm{div}(\nabla\phi \otimes \nabla\phi))dt = \Phi(t, \mathrm{u})dt + \mathrm{g}_2(t, \mathrm{u})\, dW, \quad (1.4)$$

where $\widetilde{\mathrm{p}} = \mathrm{p} - \mathcal{K}(\frac{\delta}{2}\left|\nabla\phi\right|^2 + \alpha\mathrm{F}(\phi))$, since

$$\mathcal{K}\mu\nabla\phi = \nabla(\tfrac{\delta}{2}\left|\nabla\phi\right|^2 + \alpha\mathrm{F}(\phi)) - \mathcal{K}\mathrm{div}(\nabla\phi \otimes \nabla\phi).$$

The stress tensor $\nabla\phi \otimes \nabla\phi$ is considered the main contribution modeling capillary forces due to surface tension at the interface between the two phases of the fluid.

There are a few notable works available on the stochastic Cahn-Hilliard Navier-Stokes driven by Gaussian noise or by jump noise of Lévy type (for instance [17], [12], [13], [11], [34]). In [13] and [34], the authors studied the existence of a unique strong solution for the stochastic 2D local Cahn–Hilliard Navier–Stokes system. The proof is based on the Galerkin approximation, the stopping time techniques and the principle of weak convergence in functional analysis. In [10], the authors considered the stochastic 3D globally modified Cahn-Hilliard Navier-Stokes system with multiplicative Gaussian noise. They proved the existence and uniqueness of strong solution (in the sense of partial differential equations and stochastic analysis). Moreover, they studied the asymptotic behaviour of the unique solution and obtained the existence of a probabilistic weak solution for the stochastic 3D Cahn-Hilliard Navier-Stokes system. In [12], they also studied the asymptotic stability of the unique strong solution for the 3D globally modified Cahn-Hilliard Navier-Stokes system.

In this work, our main task consists of studying optimal feedback control problem for the stochastic Cahn-Hilliard Navier-Stokes system (1.1)–(1.2). Let us now briefly mention some related literature on optimal control problems when $\mathrm{g}_2(t, \mathrm{u}) = 0$. Optimal control problems for the Cahn-Hilliard Navier-Stokes system are studied by several mathematicians and some of these results can be found in ([6], [7], [20] and the references therein). In [6], the authors have studied a distributed optimal control problem for nonlocal Cahn-Hilliard Navier-Stokes system and established the Pontryagin's maximum principle. They have characterized the optimal control using the adjoint variable. An optimal initialization problem, a type of data assimilation problem is examined in [7]. A distributed optimal control problem for the two-dimensional nonlocal Cahn-Hilliard Navier-Stokes system with degenerate mobility and singular potential is described in [20]. Optimal control problems with state constraint and robust control for local Cahn-Hilliard Navier-Stokes system are investigated in [32], [33] respectively.

Optimal control problems for the stochastic Navier-Stokes equations have been studied by several mathematicians and some of the results can be found in ([2], [3], [4], [26], [29]). In [2], the authors proved the well-posedness of local mild solution to the stochastic Navier-Stokes equations with multiplicative Lévy noise. Then, they proved the existence and uniqueness of optimal control, by analysing some cost functional depending on stopping time. Recently, they studied in [3], the optimality system and established the necessary and sufficient optimality conditions with a multiplicative noise driven by a Wiener process. It is worth mentioning that the approach in ([2], [3]) was based on the theory of semigroups. There is not much work written about the control problem of the stochastic Cahn-Hilliard system. In [36], a distributed optimal control problem associated to the stochastic pure Cahn-Hilliard equation was investigated. The control is represented by a source-term in the definition of the chemical potential. The existence of an optimal control is established thanks to probability and analytical compactness arguments. Moreover, the first order necessary conditions of optimality have also been established by means of the monotonicity and compactness arguments.

Our main interest consists in the study of the optimal feedback control for the stochastic Cahn-Hilliard Navier-Stokes system. The proof of our main result follows closely the approach used in [29], [26], where the author showed the existence of optimal and ε-optimal controls for the stochastic Navier-Stokes equations in two dimensional bounded domain, controlled by different external forces and assuming that the set of admissible controls is weak sequentially compact. Let us note that the coupling between the stochastic Navier–Stokes equations and the Cahn–Hilliard system introduces in the coupled model a highly non linear term that makes the analysis more involved compared to the references cited above. Since the considered system is non linear, we are dealing with a non-convex optimization problem. The problem of the existence of an optimal control for SPDEs is an important question in optimal control theory and often resolved by assuming that the set of admissible control is compact and by using the main theorem for minimum Problem (see [42], Theorem 38.B). We use a weaker condition to the set of admissible controls which is weak sequentially compact and we assume that the functional cost is weak sequentially lower semi continuous. The main idea in solving the control problem is the following: First, we prove that the set of admissible controls is weak sequentially compact (see Lemma 3.2) and then we prove that weak convergence of the feedback controls implies strong convergence of the corresponding solutions (see Theorems 3.10 and 3.12). Now, with the required well-posedness result of (1.1)–(1.2) in hand (see [34] for more details), the road is paved for studying optimal control problems associated with the system (1.1)–(1.2). We restrict ourselves to dimension two as the regularity of the solution required to study optimal control problems is available in dimension two only. However, in dimension three, similar to the case of the stochastic Navier-Stokes equations, the uniqueness of a weak solution for the stochastic Cahn-Hilliard Navier-Stokes system remains open. The control problem under investigation in this paper reads as follows:

Let us consider the cost functional:

$$\mathcal{J}(\Phi) = \mathbb{E}\int_0^T (\mathcal{L}[s, \mathrm{u}_\Phi(s), \phi_\Phi(s)] + \mathcal{K}_1[\Phi(s, \mathrm{u}_\Phi(s))])ds + \mathbb{E}\mathcal{K}_2[\mathrm{u}_\Phi(T), \phi_\Phi(T)] \quad (1.5)$$

for $\Phi \in \mathcal{U}$. So the optimal control problem is formulated as follows: Our control problem is to minimize (1.5) over $\mathcal{U}$, where $\mathcal{U}$ is the set of all admissible controls. Any $\Phi^* \in \mathcal{U}$ satisfying

$$\mathcal{J}(\Phi^*) = \inf\{\mathcal{J}(\Phi) : \Phi \in \mathcal{U}\} \quad (1.6)$$

is called an optimal control, $(\mathrm{u}_{\Phi^*}, \phi_{\Phi^*})$ is called state variable (see Section 3 for more details).

The structure of this paper is as follows: In Section 2, we present the mathematical setting of the stochastic Cahn-Hilliard Navier–Stokes system. We also state the existence and uniqueness of strong solution (in the sense of stochastic analysis) of the system (see Theorem 2.4). This result is crucial in proving the existence of optimal control which is tackled in the following section. In Section 3, we formulate the control problem, which is the goal of this work and we prove the existence of an optimal control. Section 4 contains the proof for the existence of ϵ-optimal controls.

2. Mathematical setting

In this section, we introduce the necessary function space needed through out the paper. We define some operators to write (1.1)–(1.2). We also state the existence and uniqueness of a strong solution result for the system (1.1)–(1.2).

Hereafter, we assume that $f \in \mathcal{C}^2(\mathbb{R})$ satisfies

$$\begin{cases} \lim\limits_{|r|\to\infty} f'(r) > 0, \\ |f''(r)| \leq c_f(1+|r|^{m-1}), \quad \forall\, r \in \mathbb{R}, \end{cases} \tag{2.1}$$

where c_f is some positive constant and $m \in [1,\infty)$ is fixed. It follows from (2.1) that

$$|f'(r)| \leq c_f(1+|r|^m), \quad |f(r)| \leq c_f(1+|r|^{m+1}), \quad \forall\, r \in \mathbb{R}. \tag{2.2}$$

Note that the derivation of the typical double-well potential f satisfies conditions similar to (2.1).

We introduce some notations and functional set up for the system (1.1) − (1.2).

If X is a real Hilbert space with inner product $(.;.)$ we will denote the induced norm by $|.|_{\mathrm{X}}$ while X^* will indicate its dual. We denote by $(.;.)$ and $|.|_{L^2}$, $d=2$, respectively, the scalar product and associated norm in $(L^2(D))^d$ and by $(\nabla \mathrm{u}, \nabla \mathrm{v})$ the scalar product in $(L^2(D))^d$ of the gradient of u and v. We set

$$\mathcal{V}_1 = \{u \in (\mathcal{C}_0^\infty(D))^d : \operatorname{div} \mathrm{u} = 0 \;\text{ in }\; D\}.$$

We denote by H_1 and V_1 the closure of $\mathcal{V}_1$ in $(L^2(D))^d$ and $(H_0^1(D))^d$ respectively. The scalar product in H_1 is denoted by $(.;.)$ and the associated norm by $|.|_{L^2}$. Moreover, the space V_1 is endowed with the scalar product

$$((\mathrm{u},\mathrm{v})) = \sum_{i=1}^{2} (\partial_{x_i}\mathrm{u}, \partial_{x_i}\mathrm{v})_{L^2}, \quad ||\mathrm{u}|| = ((\mathrm{u},\mathrm{u}))^{1/2}.$$

We now define the operator A_0 by

$$\mathrm{A}_0\mathrm{u} = -\mathcal{P}_1\Delta\mathrm{u}, \quad \forall \mathrm{u} \in \mathrm{D}(\mathrm{A}_0) = (H^2(D))^d \cap \mathrm{V}_1,$$

where $\mathcal{P}_1$ is the Leray-Helmotz projector in $(L^2(D))^d$ onto H_1. Then, A_0 is a self-adjoint positive unbounded operator in H_1 which the scalar product defined above. Furthermore, A_0^{-1} is a compact linear operator on H_1 and $|\mathrm{A}_0\,.\,|_{L^2}$ is norm on $\mathrm{D}(\mathrm{A}_0)$ that is equivalent to the H^2-norm.

Now, we define the linear positive unbounded operator on Hilbert space $\mathrm{L}_0^2(\mathrm{D})$ of L^2-function with null mean by:

$$\mathrm{A}_1\phi = -\Delta\phi, \quad \forall\phi \in \mathrm{D}(\mathrm{A}_1) = \{\phi \in \mathrm{H}^2(\mathrm{D}), \partial_n\phi = 0, \quad \text{on } \; \partial D\} \cap \mathrm{L}_0^2(D). \tag{2.3}$$

Note that A_1 is a self-adjoint, positive definite, unbounded linear operator on $L_0^2(D)$ with compact inverse. More generally, we can define A_1^s, for any $s \in \mathbb{R}$, noting that $|A_1^{s/2}.|_{L^2}$, $s > 0$, is an equivalent norm to the canonical H^s-norm on $D(A_1^{s/2}) \subset H^s(D) \cap L_0^2(D)$. Moreover, we set $H^{-s}(D) = (H^s(D))^*$, wherever $s < 0$.

Hereafter, we set $\quad H_2 = L_0^2(D),\ V_2 = D(A_1^{1/2}),\ $ and $\mathbb{H} = H_1 \times H_2$. (2.4)

The norms in H_2 and V_2 are denoted respectively by $|.|_{L^2}$ and $||.||$, where we have $||\psi|| = |A_1^{1/2}\psi|_{L^2}$. In addition, we need to introduce the bilinear operators B_0, B_1 (and their associated trilinear form b_0, b_1) as well as the coupling mapping R_0, which are defined from $D(A_0) \times D(A_0)$ into H_1, $D(A_0) \times D(A_1)$ into $L^2(D)$, and $L^2(D) \times D(A_1) \cap H^3(D)$ into H_1, respectively. More precisely, we set

$$\langle B_0(u,v), w\rangle = \int_D [(u\cdot\nabla)v]w dx = b_0(u,v,w),\ u,v,w \in D(A_0),$$

$$\langle B_1(u,\phi), \psi\rangle = \int_D [(u\cdot\nabla)\phi]\psi dx = b_1(u,\phi,\psi),\ u \in D(A_0), \phi,\psi \in D(A_1),$$

$$\langle R_0(\mu,\phi), w\rangle = \int_D \mu[\nabla\phi.w]dx = b_2(w,\phi,\mu),\ w\in D(A_0), \mu\in L^2(D), \phi\in D(A_1)\cap H^3(D).$$

Note as in [34] (see also [33], [13]) that $R_0(\mu,\phi) = \mathcal{P}\mu\nabla\phi$. We derive that b_0 and b_1 satisfying the following estimates (see [34] or [15])

$$|b_0(u,v,w)| \le C\,|u|_{L^2}^{1/2}\,\|u\|^{1/2}\,\|v\|\,|w|_{L^2}^{1/2}\,\|w\|^{1/2}, \quad \text{for all } \ u,v,w \in V_1.$$

$$|b_1(u,\phi,\psi)| \le C\,|u|_{L^2}^{1/2}\,\|u\|^{1/2}\,|A_1\phi|_{L^2}^{1/2}\,\|\phi\|^{1/2}\,|\psi|_{L^2}, \ \text{for all } u\in V_1,\ \phi\in D(A_1),\ \psi\in H_2.$$

$$\|B_0(u,v)\|_{V_1^*} \le C\,|u|_{L^2}^{1/2}\,\|u\|^{1/2}\,|v|_{L^2}^{1/2}\,\|v\|^{1/2} \quad \text{for all } \ u,v \in V_1.$$

$$\|R_0(A_1\phi,\varphi)\|_{V_1^*} \le C\,\|\varphi\|^{1/2}\,|A_1\varphi|_{L^2}^{1/2}\,|A_1\phi|_{L^2}, \quad \text{for all } \ \phi,\varphi \in D(A_1).$$

$$\|B_1(u,\phi)\|_{V_2^*} \le C\,|u|_{L^2}^{1/2}\,\|u\|^{1/2}\,\|\phi\| \quad \text{for all } \ u\in V_1,\ \phi\in D(A_1).$$

We recall that (due to the mass conservation) we have:

$$\langle\phi(t)\rangle = \langle\phi(0)\rangle = M_0,\ \forall t > 0 \tag{2.5}$$

Thus, up to a shift of the order parameter field, we can always assume that the mean of ϕ is zero at the initial time and, therefore it will remain zero for all positive times. Hereafter, we assume that:

$$\langle\phi(t)\rangle = \langle\phi(0)\rangle = 0, \quad \forall t > 0. \tag{2.6}$$

Now, we set

$$\mathbb{Y} = H_1 \times V_2. \tag{2.7}$$

The space $\mathbb{Y}$ is a complete metric space with respect to the norm:

$$|(u,\phi)|_{\mathbb{Y}}^2 = \mathcal{K}^{-1}|u|_{L^2}^2 + \epsilon|\nabla\phi|_{L^2}^2. \tag{2.8}$$

Also, we define another Hilbert space $\mathbb{V}$ by:

$$\mathbb{V} = V_1 \times D(A_1) \tag{2.9}$$

endowed with the scalar product whose associated norm is given by:

$$||(u,\phi)||_{\mathbb{V}}^2 = ||u||^2 + |A_1\phi|_{L^2}^2. \tag{2.10}$$

Let us consider the following energy functional $\mathcal{E} : \mathbb{Y} \to \mathbb{R}_+$ defined by:

$$\mathcal{E}(\theta, \psi) = |(\mathrm{u}, \phi)|^2_{\mathbb{Y}} + 2\alpha(\mathrm{F}(\psi), 1) + c_1, \tag{2.11}$$

where $c_1 > 0$ is a constant large enough and independent on (θ, ψ) such that $\mathcal{E}(\theta, \psi)$ is non-negative (note that F is bounded from below).

We can check that (see [34]) there exists a monotone non-decreasing function (independent on time and the initial condition) such that:

$$|(\theta, \psi)|^2_{\mathbb{Y}} \leq \mathcal{E}(\theta, \psi) \leq \mathcal{Q}_0(|(\theta, \psi)|^2_{\mathbb{Y}}), \quad \forall\, (\theta, \psi) \in \mathbb{Y}. \tag{2.12}$$

We will also denote by C a generic positve constant that depend on the domain D.

To simplify the notation, we set (without loss of generality) $\nu = \mathcal{K} = \alpha = \epsilon = 1$. Using the above defined operators and assuming that the external forcing term acts as a control we write the controlled system (1.1)–(1.2) as follows:

$$\begin{cases} d\mathrm{u} + (\mathrm{A}_0\mathrm{u} + B_0(\mathrm{u}, \mathrm{u}) - R_0(\mathrm{A}_1\phi, \phi) - \Phi(t, \mathrm{u}))dt = \mathrm{g}_2(t, \mathrm{u})d\mathrm{W} \quad \text{in} \;\; \mathrm{V}_1^*, \\ \frac{\partial \phi}{\partial t} + \mathrm{A}_1\mu + B_1(\mathrm{u}, \phi) = 0, \quad \text{in} \;\; \mathrm{V}_2^*, \\ \mu = \mathrm{A}_1\phi + f(\phi), \quad (\mathrm{u}, \phi)(0) = (\mathrm{u}_0, \phi_0), \end{cases} \tag{2.13}$$

$\mathbb{P}$-a-s., and for all t$\in [0, T]$.

Remark 2.1. For the sake of convenience, as in [34], we will replaced μ in $(2.13)_2$ by $\overline{\mu} = \mu - <\mu>$, that is $\overline{\mu} = \mathrm{A}_1\phi + f(\phi) - < f(\phi) >$, a.e., in $D \times (0, T)$. Obviously we have $< \overline{\mu}(t) >= 0$, $\forall\, t > 0$. □

For a separable Banach space X and $p \in [1, \infty]$, we denote by $\mathcal{M}^p_{\mathcal{F}_t}(0, T; X)$ the space of all process $\rho \in L^p(\Omega \times (0, T), d\mathbb{P} \times dt; X)$ that are $\mathcal{F}_t$-progressively measurable. The space $\mathcal{M}^p_{\mathcal{F}_t}(0, T; X)$ is Banach subspace of $L^p(\Omega \times (0, T), d\mathbb{P} \times dt; X)$. We will also denote by $L^p(\Omega; \mathcal{C}([0, T]; X))$, for $1 \leq p < \infty$, the space of all continuous and $\mathcal{F}_t$-progressively measurable $X-$valued processes $\{\rho_t; 0 \leq t \leq T\}$, satisfying

$$\mathbb{E}\Big(\sup_{t \in [0,T]} \|\rho_t\|^p_X\Big) < \infty,$$

and $\mathcal{M}^p(0, T; X)$ space of all $\mathcal{B}([0, T])$-measurable function $h : [0, T] \to X$ with

$$\int_0^T \|h(t)\|^p_X \, dt < \infty.$$

2.1. Assumptions

First, we state the conditions imposed on the stochastic evolution equation that will be considered:

(I) $(\mathbb{V}, \mathbb{Y}, \mathbb{V}^*)$ is an evolution triple (see [29], p. 97) and the embedding operator $\mathbb{V} \hookrightarrow \mathbb{Y}$ and $\mathbb{V} \hookrightarrow \mathbb{H}$ are assumed to be compact i.e., the embedding operator $\mathrm{V}_1 \hookrightarrow \mathrm{H}_1$ and $\mathrm{V}_2 \hookrightarrow \mathrm{H}_2$ are assumed to be compact.

(II) $\mathrm{g}_2 : [0,T] \times \mathrm{H}_1 \to \mathrm{H}_1$ is a mapping such that:
 (a) $|\mathrm{g}_2(t,\mathrm{u}) - \mathrm{g}_2(t,\mathrm{v})|_{L^2}^2 \leq \lambda\,|\mathrm{u}-\mathrm{v}|_{L^2}^2$ for all $t\in[0,T]$, $\mathrm{u},\mathrm{v}\in\mathrm{H}_1$, where λ is a positive constant;
 (b) $\mathrm{g}_2(t,0) = 0$, for all $t \in [0,T]$;
 (c) $\mathrm{g}_2(.,\mathrm{u}) \in \mathcal{M}^2(0,T;\mathrm{H}_1)$, $\mathrm{u} \in \mathrm{H}_1$.

(III) $\Phi : [0,T] \times \mathrm{H}_1 \to \mathrm{H}_1$ is a mapping such that:
 (a) $|\Phi(t,\mathrm{u}) - \Phi(t,\mathrm{v})|_{L^2}^2 \leq \kappa\,|\mathrm{u}-\mathrm{v}|_{L^2}^2$ for all $t\in[0,T]$, $\mathrm{u},\mathrm{v}\in\mathrm{H}_1$, where κ is a positive constant;
 (b) $\Phi(t,0) = 0$, for all $t \in [0,T]$;
 (c) $\Phi(.,\mathrm{u}) \in \mathcal{M}^2(0,T;\mathrm{H}_1)$, $\mathrm{u} \in \mathrm{H}_1$.

(IV) $(\mathrm{u}_0,\phi_0) \in L^2(\Omega,\mathcal{F}_0,\mathbb{P},\mathbb{Y})$ and satisfy $\mathbb{E}[\mathcal{E}(\mathrm{u}_0,\phi_0)]^2 < \infty$.

In this work, we understand that the stochastic process (u,ϕ) is a solution to problem (2.13) in the following sense.

Definition 2.2. Let $T > 0$ be fixed. We call a process $(\mathrm{u}_\Phi(t),\phi_\Phi(t))_{t\in[0,T]}$ from the space $\mathcal{M}^2_{\mathcal{F}_t}(0,T;\mathbb{V})$ with $\mathbb{E}\left[\mathcal{E}(\mathrm{u}_\Phi,\phi_\Phi)(t)\right] < \infty$ for all $t \in [0,T]$ a solution of the stochastic Cahn-Hilliard Navier-Stokes system if it satisfies the system:

$$\begin{cases} (\mathrm{u}_\Phi(t),\mathrm{v}) = (\mathrm{u}_0,\mathrm{v}) - \int_0^t \langle \mathrm{A}_0\mathrm{u}_\Phi(s) + B_0(\mathrm{u}_\Phi(s),\mathrm{u}_\Phi(s)),\mathrm{v}\rangle\, ds \\ \qquad + \int_0^t \langle R_0(\mathrm{A}_1\phi_\Phi(s),\phi_\Phi(s)),\mathrm{v}\rangle\, ds + \int_0^t (\Phi(s,\mathrm{u}_\Phi(s)),\mathrm{v})ds \\ \qquad + \int_0^t (\mathrm{g}_2(s,\mathrm{u}_\Phi(s)),\mathrm{v})dW(s), \\ (\phi_\Phi(t),\psi) = (\phi_0,\psi) - \int_0^t \langle \mathrm{A}_1\mu_\Phi(s) + B_1(\mathrm{u}_\Phi(s),\phi_\Phi(s)),\psi\rangle\, ds, \\ \mu_\Phi = \mathrm{A}_1\phi_\Phi + \mathrm{f}(\phi_\Phi), \end{cases} \tag{2.14}$$

for all $t \in [0,T]$, $\mathrm{v} \in \mathrm{V}_1$, $\psi \in \mathrm{V}_2$ and $\mathbb{P}$-a.s., where the stochastic integral is understood in the Itô sense.

Remark 2.3. As in [29], the condition $\mathrm{g}_2(t,0) = 0$, is given only to simplify the calculations. It can be omitted, in which case one can use the estimate $|\mathrm{g}_2(s,\mathrm{u})|_{L^2}^2 \leq 2\lambda\,|\mathrm{u}|_{L^2}^2 + 2\,|\mathrm{g}_2(s,0)|_{L^2}^2$ that follows from the Lipschitz condition. The same remark also holds for Φ. □

We need the following existence of solutions theorem.

Theorem 2.4. *Suppose that the hypotheses* (I)–(IV) *hold. Then, problem* (2.14) *has a solution* $(u_\Phi,\phi_\Phi) \in \mathcal{M}^2_{\mathcal{F}_t}(0,T;\mathbb{V})$, *which is almost surely unique and has almost surely continuous trajectories in* $\mathbb{Y}$. *Moreover, there exists a positive constant* C *(depending on* λ,κ, T, u_0,ϕ_0*) such that:*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |(u_\Phi,\phi_\Phi)(t)|_{\mathbb{Y}}^2 + \int_0^T ||(u_\Phi,\phi_\Phi)(s)||_{\mathbb{V}}^2 ds\Big] \leq C, \tag{2.15}$$

and

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |(u_\Phi,\phi_\Phi)(t)|_{\mathbb{Y}}^4\Big] + \mathbb{E}\Big[\int_0^T ||(u_\Phi,\phi_\Phi)(s)||_{\mathbb{V}}^2 ds\Big]^2 \leq C, \tag{2.16}$$

where C *is a positive constant* (*depending on* $\kappa,\lambda,T,u_0,\phi_0$).

