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Characterizations of Metric (Sub)Regularity via (Sub)Transversality

Stoyan Apostolov
Faculty of Mathematics and Informatics, Sofia University, Sofia, Bulgaria
sapostolov@fmi.uni-sofia.bg,

Mira Bivas
Faculty of Mathematics and Informatics, Sofia University, Sofia, Bulgaria; and: Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria
mira.bivas@math.bas.bg

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We obtain some classical characterizations of metric regularity and subregularity in a new and unified way using results for transversality and subtransversality from a recent paper by S. Apostolov, M. Bivas, and N. Ribarska [*Characterizations of some transversality-type properties*, Set-Valued Var. Analysis 30/3 (2022), art. no. 1041-1060] and the equivalence between (sub)transversality and (sub)regularity.

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1. Introduction

Transversality is a classical concept of mathematical analysis and differential topology. Recently, it has proven to be useful in other research areas as well. As it is stated in [16], “the transversality oriented language is extremely natural and convenient in some parts of variational analysis, including subdifferential calculus and nonsmooth optimization”.

There is no need to comment the importance of the metric regularity property in variational analysis – it is already a central concept, whose roots go back to classical results such as the implicit function theorem, Banach open mapping theorem and the theorems of Lyusternik and Graves.

In this paper, we state the relations between transversality/subtransversality and metric regularity/subregularity and use them to obtain some classical results about regularity-type properties in a new and unified way.

Transversality and metric regularity have been widely studied in the last decades – see e.g. the recent books [10] and [17]. The equivalence of transversality of sets

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and regularity of some associated maps was established 20 years ago in [14] and [15]. The other way around – reducing regularity properties to the corresponding transversality properties of two sets – has been examined recently in [5], [6] and [7] (see also [21]).

Here, we state clearly their relation and we transfer our results about transversality-type properties from [1] to the corresponding regularity-type ones, opposed to the standard approach – from regularity-type to transversality-type properties. We avoid the repeated use of Ekeland variational principle by using a rather general technical result (Lemma 4.1) in the spirit of Lemma 3.4 in [1]. This saves us from seeking for the "right" function in every particular case and enables us to give simple and straightforward proofs. Thus we obtain the classical "rate of descent" (cf., e.g., Theorem 2.50 and Theorem 2.58 in [17]) and tangential characterizations (cf., e.g., [9] and [13]) of metric regularity and subregularity in a unified way. The other key piece in our scheme is the primal space characterization of subtransversality from [1].

Another type of criteria for metric (sub)regularity of set-valued mappings are those formulated in terms of slopes (introduced in [8]) of suitable single-valued functions. This approach is initiated by Azé, Corvellec and Lucchetti in [3] and developed in [2] and [15] (see also [22] for metric subregularity). In this paper we do not consider such characterizations.

The paper is organized as follows: The second section contains some necessary definitions and known assertions. Some basic relations between subtransversality and subregularity and between transversality and regularity are obtained in Section 3. In Section 4 we state our main technical result, whose proof is given in the Appendix, and use it to characterize subregularity in primal space terms. Primal space characterizations of regularity are obtained in Section 5.

2. Preliminaries

Throughout the paper if (X, d) is a metric space, $\mathbf{B}_r(x_0)$ will denote the open ball centered at x_0 with radius r: $\mathbf{B}_r(x_0) := \{x \in Y \mid d(x, x_0) < r\}$. The closed ball will be denoted by $\overline{\mathbf{B}}_r(x_0)$.

In what follows, for given metric spaces X and Y, F is called a set-valued map between X and Y, denoted by $F : X \rightrightarrows Y$, if $F : X \to 2^Y$. The graph of F, denoted by Gr F, is defined by

$$\text{Gr } F := \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

The inverse map of F, $F^{-1} : Y \rightrightarrows X$ is defined by

$$F^{-1}(y) := \{x \in X \mid y \in F(x)\}, \text{ whenever } y \in Y.$$

We endow the Cartesian product $X \times Y$ of the metric spaces X and Y, with the metric $d((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2)$ for the sake of simplicity. The particular choice of the metric is relevant only to the constants involved. However, our goal in this paper is to derive qualitative results, so that we are not concerned with the constants.

We remind the already classical definitions.