Proof. The proof of the result can be found in [34] (see Propositions 3.2 and 3.3). □

Let $\{(w_i,\psi_i),\ i=1,2,3,...\}\subset\mathbb{V}$ be an orthonormal basis of the space $\mathbb{Y}$, where $\{w_i,\ i=1,2,3,...\}$, $\{\psi_i,\ i=1,2,3,...\}$ are eigenvectors of A_0 and A_1 respectively. For each $n\in\mathbb{N}$, we set $\mathbb{V}_n=\mathbb{Y}_n=\mathrm{span}\{(w_1,\psi_1),...,(w_n,\psi_n)\}$. Hereafter, we denote by $\mathcal{P}_n^1$ (respectively $\mathcal{P}_n^2$) the L^2-orthogonal projection from H_1 (respectively H_2) onto $\mathrm{V}_1^n=\mathrm{H}_{1,n}=\mathrm{span}\{w_1,...,w_n\}$ (respectively $\mathrm{V}_2^n=\mathrm{H}_{2,n}=\mathrm{span}\{\psi_1,...,\psi_n\}$), i.e.,

$$\mathcal{P}_n^1 w=\sum_{i=1}^n (w,w_i)w_i,\quad \mathcal{P}_n^2\psi=\sum_{i=1}^n(\psi,\psi_i)\psi_i.$$

For each $n=1,2,3,...$, we consider the sequence of finite dimensional evolution systems:

$$\begin{aligned}
&(\mathrm{u}_{n,\mathcal{P}_n^1\Phi}(t),\mathrm{v})=(\mathrm{u}_{0n},\mathrm{v})-\int_0^t\left\langle \mathrm{A}_{0n}\mathrm{u}_{n,\mathcal{P}_n^1\Phi}(s)+B_{0n}(\mathrm{u}_{n,\mathcal{P}_n^1\Phi}(s),\mathrm{u}_{n,\mathcal{P}_n^1\Phi}(s)),\mathrm{v}\right\rangle ds\\
&\qquad+\int_0^t\left\langle R_{0n}(\phi_{n,\mathcal{P}_n^1\Phi}(s),\phi_{n,\mathcal{P}_n^1\Phi}(s)),\mathrm{v}\right\rangle ds+\int_0^t\mathcal{P}_n^1(\Phi(s,\mathrm{u}_{n,\mathcal{P}_n^1\Phi}(s)),\mathrm{v})ds\\
&\qquad+\int_0^t(\mathrm{g}_{2n}(s,\mathrm{u}_{n,\mathcal{P}_n^1\Phi}(s)),\mathrm{v})dW(s), \qquad\qquad (2.17)\\
&(\phi_{n,\mathcal{P}_n^1\Phi}(t),\psi)=(\phi_{0n},\psi)-\int_0^t\left\langle \mathrm{A}_{1n}\mu_{n,\mathcal{P}_n^1\Phi}(s)+B_{1n}(\mathrm{u}_{n,\mathcal{P}_n^1\Phi}(s),\phi_{n,\mathcal{P}_n^1\Phi}(s)),\psi\right\rangle ds,\\
&\mu_{n,\mathcal{P}_n^1\Phi}=\mathrm{A}_{1n}\phi_{n,\mathcal{P}_n^1\Phi}+\mathrm{f}(\phi_{n,\mathcal{P}_n^1\Phi}),
\end{aligned}$$

for all $(\mathrm{w},\psi)\in\mathbb{V}_n,\ t\in[0,T]$ and $\mathbb{P}$-a.s. where $\Pi_n=(\mathcal{P}_n^1,\mathcal{P}_n^2)$ is the orthogonal projection of $\mathbb{Y}$ onto $\mathbb{Y}_n$

Theorem 2.5. *Suppose that the hypotheses* (I)–(IV) *hold. Then for each $n\in\mathbb{N}$, system* (2.17) *has a solution $(u_{n,\mathcal{P}_n^1\Phi},\phi_{n,\mathcal{P}_n^1\Phi})\in\mathcal{M}^2_{\mathcal{F}_t}(0,T;\mathbb{V})$, which is unique almost surely and has almost surely continuous trajectories in $\mathbb{Y}$. Moreover, we have:*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|(u_{n,\mathcal{P}_n^1\Phi},\phi_{n,\mathcal{P}_n^1\Phi})(t)|^2_{\mathbb{Y}}+\int_0^T||(u_{n,\mathcal{P}_n^1\Phi},\phi_{n,\mathcal{P}_n^1\Phi})(s)||^2_{\mathbb{V}}ds\Big]\le C,$$

and $$\mathbb{E}\Big[\sup_{t\in[0,T]}\left|(u_{n,\mathcal{P}_n^1\Phi},\phi_{n,\mathcal{P}_n^1\Phi})(t)\right|^4_{\mathbb{Y}}\Big]+\mathbb{E}\Big[\int_0^T\left\|(u_{n,\mathcal{P}_n^1\Phi},\phi_{n,\mathcal{P}_n^1\Phi})(s)\right\|^2_{\mathbb{V}}ds\Big]^2\le C,$$

where C is a positive constant (depending on $\kappa,\lambda,T,u_0,\phi_0$).

Proof. See the proof of Proposition 3.3 in [34], p. 1040. □

Remark 2.6. Let $(Q(t))_{t\in[0,T]}$ be a process in the space $\mathcal{M}^2_{\mathcal{F}_t}(0,T;\mathbb{V})$ and we define $Q_n=\Pi_nQ$. Using the properties of A_0 and A_1, we have (see [34], p. 1044)

$$\begin{aligned}
&\|Q_n(t)\|^2_{\mathbb{V}}\le C\,\|Q(t)\|^2_{\mathbb{V}}, && |Q_n(t)|^2_{\mathbb{Y}}\le|Q(t)|^2_{\mathbb{Y}},\\
&\|Q(t)-Q_n(t)\|^2_{\mathbb{V}}\le C\,\|Q(t)\|^2_{\mathbb{V}}, && |Q(t)-Q_n(t)|^2_{\mathbb{Y}}\le|Q(t)|^2_{\mathbb{Y}}.
\end{aligned}\qquad (2.18)$$

Hence, for $\mathbb{P}\times[0,T]$ a.e $(\omega,t)\in\Omega\times[0,T]$ we have

$$\lim_{n\to\infty}\|Q(\omega,t)-Q_n(\omega,t)\|^2_{\mathbb{V}}=0.$$

By the Lebesgue Dominated convergence Theorem it follows that

$$\lim_{n\to\infty}\int_0^T \|Q(t)-Q_n(t)\|_{\mathbb{V}}^2 = 0 \tag{2.19}$$

and

$$\lim_{n\to\infty}\mathbb{E}\int_0^T \|Q(t)-Q_n(t)\|_{\mathbb{V}}^2 = 0. \tag{2.20}$$

If the process $(Q(t))_{t\in[0,T]}$ has almost surely continuous trajectories in $\mathbb{Y}$, then we have

$$\lim_{n\to\infty}|Q(T)-Q_n(T)|_{\mathbb{Y}}^2 = 0, \quad \text{for a.e.}, \omega\in\Omega. \tag{2.21}$$

3. Formulation of the control problem and existence of optimal controls

3.1. Formulation of the control problem

We consider the stochastic Cahn-Hilliard Navier-Stokes system controlled by continuous feedback controls. We denote by $\mathcal{U}$ the set of all admissible controls, which we assume to be the set of all functions $\Phi : [0;T]\times \mathrm{H}_1 \to \mathrm{H}_1$ satisfying the following conditions:

$$|\Phi(0,0)|_{L^2}^2 \leq \eta, \tag{3.1}$$

and for all $s,t\in[0,T]$, $x,y\in \mathrm{H}_1$

$$|\Phi(s,x)-\Phi(t,y)|_{L^2}^2 \leq \alpha|s-t|^2+\kappa|x-y|_{L^2}^2, \tag{3.2}$$

where α,κ,η are given constants. We consider the following cost functional:

$$\mathcal{J}(\Phi) = \mathbb{E}\int_0^T (\mathcal{L}[s,\mathrm{u}_\Phi(s),\phi_\Phi(s)]+\mathcal{K}_1[\Phi(s,\mathrm{u}_\Phi(s))])ds + \mathbb{E}\mathcal{K}_2[\mathrm{u}_\Phi(T),\phi_\Phi(T)], \tag{3.3}$$

$\Phi\in\mathcal{U}$, $\mathcal{L} : [0,T]\times\mathbb{V}\to\mathbb{R}_+$, $\mathcal{K}_1 : \mathrm{H}_1\to\mathbb{R}_+$, $\mathcal{K}_2 : \mathbb{H}\to\mathbb{R}_+$. We assume that the mappings $\mathcal{K}_1,\mathcal{K}_2$ and $(\mathrm{u},\phi)\in\mathcal{M}^2(0,T;\mathbb{V})\mapsto\int_0^T(\mathcal{L}[s,\mathrm{u}(s),\phi(s)]ds\in\mathbb{R}_+$ are weak and sequentially lower semicontinuous and the integral in (3.3) exists and is finite.

Our control problem is to minimize (3.3) over $\mathcal{U}$, we denote by $(\mathcal{P})$ the problem of minimizing $\mathcal{J}$ among the admissible controls. Any $\Phi^*\in\mathcal{U}$ satisfying $\mathcal{J}(\Phi^*) = \inf\{\mathcal{J}(\Phi) : \Phi\in\mathcal{U}\}$ is called an optimal control and $(\mathrm{u}_{\Phi^*},\phi_{\Phi^*})$ is called state variable.

Remark 3.1. For each $\Phi\in\mathcal{U}$ and $\mathrm{u}\in\mathcal{M}^2(0,T;\mathrm{H}_1)$, we have (see [26] or [29])

$$\int_0^T |\Phi(s,\mathrm{u}(s))|_{L^2}^2\,ds \leq C_3\Big(1+\int_0^T|\mathrm{u}(s)|_{L^2}^2\,ds\Big),$$

where $C_3>0$ depends on α,κ,η,T and is independent of the choice of Φ and u. The assumption (III)(b) on Φ (see Section 2) is replaced by (3.1), but the Theorems 2.5 and 2.4 still hold with the only modification that the constants appearing there depend also on η. □

3.2. Existence of optimal controls

In this subsection, we prove the existence of optimal feedback controls for the optimization problem $(\mathcal{P})$.

Let (Φ_n) be a minimizing sequence for problem $(\mathcal{P})$. The following Lemma 3.2 proves that given a minimising sequence for the problem $(\mathcal{P})$, we can obtain a subsequence and a mapping $\Phi \in \mathcal{U}$, such that the subsequence converges weakly to Φ.

Lemma 3.2. *Let (Φ_n) be a minimising sequence for the problem $(\mathcal{P})$. There exists a subsequence (n') of (n) and a mapping $\Phi \in \mathcal{U}$ such that for all $t \in [0,T], x; y \in H_1$ we have*

$$\lim_{n'\to\infty} (\Phi_{n'}(t,x), y) = (\Phi(t,x), y) \tag{3.4}$$

Proof. See Lemma 4.1 in [26] □

For convenience, in the following we will denote the subsequence of indices (n') obtained in Lemma 3.2 by (n).

For $n = 1, 2, ...$, we consider the evolution system:

$$\begin{cases} (\widetilde{\mathrm{u}}_{\Phi_n}(t), \mathrm{v}) = (\mathrm{u}_0, \mathrm{v}) - \int_0^t < \mathrm{A}_0\widetilde{\mathrm{u}}_{\Phi_n}(s) + B_0(\mathrm{u}_\Phi(s), \widetilde{\mathrm{u}}_{\Phi_n}(s)), \mathrm{v} > ds \\ \qquad\qquad + \int_0^t < R_0(\mathrm{A}_1\widetilde{\phi}_{\Phi_n}(s), \phi_\Phi(s)), \mathrm{v} > ds \\ \qquad\qquad + \int_0^t (\Phi_n(s, \mathrm{u}_\Phi(s)), \mathrm{v}) ds + \int_0^t (\mathrm{g}_2(s, \mathrm{u}_\Phi(s)), \mathrm{v}) dW(s), \\ (\widetilde{\phi}_{\Phi_n}(t), \psi) = (\phi_0, \psi) - \int_0^t < \mathrm{A}_1\widetilde{\mu}_{\Phi_n}(s) + B_1(\widetilde{\mathrm{u}}_{\Phi_n}(s), \phi_\Phi(s)), \psi > ds, \\ \widetilde{\mu}_{\Phi_n} = \mathrm{A}_1\widetilde{\phi}_{\Phi_n} + \mathrm{f}(\widetilde{\phi}_{\Phi_n}), \end{cases} \tag{3.5}$$

for all $t \in [0,T]$, $\mathrm{v} \in \mathrm{V}_1$, $\psi \in \mathrm{V}_2$ and a.e. $\omega \in \Omega$.

The following result proves the unique solvability of the scheme (3.5).

Theorem 3.3. *For each $n \geq 1$, problem (3.5) has with probability one a unique solution $(\widetilde{u}_{\Phi_n}, \widetilde{\phi}_{\Phi_n}) \in L^2(\Omega \times [0,T]; \mathbb{V})$ and $(\widetilde{u}_{\Phi_n}, \widetilde{\phi}_{\Phi_n})$ has almost surely continuous trajectories in $\mathbb{H}$.*

3.3. Proof of Theorem 3.3

In this subsection, we prove the existence and uniqueness of problem (3.5). We prove Theorem 3.3 by induction. Indeed, we apply successively $(\widetilde{\mathrm{u}}_{\Phi_n}, \widetilde{\phi}_{\Phi_n}, \widetilde{\mu}_{\Phi_n}) = (\mathrm{U}, \Psi, \mu)$ and $\Phi_n = X$. The results of Lemma 3.5 and Lemma 3.7 are given below.

Now, let $(\mathrm{u}_\Phi, \phi_\Phi) \in L^2(\Omega \times [0,T]; \mathbb{V})$ be an arbitrary process fulfilling the condition $(\mathrm{u}_\Phi, \phi_\Phi) \in L^4(\Omega, \mathcal{C}([0,T]; \mathbb{Y}))$, i.e., u_Φ and ϕ_Φ are $\mathbb{Y}$-valued continuous functions and $\mathbb{E} \sup_{s\in[0,T]} |(\mathrm{u}_\Phi, \phi_\Phi)(s)|_{\mathbb{Y}}^4 < \infty$. We consider the following evolution system:

$$\begin{cases} (\mathrm{U}(t), \mathrm{v}) + \int_0^t \langle \mathrm{A}_0\mathrm{U}(s), \mathrm{v}\rangle\, ds + \int_0^t \langle B_0(\mathrm{u}_\Phi(s), \mathrm{U}(s)), \mathrm{v}\rangle\, ds = (\mathrm{u}_0, \mathrm{v}) \\ \quad + \int_0^t \langle \mathrm{R}_0(\mathrm{A}_1\Psi(s), \phi_\Phi(s)), \mathrm{v}\rangle\, ds + \int_0^t (X(s, \mathrm{u}_\Phi(s)), \mathrm{v}) ds + \int_0^t (\mathrm{g}_2(s, \mathrm{u}_\Phi(s)), \mathrm{v})\, dW(s), \\ (\Psi(t), \psi) + \int_0^t \langle \mathrm{A}_1\mu(s), \psi\rangle\, ds + \int_0^t \langle \mathrm{B}_1(\mathrm{U}(s), \phi_\Phi(s)), \psi\rangle\, ds = (\phi_0, \psi), \\ \mu = \mathrm{A}_1\Psi + \mathrm{f}(\Psi), \end{cases} \tag{3.6}$$

for all $t \in [0,T]$, $\mathrm{v} \in \mathrm{V}_1$, $\psi \in \mathrm{V}_2$, and $(\mathrm{U}_0(t), \Psi_0(t)) = (0,0)$ for all $t \in [0,T]$ and a.e. $\omega \in \Omega$.

We now give the definition of solution of problem (3.6).

Definition 3.4. For any fixed process $(\mathrm{u}_\Phi, \phi_\Phi) \in L^2(\Omega\times[0,T];\mathbb{V})$ such that $(\mathrm{u}_\Phi, \phi_\Phi)(t)$ is $\mathcal{F}_t$-adapted for all $t \in [0,T]$, $(\mathrm{u}_\Phi, \phi_\Phi) \in L^4(\Omega, \mathcal{C}([0,T];\mathbb{Y}))$, a solution of problem (3.6) is an adapted process $(\mathrm{U}, \Psi) \in L^2(\Omega \times [0,T];\mathbb{V})$ satisfying (3.6).

Lemma 3.5. *If* $(U, \Psi) \in L^2(\Omega \times [0,T];\mathbb{V})$ *is a solution of problem* (3.6)*, then it is almost surely unique.*

Proof. We assume that (U_1, Ψ_1) and (U_2, Ψ_2) are two solutions of (3.6). We set $(\mathrm{U}, \Psi, \mu) = (\mathrm{U}_1, \Psi_1, \mu_1) - (\mathrm{U}_2, \Psi_2, \mu_2)$. Then, (U, Ψ) satisfies evolution system:

$$\begin{cases} (\mathrm{U}(t), \mathrm{v}) + \int_0^t \langle \mathrm{A}_0\mathrm{U}(s), \mathrm{v}\rangle\, ds + \int_0^t \langle B_0(\mathrm{u}_\Phi(s), \mathrm{U}(s)), \mathrm{v}\rangle\, ds \\ \qquad = \int_0^t \langle \mathrm{R}_0(\mathrm{A}_1\Psi(s), \phi_\Phi(s)), \mathrm{v}\rangle\, ds \\ (\Psi(t), \psi) + \int_0^t \langle \mathrm{A}_1\mu(s), \psi\rangle\, ds + \int_0^t \langle \mathrm{B}_1(\mathrm{U}(s), \phi_\Phi(s)), \psi\rangle\, ds = 0, \\ \mu = \mathrm{A}_1\Psi + \mathrm{f}(\Psi_1) - \mathrm{f}(\Psi_2), \quad (\mathrm{U}, \Psi)(0) = (0,0) \end{cases} \tag{3.7}$$

for all $t \in [0,T], (\mathrm{v}, \psi) \in \mathrm{V}_1 \times \mathrm{V}_2$ and a.e. $\omega \in \Omega$. Let us take test function U as v in $(3.7)_1$ to obtain

$$|\mathrm{U}(t)|_{L^2}^2 + 2\int_0^t \|\mathrm{U}(s)\|^2\, ds = 2\int_0^t \langle \mathrm{R}_0(\mathrm{A}_1\Psi(s), \phi_\Phi(s)), \mathrm{U}\rangle\, ds \tag{3.8}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$, where we used the fact that $b_0(\mathrm{u}_\Phi, \mathrm{U}, \mathrm{U}) = 0$.

Let us choose $\psi = \Psi$ in $(3.7)_2$, we obtain

$$|\Psi(t)|_{L^2}^2 + 2\int_0^t (\mathrm{A}_1\mu(s), \Psi)\, ds + 2\int_0^t (\mathrm{B}_1(\mathrm{U}(s), \phi_\Phi(s)), \Psi)\, ds = 0 \tag{3.9}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$.

Adding the corresponding relation (3.9) and (3.8) together yields the following evolution equations for $|\mathrm{U}(t)|^2$ and $|\Psi(t)|_{L^2}^2$, we obtain for all $t \in [0,T]$ and a.e. $\omega \in \Omega$

$$\begin{aligned} &|\mathrm{U}(t)|_{L^2}^2 + |\Psi(t)|_{L^2}^2 + 2\int_0^t \|\mathrm{U}(s)\|^2\, ds + 2\int_0^t (\mathrm{A}_1\mu(s), \Psi)\, ds \\ &+ 2\int_0^t (\mathrm{B}_1(\mathrm{U}(s), \phi_\Phi(s)), \Psi)\, ds = 2\int_0^t \langle \mathrm{R}_0(\mathrm{A}_1\Psi(s), \phi_\Phi(s)), \mathrm{U}\rangle\, ds. \end{aligned} \tag{3.10}$$

We estimate the terms in (3.10) as follows. Observe that

$$\begin{aligned} -(\mathrm{A}_1\mu(s), \Psi) &= -(\mathrm{A}_1(\mathrm{A}_1\Psi + \mathrm{f}(\Psi_1) - \mathrm{f}(\Psi_2)), \Psi) \\ &= -|\mathrm{A}_1\Psi|_{L^2}^2 - (\mathrm{A}_1(\mathrm{f}(\Psi_1) - \mathrm{f}(\Psi_2)), \Psi) \end{aligned} \tag{3.11}$$

and

$$|(\mathrm{A}_1(\mathrm{f}(\Psi_1) - \mathrm{f}(\Psi_2)), \Psi)| \le |\mathrm{f}(\Psi_1) - \mathrm{f}(\Psi_2)|\, |\mathrm{A}_1\Psi| \le \frac{1}{4}\, |\mathrm{A}_1\Psi|_{L^2}^2 + c_f\, |\Psi|_{L^2}^2. \tag{3.12}$$

$$\begin{aligned}\text{Note that}\quad &|\langle \mathrm{R}_0(\mathrm{A}_1\Psi(s),\phi_\Phi(s)),\mathrm{U}\rangle| = |\langle \mathrm{B}_1(\mathrm{U}(s),\phi_\Phi(s)),\mathrm{A}_1\Psi\rangle|\\ &\le C\,|\mathrm{U}(s)|^{1/2}\,\|\mathrm{U}(s)\|^{1/2}\,\|\phi_\Phi(s)\|^{1/2}\,|\mathrm{A}_1\phi_\Phi(s)|_{L^2}^{1/2}\,|\mathrm{A}_1\Psi(s)|_{L^2} \qquad (3.13)\\ &\le \frac14\,|\mathrm{A}_1\Psi|_{L^2}^2+\frac14\,\|\mathrm{U}(s)\|^2+C\,\|\phi_\Phi(s)\|^2\,|\mathrm{A}_1\phi_\Phi(s)|_{L^2}^2\,|\mathrm{U}(s)|^2.\end{aligned}$$

Using Hölder's inequality and Agmon's inequality, we get

$$\begin{aligned}|(\mathrm{B}_1(\mathrm{U}(s),\phi_\Phi(s)),\Psi)| &\le C\,\|\mathrm{U}(s)\|\,|\nabla\phi_\Phi(s)|_{L^\infty}\,|\Psi(s)|_{L^2}\\ &\le \frac14\,\|\mathrm{U}(s)\|^2+C\,|\Psi(s)|_{L^2}^2\,(\|\phi_\Phi(s)\|^2+|\phi_\Phi(s)|_{H^3}^2).\end{aligned}\tag{3.14}$$

Substituting (3.11)–(3.14) in (3.10), we get

$$\begin{aligned}|\mathrm{U}(t)|_{L^2}^2+|\Psi(t)|_{L^2}^2&+\int_0^t\|\mathrm{U}(s)\|^2\,ds+\int_0^t|\mathrm{A}_1\Psi(s)|_{L^2}^2\,ds\\ &\le C\int_0^t\mathcal{K}_1(s)\,|\mathrm{U}(s)|_{L^2}^2\,ds+C\int_0^t\mathcal{K}_2(s)\,|\Psi(s)|_{L^2}^2\,ds,\end{aligned}\tag{3.15}$$

for all $t\in[0,T]$ and a.e. $\omega\in\Omega$, where

$$\mathcal{K}_1=\|\phi_\Phi(s)\|^2\,|\mathrm{A}_1\phi_\Phi(s)|_{L^2}^2\quad\text{and}\quad\mathcal{K}_2=(\|\phi_\Phi(s)\|^2+|\phi_\Phi(s)|_{H^3}^2)+c_f.$$

Now, thanks to the Theorem 2.4 and applying the Gronwall lemma to (3.15), we derive that for all $t\in[0,T]$ and a.e. $\omega\in\Omega$.

$$|\mathrm{U}(t)|_{L^2}^2+|\Psi(t)|_{L^2}^2=0.\tag{3.16}$$

Hence
$$\mathbb{P}\Big[\sup_{t\in[0,T]}\big(|\mathrm{U}(t)|_{L^2}^2+|\Psi(t)|_{L^2}^2\big)=0\Big]=1.$$

We then conclude that problem (3.6) has at most one solution. □

To prove the existence of a solution of problem (3.6), we will consider a sequence of more simple equations, whose solutions converge a.s. to the solution of problem (3.6).

We approximate the solution (U,Ψ) of equation (3.6) by the sequence $(\mathrm{U}_n,\Psi_n)_{n\ge1}$, where $(\mathrm{U}_n,\Psi_n)_{n\ge1}$ is the solution of the evolution system

$$\begin{cases}(\mathrm{U}_n(t),\mathrm{v})+\int_0^t\langle \mathrm{A}_0\mathrm{U}_n(s),\mathrm{v}\rangle\,ds+\int_0^t\langle B_0(\mathrm{u}_\Phi(s),\mathrm{U}_{n-1}(s)),\mathrm{v}\rangle\,ds=(\mathrm{u}_0,\mathrm{v})\\ +\int_0^t\langle \mathrm{R}_0(\mathrm{A}_1\Psi_{n-1}(s),\phi_\Phi(s)),\mathrm{v}\rangle\,ds+\int_0^t(X(s,\mathrm{u}_\Phi(s)),\mathrm{v})ds\\ +\int_0^t(\mathrm{g}_2(s,\mathrm{u}_\Phi(s)),\mathrm{v})dW(s),\\ (\Psi_n(t),\psi)+\int_0^t\langle \mathrm{A}_1\mu_n(s),\psi\rangle\,ds+\int_0^t\langle \mathrm{B}_1(\mathrm{U}_{n-1}(s),\phi_\Phi(s)),\Psi\rangle\,ds=(\phi_0,\psi),\\ \mu_n=\mathrm{A}_1\Psi_n+f(\Psi_n),\end{cases}\tag{3.17}$$

for all $t\in[0,T]$, $(\mathrm{v},\psi)\in\mathrm{V}_1\times\mathrm{V}_2$ and a.e. $\omega\in\Omega$, where $(\mathrm{U}_0(t),\Psi_0(t))=(0,0)$ for all $t\in[0,T]$ and a.e. $\omega\in\Omega$.

The following result gives the existence and uniqueness of problem (3.17).

Lemma 3.6. *For each $n\ge1$, there exist a unique $\mathbb{H}$-valued adapted process (U_n,Ψ_n) which is a solution of* (3.17) *such that $(U_n,\Psi_n)\in L^2(0,T;\mathbb{V})$ for a.e. $\omega\in\Omega$.*

Proof. Let (v,y) be the solution of the following evolution system:

$$\begin{cases}(\xi,\mathrm{v})+\int_0^t\langle \mathrm{A}_0\xi,\mathrm{v}\rangle=(\mathrm{u}_0,\mathrm{v})+\int_0^t(X(s,\mathrm{u}_\Phi(s)),\mathrm{v})ds+\int_0^t(\mathrm{g}_2(s,\mathrm{u}_\Phi(s)),\mathrm{v})\,dW(s),\\ (\mathrm{y},\psi)+\int_0^t\langle \mathrm{A}_1\mu_1(s),\psi\rangle=(\phi_0,\psi),\\ \mu_1=\mathrm{A}_1\mathrm{y}+f(\mathrm{y}),\end{cases}\tag{3.18}$$

for all $(\mathrm{v},\psi)\in V_1\times V_2$, $t\in[0,T]$ and a.e. $\omega\in\Omega$. Then, the existence of the solution $(\xi,\mathrm{y})\in L^2(\Omega\times[0,T];\mathbb{V})$ can be achieved similary as in [28] or [34], and it is not difficult to prove that these solution are in fact unique. We can also check that has almost surely continuous trajectories in $\mathbb{H}$ and that $\mathbb{E}\sup_{t\in[0,T]}|(\xi,\mathrm{y})(t)|^2_{\mathbb{H}}<\infty$.

Now, for the proof Lemma 3.6 we use the method of induction. For $n=1$, we have $(\xi,\mathrm{y})=(\mathrm{U}_1,\Psi_1)$ and the conlusion of our Lemma follows from the properties of (ξ,y).

Let n be such $(\mathrm{U}_{n-1},\Psi_{n-1})\in L^2(0,T;\mathbb{V})$ a.e. $\omega\in\Omega$. $(\mathrm{U}_{n-1},\Psi_{n-1})$ is the solution of (3.17) and it has almost surely trajectories in $\mathbb{Y}$.

Let $(\widetilde{\mathrm{U}}_n,\widetilde{\Psi}_n)(t)=(\mathrm{U}_n,\Psi_n)(t)-(\xi,\mathrm{y})(t)$, for all $t\in[0,T]$ and a.e $\omega\in\Omega$. $(\widetilde{\mathrm{U}}_n,\widetilde{\Psi}_n)$ satisfies

$$\begin{cases}(\widetilde{\mathrm{U}}_n(t),\mathrm{v})+\nu\int_0^t\langle \mathrm{A}_0\widetilde{\mathrm{U}}_n(s),\mathrm{v}\rangle ds+\int_0^t\langle B_0(\mathrm{u}_\Phi(s),\mathrm{U}_{n-1}(s)),\mathrm{v}\rangle\,ds\\ \qquad=\int_0^t\langle R_0(\mathrm{A}_1\Psi_{n-1}(s),\phi_\Phi(s)),\mathrm{v}\rangle\,ds,\\ (\widetilde{\Psi}_n(t),\psi)+\int_0^t\langle \mathrm{A}_1\widetilde{\mu_n}(s),\psi\rangle\,ds+\int_0^t\langle \mathrm{B}_1(\mathrm{U}_{n-1}(s),\phi_\Phi(s)),\Psi\rangle\,ds=0,\\ \widetilde{\mu_n}=\mathrm{A}_1\widetilde{\Psi}_n+f(\Psi_n)-f(\mathrm{y}),\end{cases}\tag{3.19}$$

for all $t\in[0,T]$, $(\mathrm{v},\psi)\in \mathrm{V}_1\times\mathrm{V}_2$ and a.e. $\omega\in\Omega$. Since $(\mathrm{u}_\Phi,\phi_\Phi)\in L^2(\Omega\times[0,T];\mathbb{V})$, $\mathbb{E}\sup_{t\in[0,T]}|(\mathrm{u}_\Phi,\phi_\Phi)(t)|^4_{\mathbb{Y}}<\infty$ and $(\mathrm{U}_{n-1},\Psi_{n-1})$ has the properties mentioned above, we have for a.e. $\omega\in\Omega$

$$\begin{aligned}&\int_0^T\|B_0(\mathrm{u}_\Phi(t),\mathrm{U}_{n-1}(t))\|^2_{\mathrm{V}_1^*}\,dt\le C\int_0^T|\mathrm{u}_\Phi(t)|_{L^2}\,\|\mathrm{u}_\Phi(t)\|\,|\mathrm{U}_{n-1}(t)|_{L^2}\,\|\mathrm{U}_{n-1}(t)\|\,dt\\ &\le C\Big(\sup_{t\in[0,T]}|\mathrm{u}_\Phi(t)|^2_{L^2}\int_0^T\|\mathrm{u}_\Phi(t)\|^2\,dt\Big)^{1/2}\Big(\sup_{t\in[0,T]}|\mathrm{U}_{n-1}(t)|^2_{L^2}\int_0^T\|\mathrm{U}_{n-1}(t)\|^2\,dt\Big)^{1/2}<\infty\end{aligned}$$

for a.e. $\omega\in\Omega$, where we have also used the properties of B_0.
Thus, we have $B_0(\mathrm{u}_\Phi(s),\mathrm{U}_{n-1}(s))\in L^2(0,T;\mathrm{V}_1^*)$ for a.e. $\omega\in\Omega$.
Thanks to the properties of R_0 we have

$$\begin{aligned}&\int_0^T\|\mathrm{R}_0(\mathrm{A}_1\Psi_{n-1}(t),\phi_\Phi(t))\|^2_{\mathrm{V}_1^*}\,dt\le C\int_0^T|\mathrm{A}_1\phi_\Phi(t)|_{L^2}\,\|\phi_\Phi(t)\|\,|\mathrm{A}_1\Psi_{n-1}(t)|^2_{L^2}\,dt\\ &\le C\Big(\sup_{t\in[0,T]}\|\phi_\Phi(t)\|^2\int_0^T|\mathrm{A}_1\phi_\Phi(t)|^2_{L^2}\,dt\Big)^{1/2}\Big[\Big(\int_0^T|\mathrm{A}_1\Psi_{n-1}(t)|^2_{L^2}\,dt\Big)^2\Big]^{1/2}<\infty\\ &\text{for a.e. }\omega\in\Omega.\end{aligned}\tag{3.20}$$

Thus, it follows that $\mathrm{R}_0(\mathrm{A}_1\Psi_{n-1}(s),\phi_\Phi(s))\in L^2(0,T;\mathrm{V}_1^*)$ for a.e. $\omega\in\Omega$.

Similarly, as in (3.20), we also have

$$\int_0^T \|B_1(\mathrm{U}_{n-1}(t),\phi_\Phi(t))\|^2_{\mathrm{V}_2^*}\,dt \le C\int_0^T \|\phi_\Phi(t)\|^2\,|\mathrm{U}_{n-1}(t)|\,\|\mathrm{U}_{n-1}(t)\|\,dt$$
$$\le C\Big(\sup_{t\in[0,T]}\|\phi_\Phi(t)\|^4\Big)^{1/2}\Big(\sup_{t\in[0,T]}|\mathrm{U}_{n-1}(t)|^2_{L^2}\int_0^T\|\mathrm{U}_{n-1}(t)\|^2\,dt\Big)^{1/2}<\infty$$

for a.e. $\omega\in\Omega$. Therefore $B_1(\mathrm{U}_{n-1}(t),\phi_\Phi(t))\in L^2(0,T;\mathrm{V}_2^*)$. In consequence of $X\in L^2(\Omega\times[0,T];\mathrm{H}_1)$, we get $X\in L^2(0,T;\mathrm{H}_1)$ for a.e. $\omega\in\Omega$. We consider (3.19) over

$$\widetilde{\Omega}_n=\left\{\omega\in\Omega:\begin{array}{l}B_0(\mathrm{u}_\Phi,\mathrm{U}_{n-1}),\mathrm{R}_0(\mathrm{A}_1\Psi_{n-1}(s),\phi_\Phi(s))\in L^2(0,T;\mathrm{V}_1^*),\\ B_1(\mathrm{U}_{n-1}(t),\phi_\Phi(t))\in L^2(0,T;\mathrm{V}_2^*)\ \text{ and }\ X\in L^2(0,T;\mathrm{H}_1)\},\end{array}\right\}$$

for which $\mathbb{P}(\widetilde{\Omega}_n)=1$.

For each $\omega\in\widetilde{\Omega}_n$ the existence and uniqueness of $(\widetilde{\mathrm{U}}_n(.,\omega),\widetilde{\Psi}_n(.,\omega))$ follows from the deterministic theory [41], where $(\widetilde{\mathrm{U}}_n(.,\omega),\widetilde{\Psi}_n(.,\omega))$ is approximate by the solution of the Galerkin equations. Since the solutions of the Galerkin approximation are adapted and $\mathcal{F}\times\mathcal{B}_{[0,T]}$-measurable (because $(\mathrm{u}_\Phi,\phi_\Phi)$ and $(\mathrm{U}_{n-1},\Psi_{n-1})$ have the same properties), we infer that $(\widetilde{\mathrm{U}}_n,\widetilde{\Psi}_n)$ is adapted, $\mathcal{F}\times\mathcal{B}_{[0,T]}$-measurable and we have $(\widetilde{\mathrm{U}}_n,\widetilde{\Psi}_n)\in L^2(0,T;\mathbb{V})$, $\mathbb{P}$-a.e. It also follows that $(\widetilde{\mathrm{U}}_n,\widetilde{\Psi}_n)$ has almost surely continuous trajectories in $\mathbb{H}$.

The processes $(\widetilde{\mathrm{U}}_n,\widetilde{\Psi}_n)+(\xi,\mathrm{y})=(\mathrm{U}_n,\Psi_n)$ is adapted, $\mathcal{F}\times\mathcal{B}_{[0,T]}$-measurable and we have almost surely continuous in $\mathbb{H}$ and satisfy (3.19). This completes the proof of Lemma 3.6. □

Now, all the necessary tools are collected to give the proof the existence of solutions problem (3.6.

Lemma 3.7. *There exists a process* $(U,\Psi)\in L^2(\Omega\times[0,T];\mathbb{V})$ *satisfying problem* (3.6)*, which has almost surely continuous trajectories in* $\mathbb{H}$ *and we have:*

$$\mathbb{E}\left[\sup_{t\in[0,T]}|(U,\Psi)(t)|^4_{\mathbb{H}}\right]+\mathbb{E}\left[\int_0^T||(U,\Psi)(s)||^2_{\mathbb{V}}ds\right]^2\le C,\tag{3.21}$$

where C is a positive constant.

Proof. Let $n\ge 2$ arbitraty fixed and

$$\widetilde{\theta}_{n-i}=\mathrm{U}_{n-i}-\mathrm{U}_{n-i-1},\quad \widetilde{\Psi}_{n-i}=\Psi_{n-i}-\Psi_{n-i-1},\quad \widetilde{\mu}_{n-i}=\mu_{n-i}-\mu_{n-i-1}.$$

Then for all $(\mathrm{v},\psi)\in\mathrm{V}_1\times\mathrm{V}_2$, $t\in[0,T]$ and a.e. $\omega\in\Omega$, we have

$$\begin{cases}(\widetilde{\theta}_n(t),\mathrm{v})+\int_0^t\Big\langle\mathrm{A}_0\widetilde{\theta}_n(s),\mathrm{v}\Big\rangle ds+\int_0^t\Big\langle B_0(\mathrm{u}_\Phi(s),\widetilde{\theta}_{n-1}(s)),\mathrm{v}\Big\rangle ds\\ \qquad=\int_0^t\Big\langle\mathrm{R}_0(\mathrm{A}_1\widetilde{\Psi}_{n-1}(s),\phi_\Phi(s)),\mathrm{v}\Big\rangle ds,\\ (\widetilde{\Psi}_n(t),\psi)+\int_0^t\langle\mathrm{A}_1\widetilde{\mu}_n(s),\psi\rangle\,ds+\int_0^t\Big\langle B_1(\widetilde{\theta}_{n-1}(s),\phi_\Phi(s)),\psi\Big\rangle ds=0,\\ \widetilde{\mu}_n=\mathrm{A}_1\widetilde{\Psi}_n+f(\Psi_n)-f(\Psi_{n-1}),\end{cases}\tag{3.22}$$

Let us choose the function $\widetilde{\theta}_n$ as v in $(3.22)_1$ to obtain

$$\begin{aligned}&\left|\widetilde{\theta}_n(t)\right|^2_{L^2}+2\int_0^t\left\|\widetilde{\theta}_n(s)\right\|^2 ds+2\int_0^t\left\langle B_0(\mathrm{u}_\Phi(s),\widetilde{\theta}_{n-1}(s)),\widetilde{\theta}_n\right\rangle ds\\&=2\int_0^t\left\langle \mathrm{R}_0(\mathrm{A}_1\widetilde{\Psi}_{n-1}(s),\phi_\Phi(s)),\widetilde{\theta}_n\right\rangle ds.\end{aligned}\tag{3.23}$$

Now, we estimate each term of the above equality. To estimate $b_0(\mathrm{u}_\Phi,\widetilde{\theta}_{n-1}(s),\widetilde{\theta}_n(s))$, we use Hölder's, Ladyzhenskaya's and Young's inequalities to find

$$\begin{aligned}&2\left|b_0(\mathrm{u}_\Phi(s),\widetilde{\theta}_{n-1}(s),\widetilde{\theta}_n(s))\right|\\&\leq 2C\left\|\mathrm{u}_\Phi(s)\right\|^{1/2}\left|\mathrm{u}_\Phi(s)\right|^{1/2}\left\|\widetilde{\theta}_{n-1}(s)\right\|\left\|\widetilde{\theta}_n(s)\right\|^{1/2}\left|\widetilde{\theta}_n(s)\right|^{1/2}\\&\leq\frac{1}{8}\left\|\widetilde{\theta}_{n-1}(s)\right\|^2+\frac{1}{2}\left\|\widetilde{\theta}_n(s)\right\|^2+C\left\|\mathrm{u}_\Phi(s)\right\|^2\left|\mathrm{u}_\Phi(s)\right|^2\left|\widetilde{\theta}_n(s)\right|^2_{L^2}.\end{aligned}\tag{3.24}$$

Once again using Hölder's, Ladyzhenskaya's and Young's inequalities, we obtain

$$\begin{aligned}&2\left|\left\langle \mathrm{R}_0(\mathrm{A}_1\widetilde{\Psi}_{n-1}(s),\phi_\Phi(s)),\widetilde{\theta}_n\right\rangle\right|=\left|\left\langle B_1(\widetilde{\theta}_n,\phi_\Phi(s)),\mathrm{A}_1\widetilde{\Psi}_{n-1}(s)\right\rangle\right|\\&\leq 2C\left\|\widetilde{\theta}_n(s)\right\|^{1/2}\left|\widetilde{\theta}_n(s)\right|^{1/2}\left\|\phi_\Phi(s)\right\|^{1/2}\left|\mathrm{A}_1\phi_\Phi(s)\right|^{1/2}_{L^2}\left|\mathrm{A}_1\widetilde{\Psi}_{n-1}(s)\right|_{L^2}\\&\leq\frac{1}{4}\left|\mathrm{A}_1\widetilde{\Psi}_{n-1}(s)\right|^2_{L^2}+\frac{1}{2}\left\|\widetilde{\theta}_n(s)\right\|^2+C\left\|\phi_\Phi(s)\right\|^2\left|\mathrm{A}_1\phi_\Phi(s)\right|^2_{L^2}\left|\widetilde{\theta}_n(s)\right|^2_{L^2}.\end{aligned}\tag{3.25}$$

Let us now choose ψ as $\widetilde{\Psi}_n$ in $(3.22)_2$ and reasoning as in the proof of Lemma 3.5, we derive that

$$\begin{aligned}\left|\widetilde{\Psi}_n(t)\right|^2_{L^2}+2\int_0^t\left|\mathrm{A}_1\widetilde{\Psi}_n(s)\right|^2_{L^2}ds&=-2\int_0^t\left(B_1(\widetilde{\theta}_{n-1}(s),\phi_\Phi(s)),\widetilde{\Psi}_n\right)ds\\&\quad-2\int_0^t\left(\mathrm{A}_1(f(\Psi_n)-f(\Psi_{n-1})),\widetilde{\Psi}_n\right)ds.\end{aligned}\tag{3.26}$$

Note that

$$\begin{aligned}&\left|\left(\mathrm{A}_1(f(\Psi_n)-f(\Psi_{n-1})),\widetilde{\Psi}_n\right)\right|\leq\left|f(\Psi_n)-f(\Psi_{n-1})\right|\left|\mathrm{A}_1\widetilde{\Psi}_n\right|_{L^2}\\&\leq\frac{1}{2}\left|\mathrm{A}_1\widetilde{\Psi}_n\right|^2_{L^2}+c_f\left|\widetilde{\Psi}_n\right|^2_{L^2}.\end{aligned}\tag{3.27}$$

Using Hölder's inequality and Agmon's inequality, we get

$$\begin{aligned}&2\left|\left(B_1(\widetilde{\theta}_{n-1}(s),\phi_\Phi(s)),\widetilde{\Psi}_n(s)\right)\right|\leq 2C\left\|\widetilde{\theta}_{n-1}(s)\right\|\left|\nabla\phi_\Phi(s)\right|_{L^\infty}\left|\widetilde{\Psi}_n(s)\right|_{L^2}\\&\leq\frac{1}{8}\left\|\widetilde{\theta}_{n-1}(s)\right\|^2+C\left|\widetilde{\Psi}_n(s)\right|^2_{L^2}(\left\|\phi_\Phi(s)\right\|^2+\left|\phi_\Phi(s)\right|^2_{H^3}).\end{aligned}\tag{3.28}$$

We define $Z(t)=\exp\Big(-\int_0^t Z_1(s)ds\Big)$, where

$$Z_1(s)=C\left((\left\|\phi_\Phi(s)\right\|^2+\left|\phi_\Phi(s)\right|^2_{H^3})+c_f+\left\|\phi_\Phi(s)\right\|^2\left|\mathrm{A}_1\phi_\Phi(s)\right|^2_{L^2}+\left\|\mathrm{u}_\Phi(s)\right\|^2\left|\mathrm{u}_\Phi(s)\right|^2\right)$$

for all $t\in[0,T]$ and a.e. $\omega\in\Omega$.

It is easy to see that $Z(T) < Z(t) \leq 1$, for all $t \in [0,T]$ and a.e. $\omega \in \Omega$.

We also set
$$\mathcal{Z}_n(t) = \left|\widetilde{\theta}_n(t)\right|^2_{L^2} + \left|\widetilde{\Psi}_n(t)\right|^2_{L^2}.$$

From (3.23)–(3.28), we deduce that

$$\begin{aligned} &Z(t)\mathcal{Z}_n(t) + \int_0^t Z(s)\left[\left\|\widetilde{\theta}_n(s)\right\|^2 + \left|\mathrm{A}_1\widetilde{\Psi}_n\right|^2_{L^2}\right] ds \\ &\leq \int_0^t Z(s)\left[\frac{1}{4}\left\|\widetilde{\theta}_{n-1}(s)\right\|^2 + \frac{1}{4}\left|\mathrm{A}_1\widetilde{\Psi}_{n-1}\right|^2_{L^2}\right] ds, \end{aligned} \tag{3.29}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. Hence

$$\int_0^T Z(s)\left[\left\|\widetilde{\theta}_n(s)\right\|^2 + \left|\mathrm{A}_1\widetilde{\Psi}_n\right|^2_{L^2}\right] ds \leq \int_0^T Z(s)\left[\frac{1}{4}\left\|\widetilde{\theta}_{n-1}(s)\right\|^2 + \frac{1}{4}\left|\mathrm{A}_1\widetilde{\Psi}_{n-1}\right|^2_{L^2}\right] ds, \tag{3.30}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. After successive application of (3.30), we obtain

$$\int_0^T Z(s)\left[\left\|\widetilde{\theta}_n(s)\right\|^2 + \left|\mathrm{A}_1\widetilde{\Psi}_n\right|^2_{L^2}\right] ds \leq \frac{1}{4^{n-1}}\int_0^T Z(s)\left[\left\|(\widetilde{\theta}_1, \widetilde{\Psi}_1)(s)\right\|^2_{\mathbb{V}}\right] ds. \tag{3.31}$$

Since $(\mathrm{U}_0(t), \Psi_0(t)) = (0,0)$ for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. For $k, n \in \mathbb{N}$ we infer

$$\begin{aligned} &\left[\mathbb{E}\int_0^T Z(s)(\|\mathrm{U}_{n+k}(s) - \mathrm{U}_n(s)\|^2 + |\mathrm{A}_1\Psi_{n+k}(s) - \mathrm{A}_1\Psi_n(s)|^2_{L^2})ds\right]^{1/2} \\ &\leq \frac{1}{2^{n-2}}\left(1 - \frac{1}{2^k}\right)^{1/2}\left(\mathbb{E}\int_0^T \left\|(\widetilde{\theta}_1, \widetilde{\Psi}_1)(s)\right\|^2_{\mathbb{V}} ds\right)^{1/2}. \end{aligned} \tag{3.32}$$

From the previous inequality, we deduce that $\{\sqrt{Z}(\widetilde{\theta}_n, \widetilde{\Psi}_n)\}_{n\geq 1}$ is a Cauchy sequence in the space $L^2(\Omega \times [0,T]; \mathbb{V})$. Therefore there exists $(X,Y) \in L^2(\Omega \times [0,T]; \mathbb{V})$ with

$$\mathbb{E}\int_0^T \left\|\sqrt{Z(s)(s)}(\widetilde{\theta}_n, \widetilde{\Psi}_n)(s) - (X,Y)(s)\right\|^2_{\mathbb{V}} ds \to 0, \quad \text{for} \quad n \to \infty.$$

Hence, there exists a subsequence (n_k) of n such that for a.e. $\omega \in \Omega$, we have

$$\int_0^T \left\|\sqrt{Z(s)}(\widetilde{\theta}_{n_k}, \widetilde{\Psi}_{n_k})(s) - (X,Y)(s)\right\|^2_{\mathbb{V}} ds \to 0, \quad \text{for} \quad n_k \to \infty.$$

This implies that

$$(\widetilde{\theta}_{n_k}, \widetilde{\Psi}_{n_k})(t) \to Z(t)^{-1/2}(X,Y)(t) = (\mathrm{U}, \Psi)(t) \ \text{ in } \ L^2(0,T;\mathbb{V}), \quad \mathbb{P}-a.e..$$

Therefore (U, Ψ) is an $\mathcal{F} \times \mathcal{B}_{[0,T]}$- measurable adapted process and there exists also a set $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that for all $\omega \in \Omega'$

$$\int_0^T \left\|(\widetilde{\theta}_{n_k}, \widetilde{\Psi}_{n_k})(s) - (\mathrm{U}, \Psi)(s)\right\|^2_{\mathbb{V}} ds \to 0, \quad \text{for} \quad n_k \to \infty, \tag{3.33}$$

and for each $k \in \mathbb{N}$ $(\widetilde{\theta}_{n_k}(.,\omega), \widetilde{\Psi}_{n_k}(.,\omega))$ verifies problem (3.6) for all $t \in [0,T]$.

For an arbitrary fixed $\omega \in \Omega'$ we take $n_k \to \infty$ in problem (3.6) and use the strong convergence (3.33) in $L^2(0,T;\mathbb{V})$. Therefore for this fixed $\omega \in \Omega'$ we have

$$\begin{cases} (\mathrm{U}(t),\mathrm{v}) + \int_0^t \langle \mathrm{A}_0\mathrm{U}(s),\mathrm{v}\rangle\, ds + \int_0^t \langle B_0(\mathrm{u}_\Phi(s),\mathrm{U}(s)),\mathrm{v}\rangle\, ds = (\mathrm{u}_0,\mathrm{v}) \\ + \int_0^t \langle \mathrm{R}_0(\mathrm{A}_1\Psi(s),\phi_\Phi(s)),\mathrm{v}\rangle\, ds + \int_0^t (X(s,\mathrm{u}_\Phi(s)),\mathrm{v}) ds + \int_0^t (\mathrm{g}_2(s,\mathrm{u}_\Phi(s)),\mathrm{v})\, dW(s), \\ (\Psi(t),\psi) + \int_0^t \langle \mathrm{A}_1\mu(s),\psi\rangle\, ds + \int_0^t \langle \mathrm{B}_1(\mathrm{U}(s),\phi_\Phi(s)),\psi\rangle\, ds = (\phi_0,\psi), \\ \mu = \mathrm{A}_1\Psi + \mathrm{f}(\Psi), \end{cases} \tag{3.34}$$

for all $t \in [0,T]$, $(\mathrm{v},\psi) \in \mathrm{V}_1 \times \mathrm{V}_2$ and a.e. $\omega \in \Omega$. We then conclude that (U,Ψ) is an adapted $\mathcal{F} \times \mathcal{B}_{[0,T]}$- measurable $\mathbb{V}$-valued process which satisfies (3.6) for all $t \in [0,T]$ $(\mathrm{v},\psi) \in \mathrm{V}_1 \times \mathrm{V}_2$ and a.e. $\omega \in \Omega$ and has almost trajectories in $\mathbb{H}$. The proof of the existence of solutions of problem (3.6) is now complete.