Definition 2.1. Let X and Y be metric spaces, $F : X \rightrightarrows Y$ and $(\overline{x}, \overline{y}) \in \mathrm{Gr}\ F$. We say that F is *(metrically) regular* at $(\overline{x}, \overline{y})$ if there exist $K > 0$ and $\delta > 0$ such that for all $x \in \mathbf{B}_\delta(\overline{x})$ and all $y \in \mathbf{B}_\delta(\overline{y})$ the following inequality holds:

$$d(x, F^{-1}(y)) \leq Kd(y, F(x)).$$

Definition 2.2. Let X and Y be metric spaces, $F : X \rightrightarrows Y$ and $(\overline{x}, \overline{y}) \in \mathrm{Gr}\ F$. We say that F is *(metrically) subregular* at $(\overline{x}, \overline{y})$ if there exist $K > 0$ and $\delta > 0$ such that for all $x \in \mathbf{B}_\delta(\overline{x})$ the following inequality holds:

$$d(x, F^{-1}(\overline{y})) \leq Kd(\overline{y}, F(x)).$$

Assume that A and B are subsets of the normed space X. Consider the function $H_{A,B} : X \times X \to X$ defined as

$$H_{A,B}(x_1, x_2) = \begin{cases} \{x_1 - x_2\}, & x_1 \in A,\ x_2 \in B \\ \varnothing, & \text{else} \end{cases} \tag{1}$$

Definition 2.3. Let X be a Banach space, and $A,\ B$ be closed subsets of X. Let $\overline{x} \in A \cap B$. Then A and B are called transversal at $\overline{x}$ if $H_{A,B}$ is regular at $((\overline{x}, \overline{x}), \mathbf{0})$.

Definition 2.4. Let X be a Banach space, and $A,\ B$ be closed subsets of X. Let $\overline{x} \in A \cap B$. Then A and B are called subtransversal at $\overline{x}$ if $H_{A,B}$ is subregular at $((\overline{x}, \overline{x}), \mathbf{0})$.

A characterization of transversality derived in [15] (cf. [21]) is

Proposition 2.5. *Let A and B be closed subsets of the normed space X. A and B are transversal at $\overline{x} \in A \cap B$, if and only if there exists $K > 0$ and $\delta > 0$ such that*

$$d(x, (A - a) \cap (B - b)) \leq K(d(x, A - a) + d(x, B - b))$$

for all $x \in \overline{\mathbf{B}}_\delta(\overline{x})$ and $a, b \in \overline{\mathbf{B}}_\delta(\mathbf{0})$.

One observes that only one of the sets could be translated, i.e. we may take $a = 0$ and only vary b.

When a and b are fixed to be $\mathbf{0}$ in the last definition, a similar characterization of subregularity is obtained (cf. [15]):

Proposition 2.6. *Let A and B be closed subsets of the complete metric space X. A and B are subtransversal at $\overline{x} \in A \cap B$, if and only if there exists $K > 0$ and $\delta > 0$ such that*

$$d(x, A \cap B) \leq K(d(x, A) + d(x, B)) \quad \text{for all } x \in \mathbf{B}_\delta(\overline{x}).$$

Thus we observe that A and B are transversal at $\overline{x} \in A \cap B$ if and only if the subtransversality inequality holds for $A - a$ and $B - b$ with constants K and δ for all $x \in \overline{\mathbf{B}}_\delta(\overline{x})$ and $a, b \in \overline{\mathbf{B}}_\delta(\mathbf{0})$.

It is worth noting that while the definitions of transversality and subtransversality clearly make use of the linear structure, the characterization of subtransversality, given by Proposition 2.6, is purely metric. Thus, one may think of subtransversality as a metric concept. However, all characterizations of transversality use the linear structure.

The following characterization of subtransversality is from our recent paper [1].

Definition 2.7. Let A and B be closed subsets of the metric space X and $\overline{x} \in X$. We say that A and B have *property* $(\mathcal{T})$ at $\overline{x}$ if there exist $\delta > 0$ and $M > 0$ such that $A \cap \overline{\mathbf{B}}_{\frac{\delta}{2(1+2M)}}(\overline{x}) \neq \varnothing$, $B \cap \overline{\mathbf{B}}_{\frac{\delta}{2(1+2M)}}(\overline{x}) \neq \varnothing$ and for any $x^A \in A \cap \overline{\mathbf{B}}_\delta(\overline{x})$ and $x^B \in B \cap \overline{\mathbf{B}}_\delta(\overline{x})$ with $x^A \neq x^B$ there exist $\theta > 0$, $\hat{x}^A \in A$ and $\hat{x}^B \in B$ such that

$$d(x^A, \hat{x}^A) \leq \theta M \ , \quad d(x^B, \hat{x}^B) \leq \theta M \quad \text{and} \quad d(\hat{x}^A, \hat{x}^B) \leq d(x^A, x^B) - \theta \ .$$