To prove (3.21), we apply the Itô formula to the process $|\mathrm{U}(t)|^2_{L^2}$ to obtain

$$|\mathrm{U}(t)|^2_{L^2} + 2\int_0^t \|\mathrm{U}(s)\|^2\, ds = |\mathrm{u}_0|^2_{L^2} + 2\int_0^t \langle \mathrm{R}_0(\mathrm{A}_1\Psi(s),\phi_\Phi(s)),\mathrm{U}\rangle\, ds \tag{3.35}$$

$$+ 2\int_0^t (X(s,\mathrm{u}_\Phi(s)),\mathrm{U}) ds + \int_0^t |\mathrm{g}_2(s,\mathrm{u}_\Phi(s))|^2\, ds + 2\int_0^t (\mathrm{g}_2(s,\mathrm{u}_\Phi(s)),\mathrm{U}(s)) dW(s),$$

for all $t \in [0,T]$, where we used the fact that $b_0(\mathrm{u}_\Phi(s),\mathrm{U}(s),\mathrm{U}(s)) = 0$.

Let us now, take the test function ψ as Ψ in $(3.6)_2$ and after some elementary calculations we obtain

$$\begin{aligned} &|\Psi(t)|^2_{L^2} + 2\int_0^t |\mathrm{A}_1\Psi(s)|^2_{L^2}\, ds + 2\int_0^t (\mathrm{A}_1\mathrm{f}(\Psi(s)),\Psi(s))\, ds \\ &= -2\int_0^t (\mathrm{B}_1(\mathrm{U}(s),\phi_\Phi(s)),\Psi)\, ds + 2(\phi_0,\Psi) ds. \end{aligned} \tag{3.36}$$

Define the stopping time τ^k by, for each $k > 0$

$$\tau^k := \inf\{t \geq 0; |\mathrm{U}(t)|^2_{L^2} + |\Psi(t)|^2_{L^2} \geq k^2\} \wedge T. \tag{3.37}$$

If the set is empty, taking $\tau^k = T$. Note that τ^k is increasing with $\lim\limits_{k\longrightarrow\infty} \tau^k = T$.

Now, adding (3.36) and (3.35), we obtain

$$\begin{aligned} &\left|\mathrm{U}(t\wedge\tau^k)\right|^2_{L^2} + \left|\Psi(t\wedge\tau^k)\right|^2_{L^2} + 2\int_0^{t\wedge\tau^k} \|\mathrm{U}(s)\|^2\, ds + 2\int_0^{t\wedge\tau^k} |\mathrm{A}_1\Psi(s)|^2_{L^2}\, ds \\ &= |\mathrm{u}_0|^2_{L^2} + 2(\phi_0,\Psi) + 2\int_0^t \langle \mathrm{R}_0(\mathrm{A}_1\Psi(s),\phi_\Phi(s)),\mathrm{U}\rangle\, ds - 2\int_0^t (\mathrm{A}_1\mathrm{f}(\Psi(s)),\Psi(s))\, ds \\ &\quad + 2\int_0^{t\wedge\tau^k} (X(s,\mathrm{u}_\Phi(s)),\mathrm{U}(s) ds + \int_0^t |\mathrm{g}_2(s,\mathrm{u}_\Phi(s))|^2\, ds \\ &\quad - 2\int_0^t (\mathrm{B}_1(\mathrm{U}(s),\phi_\Phi(s)),\Psi)\, ds + 2\int_0^{t\wedge\tau^k} (\mathrm{g}_2(s,\mathrm{u}_\Phi),\mathrm{U}) dW(s). \end{aligned} \tag{3.38}$$

Now, we are going to estimate each term of the right hand-side of (3.38) as follows:

Using the Cauchy–Schwarz inequality, Young's inequality and Assumption (III), we obtain

$$2\left|(X(s,\mathrm{u}_\Phi(s)),\mathrm{U}(s)\right| \leq \frac{1}{3}\left\|\mathrm{U}(s)\right\|^2 + C_1\left|\mathrm{u}_\Phi(s)\right|_{L^2}^2, \tag{3.39}$$

where C_1 is a positive constant depending on κ. Using the Cauchy–Schwarz inequality and the Young inequality and from the hypothesis **(II)** in Assumption 2.1 we deduce that

$$2\left|(\phi_0,\Psi)\right| \leq \frac{1}{3}\left|\mathrm{A}_1\Psi\right|_{L^2}^2 + C\left|\phi_0\right|_{L^2}^2, \quad \left|(\mathrm{g}_2(s,\mathrm{u}_\Phi(s))\right|^2 \leq \lambda\left|\mathrm{u}_\Phi(s)\right|_{L^2}^2. \tag{3.40}$$

Note that

$$\begin{aligned}2\left|(\mathrm{A}_1(\mathrm{f}(\Psi(s)),\Psi(s))\right| &\leq 2\left|\mathrm{f}(\Psi(s))\right|\left|\mathrm{A}_1\Psi(s)\right| \\ &\leq \frac{1}{3}\left|\mathrm{A}_1\Psi(s)\right|_{L^2}^2 + C\left(1+\left|\Psi(s)\right|_{L^2}^{2m+2}\right).\end{aligned} \tag{3.41}$$

Thanks to the properties of the operator b_1, we have

$$\begin{aligned}&2\left|\langle \mathrm{R}_0(\mathrm{A}_1\Psi(s),\phi_\Phi(s)),\mathrm{U}\rangle\right| = 2\left|\langle \mathrm{B}_1(\mathrm{U}(s),\phi_\Phi(s)),\mathrm{A}_1\Psi\rangle\right| \\ &\leq 2C\left|\mathrm{U}(s)\right|^{1/2}\left\|\mathrm{U}(s)\right\|^{1/2}\left\|\phi_\Phi(s)\right\|^{1/2}\left|\mathrm{A}_1\phi_\Phi(s)\right|_{L^2}^{1/2}\left|\mathrm{A}_1\Psi(s)\right|_{L^2} \\ &\leq \frac{1}{3}\left|\mathrm{A}_1\Psi(s)\right|_{L^2}^2 + \frac{1}{3}\left\|\mathrm{U}(s)\right\|^2 + C\left\|\phi_\Phi(s)\right\|^2\left|\mathrm{A}_1\phi_\Phi(s)\right|_{L^2}^2\left|\mathrm{U}(s)\right|_{L^2}^2.\end{aligned} \tag{3.42}$$

Using Hölder's inequality and Agmon's inequality, we get

$$\begin{aligned}2\left|(\mathrm{B}_1(\mathrm{U}(s),\phi_\Phi(s)),\Psi)\right| &\leq 2C\left\|\mathrm{U}(s)\right\|\left|\nabla\phi_\Phi(s)\right|_{L^\infty}\left|\Psi(s)\right|_{L^2} \\ &\leq \frac{1}{3}\left\|\mathrm{U}(s)\right\|^2 + C\left|\Psi(s)\right|_{L^2}^2(\left\|\phi_\Phi(s)\right\|^2 + \left|\phi_\Phi(s)\right|_{H^3}^2).\end{aligned} \tag{3.43}$$

We use (3.39)–(3.43) in (3.38) to obtain

$$\begin{aligned}&\left|\mathrm{U}(t\wedge\tau^k)\right|_{L^2}^2 + \left|\Psi(t\wedge\tau^k)\right|_{L^2}^2 + \int_0^{t\wedge\tau^k}(\left\|\mathrm{U}(s)\right\|^2 + \left|\mathrm{A}_1\Psi(s)\right|_{L^2}^2)ds \\ &\leq C\int_0^{t\wedge\tau^k}\mathcal{K}_1(s)\left|\mathrm{U}(s)\right|_{L^2}^2 ds + \int_0^{t\wedge\tau^k}\mathcal{K}_2(s)\left|\Psi(s)\right|_{L^2}^2 ds \\ &\quad + (C_1+\lambda)\int_0^T\left|\mathrm{u}_\Phi(s)\right|_{L^2}^2 ds + C\mathbb{E}\left[\mathcal{E}(\mathrm{u}(0),\phi(0))\right] \\ &\quad + 2\int_0^{t\wedge\tau^k}(\mathrm{g}_2(s,\mathrm{u}_\Phi(s)),\mathrm{U}(s))dW(s) + C\int_0^{t\wedge\tau^k}\left(1+\left|\Psi(s)\right|_{L^2}^{2m+2}\right)ds\end{aligned} \tag{3.44}$$

Now, raising both side to the power 2, taking supremum over $s \in [0, t\wedge\tau_n^k]$ and taking mathematical expectation, we infer that

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,t\wedge\tau^k]}\left(\left|\mathrm{U}(s)\right|_{L^2}^2 + \left|\Psi(s)\right|_{L^2}^2\right)^2 + \mathbb{E}\Big(\int_0^{t\wedge\tau^k}(\left\|\mathrm{U}(s)\right\|^2 + \left|\mathrm{A}_1\Psi(s)\right|_{L^2}^2)ds\Big)^2 \\ &\leq C\mathbb{E}\left[\mathcal{E}(\mathrm{u}(0),\phi(0))\right]^2 + \mathbb{E}\Big(\int_0^{t\wedge\tau^k}\mathcal{K}_1(s)\left|\mathrm{U}(s)\right|_{L^2}^2 ds\Big)^2 \\ &\quad + C\mathbb{E}\Big(\int_0^T\left|\mathrm{u}_\Phi(s)\right|_{L^2}^2 ds\Big)^2 + C\mathbb{E}\Big(\int_0^{t\wedge\tau^k}(1+\left|\Psi(s)\right|_{L^2}^{2m+2})ds\Big)^2 \\ &\quad + 2C\mathbb{E}\sup_{s\in[0,t\wedge\tau^k]}\Big|\int_0^s(\mathrm{g}_2(r,\mathrm{u}_\Phi(r)),\mathrm{U}(r)dW(r)\Big|^2,\end{aligned} \tag{3.45}$$

where $\mathcal{K}_1 = \left\|\phi_\Phi(s)\right\|^2\left|\mathrm{A}_1\phi_\Phi(s)\right|_{L^2}^2$ and $\mathcal{K}_2 = (\left\|\phi_\Phi(s)\right\|^2 + \left|\phi_\Phi(s)\right|_{H^3}^2)$.

Using Burkhölder-Davis-Gundy's inequality, we derive that

$$2\mathbb{E}\sup_{s\in[0,t\wedge\tau^k]}\Big|\int_0^s (\mathrm{g}_2(r,\mathrm{u}_\Phi(r)),\mathrm{U}(r)dW(r)\Big|^2 \leq C\mathbb{E}\Big(\int_0^{t\wedge\tau^k}|\mathrm{g}_2(r,\mathrm{u}_\Phi(r))|_{L^2}^2\,|\mathrm{U}(r)|_{L^2}^2\,dr\Big)$$

$$\leq C\mathbb{E}\Big[\sup_{s\in[0,t\wedge\tau^k]}|\mathrm{U}(s)|_{L^2}^2\int_0^{t\wedge\tau^k}|\mathrm{g}_2(r,\mathrm{u}_\Phi(r))|_{L^2}^2\,dr\Big]$$

$$\leq \frac{1}{2}\mathbb{E}\sup_{s\in[0,t\wedge\tau^k]}|\mathrm{U}(s)|_{L^2}^2 + C\mathbb{E}\int_0^T|\mathrm{u}_\Phi(t)|_{L^2}^2\,dt$$

$$\leq \frac{1}{2}\mathbb{E}\sup_{s\in[0,t\wedge\tau^k]}\big(|\mathrm{U}(s)|_{L^2}^2+|\Psi(s)|_{L^2}^2\big)^2 + C\mathbb{E}\int_0^T|\mathrm{u}_\Phi(t)|_{L^2}^4\,dt. \tag{3.46}$$

Thanks to the Hölder inequality and the Theorem 2.4, we get

$$\begin{aligned}\mathbb{E}\Big(\int_0^{t\wedge\tau^k}\mathcal{K}_1(s)\,|\mathrm{U}(s)|_{L^2}^2\,ds\Big)^2 &\leq C\mathbb{E}\int_0^{t\wedge\tau^k}|\mathrm{U}(s)|_{L^2}^4\,ds,\\ \mathbb{E}\Big(\int_0^{t\wedge\tau^k}\mathcal{K}_2(s)\,|\Psi(s)|_{L^2}^2\,ds\Big)^2 &\leq C\mathbb{E}\int_0^{t\wedge\tau^k}|\Psi(s)|_{L^2}^4\,ds.\end{aligned} \tag{3.47}$$

From (3.46) we obtain

$$\mathbb{E}\sup_{s\in[0,t\wedge\tau^k]}[\mathcal{Z}(s)]^2 + 2\mathbb{E}\Big(\int_0^{t\wedge\tau^k}(\|\mathrm{U}(s)\|^2+|\mathrm{A}_1\Psi(s)|^2)ds\Big)^2 \leq C\mathbb{E}\,[\mathcal{E}(\mathrm{u}(0),\phi(0))]^2$$

$$+C\int_0^T|\mathrm{u}_\Phi(t)|_{L^2}^4\,dt + C\mathbb{E}\int_0^{t\wedge\tau^k}\big(1+|\Psi(s)|_{L^2}^4+|\mathrm{U}(s)|_{L^2}^4+|\Psi(s)|_{L^2}^{4m+4}\big)\,ds, \tag{3.48}$$

where $\mathcal{Z}(s) = |\mathrm{U}(s)|_{L^2}^2 + |\Psi(s)|_{L^2}^2$.

Ignoring the last two nonnegatives terms on left-hand side of (3.48) and applying Gronwall's inequality (see [30], Lemma 2, p. 7340) yields

$$\mathbb{E}\sup_{s\in[0,t\wedge\tau^k]}\big(|\mathrm{U}(s)|_{L^2}^2+|\Psi(s)|_{L^2}^2\big)^2 \leq C. \tag{3.49}$$

Using the fact τ^k is increasing to T as $k\to\infty$, we derive that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|(\mathrm{U}(t),\Psi(t))|_{\mathbb{H}}^4\Big] \leq C. \tag{3.50}$$

Substituting (3.50) in (3.48) and using the fact τ^k is increasing to T as $k\to\infty$, we derive that

$$\mathbb{E}\Big(\int_0^T\|(\mathrm{U}(t),\Psi(t))\|_{\mathbb{V}}^2\,dt\Big)^2 \leq C. \tag{3.51}$$

Combining (3.51) and (3.50), we find (3.21). This complete the proof of Lemma 3.7 and ends the proof of Theorem 3.3 □

For $n = 1, 2, ...$, we consider the n-dimensional evolution system:

$$\begin{cases} (\widetilde{\mathrm{u}}_{n,\Phi_n}(t), \mathrm{v}) = (\mathrm{u}_{0n}, \mathrm{v}) - \int_0^t < \mathrm{A}_0\widetilde{\mathrm{u}}_{n,\Phi_n}(s) + \mathrm{B}_{0n}(\mathcal{P}_n^1 \mathrm{u}_\Phi(s), \widetilde{\mathrm{u}}_{n,\Phi_n}(s)), \mathrm{v} > ds \\ + \int_0^t < \mathrm{R}_{0n}(\mathrm{A}_1\widetilde{\phi}_{n,\Phi_n}, \mathcal{P}_n^2\phi(s)), \mathrm{v} > ds + \int_0^t (\mathcal{P}_n^1\Phi_n(s, \mathrm{u}_\Phi(s)), \mathrm{v})\, ds \\ + \int_0^t (\mathcal{P}_n^1 \mathrm{g}_2(s, \mathrm{u}_\Phi(s)), \mathrm{v})\, dW(s), \\ (\widetilde{\phi}_{n,\Phi_n}(t), \psi) = (\phi_{0n}, \psi) - \int_0^t < \mathrm{A}_1\widetilde{\mu}_{n,\Phi_n}(s) + \mathrm{B}_{1n}(\mathcal{P}_n^1 \mathrm{u}(s), \mathcal{P}_n^2\phi(s)), \psi > ds, \\ \widetilde{\mu}_{n,\Phi_n} = \mathrm{A}_1\widetilde{\phi}_{n,\Phi_n} + \mathrm{f}(\widetilde{\phi}_{n,\Phi_n}), \end{cases} \tag{3.52}$$

for all $t \in [0, T]$, $(\mathrm{v}, \psi) \in \mathbb{Y}_n$ and and a.e. $\omega \in \Omega$. System (3.52) is a system of stochastic differential equations is a finite dimensional Banach spaces with locally and globally Lipschitz coefficients. Hence by a well known result about existence and uniqueness of solution to stochastic differential equations (see [31], Theorem 5.5, p. 45), there exists a solution $(\widetilde{\mathrm{u}}_{n,\Phi_n}, \widetilde{\phi}_{n,\Phi_n}) \in \mathcal{M}^2_{\mathcal{F}_t}(0, T; \mathbb{V}_n)$ of (3.52) and almost surely continuous trajectories in $\mathbb{H}$; this solution is almost surely unique. Using the same idea as in the estimates (3.21) we obtain the following estimation

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |(\widetilde{\mathrm{u}}_{n,\Phi_n}, \widetilde{\phi}_{n,\Phi_n})(t)|^4_{\mathbb{H}}\Big] + \mathbb{E}\Big[\int_0^T ||(\widetilde{\mathrm{u}}_{n,\Phi_n}, \widetilde{\phi}_{n,\Phi_n})(s)||^2_{\mathbb{V}} ds\Big]^2 \le C, \tag{3.53}$$

where C is a constant independent of n.

We need the following results in the sequel.

Theorem 3.8. [40] (see Theorem 4.1, p. 132) *Let B_0, B, B_1 be reflexive Banach space such that the embeddings $B_0 \subset B \subset B_1$ are continuous and the embedding $B_0 \subset B$ is compact. Let $1 \le q \le \infty$ and M be a bounded set in $L^q(0, T; B_0)$ consisting of functions $u(t)$ equi-continuous in $\mathcal{C}(0, T; B_1)$. Then, M is relatively compact in $L^q(0, T; B)$ and $\mathcal{C}(0, T; B_1)$.*

Lemma 3.9. [29] *Let $\{Q_m; m \ge 1\} \subset \mathcal{M}^2_{\mathcal{F}_t}(0, T; \mathbb{R})$ be a sequence of continuous real processes, and let $\{\tau_n; n \ge 1\}$ be a sequence of $\mathcal{F}_t$-stopping times such that $\tau_n \uparrow T$, $\sup_{m\ge1} \mathbb{E}|Q_m(T)|^2 < \infty$, and $\lim_{m\to\infty} \mathbb{E}|Q_m(\tau_n)| = 0$ for $n \ge 1$. Then*

$$\lim_{m\to\infty} \mathbb{E}|Q_m(T)| = 0.$$

Theorem 3.10. *The following convergences hold:*

$$\lim_{n\to\infty} \mathbb{E} \int_0^T ||(u_\Phi, \phi_\Phi)(s) - (\widetilde{u}_{\Phi_n}, \widetilde{\phi}_{\Phi_n})(s)||^2_{\mathbb{V}} ds = 0,$$
$$\lim_{n\to\infty} \mathbb{E}|(u_\Phi, \phi_\Phi)(T) - (\widetilde{u}_{\Phi_n}, \widetilde{\phi}_{\Phi_n})(T)|^2_{\mathbb{H}} = 0,$$
$$\lim_{n\to\infty} \mathbb{E} \int_0^T ||(u_\Phi, \phi_\Phi)(s) - (\widetilde{u}_{n,\Phi_n}, \widetilde{\phi}_{n,\Phi_n})(s)||^2_{\mathbb{V}} ds = 0,$$
$$\lim_{n\to\infty} \mathbb{E}|(u_\Phi, \phi_\Phi)(T) - (\widetilde{u}_{n,\Phi_n}, \widetilde{\phi}_{n,\Phi_n})(T)|^2_{\mathbb{H}} = 0.$$

Proof. We consider the evolution system

$$\begin{cases} (\mathrm{z}(t), \mathrm{v}) + \int_0^t \langle \mathrm{A}_0 \mathrm{z}(s), \mathrm{v}\rangle = (\mathrm{u}_0, \mathrm{v}) + \int_0^t (\mathrm{g}_2(s, \mathrm{u}_\Phi(s), \mathrm{v}) dW(s), \\ (\mathrm{y}(t), \psi) + \int_0^t \langle \mathrm{A}_1 \mu(s), \psi\rangle = (\phi_0, \psi), \quad \mu_1 = \mathrm{A}_1 \mathrm{y} + \mathrm{f}(\mathrm{y}), \end{cases} \tag{3.54}$$

for all $t \in [0, T]$, $(\mathrm{v}, \psi) \in \mathrm{V}_1 \times \mathrm{V}_2$ and a.e., $\omega \in \Omega$.

By a similar argument as in the case of system (3.18), there exists a unique solution $(\mathrm{z},\mathrm{y}) \in \mathcal{M}^2_{\mathbb{F}_t}(0,T;\mathbb{V})$ of (3.54) which has a.e continuous trajectories in $\mathbb{H}$. By using Gronwall's lemma, we get the estimate

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |(\mathrm{z},\mathrm{y})(t)|^2_{\mathbb{H}}\Big] + \mathbb{E}\int_0^T ||(\mathrm{z},\mathrm{y})(t)||^2_{\mathbb{V}} dt \le C, \tag{3.55}$$

where C is a positive constant. From Theorem 2.4, we have

$$\mathbb{E}\sup_{t\in[0,T]} |(\mathrm{u}_\Phi(t),\phi_\Phi(t))|^2_{\mathbb{H}} < \infty, \quad \mathbb{E}\int_0^T ||(\mathrm{u}_\Phi(t),\phi_\Phi(t))||^2_{\mathbb{V}} dt < \infty.$$

Hence, there exists $k_2(\omega)>0$ independent of n, such that for all $n\in\mathbb{N}$ and a.e. $\omega\in\Omega$

$$\begin{aligned} &\sup_{t\in[0,T]} |\mathcal{P}_n(\mathrm{z}(t),\mathrm{y}(t))|^2_{\mathbb{H}} \le \sup_{t\in[0,T]} |(\mathrm{z}(s),\mathrm{y}(t))|^2_{\mathbb{H}} < k_2(\omega),\\ &\int_0^T ||\mathcal{P}_n(\mathrm{z}(t),\mathrm{y}(t))||^2_{\mathbb{V}} dt \le \int_0^T ||(\mathrm{z}(t),\mathrm{y}(t))||^2_{\mathbb{V}} dt < k_2(\omega), \end{aligned} \tag{3.56}$$

$$\begin{aligned} &\sup_{t\in[0,T]} |\mathcal{P}_n(\mathrm{u}_\Phi(t),\phi_\Phi(t))|^2_{\mathbb{H}} \le \sup_{t\in[0,T]} |(\mathrm{u}_\Phi(t),\phi_\Phi(t))|^2_{\mathbb{H}} < k_2(\omega),\\ &\int_0^T ||\mathcal{P}_n(\mathrm{u}_\Phi(t),\phi_\Phi(t))||^2_{\mathbb{V}} dt \le \int_0^T ||(\mathrm{u}_\Phi(t),\phi_\Phi(t))||^2_{\mathbb{V}} dt < k_2(\omega). \end{aligned} \tag{3.57}$$

Using stochastic integral properties and (2.16), we obtain for all $s,t\in[0,T]$, $t>s$

$$\begin{aligned} &\mathbb{E}\Big\|\int_s^t \mathrm{g}_2(r,\mathrm{u}_\phi(r))dW(r)\Big\|^4_{\mathrm{V}^*_{div}} \le k_3\mathbb{E}\Big(\int_s^t |\mathrm{g}_2(r,\mathrm{u}_\Phi(r))|^2_{L^2}dr\Big)^2\\ &\le k_4(t-s)^2\mathbb{E}\sup_{r\in[0,T]}|\mathrm{u}_\Phi(r)|^4_{L^2} \le k_4(t-s)^2\mathbb{E}\sup_{r\in[0,T]}|(\mathrm{u}_\Phi(r),\phi_\Phi(r))|^4_{\mathbb{H}} < \infty \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}\Big\|\int_s^t \mathcal{P}^1_n\mathrm{g}_2(r,\mathrm{u}_\Phi(r))dW(r)\Big\|^4_{\mathrm{V}^*_{div}} \le k_3\mathbb{E}\Big(\int_s^t |\mathcal{P}^1_n\mathrm{g}_2(r,\mathrm{u}_\Phi(r))|^2_{L^2}dr\Big)^2\\ &\le k_4(t-s)^2\mathbb{E}\sup_{r\in[0,T]}|\mathrm{u}_\Phi(r)|^4_{L^2} \le k_4(t-s)^2\mathbb{E}\sup_{r\in[0,T]}|(\mathrm{u}_\Phi(r),\phi_\Phi(r))|^4_{\mathbb{H}} < \infty \end{aligned}$$

where k_3, k_4 are positive constants.

By *Kolmogorov's continuity criterion* (see [25], Theorem 1.4.1, p. 31) it follows that there exists a random variable $\chi(\omega)$ such that

$$\Big\|\int_s^t ||\mathrm{g}_2(r,\mathrm{u}_\Phi(r))dW(r)\Big\|^2_{V_1^*} \le \chi(\omega)|t-s|^{2\gamma}, \tag{3.58}$$

$$\Big\|\int_s^t \mathcal{P}^1_n\mathrm{g}_2(r,\mathrm{u}_\Phi(r))dW(r)\Big\|^2_{V_1^*} \le \chi(\omega)|t-s|^{2\gamma}, \tag{3.59}$$

for $\gamma \in (0,\frac{1}{4})$ and for every $t,s\in[0,T]$, a.e. $\omega\in\Omega$.

Let $\widetilde{\Omega}\subset\Omega$ with $\mathbb{P}(\widetilde{\Omega})=1$ be such that for all $\omega\in\widetilde{\Omega}$ we have:

1. The equations (2.14) and (3.54) hold for all $t\in[0,T]$, $(\mathrm{v},\psi)\in\mathrm{V}_1\times\mathrm{V}_2$;

2. For each $n=1,2,...$ the equations (3.5) and (3.52) hold for all $t\in[0,T], (\mathrm{v},\psi)\in \mathrm{V}_1\times \mathrm{V}_2,$ respectively, $(\mathrm{v},\psi)\in\mathbb{Y}_n$;
3. The inequalities in (3.56), (3.57), (3.58) and (3.59) hold.

From (3.5), (3.54), (3.56), (3.57), it follows that for all $\omega\in\widetilde{\Omega}$ we have:

$$\begin{cases}(\widetilde{\mathrm{u}}_{\Phi_n}(t)-\mathrm{z}(t),\mathrm{v})+\int_0^t\left\langle(\mathrm{A}_0\widetilde{\mathrm{u}}_{\Phi_n}(s)-\mathrm{A}_0\mathrm{z}(s))+\mathrm{B}_0(\mathrm{u}_\Phi(s),\widetilde{\mathrm{u}}_{\Phi_n}(s)),\mathrm{v}\right\rangle ds\\ \quad +\int_0^t\left\langle \mathrm{R}_0(\mathrm{A}_1\widetilde{\phi}_{\Phi_n}(s),\phi(s)),\mathrm{w}\right\rangle ds-\int_0^t(\Phi_n(s,\mathrm{u}_\Phi(s)),\mathrm{v})ds=0,\\ (\widetilde{\phi}_{\Phi_n}(t)-\mathrm{y},\psi)+\int_0^t\left\langle(\mathrm{A}_1\widetilde{\mu}_{\Phi_n}(s)-\mathrm{A}_1\mu_1(s))+\mathrm{B}_1(\widetilde{\mathrm{u}}_{\Phi_n}(s),\phi(s)),\psi\right\rangle ds=0,\\ \widetilde{\mu}_{\Phi_n}-\mu_1=\mathrm{A}_1\widetilde{\phi}_{\Phi_n}-\mathrm{A}_1\mathrm{y}+\mathrm{f}(\widetilde{\phi}_{\Phi_n})-\mathrm{f}(\mathrm{y}).\end{cases}\tag{3.60}$$

Let $\theta_n(t)=\widetilde{\mathrm{u}}_{\Phi_n}(t)-\mathrm{z}(t)$ and $\Psi_n(t)=\widetilde{\phi}_{\Phi_n}(t)-\mathrm{y}(t)$ for all $t\in[0,T]$. Hence we have:

$$\begin{cases}(\theta_n(t),\mathrm{v})+\int_0^t\left\langle(\mathrm{A}_0\theta_n(s))+\mathrm{B}_0(\mathrm{u}_\Phi(s),\widetilde{\mathrm{u}}_{\Phi_n}(s)),\mathrm{v}\right\rangle ds\\ \quad -\int_0^t\left\langle \mathrm{R}_0(\mathrm{A}_1\widetilde{\phi}_{\Phi_n}(s),\phi(s)),\mathrm{v}\right\rangle ds=\int_0^t(\Phi_n(s,\mathrm{u}_\Phi(s)),\mathrm{v})ds,\\ (\Psi_n,\psi)+\int_0^t\left\langle \mathrm{A}_1\mu_n+\mathrm{B}_1(\widetilde{\mathrm{u}}_{\Phi_n}(s),\phi(s)),\psi\right\rangle ds=0,\\ \mu_n=\mathrm{A}_1\Psi_n+f(\widetilde{\phi}_{\Phi_n})-f(\mathrm{y}).\end{cases}\tag{3.61}$$

Taking $\mathrm{v}=\theta_n$ in $(3.61)_1$, $\psi=\Psi_n$ in $(3.61)_2$ and proceeding as in the proof of Lemma 3.5, we get for all $\omega\in\widetilde{\Omega}$

$$\begin{aligned}&|\theta_n(t)|_{L^2}^2+|\Psi_n(t)|_{L^2}^2+2\int_0^t\|\theta_n(s)\|^2ds+2\int_0^t|\mathrm{A}_1\Psi_n(s)|_{L^2}^2ds\\ &+2\int_0^t\langle B_0(\mathrm{u}_\Phi(s),\widetilde{\mathrm{u}}_{\Phi_n}(s)),\theta_n(s)\rangle ds=2\int_0^t\left\langle \mathrm{R}_0(\mathrm{A}_1\widetilde{\phi}_{\Phi_n}(s),\phi_\Phi(s)),\theta_n(s)\right\rangle ds\\ &+2\int_0^t(\Phi_n(s,\mathrm{u}_\Phi(s)),\theta_n(s))ds-2\int_0^t(\mathrm{B}_1(\widetilde{\mathrm{u}}_{\Phi_n}(s),\phi(s)),\Psi_n)\,ds\\ &-2\int_0^t\left(\mathrm{A}_1(f(\widetilde{\phi}_{\Phi_n})-f(\mathrm{y})),\Psi_n\right)ds.\end{aligned}\tag{3.62}$$

Using the fact that $\mathbb{V}\hookrightarrow\mathbb{Y}$, we estimate some terms of the above equality. Thanks to the properties of B_0, we derive that

$$\begin{aligned}&2\left|\langle B_0(\mathrm{u}_\Phi(s),\widetilde{\mathrm{u}}_{\Phi_n}(s)),\theta_n(s)\rangle\right|=2\left|\langle \mathrm{B}_0(\mathrm{u}_\Phi(s),\theta_n(s),\mathrm{z})\rangle\right|\\ &\le 2C\|\mathrm{u}_\Phi(s)\|^{1/2}|\mathrm{u}_\Phi(s)|_{L^2}^{1/2}\|\theta_n(s)\|\,\|\mathrm{z}(s)\|^{1/2}|\mathrm{z}(s)|_{L^2}^{1/2}\\ &\le\frac{1}{3}\|\theta_n(s)\|^2+C\|\mathrm{u}_\Phi(s)\|\,|\mathrm{u}_\Phi(s)|_{L^2}\|\mathrm{z}(s)\|\,|\mathrm{z}(s)|_{L^2}.\end{aligned}\tag{3.63}$$

Owing to the assumptions on Φ and Young's inequality, we have

$$2\left|(\Phi_n(s,\mathrm{u}_\Phi),\theta_n(s))\right|\le 2\kappa|\mathrm{u}_\Phi|_{L^2}\|\theta_n(s)\|\le\frac{1}{3}\|\theta_n(s)\|^2+C\kappa\|\mathrm{u}_\Phi(s)\|^2.\tag{3.64}$$

Note that

$$2\left|\left(\mathrm{A}_1(f(\widetilde{\phi}_{\Phi_n})-f(\mathrm{y})),\Psi_n\right)\right|\le\frac{1}{2}|\mathrm{A}_1\Psi_n(s)|_{L^2}^2+c_f|\Psi_n(s)|_{L^2}^2.\tag{3.65}$$

Thanks to the properties of B_1 and the Young inequality, the following estimates holds:

$$\begin{aligned}&2\left|(\mathrm{B}_1(\widetilde{\mathrm{u}}_{\Phi_n}(s),\phi(s)),\Psi_n)\right| = 2\left|(\mathrm{B}_1(\widetilde{\mathrm{u}}_{\Phi_n}(s),\Psi_n(s),\phi(s)))\right| \\ &\le C\left|\widetilde{\mathrm{u}}_{\Phi_n}(s)\right|_{L^2}^{1/2}\left\|\widetilde{\mathrm{u}}_{\Phi_n}(s)\right\|^{1/2}\left\|\Psi_n(s)\right\|^{1/2}\left|\mathrm{A}_1\Psi_n(s)\right|_{L^2}^{1/2}\left|\phi(s)\right|_{L^2} \\ &\le 2C\left\|\widetilde{\mathrm{u}}_{\Phi_n}(s)\right\|\left|\mathrm{A}_1\Psi_n(s)\right|_{L^2}\left|\phi(s)\right|_{L^2} \le \frac{1}{2}\left|\mathrm{A}_1\Psi_n(s)\right|_{L^2}^2 + C\left\|\widetilde{\mathrm{u}}_{\Phi_n}(s)\right\|^2\left|\phi(s)\right|_{L^2}^2\end{aligned} \tag{3.66}$$

and

$$\begin{aligned}&2\left|\left\langle \mathrm{R}_0(\mathrm{A}_1\widetilde{\phi}_{\Phi_n}(s),\phi_\Phi(s)),\theta_n(s)\right\rangle\right| = 2\left|b_1(\theta_n(s),\phi_\Phi(s),\mathrm{A}_1\widetilde{\phi}_{\Phi_n}(s))\right| \\ &\le \frac{1}{3}\left\|\theta_n(s)\right\|^2 + C\left|\phi_\Phi(s)\right|\left|\mathrm{A}_1\phi_\Phi(s)\right|_{L^2}\left|\mathrm{A}_1\widetilde{\phi}_{\Phi_n}(s)\right|_{L^2}^2.\end{aligned} \tag{3.67}$$

Let $\mathcal{K}(t) = \exp\left(-c_f t\right)$. We substitute (3.63)–(3.67) in (3.62) to obtain

$$\begin{aligned}&\mathcal{K}(t)\left[\left|\theta_n(t)\right|_{L^2}^2 + \left|\Psi_n(t)\right|_{L^2}^2\right] + \nu\int_0^t \mathcal{K}(s)\left\|\theta_n(s)\right\|^2 ds + \int_0^t \mathcal{K}(s)\left|\mathrm{A}_1\Psi_n(s)\right|_{L^2}^2 ds \\ &\le C\int_0^t \mathcal{K}(s)\left[\kappa\left\|\mathrm{u}_\Phi(s)\right\|^2 + \left\|\mathrm{u}_\Phi(s)\right\|\left|\mathrm{u}_\Phi(s)\right|\left\|\mathrm{z}(s)\right\|\left|\mathrm{z}(s)\right|\right] ds \\ &\quad + C\int_0^t \mathcal{K}(s)\left|\phi_\Phi(s)\right|\left|\mathrm{A}_1\phi_\Phi(s)\right|_{L^2}\left|\mathrm{A}_1\widetilde{\phi}_{\Phi_n}(s)\right|_{L^2}^2 ds + C\int_0^t \mathcal{K}(s)\left\|\widetilde{\mathrm{u}}_{\Phi_n}(s)\right\|^2\left|\phi(s)\right|_{L^2}^2 ds.\end{aligned} \tag{3.68}$$

Thanks to the Hölder inequality, Theorem 2.4, (3.21) and the fact que $0 < \mathcal{K}(t) \le 1$ we can prove that each term of the right hand-sides of (3.68) is finite for all $\omega \in \widetilde{\Omega}$. Hence, using the fact que $0 < \mathcal{K}(t) \le 1$, it follows that for all $\omega \in \widetilde{\Omega}$, we have

$$\sup_{t\in[0,T]}\left[\left|\theta_n(t)\right|_{L^2}^2 + \left|\Psi_n(t)\right|_{L^2}^2\right] + \int_0^t \left\|\theta_n(s)\right\|^2 ds + \int_0^t \left|\mathrm{A}_1\Psi_n(s)\right|_{L^2}^2 ds \le k_3(k_2(\omega)), \tag{3.69}$$

where k_3 is a positive constant (independant of n and ω).

Thus, we conclude that for all $\omega \in \widetilde{\Omega}$, we have

$$\begin{aligned}&\sup_{t\in[0,T]}\left[\left|(\widetilde{\mathrm{u}}_{\Phi_n},\widetilde{\phi}_{\Phi_n})(t) - (\mathrm{z},\mathrm{y})(t)\right|_{\mathbb{H}}^2\right] + \int_0^T \left\|(\widetilde{\mathrm{u}}_{\Phi_n},\widetilde{\phi}_{\Phi_n})(t) - (\mathrm{z},\mathrm{y})(t)\right\|_{\mathbb{V}}^2 dt \\ &\le k_3(k_2(\omega)).\end{aligned} \tag{3.70}$$

Analogously, using (3.52) and (3.54) we have

$$\begin{aligned}&\sup_{t\in[0,T]}\left|(\widetilde{\mathrm{u}}_{n,\Phi_n},\widetilde{\phi}_{n,\Phi_n})(t) - \Pi_n(\mathrm{z},\mathrm{y})(t)\right|_{\mathbb{H}}^2 + \int_0^T \left\|(\widetilde{\mathrm{u}}_{n,\Phi_n},\widetilde{\phi}_{n,\Phi_n})(t) - \Pi_n(\mathrm{z},\mathrm{y})(t))\right\|_{\mathbb{V}}^2 dt \\ &\le k_3(k_2(\omega)).\end{aligned} \tag{3.71}$$

Hence, for all $n \in \mathbb{N}$, we have

$$\sup_{t\in[0,T]}\left|(\widetilde{\mathrm{u}}_{n,\Phi_n},\widetilde{\phi}_{n,\Phi_n})(t)\right|_{\mathbb{H}}^2 + \int_0^T \left\|(\widetilde{\mathrm{u}}_{n,\Phi_n},\widetilde{\phi}_{n,\Phi_n})(t)\right\|_{\mathbb{V}}^2 dt \le k_4(\omega),\ \omega\in\widetilde{\Omega}, \tag{3.72}$$

and $$\sup_{t\in[0,T]}\left|(\widetilde{\mathrm{u}}_{\Phi_n},\widetilde{\phi}_{\Phi_n})(t)\right|_{\mathbb{H}}^2 + \int_0^T \left\|(\widetilde{\mathrm{u}}_{\Phi_n},\widetilde{\phi}_{\Phi_n})(t)\right\|_{\mathbb{V}}^2 dt \le k_4(\omega), \quad \omega \in \widetilde{\Omega}, \tag{3.73}$$

where k_4 is positive and independent of n.

Let $\omega \in \widetilde{\Omega}$. For this ω we consider the sets

$$S = \{(\widetilde{\mathrm{u}}_{\Phi_n}, \widetilde{\phi}_{\Phi_n})(\omega, .)|, n = 1, 2, 3, ...\}, \quad \widetilde{S} = \{(\widetilde{\mathrm{u}}_{n,\Phi_n}, \widetilde{\phi}_{n,\Phi_n})(\omega, .)|, n = 1, 2, 3, ...\}.$$

For each of these sets we want to apply the *Dubinsky Theorem* (Theorem 3.8). By (3.73) and (3.72) we get that $S \subset \mathcal{M}^2(0,T;\mathbb{V})$ and $\widetilde{S} \subset \mathcal{M}^2(0,T;\mathbb{V})$ are bounded. We have to verify that S, respectively, $\widetilde{S}$, are equi-continuous in $\mathcal{C}([0;T];\mathbb{V}^*)$. From (3.5) and the Schwarz inequality, we have

$$\begin{aligned}
&\|\widetilde{\mathrm{u}}_{\Phi_n}(t) - \widetilde{\mathrm{u}}_{\Phi_n}(s)\|^2_{\mathrm{V}_1^*} + \left\|\widetilde{\phi}_{\Phi_n}(t) - \widetilde{\phi}_{\Phi_n}(s)\right\|^2_{\mathrm{V}_2^*} \\
&\leq (t-s)\int_s^t (||\mathrm{A}_0\widetilde{\mathrm{u}}_{\Phi_n}||^2_{\mathrm{V}_1^*} + ||\mathrm{A}_1\widetilde{\mu}_{\Phi_n}||^2_{\mathrm{V}_2^*} + ||\mathrm{B}_0(\mathrm{u}_\Phi(r), \widetilde{\mathrm{u}}_{\Phi_n}(r))||^2_{\mathrm{V}_2^*} \\
&\quad + ||\mathrm{R}_0(\mathrm{A}_1\widetilde{\phi}_{\Phi_n}(r), \phi(r))||^2_{\mathrm{V}_1^*})dr + (t-s)\int_s^t (||\mathrm{B}_1(\widetilde{\mathrm{u}}_{\Phi_n}(r), \phi(r))||^2_{\mathrm{V}_2^*})dr \\
&\quad + ||\Phi_n(r, \mathrm{u}_\Phi(r))||^2_{\mathrm{V}_1^*}dr + \left\|\int_s^t \mathrm{g}_2(r, \mathrm{u}_\Phi(r))dW(r)\right\|^2_{\mathrm{V}_1^*},
\end{aligned}$$

for each $t, s \in [0,T]$, $t > s$. By (3.58), (3.73), and the properties of A_0, B_0, R_0, A_1, B_1, Φ_n we obtain

$$\|\widetilde{\mathrm{u}}_{\Phi_n}(t) - \widetilde{\mathrm{u}}_{\Phi_n}(s)\|^2_{\mathrm{V}_1^*} + \left\|\widetilde{\phi}_{\Phi_n}(t) - \widetilde{\phi}_{\Phi_n}(s)\right\|^2_{\mathrm{V}_2^*} \leq k_7(\omega)(t-s) + \chi(\omega)(t-s)^{2\gamma},$$

for $\gamma \in (0, \frac{1}{4})$ and for every $t, s \in [0,T], t > s$, where $k_5(\omega) > 0$ is independent of n. Consequently, S is equi-continuous in $\mathcal{C}([0,T];\mathbb{V}^*)$. Analogously, we can prove that $\widetilde{S}$ is equi-continuous in $\mathcal{C}([0,T];\mathbb{V}^*)$. Now, using the *Dubinsky Theorem* (Theorem 3.8), it follows that S and $\widetilde{S}$ are relatively compact in $\mathcal{M}^2(0,T;\mathbb{H})$ and hence there exists a subsequence (n') of (n) and $(\widetilde{\mathrm{u}}, \widetilde{\phi})$, $(\mathrm{u}^*, \phi^*) \in \mathcal{M}^2(0,T;\mathbb{H})$ such that

$$\lim_{n'\to\infty}\int_0^T ||(\widetilde{\mathrm{u}}_{\Phi_{n'}}, \widetilde{\phi}_{\Phi_{n'}})(s) - (\widetilde{\mathrm{u}}, \widetilde{\phi})(s)||^2_{\mathbb{V}}ds = 0 \tag{3.74}$$

and

$$\lim_{n'\to\infty}\int_0^T ||(\widetilde{\mathrm{u}}_{n',\Phi_{n'}}, \widetilde{\phi}_{n',\Phi_{n'}}) - (\mathrm{u}^*, \phi^*)(s)||^2_{\mathbb{V}}ds = 0. \tag{3.75}$$

Let $\theta_{n'}(t) = \widetilde{\mathrm{u}}_{\Phi_{n'}}(t) - \mathrm{u}_\Phi(t)$, $\Psi_{n'}(t) = \widetilde{\phi}_{\Phi_{n'}}(t) - \phi_\Phi(t)$ and $\mu_{n'} = \mathrm{A}_1\Psi_{n'} + f(\widetilde{\phi}_{\Phi_{n'}}) - f(\phi_\Phi)$ for all $t \in [0,T]$. We use (3.5) with (n') and (2.14), the genaralized chain rule, and the properties of $\mathrm{A}_0, \mathrm{A}_1, \mathrm{B}_0$ and B_1 (see Lemma 3.5) to obtain

$$\begin{aligned}
&|\theta_{n'}(t)|^2_{L^2} + |\Psi_{n'}(t)|^2_{L^2} + \int_0^t \|\theta_{n'}(s)\|^2 ds + \int_0^t |\mathrm{A}_1\Psi_{n'}(s)|^2_{L^2} ds \\
&\leq C\int_0^t \mathcal{K}_1(s)\left(|\theta_{n'}(s)|^2_{L^2} + |\Psi_{n'}(s)|^2_{L^2}\right) \\
&\quad + 2\int_0^t (\Phi_{n'}(s, \mathrm{u}_\Phi(s)) - \Phi(s, \mathrm{u}_\Phi(s)), \widetilde{\mathrm{u}}_{\Phi_{n'}}(s) - \widetilde{\mathrm{u}}(s))ds \\
&\quad + 2\int_0^t (\Phi_{n'}(s, \mathrm{u}_\Phi(s)) - \Phi(s, \mathrm{u}_\Phi(s)), \widetilde{\mathrm{u}}(s) - \mathrm{u}_\Phi(s))ds,
\end{aligned} \tag{3.76}$$

where $\mathcal{K}_1(s) = C\left(\|\phi_\Phi(s)\|^2 |\mathrm{A}_1\phi_\Phi(s)|^2_{L^2} + \|\phi_\Phi(s)\|^2 + |\phi_\Phi(s)|^2_{H^3}\right)$.

Let us define

$$\mathcal{Z}(t) = |\theta_{n'}(t)|^2_{L^2} + |\Psi_{n'}(t)|^2_{L^2}, \quad y(t) = \exp\Big(-c_f t - \int_0^t \mathcal{K}_1(s)ds \Big),$$

for all $t \in [0,T]$ and a.e., $\omega \in \widetilde{\Omega}$. Note that $0 < y(T) \leq y(t) \leq 1$ for all $t \in [0,T]$ and a.e., $\omega \in \widetilde{\Omega}$. From (3.76), we derive that

$$\begin{aligned} y(T)\mathcal{Z}(T) &+ \int_0^T y(s)\, \|\theta_{n'}(s)\|^2\, ds + \int_0^T y(s)\, |A_1\Psi_{n'}(s)|^2_{L^2}\, ds \\ &\leq 2\int_0^T y(s)(\Phi_{n'}(s,\mathrm{u}_\Phi(s)) - \Phi(s,\mathrm{u}_\Phi(s)), \widetilde{\mathrm{u}}_{\Phi_{n'}}(s) - \widetilde{\mathrm{u}}(s))ds \\ &\quad + 2\int_0^T y(s)(\Phi_{n'}(s,\mathrm{u}_\Phi(s)) - \Phi(s,\mathrm{u}_\Phi(s)), \widetilde{\mathrm{u}}(s) - \mathrm{u}_\Phi(s))ds. \end{aligned} \tag{3.77}$$

Thanks to the Lemma 3.2, (3.74), the properties of $\Phi_n, \Phi \in \mathcal{U}$, the dominated convergence theorem and the fact that $0 < y(T) \leq y(t) \leq 1$, we obtain that the right side of the inequality in (3.77) tends to zero. Hence, we obtain

$$\begin{aligned} &\lim_{n'\to\infty} \Big|(\widetilde{\mathrm{u}}_{\Phi_{n'}}, \widetilde{\phi}_{\Phi_{n'}})(T) - (\mathrm{u}_\Phi, \phi_\Phi)(T)\Big|^2_{\mathbb{H}} = 0, \\ &\lim_{n'\to\infty} \int_0^T \Big\|(\widetilde{\mathrm{u}}_{\Phi_{n'}}, \widetilde{\phi}_{\Phi_{n'}})(s) - (\mathrm{u}_\Phi, \phi_\Phi)(s)\Big\|^2_{\mathbb{V}}\, d = 0. \end{aligned} \tag{3.78}$$

Every subsequence of $(\widetilde{\mathrm{u}}_{\Phi_n}(\omega,.), \widetilde{\phi}_{\Phi_n}(\omega,.))$ has a further subsequence, which converges in the space $\mathcal{M}^2(0;T,\mathbb{V})$ to the same limit $(\mathrm{u}_\Phi(\omega,.), \phi_\Phi(\omega,.))$ (because we can repeat all the arguments above). Applying a convergence result (see [41], Proposition 10.13, p. 480), it follows that the whole sequence $(\widetilde{\mathrm{u}}_{\Phi_n}(\omega,.), \widetilde{\phi}_{\Phi_n}(\omega,.))$ converges to $(\mathrm{u}_\Phi(\omega,.), \phi_\Phi(\omega,.))$ in the space $\mathcal{M}^2(0,T;\mathbb{V})$.

Analogously, we conclude that the whole sequence $(\widetilde{\mathrm{u}}_{\Phi_n}(\omega,T), \widetilde{\phi}_{\Phi_n}(\omega,T))$ converges to $(\mathrm{u}_\Phi(\omega,T), \phi_\Phi(\omega,T))$ in $\mathbb{H}$.