Equivalently, A and B have *property* $(\mathcal{T})$ at $\overline{x}$ if and only if there exist $\delta > 0$ and $M > 0$ such that $A \cap \overline{\mathbf{B}}_{\frac{\delta}{2(1+2M)}}(\overline{x}) \neq \varnothing$, $B \cap \overline{\mathbf{B}}_{\frac{\delta}{2(1+2M)}}(\overline{x}) \neq \varnothing$ and for any $x^A \in A \cap \overline{\mathbf{B}}_\delta(\overline{x})$ and $x^B \in B \cap \overline{\mathbf{B}}_\delta(\overline{x})$ with $x^A \neq x^B$ there exist $\hat{x}^A \in A$ and $\hat{x}^B \in B$ such that

$$d(\hat{x}^A, \hat{x}^B) \leq d(x^A, x^B) - \frac{1}{M} \max\{d(x^A, \hat{x}^A), d(x^B, \hat{x}^B)\}$$

and $\max\{d(x^A, \hat{x}^A), d(x^B, \hat{x}^B)\} > 0$.

Proposition 2.8. (Corollary 3.6 in [1]) *If $\overline{x} \in A \cap B$, where A and B are closed subsets of the complete metric space X, then A and B have property $(\mathcal{T})$ at $\overline{x}$ if and only if A and B are subtransversal at $\overline{x}$.*

3. Basic relations between (sub)transversality and (sub)regularity

In this section we obtain some rather straightforward but nevertheless important relations between subtransversality and subregularity and between transversality and regularity.

The next theorem shows that regularity and subregularity could be characterized in terms of transversality and subtransversality. The same sets as in the formulations below appear in the papers [6](Theorem 5.2), [7] (Theorem 4.2) and [5] (Theorem 4) as well.

Theorem 3.1. *Let $F : X \rightrightarrows Y$ be a set-valued map between the metric spaces X and Y, and $(\overline{x}, \overline{y}) \in Gr\ F$. Define the sets $A := Gr\ F$ and $B := X \times \{\overline{y}\}$. Then F is subregular at $(\overline{x}, \overline{y})$ if and only if A and B are subtransversal at $(\overline{x}, \overline{y})$.*

Proof. Let the sets be subtransversal, that is there are $\delta > 0$ and $K_1 > 0$ such that

$$d((x, y), A \cap B) \leq K_1(d((x, y), A) + d((x, y), B))$$

for all $(x, y) \in \overline{\mathbf{B}}_\delta((\overline{x}, \overline{y}))$. Observe that $A \cap B = \{(\hat{x}, \overline{y}) \mid \hat{x} \in F^{-1}(\overline{y})\}$. Let $x \in \overline{\mathbf{B}}_\delta(\overline{x})$. Then

$$d((x, \overline{y}), A \cap B) = d(x, F^{-1}(\overline{y}))\,.$$

On the other hand $d((x, \overline{y}), A) \leq d(\overline{y}, F(x))$ and $d((x, \overline{y}), B) = 0$, whence subtransversality implies

$$d(x, F^{-1}(\overline{y})) \leq K_1 d(\overline{y}, F(x)),$$

hence F is subregular at $(\overline{x}, \overline{y})$ with constants K_1 and δ.

For the reverse direction, let F be subregular at $(\overline{x}, \overline{y})$, that is there are $\delta > 0$ and $K_2 > 0$ such that

$$d(x, F^{-1}(\overline{y})) \leq K_2 d(\overline{y}, F(x))\,.$$

Take $(x, y) \in \overline{\mathbf{B}}_{\delta/3}((\overline{x}, \overline{y}))$ and $\varepsilon \in (0, \delta/3)$. Observe that $d((x, y), B) = d(y, \overline{y})$. Let $(x', y') \in A$ be such that $d(x, x') + d(y, y') \le d((x, y), A) + \varepsilon$. Note that

$$\begin{aligned} d(x', \overline{x}) &\le d((x', y'), (\overline{x}, \overline{y})) \le d((x', y'), (x, y)) + d((x, y), (\overline{x}, \overline{y})) \\ &\le d((x, y), A) + \varepsilon + d((x, y), (\overline{x}, \overline{y})) \le \varepsilon + 2d((x, y), (\overline{x}, \overline{y})) \le \delta. \end{aligned}$$