Our arguments above work for an arbitrary fixed $\omega \in \widetilde{\Omega}$. Hence, (3.78) holds for a.e. $\omega \in \Omega$ and for the whole sequence (n). Taking into consideration that the processes $(\widetilde{\mathrm{u}}_{\Phi_n}(t), \widetilde{\phi}_{\Phi_n}(t))_{t\in[0,T]}$ and $(\mathrm{u}_\Phi(t), \phi_\Phi(t))_{t\in[0,T]}$ are uniformly integrable (see Theorem 2.4 and (3.55)), it follows that

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}|(\mathrm{u}_\Phi, \phi_\Phi)(T) - (\widetilde{\mathrm{u}}_{\Phi_n}, \widetilde{\phi}_{\Phi_n})(T)|^2_{\mathbb{H}} = 0, \\ &\lim_{n\to\infty} \mathbb{E}\int_0^T ||(\mathrm{u}_\Phi, \phi_\Phi)(s) - (\widetilde{\mathrm{u}}_{\Phi_n}, \widetilde{\phi}_{\Phi_n})(s)||^2_{\mathbb{V}} ds = 0. \end{aligned}$$

Now, we prove the convergence for the sequence $(\widetilde{\mathrm{u}}_{n,\Phi_n}, \widetilde{\phi}_{n,\Phi_n})$.

Let $\theta_n(t) = \widetilde{\mathrm{u}}_{n,\Phi_n}(t) - \mathcal{P}^1_n \mathrm{u}_\Phi(t)$ and $\Psi_n(t) = \widetilde{\phi}_{n,\Phi_n}(t) - \mathcal{P}^2_n \phi_\Phi(t)$ for all $t \in [0,T]$.

We use (3.52) and (2.14), the generalized chain rule, and the properties of $\mathrm{A}_0, \mathrm{A}_1$ and B_0 to obtain

$$|\theta_n(t)|^2_{L^2} + |\Psi_n(t)|^2_{L^2} + 2\int_0^t \|\theta_n(s)\|^2\, ds + 2\int_0^t |\mathrm{A}_1\Psi_n(s)|^2_{L^2}\, ds$$

$$
\begin{aligned}
&= -2\int_0^t \left\langle B_0(\mathcal{P}_n^1 \mathrm{u}_\Phi(s), \mathcal{P}_n^1 \mathrm{u}_\Phi(s)) - \mathrm{B}_0(\mathrm{u}_\Phi(s), \mathrm{u}_\Phi(s)), \theta_n \right\rangle ds \\
&+ 2\int_0^t \left\langle \mathrm{R}_0(\mathrm{A}_1 \widetilde{\phi}_{n,\Phi_n}(s), \mathcal{P}_n^2 \phi_\Phi(s)) - \mathrm{R}_0(\mathrm{A}_1 \phi_\Phi(s), \phi_\Phi(s)), \theta_n \right\rangle ds \\
&- 2\int_0^t \left(\mathrm{B}_1(\widetilde{\mathrm{u}}_{n,\Phi_n}(s), \mathcal{P}_n^2 \phi_\Phi(s)) - \mathrm{B}_1(\mathrm{u}_\Phi(s), \phi_\Phi(s)), \Psi_n \right) ds \\
&- 2\int_0^t \left(\mathrm{A}_1(f(\widetilde{\phi}_{n,\Phi_n}) - f(\phi_\Phi)), \Psi_n \right) ds \\
&+ 2\int_0^t (\mathcal{P}_n^1 \Phi_n(s, \mathrm{u}_\Phi(s)) - \Phi(s, \mathrm{u}_\Phi(s)), \theta_n(s)) ds.
\end{aligned} \tag{3.79}
$$

Note that
$$
\begin{aligned}
&2\left| \left(\mathrm{A}_1(f(\widetilde{\phi}_{n,\Phi_n}) - f(\phi_\Phi)), \Psi_n \right) \right| \\
&\le \frac{1}{3} \left| \mathrm{A}_1 \Psi_n(s) \right|_{L^2}^2 + C \left| \Psi_n(s) \right|_{L^2}^2 + C \left| \mathcal{P}_n^2 \phi_\Phi(s) - \phi_\Phi(s) \right|_{L^2}^2 .
\end{aligned} \tag{3.80}
$$

Using the properties of the operator B_0 and the Young inequality, we get

$$
\begin{aligned}
&2\left| \left\langle B_0(\mathcal{P}_n^1 \mathrm{u}_\Phi(s), \mathcal{P}_n^1 \mathrm{u}_\Phi(s)) - B_0(\mathrm{u}_\Phi(s), \mathrm{u}_\Phi(s)), \theta_n \right\rangle \right| \\
&\le \frac{1}{4} \left\| \theta_n(s) \right\|^2 + C \left\| B_0(\mathcal{P}_n^1 \mathrm{u}_\Phi(s), \mathcal{P}_n^1 \mathrm{u}_\Phi(s)) - B_0(\mathrm{u}_\Phi(s), \mathrm{u}_\Phi(s)) \right\|_{\mathrm{V}_1^*}^2 .
\end{aligned} \tag{3.81}
$$

Observe that

$$
\begin{aligned}
&2\left\langle \mathrm{R}_0(\mathrm{A}_1 \widetilde{\phi}_{n,\Phi_n}(s), \mathcal{P}_n^2 \phi_\Phi(s)) - \mathrm{R}_0(\mathrm{A}_1 \phi_\Phi(s), \phi_\Phi(s)), \theta_n \right\rangle \\
&= 2\left\langle \mathrm{R}_0(\mathrm{A}_1 \Psi_n(s), \mathcal{P}_n^2 \phi_\Phi(s)), \theta_n \right\rangle \\
&\quad + 2\left\langle \mathrm{R}_0(\mathrm{A}_1 \mathcal{P}_n^2 \phi_\Phi(s), \mathcal{P}_n^2 \phi_\Phi(s)) - \mathrm{R}_0(\mathrm{A}_1 \phi_\Phi(s), \phi_\Phi(s)), \theta_n \right\rangle
\end{aligned} \tag{3.82}
$$

and

$$
\begin{aligned}
&2\left| \left\langle \mathrm{R}_0(\mathrm{A}_1 \Psi_n(s), \mathcal{P}_n^2 \phi_\Phi(s)), \theta_n \right\rangle \right| \\
&\le \frac{1}{4} \left\| \theta_n(s) \right\|^2 + \frac{1}{3} \left| \mathrm{A}_1 \Psi_n(s) \right|_{L^2}^2 + \left\| \mathcal{P}_n^2 \phi_\Phi(s) \right\|^2 \left| \mathrm{A}_1 \mathcal{P}_n^2 \phi_\Phi(s) \right|_{L^2}^2 \left| \theta_n \right|_{L^2}^2 ,
\end{aligned} \tag{3.83}
$$

and

$$
\begin{aligned}
&2\left| \left\langle \mathrm{R}_0(\mathrm{A}_1 \mathcal{P}_n^2 \phi_\Phi(s), \mathcal{P}_n^2 \phi_\Phi(s)) - \mathrm{R}_0(\mathrm{A}_1 \phi_\Phi(s), \phi_\Phi(s)), \theta_n \right\rangle \right| \\
&\le \frac{1}{4} \left\| \theta_n(s) \right\|^2 + C \left\| \mathrm{R}_0(\mathrm{A}_1 \mathcal{P}_n^2 \phi_\Phi(s), \mathcal{P}_n^2 \phi_\Phi(s)) - \mathrm{R}_0(\mathrm{A}_1 \phi_\Phi(s), \phi_\Phi(s)) \right\|_{\mathrm{V}_1^*}^2 .
\end{aligned} \tag{3.84}
$$

Similarly, using Hölder's inequality and Agmon's inequality, we obtain

$$
\begin{aligned}
&\left(\mathrm{B}_1(\widetilde{\mathrm{u}}_{n,\Phi_n}(s), \mathcal{P}_n^2 \phi_\Phi(s)) - \mathrm{B}_1(\mathrm{u}_\Phi(s), \phi_\Phi(s)), \Psi_n \right) \\
&\le \frac{1}{4} \left\| \theta_n(s) \right\|^2 + \frac{1}{3} \left| \mathrm{A}_1 \Psi_n(s) \right|_{L^2}^2 + (\left\| \mathcal{P}_n^2 \phi_\Phi(s) \right\|^2 + \left| \mathcal{P}_n^2 \phi_\Phi(s) \right|_{H^3}^2) \left| \Psi_n \right|_{L^2}^2 \\
&\quad + C|\mathrm{B}_1(\mathcal{P}_n^1 \mathrm{u}_\Phi(s), \mathcal{P}_n^2 \phi_\Phi(s)) - \mathrm{B}_1(\mathrm{u}_\Phi(s), \phi_\Phi(s))|_{L^2}^2 .
\end{aligned} \tag{3.85}
$$

Now, we define

$$
\begin{aligned}
&\mathcal{Z}(t) = \left| \theta_n(t) \right|_{L^2}^2 + \left| \Psi_n(t) \right|_{L^2}^2 , \quad y(t) = \exp\Big(-Ct - \int_0^t \mathcal{K}_1(s) ds \Big), \\
&\mathcal{K}_1(t) = C \left(\left\| \mathcal{P}_n^2 \phi_\Phi \right\|^2 \left| \mathrm{A}_1 \mathcal{P}_n^2 \phi_\Phi \right|_{L^2}^2 + (\left\| \mathcal{P}_n^2 \phi_\Phi(s) \right\|^2 + \left| \mathcal{P}_n^2 \phi_\Phi(s) \right|_{H^3}^2) \right)
\end{aligned}
$$

for all $t \in [0, T]$ and a.e., $\omega \in \widetilde{\Omega}$.

Note that $0 < y(T) \leq y(t) \leq 1$ for all $t \in [0,T]$ and a.e., $\omega \in \widetilde{\Omega}$. From (3.79)–(3.85), we derive that

$$\begin{aligned}
&y(T)\mathcal{Z}(T) + \int_0^t \|\theta_n(s)\|^2\,ds + \int_0^t |\mathrm{A}_1\Psi_n(s)|_{L^2}^2\,ds \\
&\leq C\int_0^T y(s)\left\|B_0(\mathcal{P}_n^1\mathrm{u}_\Phi(s),\mathcal{P}_n^1\mathrm{u}_\Phi(s)) - B_0(\mathrm{u}_\Phi(s),\mathrm{u}_\Phi(s))\right\|_{\mathrm{V}_1^*}^2 ds \\
&\quad + C\int_0^T y(s)\left\|R_0(\mathrm{A}_1\mathcal{P}_n^2\phi_\Phi(s),\mathcal{P}_n^2\phi_\Phi(s)) - R_0(\mathrm{A}_1\phi_\Phi(s),\phi_\Phi(s))\right\|_{\mathrm{V}_1^*}^2 ds \\
&\quad + C\int_0^T y(s)|B_1(\mathcal{P}_n^1\mathrm{u}_\Phi(s),\mathcal{P}_n^2\phi_\Phi(s)) - B_1(\mathrm{u}_\Phi(s),\phi_\Phi(s))|_{L^2}^2 ds \\
&\quad + 2\int_0^T y(s)(\mathcal{P}_n^1\Phi_n(s,\mathrm{u}_\Phi(s)) - \Phi(s,\mathrm{u}_\Phi(s)),\widetilde{\mathrm{u}}_{n,\Phi_n} - \mathcal{P}_n^1\mathrm{u}_\Phi(s))ds \\
&\quad + 2C\int_0^T y(s)\left|\mathcal{P}_n^2\phi_\Phi(s) - \phi_\Phi(s)\right|_{L^2}^2 ds.
\end{aligned} \tag{3.86}$$

By the properties of B_0, we have:

$$\begin{aligned}
&\int_0^T ||B_0(\mathcal{P}_n^1\mathrm{u}_\Phi(s),\mathcal{P}_n^1\mathrm{u}_\Phi(s)) - B_0(\mathrm{u}_\Phi(s),\mathrm{u}_\Phi(s))||_{\mathrm{V}_1^*}^2 \\
&\leq 2C\Big(\sup_{t\in[0,T]}|\mathrm{u}_\Phi(t) - \mathcal{P}_n^1\mathrm{u}_\Phi(t)|^2\int_0^T ||\mathrm{u}_\Phi(s)||^2 ds\Big)^{1/2} \\
&\qquad \Big(\sup_{t\in[0,T]}|\mathrm{u}_\Phi(t)|^2\int_0^T ||\mathrm{u}_\Phi(s) - \mathcal{P}_n^1\mathrm{u}_\Phi(s)||^2 ds\Big)^{1/2}.
\end{aligned}$$

Thanks to (2.19) it follows that

$$\lim_{n\to\infty}\int_0^T y(s)||B_0(\mathcal{P}_n^1\mathrm{u}_\Phi(s),\mathcal{P}_n^1\mathrm{u}_\Phi(s)) - B_0(\mathrm{u}_\Phi(s),\mathrm{u}_\Phi(s))||_{\mathrm{V}_1^*}^2 = 0. \tag{3.87}$$

Similarly, using the proprities of R_0 we have

$$\begin{aligned}
&\left\|R_0(\mathrm{A}_1\mathcal{P}_n^2\phi_\Phi(s),\mathcal{P}_n^2\phi_\Phi(s)) - R_0(A_1\phi_\Phi(s),\phi_\Phi(s))\right\|_{\mathrm{V}_1^*}^2 \\
&\leq \left\|R_0((\mathrm{A}_1\mathcal{P}_n^2\phi_\Phi(s) - \mathrm{A}_1\phi(s)),\phi_\Phi(s))\right\|_{\mathrm{V}_1^*}^2 \\
&\quad + \left\|R_0(\mathrm{A}_1\mathcal{P}_n^2\phi_\Phi(s),\mathcal{P}_n^2\phi_\Phi(s) - \phi_\Phi(s))\right\|_{\mathrm{V}_1^*}^2 \\
&\leq C\|\phi_\Phi(s)\|\,|\mathrm{A}_1\phi_\Phi(s)|_{L^2}\left|\mathrm{A}_1(\mathcal{P}_n^2\phi_\Phi(s) - \phi(s))\right|_{L^2}^2 \\
&\quad + C\left|\mathrm{A}_1(\mathcal{P}_n^2\phi_\Phi(s) - \phi(s))\right|_{L^2}\left\|\mathcal{P}_n^2\phi_\Phi(s) - \phi(s)\right\|\left|\mathrm{A}_1\mathcal{P}_n^2\phi_\Phi(s)\right|_{L^2}^2.
\end{aligned} \tag{3.88}$$

Hence by Hölder's inequality and (2.19) we conclude that

$$\lim_{n\to\infty}\int_0^T y(s)\left\|R_0(\mathrm{A}_1\mathcal{P}_n^2\phi_\Phi(s),\mathcal{P}_n^2\phi_\Phi(s)) - R_0(\mathrm{A}_1\phi(s),\phi(s))\right\|_{\mathrm{V}_1^*}^2 ds = 0. \tag{3.89}$$

By using the properties of B_1 and (2.19) we can also prove that

$$\lim_{n\to\infty}\int_0^T y(s)|B_1(\mathcal{P}_n^1\mathrm{u}_\Phi(s),\mathcal{P}_n^2\phi_\Phi(s))-B_1(\mathrm{u}(s),\phi(s))|_{L^2}^2ds=0. \tag{3.90}$$

Note that $\displaystyle\int_0^T y(s)(\mathcal{P}_n^1\Phi_n(s,\mathrm{u}_\Phi(s))-\Phi(s,\mathrm{u}_\Phi(s)),\widetilde{\mathrm{u}}_{n,\Phi_n}-\mathcal{P}_n^1\mathrm{u}_\Phi(s))ds$

$$\begin{aligned}&=\int_0^T y(s)(\Phi_n(s,\mathrm{u}_\Phi(s))-\Phi(s,\mathrm{u}_\Phi(s)),\widetilde{\mathrm{u}}_{n,\Phi_n}-\mathrm{u}_\Phi(s))ds\\&\quad+\int_0^T y(s)(\Phi_n(s,\mathrm{u}_\Phi(s))-\Phi(s,\mathrm{u}_\Phi(s)),\mathrm{u}_\Phi(s)-\mathcal{P}_n^1\mathrm{u}_\Phi(s))ds.\end{aligned}$$

By using this equality for (n'), (2.19), (3.57), (3.72), (3.75), the properties of Φ_n, Φ, Lemma 3.2 and the dominated convergence theorem we get

$$\lim_{n'\to\infty}\int_0^T y(s)(\mathcal{P}_{n'}^1\Phi_{n'}(s,\mathrm{u}_\Phi(s))-\Phi(s,\mathrm{u}_\Phi(s)),\widetilde{\mathrm{u}}_{n',\Phi_{n'}}-\mathcal{P}_{n'}^1\mathrm{u}_\Phi(s))ds=0. \tag{3.91}$$

From (3.87), (3.89), (3.90) and (3.91) we obtain that the right side of the inequality in (3.86) tends to zero. Therefore,

$$\begin{aligned}&\lim_{n'\to\infty}|(\widetilde{\mathrm{u}}_{n,\Phi_n},\widetilde{\phi}_{n,\Phi_n})(T)-\Pi_n(\mathrm{u}_\Phi,\phi_\Phi)(T)|_{\mathbb{H}}^2=0,\\&\lim_{n'\to\infty}\int_0^T ||(\widetilde{\mathrm{u}}_{n',\Phi_{n'}},\widetilde{\phi}_{n',\Phi n'})(s)-\Pi_{n'}(\mathrm{u}_\Phi,\phi_\Phi)(s)||_{\mathbb{V}}^2ds=0.\end{aligned} \tag{3.92}$$

Hence, by (2.19) and (2.21) we have

$$\begin{aligned}&\lim_{n'\to\infty}\left|(\widetilde{\mathrm{u}}_{n',\Phi_{n'}},\widetilde{\phi}_{n',\Phi_{n'}})(T)-(\mathrm{u}_\Phi,\phi_\Phi)(T)\right|_{\mathbb{H}}^2=0,\\&\lim_{n'\to\infty}\int_0^T\left\|(\widetilde{\mathrm{u}}_{n',\Phi_{n'}},\widetilde{\phi}_{n',\Phi_{n'}})(s)-(\mathrm{u}_\Phi,\phi_\Phi)(s)\right\|_{\mathbb{V}}^2ds=0.\end{aligned} \tag{3.93}$$

Every sub-sequence of $(\widetilde{\mathrm{u}}_{n,\Phi_n}(\omega,.),\widetilde{\phi}_{n,\Phi_n}(\omega,.))$ has a further subsequence, which converges in the space $\mathcal{M}^2(0;T,\mathbb{V})$ to the same limit $(\mathrm{u}_\Phi(\omega,.),\phi_\Phi(\omega,.))$ (because we can repeat all the arguments above). Applying a convergence result (see [41], Proposition 10.13, p. 480), it follows that the whole sequence $(\widetilde{\mathrm{u}}_{n,\Phi_n}(\omega,.),\widetilde{\phi}_{n,\Phi_n}(\omega,.))$ converges to $(\mathrm{u}_\Phi(\omega,.),\phi_\Phi(\omega,.))$ in the space $\mathcal{M}^2(0,T;\mathbb{V})$. Analogously, we conclude that the whole sequence $(\widetilde{\mathrm{u}}_{n,\Phi_n}(\omega,T),\widetilde{\phi}_{n,\Phi_n}(\omega,T))$ converges to $(\mathrm{u}_\Phi(\omega,T),\phi_\Phi(\omega,T))$ in $\mathbb{H}$.

Our arguments above work for an arbitrary fixed $\omega\in\widetilde{\Omega}$. Hence, (3.93) holds for a.e. $\omega\in\Omega$ and for the whole sequence (n). Taking into consideration that the processes $(\widetilde{\mathrm{u}}_{n,\Phi_n}(t),\widetilde{\phi}_{n,\Phi_n}(t))_{t\in[0,T]}$ and $(\mathrm{u}_\Phi(t),\phi_\Phi(t))_{t\in[0,T]}$ are uniformly integrable (see Theorem 2.5 and (3.53)) it follows that the conclusions of this theorem hold. □

Notation 3.11. For more simplified writing, we use the notation $(\mathrm{u}_{n,\Phi_n},\phi_{n,\Phi_n})$ instead of $(\mathrm{u}_{n,\mathcal{P}_n^1\Phi_n},\phi_{n,\mathcal{P}_n^1\Phi_n})$, where $(\mathrm{u}_{n,\mathcal{P}_n^1\Phi_n},\phi_{n,\mathcal{P}_n^1\Phi_n})$ is the solution of (2.17) using the feedback $\mathcal{P}_n^1\Phi_n$ (see Section 2).

Theorem 3.12. *The following convergences hold:*

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E} \int_0^T \|(u_\Phi, \phi_\Phi)(s) - (u_{\Phi_n}, \phi_{\Phi_n})(s)\|_{\mathbb{V}}^2 \, ds = 0, \\
&\lim_{n\to\infty} \mathbb{E} \,|(u_\Phi, \phi_\Phi)(T) - (u_{\Phi_n}, \phi_{\Phi_n})(T)|_{\mathbb{H}}^2 = 0, \\
&\lim_{n\to\infty} \mathbb{E} \int_0^T \|(u_\Phi, \phi_\Phi)(s) - (u_{n,\Phi_n}, \phi_{n,\Phi_n})(s)\|_{\mathbb{V}}^2 \, ds = 0, \\
&\lim_{n\to\infty} \mathbb{E} \,|(u_\Phi, \phi_\Phi)(T) - (u_{n,\Phi_n}, \phi_{n,\Phi_n})(T)|_{\mathbb{H}}^2 = 0.
\end{aligned} \tag{3.94}$$

Proof. For the sake of convenience, we use the abbreviations $(\mathrm{u}, \phi, \mu) := (\mathrm{u}_\Phi, \phi_\Phi, \mu_\Phi)$ to have

$$\begin{cases}
(\widetilde{\mathrm{u}}_{\Phi_n}(t) - \mathrm{u}_{\Phi_n}(t), \mathrm{v}) + \int_0^t \langle \mathrm{A}_0(\widetilde{\mathrm{u}}_{\Phi_n}(s) - \mathrm{u}_{\Phi_n}(s)), \mathrm{v}\rangle \, ds \\
\quad + \int_0^t \langle \mathrm{B}_0(\mathrm{u}(s), \widetilde{\mathrm{u}}_{\Phi_n}(s)) - \mathrm{B}_0(\mathrm{u}_{\Phi_n}(s), \mathrm{u}_{\Phi_n}(s)), \mathrm{v}\rangle \, ds \\
\quad + \int_0^t \Big\langle \mathrm{R}_0(\mathrm{A}_1\widetilde{\phi}_{\Phi_n}(s), \phi(s)) - \mathrm{R}_0(\mathrm{A}_1\phi_{\Phi_n}(s), \phi_{\Phi_n}(s)), \mathrm{v}\Big\rangle \, ds \\
= \int_0^t (\Phi_n(s, \mathrm{u}(s)) - \Phi_n(s, \mathrm{u}_{\Phi_n}(s)), \mathrm{v}) ds + \int_0^t (\mathrm{g}_2(s, \mathrm{u}(s)) - \mathrm{g}_2(s, \mathrm{u}_{\Phi_n}(s)), \mathrm{v}) dW(s), \\
(\widetilde{\phi}_{\Phi_n}(t) - \phi_{\Phi_n}(t), \psi) + \int_0^t \langle \mathrm{A}_1(\widetilde{\mu}_{\Phi_n}(s) - \mu_{\Phi_n}(s)), \psi\rangle \, ds \\
\quad + \int_0^t \langle \mathrm{B}_1(\widetilde{\mathrm{u}}_{\Phi_n}(s), \phi(s)) - \mathrm{B}_1(\mathrm{u}_{\Phi_n}(s), \phi_{\Phi_n}(s)), \psi\rangle \, ds = 0, \\
\widetilde{\mu}_{\Phi_n} - \mu_{\Phi_n} = \mathrm{A}_1(\widetilde{\phi}_{\Phi_n} - \phi_{\Phi_n}) + \mathrm{f}(\widetilde{\phi}_{\Phi_n}) - \mathrm{f}(\phi_{\Phi_n}).
\end{cases} \tag{3.95}$$

Let $\theta_n(t) = \widetilde{\mathrm{u}}_{\Phi_n}(t) - \mathrm{u}_{\Phi_n}(t)$ and $\Psi_n(t) = \widetilde{\phi}_{\Phi_n}(t) - \phi_{\Phi_n}(t)$.