Then

$$\begin{aligned} d((x, y), A \cap B) &= d(x, F^{-1}(\overline{y})) + d(y, \overline{y}) \le d(x', F^{-1}(\overline{y})) + d(x, x') + d(y, \overline{y}) \\ &\le K_2 d(\overline{y}, F(x')) + d(x, x') + d(y, \overline{y}) \le K_2 d(y', \overline{y}) + d(x, x') + d(y, \overline{y}) \\ &\le K_2 d(\overline{y}, y) + K_2 d(y, y') + d(x, x') + d(y, \overline{y}) \\ &\le (K_2 + 1) d((x, y), B) + (K_2 + 1) d((x, y), A) + (K_2 + 1)\varepsilon \end{aligned}$$

Letting $\varepsilon \to 0$ proves subtransversality with constants $K_1 = K_2 + 1$ and $\delta/3$. □

Corollary 3.2. *Let $F : X \rightrightarrows Y$, X and Y be metric spaces, and $(\overline{x}, \overline{y}) \in Gr\ F$ as above. Define the sets $A := Gr\ F$ and $B_y := X \times \{y\}$. Then F is regular at $(\overline{x}, \overline{y})$ if and only if there are constants $\delta > 0$ and $K > 0$ such that for any $(x, y) \in \overline{\mathbf{B}}_\delta((\overline{x}, \overline{y}))$ and any $\hat{y} \in \overline{\mathbf{B}}_\delta(\overline{y})$*

$$d((x, y), A \cap B_{\hat{y}}) \le K(d((x, y), A) + d((x, y), B_{\hat{y}}))\,. \tag{2}$$

If in addition X and Y are normed spaces, then this is also equivalent to A and $B := B_{\overline{y}}$ being transversal at $(\overline{x}, \overline{y})$.

Proof. Observe that in the first part of the proof above, we never made explicit use of the fact that $(\overline{x}, \overline{y}) \in A \cap B$. Pick $\hat{y} \in \overline{\mathbf{B}}_\delta(\overline{y})$. The inequality (3) is satisfied with $B_{\hat{y}}$ instead of B, so that, according to Theorem 3.1, we arrive at

$$d(x, F^{-1}(\hat{y})) \le K d(\hat{y}, F(x)) \ \text{ for all } x \in \overline{\mathbf{B}}_\delta(\overline{x}).$$

Thus, we obtain regularity at $(\overline{x}, \overline{y})$.

For the other direction, again take $\hat{y} \in \overline{\mathbf{B}}_\delta(\overline{y})$. Since $d(x', F^{-1}(\hat{y})) \le K d(\hat{y}, F(x'))$, for x' near $\overline{x}$, as in the proof of Theorem 3.1, we obtain

$$d((x, y), A \cap B_{\hat{y}}) \le (K + 1)(d((x, y), A) + d((x, y), B_{\hat{y}})),$$

for all $(x, y) \in \overline{\mathbf{B}}_{\delta/3}((\overline{x}, \overline{y}))$.

If the spaces are normed, then $B_{\hat{y}} = B + (x, \hat{y} - \overline{y})$ for any x. Thus the inequality (2) is the inequality defining transversality. □

4. Primal space characterizations of subregularity

In this section we obtain primal space characterizations of subregularity via subtransversality. We are going to use the characterization of subtransversality from [1] given also in the Preliminaries.

The lemma below is our main technical tool, allowing to pass from a local inequality to a global one. It is a slight generalization of Lemma 3.4 in [1] and its proof is given in the Appendix.

Lemma 4.1. *Let A and B be closed subsets of the complete metric space X and $\overline{x} \in X$. Let $f : X \times X \to [0, +\infty)$ be lower semicontinuous such that $f(x, y) \leq d(x, y)$ for all $x, y \in X$. Assume that there exist $\delta > 0$ and $M > 0$ such that for any $x^A \in A \cap \overline{\mathbf{B}}_\delta(\overline{x})$ and $x^B \in B \cap \overline{\mathbf{B}}_\delta(\overline{x})$ with $f(x^A, x^B) \neq 0$, there are $\theta > 0$, $\hat{x}^A \in A$ and $\hat{x}^B \in B$, such that $d(x^A, \hat{x}^A) \leq M\theta$, $d(x^B, \hat{x}^B) \leq M\theta$ and*

$$f(\hat{x}^A, \hat{x}^B) \leq f(x^A, x^B) - \theta.$$

Fix $x^A \in A \cap \overline{\mathbf{B}}_{\frac{\delta}{1+2M}}(\overline{x})$ and $x^B \in B \cap \overline{\mathbf{B}}_{\frac{\delta}{1+2M}}(\overline{x})$.
Then there exist $\tilde{x}^A \in A$ and $\tilde{x}^B \in B$, such that

$$f(\tilde{x}^A, \tilde{x}^B) = 0, \quad d(\tilde{x}^A, x^A) \leq M f(x^A, x^B) \quad \text{and} \quad d(\tilde{x}^B, x^B) \leq M f(x^A, x^B).$$

The next theorem is a primal space characterization of subregularity (cf. Theorem 2.58 in [17] or Corollaries 5.8 and 5.9 in [22]).

Theorem 4.2. *Let $F : X \rightrightarrows Y$ be with closed graph and $(\overline{x}, \overline{y}) \in Gr\ F$, where X and Y are complete metric spaces. Then F is subregular at $(\overline{x}, \overline{y}) \in Gr\ F$ if and only if there exist constants $\delta > 0$ and $\tau > 0$ such that for all $(x, y) \in Gr\ F \cap \overline{\mathbf{B}}_\delta((\overline{x}, \overline{y}))$, there is $(\hat{x}, \hat{y}) \in Gr\ F \setminus \{(x, y)\}$, such that*

$$d(\hat{y}, \overline{y}) \leq d(y, \overline{y}) - \tau d((x, y), (\hat{x}, \hat{y}))\,.$$

Proof. According to Theorem 3.1, F is subregular at $(\overline{x}, \overline{y})$ if and only if the sets $A := \mathrm{Gr}\ F$ and $B := X \times \{\overline{y}\}$ are subtransversal at that point. Assume that they are subtransversal. Then, according to Proposition 2.8, *property* $(\mathcal{T})$ holds with some constants δ and M. Take $(x, y) \in A \cap \overline{\mathbf{B}}_\delta((\overline{x}, \overline{y}))$. Then $(x, \overline{y}) \in B \cap \overline{B}_\delta(\overline{x}, \overline{y})$ and thus there exist $(\hat{x}, \hat{y}) \in A$ and $(\hat{x}_B, \overline{y}) \in B$ such that

$$d((\hat{x}, \hat{y}), (\hat{x}_B, \overline{y})) \leq d((x, y), (x, \overline{y})) - \frac{1}{M} \max\{d((x, y), (\hat{x}, \hat{y})), d(x, \hat{x}_B)\}$$

and $\max\{d((x, y), (\hat{x}, \hat{y})), d(x, \hat{x}_B)\} > 0$. If we assume that $(\hat{x}, \hat{y}) = (x, y)$, then in particular $\hat{y} = y$. Thus

$$d((x, y), (x, \overline{y})) - \frac{1}{M} \max\{d((x, y), (\hat{x}, \hat{y})), d(x, \hat{x}_B)\} < d(y, \overline{y})$$

and
$$d(y, \overline{y}) \leq d((\hat{x}, \hat{y}), (\hat{x}_B, \overline{y})).$$

This contradicts the earlier inequality, hence $(\hat{x}, \hat{y}) \neq (x, y)$.

From here we obtain for $\tau := \dfrac{1}{M}$

$$\begin{aligned} d(\hat{y}, \overline{y}) &\leq d((\hat{x}, \hat{y}), (\hat{x}_B, \overline{y})) \leq d((x, y), (x, \overline{y})) - \frac{1}{M} \max\{d((x, y), (\hat{x}, \hat{y})), d(x, \hat{x}_B)\} \\ &\leq d(y, \overline{y}) - \tau d((x, y), (\hat{x}, \hat{y}))\,, \end{aligned}$$