Now, using Itô's formula in $|\theta_n(t)|_{L^2}^2$ we obtain for all $t \in [0, T]$

$$\begin{aligned}
&|\theta_n(t)|_{L^2}^2 + 2\int_0^t \|\theta_n(s)\|^2 \, ds + 2\int_0^t \langle B_0(\mathrm{u}(s), \widetilde{\mathrm{u}}_{\Phi_n}(s)) - B_0(\mathrm{u}_{\Phi_n}(s), \mathrm{u}_{\Phi_n}(s)), \theta_n\rangle \, ds \\
&\quad - 2\int_0^t \Big\langle R_0(\mathrm{A}_1\widetilde{\phi}_{\Phi_n}(s), \phi(s)) - R_0(\mathrm{A}_1\phi_{\Phi_n}(s), \phi_{\Phi_n}(s)), \theta_n\Big\rangle \, ds \qquad (3.96) \\
&= 2\int_0^t (\Phi_n(s, \mathrm{u}(s)) - \Phi_n(s, \mathrm{u}_{\Phi_n}(s)), \theta_n) ds + 2\int_0^t (\mathrm{g}_2(s, \mathrm{u}(s)) - \mathrm{g}_2(s, \mathrm{u}_{\Phi_n}(s)), \theta_n) dW(s), \\
&\quad + \int_0^t |\mathrm{g}_2(s, \mathrm{u}(s)) - \mathrm{g}_2(s, \mathrm{u}_{\Phi_n}(s))|_{L^2}^2 \, ds.
\end{aligned}$$

Taking $\psi = \Psi_n$ in $(3.95)_2$ and by the same arguments as the proof of uniqueness (see Lemma 3.5), we have

$$\begin{aligned}
&|\Psi_n(t)|_{L^2}^2 + 2\int_0^t |\mathrm{A}_1\Psi_n(s)|_{L^2}^2 \, ds = -2\int_0^t \Big(\mathrm{A}_1(\mathrm{f}(\widetilde{\phi}_{\Phi_n}) - \mathrm{f}(\phi_{\Phi_n})), \Psi_n\Big) \, ds \\
&= -2\int_0^t (B_1(\widetilde{\mathrm{u}}_{\Phi_n}(s), \phi(s)) - B_1(\mathrm{u}_{\Phi_n}(s), \phi_{\Phi_n}(s)), \Psi_n) \, ds.
\end{aligned} \tag{3.97}$$

Adding (3.97) and (3.96) we derive that for all $t \in [0, T]$

$$|\theta_n(t)|_{L^2}^2 + |\Psi_n(t)|_{L^2}^2 + 2\int_0^t \|\theta_n(s)\|^2 \, ds + 2\int_0^t |\mathrm{A}_1\Psi_n(s)|_{L^2}^2 \, ds$$

$$
\begin{aligned}
&= -2\int_0^t \langle B_0(\mathrm{u}(s), \widetilde{\mathrm{u}}_{\Phi_n}(s)) - B_0(\mathrm{u}_{\Phi_n}(s), \mathrm{u}_{\Phi_n}(s)), \theta_n\rangle \, ds \\
&\quad + 2\int_0^t \left\langle R_0(\mathrm{A}_1\widetilde{\phi}_{\Phi_n}(s), \phi(s)) - R_0(\mathrm{A}_1\phi_{\Phi_n}(s), \phi_{\Phi_n}(s)), \theta_n\right\rangle ds \\
&\quad + 2\int_0^t (\Phi_n(s, \mathrm{u}(s)) - \Phi_n(s, \mathrm{u}_{\Phi_n}(s)), \theta_n) ds \qquad (3.98) \\
&\quad + 2\int_0^t (\mathrm{g}_2(s, \mathrm{u}(s)) - \mathrm{g}_2(s, \mathrm{u}_{\Phi_n}(s)), \theta_n) dW(s) \\
&\quad + \int_0^t |\mathrm{g}_2(s, \mathrm{u}(s)) - \mathrm{g}_2(s, \mathrm{u}_{\Phi_n}(s))|_{L^2}^2 \, ds \\
&\quad - 2\int_0^t (B_1(\widetilde{\mathrm{u}}_{\Phi_n}(s), \phi(s)) - B_1(\mathrm{u}_{\Phi_n}(s), \phi_{\Phi_n}(s)), \Psi_n) \, ds \\
&\quad - 2\int_0^t \left(\mathrm{A}_1(\mathrm{f}(\widetilde{\phi}_{\Phi_n}) - \mathrm{f}(\phi_{\Phi_n})), \Psi_n\right) ds.
\end{aligned}
$$

Now, we estimate the right hand side by one by one using Hölder's, Young's, Agmon's and Gagliardo-Nirenberg inequalities. Note that

$$
\begin{aligned}
&2\langle B_0(\mathrm{u}(s), \widetilde{\mathrm{u}}_{\Phi_n}(s)) - B_0(\mathrm{u}_{\Phi_n}(s), \mathrm{u}_{\Phi_n}(s)), \theta_n(s)\rangle \\
&\quad = 2\langle B_0(\mathrm{u}(s) - \mathrm{u}_{\Phi_n}(s)), \widetilde{\mathrm{u}}_{\Phi_n} - \mathrm{u}(s)), \theta_n(s)\rangle \qquad (3.99) \\
&\quad + 2\langle B_0(\mathrm{u}(s) - \widetilde{\mathrm{u}}_{\Phi_n}(s, \mathrm{u})), \theta_n(s)\rangle + 2\langle B_0(\theta_n(s), \mathrm{u}(s)), \theta_n(s)\rangle .
\end{aligned}
$$

Thanks to the properties of B_0 and the Young inequality we obtain

$$
\begin{aligned}
&2\langle B_0(\mathrm{u}(s) - \mathrm{u}_{\Phi_n}(s), \widetilde{\mathrm{u}}_{\Phi_n} - \mathrm{u}(s)), \theta_n(s)\rangle \\
&\leq \frac{1}{8}\|\theta_n(s)\|^2 + C\|\mathrm{u}(s) - \mathrm{u}_{\Phi_n}(s)\|^2 \|\widetilde{\mathrm{u}}_{\Phi_n}(s) - \mathrm{u}(s)\|^2, \qquad (3.100)
\end{aligned}
$$

$$
2\langle B_0(\mathrm{u}(s) - \widetilde{\mathrm{u}}_{\Phi_n}(s), \mathrm{u}(s)), \theta_n(s)\rangle \leq \frac{1}{8}\|\theta_n(s)\|^2 + C\|\mathrm{u}(s)\|^2 \|\mathrm{u}(s) - \widetilde{\mathrm{u}}_{\Phi_n}(s)\|^2 \qquad (3.101)
$$

and

$$
2\langle B_0(\theta_n(s), \mathrm{u}(s)), \theta_n(s)\rangle \leq \frac{1}{8}\|\theta_n(s)\|^2 + C\|\mathrm{u}(s)\|^2 |\theta_n(s)|_{L^2}^2 . \qquad (3.102)
$$

Due to the assumptions (II) and (III) we have

$$
\begin{aligned}
&|\mathrm{g}_2(s, \mathrm{u}(s)) - \mathrm{g}_2(s, \mathrm{u}_{\Phi_n}(s))|_{L^2}^2 \leq |\mathrm{g}_2(s, \mathrm{u}(s)) - \mathrm{g}_2(s, \widetilde{\mathrm{u}}_{\Phi_n}(s))|_{L^2}^2 \\
&\quad + |\mathrm{g}_2(s, \widetilde{\mathrm{u}}_{\Phi_n}(s)) - \mathrm{g}_2(s, \mathrm{u}_{\Phi_n}(s))|_{L^2}^2 \leq \lambda|\mathrm{u}(s) - \widetilde{\mathrm{u}}_{\Phi_n}(s)|_{L^2}^2 + \lambda|\theta_n(s)|_{L^2}^2 \qquad (3.103) \\
&\leq \lambda\|\mathrm{u}(s) - \widetilde{\mathrm{u}}_{\Phi_n}(s)\|^2 + \lambda|\theta_n(s)|_{L^2}^2 ,
\end{aligned}
$$

$$
\begin{aligned}
&2|(\Phi_n(s, \mathrm{u}(s)) - \Phi_n(s, \mathrm{u}_{\Phi_n}(s)), \theta_n(s))| \leq \sqrt{\kappa}|\mathrm{u}(s) - \widetilde{\mathrm{u}}_{\Phi_n}(s)|_{L^2}\|\theta_n(s)\| \qquad (3.104) \\
&\quad + C\sqrt{\kappa}|\theta_n(s)|_{L^2}\|\theta_n(s)\| \leq \frac{1}{8}\|\theta_n(s)\|^2 + C\kappa|\theta_n(s)|_{L^2}^2 + C\kappa\|\mathrm{u}(s) - \widetilde{\mathrm{u}}_{\Phi_n}(s)\|^2 .
\end{aligned}
$$

Note also that

$$
2\left|\left(\mathrm{A}_1(\mathrm{f}(\widetilde{\phi}_{\Phi_n}) - \mathrm{f}(\phi_{\Phi_n})), \Psi_n\right)\right| \leq \frac{1}{2}|\mathrm{A}_1\Psi_n(s)|_{L^2}^2 + C|\Psi_n(t)|_{L^2}^2 . \qquad (3.105)
$$

Now it is not difficult to see that:

$$\begin{aligned}
&\left\langle R_0(\mathrm{A}_1\widetilde{\phi}_{\Phi_n}(s),\phi(s)) - R_0(\mathrm{A}_1\phi_{\Phi_n}(s),\phi_{\Phi_n}(s)),\theta_n\right\rangle \\
&= \left\langle R_0(\mathrm{A}_1(\widetilde{\phi}_{\Phi_n}(s)-\phi(s)),\phi(s)),\theta_n(s)\right\rangle \\
&\quad + \left\langle R_0(\mathrm{A}_1\phi(s),\phi(s) - R_0(\mathrm{A}_1\widetilde{\phi}_{\Phi_n}(s),\widetilde{\phi}_{\Phi_n}(s)),\theta_n(s)\right\rangle \\
&\quad + \left\langle R_0(\widetilde{\phi}_{\Phi_n}(s),\widetilde{\phi}_{\Phi_n}(s) - R_0(\mathrm{A}_1\phi_{\Phi_n}(s),\phi_{\Phi_n}(s),\theta_n(s)\right\rangle
\end{aligned} \tag{3.106}$$

and

$$\begin{aligned}
&(B_1(\widetilde{\mathrm{u}}_{\Phi_n}(s),\phi(s)) - B_1(\mathrm{u}_{\Phi_n}(s),\phi_{\Phi_n}(s)),\Psi_n) \\
&= (\mathrm{B}_1((\widetilde{\mathrm{u}}_{\Phi_n}(s)-\mathrm{u}(s)),\phi(s)),\Psi_n) + (B_1(\mathrm{u}(s),\phi(s)) - B_1(\widetilde{\mathrm{u}}_{\Phi_n}(s),\widetilde{\phi}_{\Phi_n}(s)),\Psi_n) \\
&\quad + (B_1(\widetilde{\mathrm{u}}_{\Phi_n}(s),\widetilde{\phi}_{\Phi_n}(s)) - B_1(\mathrm{u}_{\Phi_n}(s),\phi_{\Phi_n}(s)),\Psi_n).
\end{aligned} \tag{3.107}$$

Thanks to the Hölder inequality and Agmon's inequality, we get,

$$\begin{aligned}
&2\left|(B_1((\widetilde{\mathrm{u}}_{\Phi_n}(s)-\mathrm{u}(s)),\phi(s)),\Psi_n)\right| \\
&\leq 2C\left\|(\widetilde{\mathrm{u}}_{\Phi_n}(s)-\mathrm{u}(s))\right\|\left|\nabla\phi(s)\right|_{L^\infty}\left|\Psi_n\right|_{L^2} \\
&\leq \frac{1}{6}\left\|(\widetilde{\mathrm{u}}_{\Phi_n}(s)-\mathrm{u}(s))\right\|^2 + C\left|\Psi_n\right|_{L^2}^2(\left|\nabla\phi(s)\right|_{L^2}^2 + \left|\phi(s)\right|_{H^3}^2)
\end{aligned} \tag{3.108}$$

and

$$\begin{aligned}
&2(B_1(\mathrm{u}(s),\phi(s)) - B_1(\widetilde{\mathrm{u}}_{\Phi_n}(s),\widetilde{\phi}_{\Phi_n}(s)),\Psi_n) \\
&= 2(B_1(\mathrm{u}(s)-\widetilde{\mathrm{u}}_{\Phi_n}(s),\widetilde{\phi}_{\Phi_n}(s)),\Psi_n) + 2(B_1(\mathrm{u}(s),\phi(s)-\widetilde{\phi}_{\Phi_n}(s)),\Psi_n) \\
&\leq \frac{1}{6}\left\|\mathrm{u}(s)-\widetilde{\mathrm{u}}_{\Phi_n}(s)\right\|^2 + \frac{1}{6}\left|\mathrm{A}_1(\phi(s)-\widetilde{\phi}_{\Phi_n}(s))\right|_{L^2}^2 + C\left\|\mathrm{u}(s)\right\|^2\left|\Psi_n\right|_{L^2}^2 \\
&\quad + C\left|\Psi_n\right|_{L^2}^2(\left|\nabla\widetilde{\phi}_{\Phi_n}(s)\right|_{L^2}^2 + \left|\widetilde{\phi}_{\Phi_n}(s)\right|_{H^3}^2).
\end{aligned} \tag{3.109}$$

Similarly,

$$\begin{aligned}
&2\left(B_1(\widetilde{\mathrm{u}}_{\Phi_n}(s),\widetilde{\phi}_{\Phi_n}(s)) - B_1(\mathrm{u}_{\Phi_n}(s),\phi_{\Phi_n}(s)),\Psi_n\right) \\
&= 2\left(B_1(\theta_n(s),\phi_{\Phi_n}(s)),\Psi_n\right) + 2\left(B_1(\widetilde{\mathrm{u}}_{\Phi_n}(s),\Psi_n(s)),\Psi_n\right) \\
&\leq \frac{1}{8}\left\|\theta_n(s)\right\|^2 + C\left|\Psi_n\right|_{L^2}^2(\left|\nabla\phi_{\Phi_n}(s)\right|_{L^2}^2 + \left|\phi_{\Phi_n}(s)\right|_{H^3}^2).
\end{aligned} \tag{3.110}$$

Similarly, we have

$$2\left\langle R_0(\mathrm{A}_1(\widetilde{\phi}_{\Phi_n}(s)-\phi(s)),\phi(s)),\theta_n(s)\right\rangle \tag{3.111}$$

$$\leq \frac{1}{8}\left\|\theta_n(s)\right\|^2 + \frac{1}{6}\left|\mathrm{A}_1(\widetilde{\phi}_{\Phi_n}(s)-\phi(s))\right|_{L^2}^2 + C\left\|\phi(s)\right\|^2\left|\mathrm{A}_1\phi(s)\right|_{L^2}^2\left|\theta_n(s)\right|_{L^2}^2,$$

$$\begin{aligned}
&2\left\langle R_0(\mathrm{A}_1\phi(s),\phi(s) - R_0(\mathrm{A}_1\widetilde{\phi}_{\Phi_n}(s),\widetilde{\phi}_{\Phi_n}(s)),\theta_n(s)\right\rangle \\
&\leq \frac{1}{8}\left\|\theta_n(s)\right\|^2 + \frac{1}{6}\left|\mathrm{A}_1(\phi(s)-\widetilde{\phi}_{\Phi_n}(s))\right|_{L^2}^2 + C\left|\mathrm{A}_1\phi(s)\right|_{L^2}^2\left|\mathrm{A}_1(\phi(s)-\widetilde{\phi}_{\Phi_n}(s))\right|_{L^2}^2 \\
&\quad + C\left\|\widetilde{\phi}_{\Phi_n}(s)\right\|^2\left|\mathrm{A}_1\widetilde{\phi}_{\Phi_n}(s)\right|_{L^2}^2\left|\theta_n(s)\right|_{L^2}^2
\end{aligned} \tag{3.112}$$

and
$$
\begin{aligned}
&2\left\langle R_0(\widetilde{\phi}_{\Phi_n}(s),\widetilde{\phi}_{\Phi_n}(s))-R_0(\mathrm{A}_1\phi_{\Phi_n}(s),\phi_{\Phi_n}(s)),\theta_n(s)\right\rangle\\
&\leq \frac{1}{8}\left\|\theta_n(s)\right\|^2+\frac{1}{2}\left|\mathrm{A}_1\Psi_n(s)\right|^2_{L^2}+C\left|\theta_n\right|^2_{L^2}\left|\mathrm{A}_1\phi_{\Phi_n}\right|^2_{L^2}\left\|\phi_{\Phi_n}\right\|^2 \\
&+C\left|\mathrm{A}_1\widetilde{\phi}_{\Phi_n}(s)\right|^2_{L^2}(\left|\theta_n\right|^2_{L^2}+\left|\Psi_n\right|^2_{L^2}).
\end{aligned}\tag{3.113}
$$

Let

$$
Z(t)=\left|\theta(t)\right|^2_{L^2}+\left|\Psi_n(t)\right|^2_{L^2},\quad Y_2=\lambda+C\kappa+C,\quad K(t)=\left\|\theta(t)\right\|^2+\left|\mathrm{A}_1\Psi_n(t)\right|^2_{L^2}
$$

$$
\begin{aligned}
Y_1=&\,C\Big(2\left\|\mathrm{u}(s)\right\|^2+(\left|\nabla\phi_{\Phi_n}(s)\right|^2_{L^2}+\left|\phi_{\Phi_n}(s)\right|^2_{H^3})+(\left|\nabla\widetilde{\phi}_{\Phi_n}(s)\right|^2_{L^2}+\left|\widetilde{\phi}_{\Phi_n}(s)\right|^2_{H^3})\Big)\\
&+C\Big((\left|\nabla\phi(s)\right|^2_{L^2}+\left|\phi(s)\right|^2_{H^3})+\left|\mathrm{A}_1\phi_{\Phi_n}\right|^2_{L^2}\left\|\phi_{\Phi_n}\right\|^2+\left|\mathrm{A}_1\widetilde{\phi}_{\Phi_n}(s)\right|^2_{L^2}\left\|\widetilde{\phi}_{\Phi_n}(s)\right\|^2\Big)\\
&+C\left|\mathrm{A}_1\phi(s)\right|^2_{L^2}\left\|\phi(s)\right\|^2,\quad \sigma(t)=\exp\Big(-Y_2t-\int_0^t Y_1(s)ds\Big).
\end{aligned}
$$

Note that $0<y(T)\leq y(t)\leq 1$ for all $t\in[0,T]$. Using (3.96)–(3.113), it follows from Itô's formula that

$$
\begin{aligned}
&\sigma(t)Z(t)+\int_0^t\sigma(s)K(s)ds\\
&\leq C\int_0^t\sigma(s)\left[\left\|\mathrm{u}(s)-\mathrm{u}_{\Phi_n}(s)\right\|^2+\left\|\mathrm{u}(s)\right\|^2+\frac{1}{3}\right]\left\|\mathrm{u}(s)-\widetilde{\mathrm{u}}_{\Phi_n}(s)\right\|^2ds\\
&\quad+C(\lambda+\kappa)\int_0^t\sigma(s)\left\|\mathrm{u}(s)-\widetilde{\mathrm{u}}_{\Phi_n}(s)\right\|^2ds\\
&\quad+C\int_0^t\sigma(s)\Big(\left|\mathrm{A}_1\phi(s)\right|^2_{L^2}+\frac{1}{2}\Big)\left|\mathrm{A}_1(\phi(s)-\widetilde{\phi}_{\Phi_n}(s))\right|^2_{L^2}ds\\
&\quad+2\int_0^t\sigma(s)(\mathrm{g}_2(s,\mathrm{u}(s))-\mathrm{g}_2(s,\mathrm{u}_{\Phi_n}(s)),\theta_n(s))dW(s).
\end{aligned}\tag{3.114}
$$

Let $n\geq 1$ be fixed. For each $m\geq 1$, we consider the $\mathcal{F}_t$-stopping time τ_m defined by:

$$
\tau_m=\min(\tau_m^1,\tau_m^2),\tag{3.115}
$$

where
$$
\begin{aligned}
\tau_m^1&=\inf\{t\in[0,T],\left|(\mathrm{u},\phi)\right|^2_{\mathbb{H}}+\int_0^t\left\|(\mathrm{u},\phi)\right\|^2_{\mathbb{V}}\geq m^2\}\wedge T,\\
\tau_m^2&=\inf\{t\in[0,T],\left|(\widetilde{\mathrm{u}}_{\Phi_n},\widetilde{\phi}_{\Phi_n})\right|^2_{\mathbb{H}}+\int_0^t\left\|(\widetilde{\mathrm{u}}_{\Phi_n},\widetilde{\phi}_{\Phi_n})\right\|^2_{\mathbb{V}}\geq m^2\}\wedge T.
\end{aligned}
$$

Now, taking the expectation (3.114) and after some elementary calculation, we get

$$
\begin{aligned}
&\mathbb{E}\sigma(\tau_m)Z(\tau_m)+\mathbb{E}\int_0^{\tau_m}\sigma(s)K(s)ds\\
&\leq C\mathbb{E}\int_0^T\sigma(s)\left[\left\|\mathrm{u}(s)-\mathrm{u}_{\Phi_n}(s)\right\|^2+\left\|\mathrm{u}(s)\right\|^2+\frac{1}{3}\right]\left\|\mathrm{u}(s)-\widetilde{\mathrm{u}}_{\Phi_n}(s)\right\|^2ds\\
&\quad+C(\lambda+\kappa)\mathbb{E}\int_0^T\sigma(s)\left\|\mathrm{u}(s)-\widetilde{\mathrm{u}}_{\Phi_n}(s)\right\|^2ds\\
&\quad+C\mathbb{E}\int_0^T\sigma(s)\left(\left|\mathrm{A}_1\phi(s)\right|^2_{L^2}+\frac{1}{2}\right)\left|\mathrm{A}_1(\phi(s)-\widetilde{\phi}_{\Phi_n}(s))\right|^2_{L^2}ds.
\end{aligned}\tag{3.116}
$$

Thanks to the Theorem 3.10, 2.4 and (2.19) we can prove that, the right side of (3.116) goes to 0 as $n \to \infty$. Since $0 < y(T) \le y(t) \le 1$, we derive that

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E} Z(\tau_m) = \lim_{n\to\infty} \mathbb{E}\Big(\Big|(\widetilde{\mathrm{u}}_{\Phi_n}, \widetilde{\phi}_{\Phi_n})(\tau_m) - (\mathrm{u}_{\Phi_n}, \phi_{\Phi_n})(\tau_m)\Big|^2_{\mathbb{H}} \Big) = 0, \\ &\lim_{n\to\infty} \mathbb{E} \int_0^{\tau_m} \sigma(s) K(s) ds \\ &= \lim_{n\to\infty} \mathbb{E} \int_0^{\tau_m} \sigma(s) \Big\|(\widetilde{\mathrm{u}}_{\Phi_n}, \widetilde{\phi}_{\Phi_n})(s) - (\mathrm{u}_{\Phi_n}, \phi_{\Phi_n})(s)\Big\|^2_{\mathbb{V}} ds = 0. \end{aligned} \tag{3.117}$$

Thanks to the Theorem 3.10 and the triangle inequality, we obtain

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E} \left|(\mathrm{u}, \phi)(\tau_m) - (\mathrm{u}_{\Phi_n}, \phi_{\Phi_n})(\tau_m)\right|^2_{\mathbb{H}} = 0, \\ &\lim_{n\to\infty} \mathbb{E} \int_0^{\tau_m} \|(\mathrm{u}, \phi)(s) - (\mathrm{u}_{\Phi_n}, \phi_{\Phi_n})(s)\|^2_{\mathbb{V}} \, ds = 0. \end{aligned} \tag{3.118}$$

By Lemma 3.9, applied to $\tau_m := T$,

$$Q_n(\tau_m); = |(\mathrm{u}, \phi)(\tau_m) - (\mathrm{u}_{\Phi_n}, \phi_{\Phi_n})(\tau_m)|^2_{\mathbb{H}}$$

respectively, $\quad Q_n(\tau_m) := \int_0^{\tau_m} \|(\mathrm{u}, \phi)(s) - (\mathrm{u}_{\Phi_n}, \phi_{\Phi_n})(s)\|^2_{\mathbb{V}} \, ds,$

we get

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E} \left|(\mathrm{u}, \phi)(T) - (\mathrm{u}_{\Phi_n}, \phi_{\Phi_n})(T)\right|^2_{\mathbb{H}} = 0, \\ &\lim_{n\to\infty} \mathbb{E} \int_0^{T} \|(\mathrm{u}, \phi)(s) - (\mathrm{u}_{\Phi_n}, \phi_{\Phi_n})(s)\|^2_{\mathbb{V}} \, ds = 0. \end{aligned} \tag{3.119}$$

Similary, we prove that

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E} \int_0^{T} ||(\mathrm{u}_{\Phi}, \phi_{\Phi})(s) - (\mathrm{u}_{n,\Phi_n}, \phi_{n,\Phi_n})(s)||^2_{\mathbb{V}} ds = 0, \\ &\lim_{n\to\infty} \mathbb{E} |(\mathrm{u}_{\Phi}, \phi_{\Phi})(T) - (\mathrm{u}_{n,\Phi_n}, \phi_{n,\Phi_n})(T)|^2_{\mathbb{H}} = 0. \end{aligned} \tag{3.120}$$

This ends the proof of Theorem 3.12. □

Finally, we are in a position to formulate the main result of the present work

Theorem 3.13. *There exists an optimal feedback control for problem* $(\mathcal{P})$.