Now assume that there exist constants $\delta > 0$ and $\tau > 0$ with the property that for all $(x, y) \in \mathrm{Gr}\ F \cap \overline{\mathbf{B}}_\delta((\overline{x}, \overline{y}))$, there is $(\hat{x}, \hat{y}) \in \mathrm{Gr}\ F$, such that

$$d(\hat{y}, \overline{y}) \leq d(y, \overline{y}) - \tau d((x, y), (\hat{x}, \hat{y}))\,.$$

Let the function $f : (X \times Y) \times (X \times Y) \to [0, +\infty)$ be given by

$$f((x_1, y_1), (x_2, y_2)) = d(y_1, y_2)\,.$$

Let $(x, y) \in A \cap \overline{\mathbf{B}}_\delta((\overline{x}, \overline{y}))$ and $(x_B, \overline{y}) \in B$ with $d(x_B, \overline{x}) \le \delta$ be arbitrary. Then, there exists a point $(\hat{x}, \hat{y}) \in A$, satisfying the inequality. For the points $(\hat{x}, \hat{y}) \in A$ and $(x_B, \overline{y}) \in B$ we estimate

$$\begin{aligned} f((\hat{x}, \hat{y}), (x_B, \overline{y})) &= d(\hat{y}, \overline{y}) \le d(y, \overline{y}) - \tau d((x, y), (\hat{x}, \hat{y})) \\ &= f((x, y), (x_B, \overline{y})) - \tau \max\{d((x, y), (\hat{x}, \hat{y})), d((x_B, \overline{y}), (x_B, \overline{y}))\}\,, \end{aligned}$$

which means that we can apply Lemma 4.1 for A and B at $(\overline{x}, \overline{y})$ with function f and constants δ and $M := \frac{1}{\tau}$ if the starting points are sufficiently close to $(\overline{x}, \overline{y})$.

Let $\hat{\delta} = \frac{\tau}{(\tau + 2)(\tau + 1)}\delta$ and take $x \in \overline{\mathbf{B}}_{\hat{\delta}}(\overline{x})$. If $d(\overline{y}, F(x)) \ge \tau\hat{\delta}$, then

$$\frac{1}{\tau} d(\overline{y}, F(x)) \ge \hat{\delta} \ge d(x, \overline{x}) \ge d(x, F^{-1}(\overline{y})).$$

Otherwise, take $\varepsilon \in \big(0, \tau\hat{\delta} - d(\overline{y}, F(x))\big)$. Take $y \in F(x)$ for which

$$d(y, \overline{y}) \le d(\overline{y}, F(x)) + \varepsilon \le \tau\hat{\delta} < \frac{\tau}{\tau + 2}\delta.$$

For $M = 1/\tau$, we have $d(x, \overline{x}) \le \frac{\delta}{1 + 2M}$ and $d(y, \overline{y}) \le \frac{\delta}{1 + 2M}$.

Applying Lemma 4.1 to $(x, y) \in A$ and $(x, \overline{y}) \in B$, it follows that there exist points $(\tilde{x}, \tilde{y}) \in A$ and $(\tilde{x}_B, \overline{y}) \in B$ such that $f((\tilde{x}, \tilde{y}), (\tilde{x}_B, \overline{y})) = 0$, hence $\tilde{y} = \overline{y}$, and $d((\tilde{x}, \overline{y}), (x, y)) \le Mf((x, y), (x, \overline{y})) = Md(y, \overline{y})$. Using that $\tilde{x} \in F^{-1}(\overline{y})$ and the choice of y, we obtain that

$$\begin{aligned} d(x, F^{-1}(\overline{y})) &\le d(x, \tilde{x}) \le d((\tilde{x}, \overline{y}), (x, y)) \le Md(y, \overline{y}) \\ &\le Md(\overline{y}, F(x)) + M\varepsilon \end{aligned}$$

Letting $\varepsilon \to 0$, we obtain $d(x, F^{-1}(\overline{y})) \le Md(\overline{y}, F(x))$ for all $x \in \overline{\mathbf{B}}_{\hat{\delta}}(\overline{x})$. We have verified that F is subregular at $(\overline{x}, \overline{y})$ by definition. □

5. Primal space characterizations of transversality and regularity

We continue to obtain necessary and sufficient primal space conditions for metric regularity. The following theorem is a classical "rate of descent" characterization (cf. Theorem 2.50 in [17] or Theorem 7 in [19]).