Proof. Let (Φ_n) be a minimizing sequence for problem $(\mathcal{P})$ to which we apply Lemma 3.2 and Theorem 3.12. Therefore, there exists a subsequence (n') of (n) and $\Phi \in \mathcal{U}$ such that for all $t \in [0, T]$, $y \in$ H_1 and a.e., $\omega \in \Omega$ the following hold:

$$(\Phi(t, \mathrm{u}_{\Phi}(t)), y) = \lim_{n'\to\infty} (\Phi_{n'}(t, \mathrm{u}_{\Phi_{n'}}(t)), y),$$

and

$$\begin{aligned} &\lim_{n'\to\infty} \mathbb{E} \int_0^{T} ||(\mathrm{u}_{\Phi}, \phi_{\Phi})(s) - (\mathrm{u}_{\Phi_{n'}}, \phi_{\Phi_{n'}})(s)||^2_{\mathbb{V}} ds = 0, \\ &\lim_{n'\to\infty} \mathbb{E} |(\mathrm{u}_{\Phi}, \phi_{\Phi})(T) - (\mathrm{u}_{\Phi_{n'}}, \phi_{\Phi_{n'}})(T)|^2_{\mathbb{H}} = 0. \end{aligned}$$

From Theorem 3.12, Fatou's Lemma and the weak sequentially semi-continuity properties of $\mathcal{L}, \mathcal{K}_1, \mathcal{K}_2$ (see Subsection 3.1) we have

$$\mathcal{J}(\Phi) \leq \liminf_{n' \to \infty} \mathcal{J}(\Phi_{n'}).$$

But (Φ_n) is a minimizing sequence for problem $(\mathcal{P})$. Hence

$$\mathcal{J}(\Phi) = \min_{\Psi \in \mathcal{U}} \mathcal{J}(\Psi),$$

and, therefore, $\Phi \in \mathcal{U}$ is an optimal feedback control for problem $(\mathcal{P})$. $\square$

4. Existence of ϵ-optimal feedback controls

For $(\mathcal{P})$ we formulate the corresponding n-dimensional control problem

$$\begin{cases} \mathcal{J}_n(\Phi_n) \to \min, \\ \Phi_n \in \mathcal{U}_n, \end{cases} \tag{4.1}$$

where
$$\mathcal{J}_n(\Phi_n) = \mathbb{E} \int_0^T (\mathcal{L}[s, \mathrm{u}_{n,\Phi_n}(s), \phi_{n,\Phi_n}(s)] + \mathcal{K}_1[\Phi_n(s, \mathrm{u}_{n,\Phi_n}(s))])ds \\ + \mathbb{E}\, \mathcal{K}_2[\mathrm{u}_{n,\Phi_n}(T), \phi_{n,\Phi_n}(T)], \quad \Phi_n \in \mathcal{U}_n$$

and
$$\mathcal{U}_n := \{\Phi_n : [0,T] \times \mathrm{H}_n \to \mathrm{H}_n | \Phi_n = \mathcal{P}_n^1 \Phi,\ \Phi \in \mathcal{U}\}.$$

Here $(\mathrm{u}_{n,\Phi_n}, \phi_{n,\Phi_n})$ is the solution of (2.17) using the feedback control $\Phi_n \in \mathcal{U}_n$ and $\mathcal{L}, \mathcal{K}_1, \mathcal{K}_2$ are mappings satisfying the assumptions given in subsection 3.1. Further, we will assume that these mappings also satisfy the following properties:

$$|\mathcal{L}(t, (\mathrm{u}, \phi)) - \mathcal{L}(t, (\mathrm{v}, \psi))| \leq C_{\mathcal{L}} ||(\mathrm{u}, \phi) - (\mathrm{v}, \psi)||_{\mathbb{V}}(||(\mathrm{u}, \phi)||_{\mathbb{V}} + ||(\mathrm{v}, \psi)||_{\mathbb{V}}), \\ \text{for all } (\mathrm{v}, \psi), (\mathrm{u}, \phi) \in \mathbb{V},\ t \in [0, T],$$

$$|\mathcal{K}_1(\mathrm{u}) - \mathcal{K}_1(\mathrm{v})| \leq C_{\mathcal{K}_1} |\mathrm{u} - \mathrm{v}|_{L^2}(|\mathrm{u}|_{L^2} + |\mathrm{v}|_{L^2}), \text{ for all } \mathrm{u}, \mathrm{v} \in \mathrm{H}_1,$$

$$|\mathcal{K}_2(\mathrm{u}, \phi) - \mathcal{K}_2(\mathrm{v}, \psi)| \leq C_{\mathcal{K}_2} |(\mathrm{u}, \phi) - (\mathrm{v}, \psi)|_{\mathbb{H}}(|(\mathrm{u}, \phi)|_{\mathbb{H}} + |(\mathrm{v}, \psi)|_{\mathbb{H}}), \\ \text{for all } (\mathrm{u}, \phi), (\mathrm{v}, \psi) \in \mathbb{H}.$$

Theorem 4.1. *Let (Φ_n) be a sequence in $\mathcal{U}$ such that $\Phi_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$. There exists a subsequence (n_k) of (n) such that*

$$\lim_{k \to \infty} \mathbb{E} \int_0^T ||(u_{\Phi_{n_k}}, \phi_{\Phi_{n_k}})(s) - (u_{n_k,\Phi_{n_k}}, \phi_{n_k,\Phi_{n_k}})(s)||_{\mathbb{V}}^2 ds = 0,$$

$$\lim_{k \to \infty} \mathbb{E}|(u_{\Phi_{n_k}}, \phi_{\Phi_{n_k}})(T) - (u_{n_k,\Phi_{n_k}}, \phi_{n_k,\Phi_{n_k}})(T)|_{\mathbb{H}}^2 = 0.$$

where $(u_{\Phi_{n_k}}, \phi_{\Phi_{n_k}})$ and $(u_{n_k,\Phi_{n_k}}, \phi_{n_k,\Phi_{n_k}})$ are the solutions of (2.14) *and* (2.17), *respectively, using the feedback control Φ_{n_k}.*

Proof. Lemma 3.2 applied to the sequence (Φ_n) implies the existence of a subsequence (n_k) of (n) and $\Phi \in \mathcal{U}$ such that for all $t \in [0,T]$, $x, y \in \mathrm{H}_1$,

$$(\Phi(t,x), y) = \lim_{k\to\infty} (\Phi_{n_k}(t,x), y).$$

By Theorem 3.12, we deduce that

$$\begin{aligned} &\lim_{k\to\infty} \mathbb{E}\int_0^T ||(\mathrm{u}_\Phi, \phi_\Phi)(s) - (\mathrm{u}_{\Phi_{n_k}}, \phi_{\Phi_{n_k}})(s)||_{\mathbb{V}}^2 ds = 0, \\ &\lim_{k\to\infty} \mathbb{E}|(\mathrm{u}_\Phi, \phi_\Phi)(T) - (\mathrm{u}_{\Phi_{n_k}}, \phi_{\Phi_{n_k}})(T)|_{\mathbb{H}}^2 = 0, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &\lim_{k\to\infty} \mathbb{E}\int_0^T ||(\mathrm{u}_\Phi, \phi_\Phi)(s) - (\mathrm{u}_{n_k,\Phi_{n_k}}, \phi_{n_k,\Phi_{n_k}})(s)||_{\mathbb{V}}^2 ds = 0, \\ &\lim_{k\to\infty} \mathbb{E}|(\mathrm{u}_\Phi, \phi_\Phi)(T) - (\mathrm{u}_{n_k,\Phi_{n_k}}, \phi_{n_k,\Phi_{n_k}})(T)|_{\mathbb{H}}^2 = 0. \end{aligned} \tag{4.3}$$

From (4.2) and (4.3), we have

$$\begin{aligned} &\lim_{k\to\infty} \mathbb{E}\int_0^T ||(\mathrm{u}_{\Phi_{n_k}}, \phi_{\Phi_{n_k}})(s) - (\mathrm{u}_{n_k,\Phi_{n_k}}, \phi_{n_k,\Phi_{n_k}})(s)||_{\mathbb{V}}^2 ds = 0, \\ &\lim_{k\to\infty} \mathbb{E}|(\mathrm{u}_{\Phi_{n_k}}, \phi_{\Phi_{n_k}})(T) - (\mathrm{u}_{n_k,\Phi_{n_k}}, \phi_{n_k,\Phi_{n_k}})(T)|_{\mathbb{H}}^2 = 0. \end{aligned} \tag{4.4}$$

□

Theorem 4.2. *Let $\Phi \in \mathcal{U}$ be an optimal control for problem $(\mathcal{P})$. Then there exists a sequence (n_k) such that for arbitrary $\epsilon > 0$ we find $n_\epsilon \in \mathbb{N}$ with the property: for all $n_k \geq n_\epsilon$*

$$|\mathcal{J}_{n_k}(\Phi_{n_k}) - \mathcal{J}(\Phi)| < \epsilon, \quad \mathcal{J}(\Phi_{n_k}) - \mathcal{J}(\Phi) < \epsilon,$$

where $\Phi_{n_k} \in \mathcal{U}_{n_k}$ is an optimal control for the n_k-dimensional control problem (2.17)*, hence Φ_{n_k} is an ϵ-optimal control for problem $(\mathcal{P})$.*

Proof. Let $n \in \mathbb{N}$, $\Psi_n \in \mathcal{U}_n$, $\Psi \in \mathcal{U}$ be arbitrary. Taking into consideration the assumptions on $\mathcal{L}$, $\mathcal{K}_1$, $\mathcal{K}_2$ given at the beginning of this section, Theorems 2.4, 2.5, Remark 3.1 and the Schwarz inequality for integrals, we have

$$\begin{aligned} &\mathbb{E}\int_0^T |\mathcal{L}[s, (\mathrm{u}_{n,\Psi_n}(s), \phi_{n,\Psi_n}(s))] - \mathcal{L}[s, (\mathrm{u}_\Psi(s), \phi_\Psi(s))]| ds \\ &\leq CC_{\mathcal{L}}\Big(\mathbb{E}\int_0^T ||(\mathrm{u}_{n,\Psi_n}(s), \phi_{n,\Psi_n}(s)) - (\mathrm{u}_\Psi(s), \phi_\Psi(s))||_{\mathbb{V}}^2 ds\Big)^{1/2}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} &\mathbb{E}\int_0^T |\mathcal{K}_1[\Psi(s, \mathrm{u}_{n,\Psi_n}(s))] - \mathcal{K}_1[\Psi(s, \mathrm{u}_\Psi(s))]| ds \\ &\leq CC_{\mathcal{K}_1}\Big[\Big(\kappa\mathbb{E}\int_0^T |\mathrm{u}_{n,\Psi_n}(s) - \mathrm{u}_\Psi(s)|_{\mathrm{L}^2}^2 ds\Big)^{1/2}\Big] \\ &\quad + CC_{\mathcal{K}_1}\left(\int_0^T |\Psi(s, \mathrm{u}_{n,\Psi_n}(s)) - \Psi(s, \mathrm{u}_\Psi(s))|_{\mathrm{L}^2}^2 ds\right)^{1/2}, \end{aligned} \tag{4.6}$$

$$\text{and}\qquad \mathbb{E}\int_0^T |\mathcal{K}_2[(\mathrm{u}_{n,\Psi_n}(T),\phi_{n,\Psi_n}(T))]-\mathcal{K}_2[(\mathrm{u}_\Psi(T),\phi_\Psi(T))]|$$
$$\leq CC_{\mathcal{K}_2}\left(\mathbb{E}\int_0^T ||(\mathrm{u}_{n,\Psi_n}(T),\phi_{n,\Psi_n}(T))-(\mathrm{u}_\Psi(T),\phi_\Psi(T))||^2_{\mathbb{H}}\right)^{1/2}. \tag{4.7}$$

Let $\epsilon > 0$ and take $\epsilon^* := \dfrac{\epsilon^2}{2C(C_{\mathcal{L}}+C_{\mathcal{K}_1}(\sqrt{\kappa}+1)+C_{\mathcal{K}_2})}$.

The properties of $\mathcal{P}^1_n$ (see Remark 2.6) imply

$$\Phi(t,x)=\lim_{n\to\infty}\mathcal{P}^1_n\Phi(t,x)\quad\text{and}\quad(\Phi(t,x),y)=\lim_{n\to\infty}(\mathcal{P}^1_n\Phi(t,x),y),$$

for all $t\in[0,T]$, $x,y\in \mathrm{H}_1$. Using the ideas from Theorems 3.10 and 3.12, it follows that there exists an $m_\epsilon>0$ such that for all $m\geq m_\epsilon$ the following hold:

$$\mathbb{E}\int_0^T ||(\mathrm{u}_\Phi,\phi_\Phi)(s)-(\mathrm{u}_{m,\mathcal{P}^1_m\Phi},\phi_{m,\mathcal{P}^1_m\Phi})(s)||^2_{\mathbb{V}}ds+\mathbb{E}\int_0^T|\mathrm{u}_\Phi(s)-\mathrm{u}_{m,\mathcal{P}^1_m\Phi}(s)|^2_{L^2}ds$$
$$+\mathbb{E}|(\mathrm{u}_\Phi,\phi_\Phi)(T)-(\mathrm{u}_{m,\mathcal{P}^1_m\Phi},\phi_{m,\mathcal{P}^1_m\Phi})(T)|^2_{\mathbb{H}}\ \leq\ \epsilon^*, \tag{4.8}$$

$$\text{and}\qquad \mathbb{E}\int_0^T|\mathcal{P}^1_n\Phi(s,\mathrm{u}_{m,\mathcal{P}^1_m\Phi}(s))-\Phi(s,\mathrm{u}_\Phi(s))|^2_{L^2}ds\ \leq\ \epsilon^*. \tag{4.9}$$

Let (Φ_n) be an optimal control for the n-dimensional control problem (4.1). By Theorem 4.1 applied on (Φ_n), there exists a subsequence (n_k) of (n) and $n_\epsilon\leq m_\epsilon$ such that for all $n_k\geq n_\epsilon$ we have

$$\mathbb{E}\int_0^T||(\mathrm{u}_{\Phi_{n_k}},\phi_{\Phi_{n_k}})(s)-(\mathrm{u}_{n_k,\Phi_{n_k}},\phi_{n_k,\Phi_{n_k}})(s)||^2_{\mathbb{V}}ds+\mathbb{E}\int_0^T|\mathrm{u}_{\Phi_{n_k}}(s)-\mathrm{u}_{n_k,\Phi_{n_k}}(s)|^2_{L^2}ds$$
$$+\mathbb{E}|(\mathrm{u}_{\Phi_{n_k}},\phi_{\Phi_{n_k}})(T)-(\mathrm{u}_{n_k,\Phi_{n_k}},\phi_{n_k,\Phi_{n_k}})(T)|^2_{\mathbb{H}}\ \leq\ \epsilon^*, \tag{4.10}$$

First case: $\mathcal{J}_{n_k}(\Phi_{n_k})-\mathcal{J}(\Phi)\geq 0$.

Then, by using the properties (4.5)–(4.7) for $\Psi_{n_k}=\mathcal{P}^1_{n_k}\Phi$, $\Psi=\Phi$ and (4.10) we have for all $n_k\geq n_\epsilon$

$$0\leq\mathcal{J}_{n_k}(\Phi_{n_k})-\mathcal{J}(\Phi)\leq\mathcal{J}_{n_k}(\mathcal{P}^1_{n_k}\Phi_{n_k})-\mathcal{J}(\Phi)$$
$$\leq C(C_{\mathcal{L}}+C_{\mathcal{K}_1}(\sqrt{\kappa}+1)+C_{\mathcal{K}_2})\sqrt{\epsilon^*}=\frac{\epsilon}{2}.$$

Second case: $\mathcal{J}_{n_k}(\Phi_{n_k})-\mathcal{J}(\Phi)<0$.

Then, by using the properties (4.5)–(4.7) for $\Psi_{n_k}=\Phi_{n_k}$, $\Psi=\Phi_{n_k}$ and (4.10) we have for all $n_k\geq n_\epsilon$

$$0<-\mathcal{J}_{n_k}(\Phi_{n_k})+\mathcal{J}(\Phi)$$
$$\leq\mathcal{J}(\Phi_{n_k})-\mathcal{J}_{n_k}(\Phi_{n_k})\leq C(C_{\mathcal{L}}+C_{\mathcal{K}_1}\sqrt{\kappa}+C_{\mathcal{K}_2})\sqrt{\epsilon^*}\ <\ \frac{\epsilon}{2}. \tag{4.11}$$

Hence, for all $n_k\geq n_\epsilon$, we have:

$$|\mathcal{J}_{n_k}(\Phi_{n_k})-\mathcal{J}(\Phi)|<\frac{\epsilon}{2}<\epsilon.$$

Using this inequality and (4.11) we get

$$0 \leq \mathcal{J}(\Phi_{n_k}) - \mathcal{J}(\Phi) \leq |\mathcal{J}(\Phi_{n_k}) - \mathcal{J}_{n_k}(\Phi_{n_k})| + |\mathcal{J}_{n_k}(\Phi_{n_k} - \mathcal{J}(\Phi))|$$
$$< C(C_{\mathcal{L}} + C_{\mathcal{K}_1}\sqrt{\kappa} + +C_{\mathcal{K}_2})\sqrt{\epsilon^*} + \frac{\epsilon}{2} < \epsilon.$$

The sequence (n_k), $n_k \geq n_\epsilon$ and Φ_{n_k} obtained above satisfy the conclusion of theorem. □

Remark 4.3. In [29] and [26] the author investigate the stochatic Navier-Stokes equations controlled by different external forces. They gives a cost functional. For our problem this would be

$$\mathcal{L}[t, (\mathrm{u}_\Phi(t), \phi_\Phi(t))] := ||\mathrm{u}_\Phi(t)||^2 + |\mathrm{A}_1\phi_\Phi(t)|^2_{L^2}, \quad \mathcal{K}_1[\Phi(t)] := |\Phi(t)|^2_{L^2}, \quad \mathcal{K}_2 := 0$$

and $\mathcal{U} := L^2(0, T; \mathrm{H}_1)$.

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References

[1] F. Abergel, R. Temam: *On some control problems in fluid mechanics*, Theor. Comput. Fluid Dynamics 1 (1990) 303–325.

[2] P. Benner, C. Trautwein: *Optimal control problems constrained by the stochastic Navier–Stokes equations with multiplicative Lévy noise*, Math. Nachrichten 292/7 (2019) 1444–1461.

[3] P. Benner, C. Trautwein: *A stochastic maximum principle for control problems constrained by the stochastic Navier–Stokes equations*, Appl. Math. Optimization 84 (2021) 1001–1054.

[4] P. Benner, C. Trautwein: *Optimal distributed and tangential boundary control for the unsteady stochastic Stokes equations*, ESAIM, Control Optim. Calc. Var. 26 (2020), art.no. 62, 32p.

[5] A. Bensoussan: *Stochastic maximum principle for distributed parameter systems*, J. Franklin Inst. 315 (1983) 387–406.

[6] T. Biswas, S. Dharmatti, M. T. Mohan: *Pontryagin maximum principle and second order optimality conditions for optimal control problems governed by 2D nonlocal Cahn-Hilliard Navier-Stokes equations*, Analysis 40/3 (2020) 127–150.

[7] T. Biswas, S. Dharmatti, M. T. Mohan, *Maximum principle and data assimilation problem for the optimal control problems governed by 2D nonlocal Cahn-Hilliard-Navier-Stokes equations*, J. Math. Fluid Mech. 22 (2020) 1–42.

[8] C. Cao, G. Gal: *Global solutions for the 2D NS-CH model for a two-phase flow of viscous, incompressible fluids with mixed partial viscosity and mobility*, Nonlinearity 25/11 (2012) 3211–3234.

[9] G. Da Prato: *Some results on Bellman equation in Hilbert spaces*, SIAM J. Control Optim. 23 (1985) 61–71.

[10] G. Deugoué, T. T. Medjo: *Convergence of the solutions of the stochastic 3D globally modified Cahn-Hilliard-Navier-Stokes equations*, J. Diff. Equations 265 (2018) 545–592.

[11] G. Deugoué, T. T. Medjo: *The exponential behavior of a stochastic globally modified Cahn-Hilliard-Navier-Stokes*, J. Math. Analysis Appl. 460/1 (2018) 140–163.

[12] G. Deugoué, T. T. Medjo: *The exponential behavior of a stochastic globally modified Cahn-Hilliard-Navier-Stokes model with multiplicative noise*, J. Math. Anal. Appl. 460/1 (2018) 140–163.

[13] G. Deugoué, T. T. Medjo: *On a stochastic 2D Cahn-Hilliard-Navier-Stokes system driven by jump noise*, Comm. Stochastic Analysis 13/1 (2019), art. no. 5.

[14] G. Deugoué, B. J. Moghomye, T. T. Medjo: *Approximation of a stochastic two-phase flow model by a splitting-up method*, Comm. Pures Appl. Analysis 20/3 (2021) 1135–1170.

[15] G. Deugoué, B. J. Moghomye, T. T. Medjo: *Splitting-up scheme for the stochastic Cahn–Hilliard Navier–Stokes model*, Stochastics Dynamics 21/1 (2021), art.no. 2150005, 46p.

[16] G. Deugoué, A. N. Ngana, T. T. Medjo: *Martingale solutions to stochastic nonlocal Cahn-Hilliard-Navier-Stokes equations with multiplicative noise of jump type*, Physica D: Nonlinear Phenomena 398 (2019) 23–68.

[17] G. Deugoué, A. N. Ngana, T. T. Medjo: *Strong solutions for the stochastic Cahn-Hilliard-Navier-Stokes system*, J. Diff. Equations 275 (2020) 27–76.

[18] I. Ekeland, R. Temam: *Convex Analysis and Variational Problems*, SIAM Series Classics in Applied Mathematics, Society for Industrial and Applied Mathematics, Philadelphia (1999).

[19] S. Frigeri, C. G. Gal, M. Grasselli: *On nonlocal Cahn-Hilliard-Navier-Stokes systems in two dimensions*, J. Nonlinear Sci. 24/4 (2016) 847–893.

[20] S. Frigeri, M. Grasselli, J. Sprekels: *Optimal distributed control of two-dimensional nonlocal Cahn-Hilliard-Navier-Stokes systems with degenerate mobility and singular potential*, Appl. Math. Optim. 81/3 (2020) 899–931.

[21] C. Gal, M. Grassell: *Asymptotic behavior of a Cahn-Hilliard-Navier-Stokes system in 2D*, Ann. Inst. H. Poincaré Anal. Non Linéaire 27/1 (2010) 401–436.

[22] N. Glatt-Holtz, Z. Mohammed: *Strong pathwise solutions of the stochastic Navier-Stokes*, Adv. Diff. Equations 14/5-6 (2009) 567–600.

[23] Y. Hu, S. Peng, *Maximum principle for optimal control of stochastic system of functional type*, Stochastic Anal. Appl. 14 (1996) 283–301.

[24] A. I. Komech, M. I. Vishik: *Statistical solutions of the Navier–Stokes and Euler equations (Russian)*, Adv. Mech. 5 (1982) 65–120.

[25] H. Kunita: *Stochastic Flows and Stochastic Differential Equations*, Cambridge University Press, Cambridge (1997).

[26] H. Lisei: *Approximation and Optimal Control of the Stochastic Navier-Stokes Equation*, Dissertation, Martin-Luther University, Halle-Wittenberg (1999).

[27] H. Lisei: *Galerkin approximation and the strong solution of the stochastic Navier–Stokes equation*, J. Appl. Math Stoch. Anal. 13 (2000) 239–259.

[28] H. Lisei: *Approximation of the solution of the stochastic Navier-Stokes equation*, Optimization 49/1-2 (2001) 15–38.

[29] H. Lisei: *Existence of optimal and ϵ-optimal controls for the stochastic Navier-Stokes equation*, Nonlinear Analysis 51 (2002) 95–119.

[30] I. N'Doye, M. Zasadzinski, M. Darouach, N.-E. Radhy, A. Bouaziz: *Exponential stabilization of a class of nonlinear systems: A generalized Gronwall–Bellman lemma approach*, Nonlinear Anal. Theory Methods Appl. 74/18 (2011) 7333–7341.

[31] B. Oksendal: *Stochastic Differential Equations*, Springer, Berlin (1985).

[32] T. T. Medjo: *Optimal control of a Cahn-Hilliard-Navier-Stokes model with state constraints*, J. Convex Analysis 22/4 (2015) 1135–1172.

[33] T. T. Medjo: *Robust control of a Cahn-Hilliard-Navier-Stokes model*, Comm. Pure Appl. Analysis 15/6 (2016) 2075–2101.

[34] T. T. Medjo: *On the existence and uniqueness of solution to a stochastic 2D Cahn-Hilliard-Navier-Stokes model*, J. Diff. Equations 263/2 (2017) 1028–1054.

[35] T. T. Medjo: *On the existence and uniqueness of solution to a stochastic 2D Allen-Cahn-Navier-Stokes model*, Stoch. Dyn. 19/1 (2019), art. no. 1950007, 28 p.

[36] L. Scarpa: *Optimal distributed control of a stochastic Cahn-Hilliard equation*, SIAM J. Control Optim. 57/5 (2019) 3571–3601.

[37] S. S. Sritharan (ed.): *Optimal Control of Viscous Flow*, SIAM Frontiers in Applied Mathematics, Society for Industrial and Applied Mathematics, Philadelphia (1997).

[38] S. S. Sritharan: *Deterministic and stochastic control of Navier-Stokes equation with linear, monotone, and hyperviscosities*, Appl. Math. Optim. 41 (2000) 255–308.

[39] C. Tudor: *ϵ-Optimal and optimal control for the stochastic linear-quadratic problem*, Math. Nachrichten 145 (1990) 135–149.

[40] M. I. Vishik, A. V. Fursikov: *Mathematical Problems of Statistical Hydromechanics*, Kluwer, London (1980).

[41] E. Zeidler: *Nonlinear Functional Analysis and its Applications. I: Fixed-Point Theorems*, Springer, New York (1986).

[42] E. Zeidler: *Nonlinear Functional Analysis and its Applications. III: Variational Methods and Optimization*, Springer, New York (1985).