Theorem 5.1. *Let $F : X \rightrightarrows Y$ be with closed graph and $(\overline{x}, \overline{y}) \in Gr\ F$, where X and Y are complete metric spaces. Then F is regular at $(\overline{x}, \overline{y}) \in Gr\ F$ if and only if there exist $\delta > 0$ and $\tau > 0$ such that for all $(x, y) \in Gr\ F \cap \overline{\mathbf{B}}_\delta((\overline{x}, \overline{y}))$ and all $v \in \overline{\mathbf{B}}_\delta(\overline{y})$, there is $(\hat{x}, \hat{y}) \in Gr\ F \setminus \{(x, y)\}$, such that*

$$d(\hat{y}, v) \le d(y, v) - \tau d((x, y), (\hat{x}, \hat{y}))\,.$$

Proof. According to Corollary 3.2, F is regular at $(\overline{x},\overline{y})$ if and only if there are constants $\delta>0$ and $K>0$ such that for any $(x,y)\in\overline{\mathbf{B}}_\delta((\overline{x},\overline{y}))$ and any $v\in\overline{\mathbf{B}}_\delta(\overline{y})$ it holds

$$d((x,y),A\cap B_v)\le K(d((x,y),A)+d((x,y),B_v))\,,$$

where $B_v:=X\times\{v\}$.

Let F be regular at $(\overline{x},\overline{y})$. Fix $\hat\delta:=\frac{\delta}{4K+10}$, $(x,y)\in A\cap\overline{\mathbf{B}}_{\hat\delta}((\overline{x},\overline{y}))$ and $v\in\overline{\mathbf{B}}_{\hat\delta}(\overline{y})$. According to Proposition 2.8, A and B_v have *property* $(\mathcal{T})$ at $(\overline{x},\overline{y})$ with constants $\hat\delta$ and M. Hence, there exist $(\hat x,\hat y)\in A$ and $(\hat x^B,v)\in B_v$ such that

$$d((\hat x,\hat y),(\hat x^B,v))\le d((x,y),(x,v))-\frac{1}{M}\max\{d((x,y),(\hat x,\hat y)),d(x,\hat x^B)\}$$

and
$$\max\{d((x,y),(\hat x,\hat y)),d(x,\hat x_B)\}>0.$$

If we assume that $(\hat x,\hat y)=(x,y)$, then in particular $\hat y=y$. Thus

$$d((x,y),(x,v))-\frac{1}{M}\max\{d((x,y),(\hat x,\hat y)),d(x,\hat x_B)\}<d(y,v)$$

and
$$d(y,v)\le d((\hat x,\hat y),(\hat x_B,v)).$$

This contradicts the earlier inequality, hence $(\hat x,\hat y)\neq(x,y)$. From here we obtain

$$d(\hat y,v)\le d(y,v)-\tau d((x,y),(\hat x,\hat y))\quad\text{for }\tau:=\frac{1}{M}.$$

Now, assume that there exist constants $\delta>0$ and $\tau>0$ with the property that for all $(x,y)\in\operatorname{Gr}F\cap\overline{\mathbf{B}}_\delta((\overline{x},\overline{y}))$ and all $v\in\overline{\mathbf{B}}_\delta(\overline{y})$, there is $(\hat x,\hat y)\in\operatorname{Gr}F\setminus\{(x,y)\}$, such that

$$d(\hat y,v)\le d(y,v)-\tau d((x,y),(\hat x,\hat y))\,.$$

Let the function $f:(X\times Y)\times(X\times Y)\to[0,+\infty)$ be given by

$$f((x_1,y_1),(x_2,y_2))=d(y_1,y_2)\,.$$

Let us fix $(x,y)\in A\cap\overline{\mathbf{B}}_{\delta/2}((\overline{x},\overline{y}))$, $v\in\overline{\mathbf{B}}_{\delta/2}(\overline{y})$ and $(x^B,v)\in B_v$ with $d(x_B,\overline{x})\le\delta/2$. Then, there exists a point $(\hat x,\hat y)\in A$, satisfying the above inequality.

For the points $(\hat x,\hat y)\in A$ and $(x_B,v)\in B_v$ we estimate

$$\begin{aligned}f((\hat x,\hat y),(x_B,v))&=d(\hat y,v)\le d(y,v)-\tau d((x,y),(\hat x,\hat y))\\&=f((x,y),(x_B,v))-\tau\max\{d((x,y),(\hat x,\hat y)),d((x_B,v),(x_B,v))\}\,,\end{aligned}$$

which means that we can apply Lemma 4.1 for A and B_v at $(\overline{x},\overline{y})$ with function f and constants $\delta/2$ and $M:=\dfrac{1}{\tau}$ if the starting points are sufficiently close to $(\overline{x},\overline{y})$.

Let $\hat\delta:=\dfrac{\tau}{4(\tau+2)(\tau+1)}\delta$ and take $v\in\overline{\mathbf{B}}_{\hat\delta}(\overline{y})$ and $x\in\overline{\mathbf{B}}_{\hat\delta}(\overline{x})$. Applying Lemma 4.1 for $(\overline{x},\overline{y})\in A$ and $(\overline{x},v)\in B_v$ we arrive at a point $x_v\in F^{-1}(v)$ with the property $d(x_v,\overline{x})\le Md(\overline{y},v)\le\hat\delta/\tau$.

If $d(v, F(x)) \geq \hat{\delta}(1+\tau)$, then

$$\frac{1}{\tau} d(v, F(x)) \geq \hat{\delta} + \frac{\hat{\delta}}{\tau} \geq d(x, \overline{x}) + d(x_v, \overline{x}) \geq d(x, x_v) \geq d(x, F^{-1}(v))$$

Otherwise, take $\varepsilon \in \big(0, \hat{\delta}(\tau+1) - d(\overline{y}, F(x))\big)$. Take $y \in F(x)$ for which

$$d(y, v) \leq d(v, F(x)) + \varepsilon \leq \hat{\delta}(\tau+1) \leq \frac{\tau}{4(\tau+2)}\delta.$$

Recall that $M = 1/\tau$, hence $d(x, \overline{x}) \leq \dfrac{\delta/2}{1+2M}$ and

$$d(y, \overline{y}) \leq d(y, v) + d(v, \overline{y}) \leq \frac{\tau}{2(\tau+2)}\delta = \frac{\delta/2}{1+2M}.$$

Applying Lemma 4.1 to $(x, y) \in A$ and $(x, v) \in B_v$, it follows that there exist points $(\tilde{x}, \tilde{y}) \in A$ and $(\tilde{x}_B, v) \in B_v$ such that $f((\tilde{x}, \tilde{y}), (\tilde{x}_B, \overline{y})) = 0$, hence $\tilde{y} = v$, and $d((\tilde{x}, v), (x, y)) \leq Mf((x, y), (x, v)) = Md(y, v)$. Using that $\tilde{x} \in F^{-1}(v)$ and the choice of y, we obtain that

$$\begin{aligned} d(x, F^{-1}(v)) &\leq d(x, \tilde{x}) \leq d((\tilde{x}, v), (x, y)) \leq Md(y, v) \\ &\leq Md(v, F(x)) + M\varepsilon \end{aligned}$$

Letting $\varepsilon \to 0$, it follows that $d(x, F^{-1}(v)) \leq Md(v, F(x))$ for all $x \in \overline{\mathbf{B}}_{\hat{\delta}}(\overline{x})$ and $v \in \overline{\mathbf{B}}_{\hat{\delta}}(\overline{y})$. We have verified that F is regular at $(\overline{x}, \overline{y})$ by definition. □

Using the above theorem, we establish a characterization of metric regularity of a set-valued map $F : X \rightrightarrows Y$, X – complete metric space and Y – Banach space, using its *first order (contingent) variation* $F^{(1)}(x, y)$. This was first done in [13] (see also Theorem 4.13 and Remark 4.14(c) in [2] for a proof in Banach spaces or [19] for an alternative proof). Given $(x, y) \in \mathrm{Gr}F$, define $F^{(1)} : X \times Y \rightrightarrows Y$ by

$$F^{(1)}(x, y) := \limsup_{t \to 0_+} \frac{F(\overline{\mathbf{B}}_t(x)) - y}{t},$$

where lim sup stands for the Kuratowski limit superior of sets. Equivalently, we have $v \in F^{(1)}(x, y)$ exactly when there exist sequences $t_n \to 0_+$, $v_n \to v$ and $(x_n, y_n) \in \mathrm{Gr}\ F$ such that $d(x_n, x) \leq t_n$ and $y_n = y + t_n v_n$.

Our proof is done via a sequential characterization of metric regularity, which we have not seen stated anywhere in the literature.

Corollary 5.2. *Let us consider $F : X \rightrightarrows Y$ with closed graph, where X is a complete metric space and Y is a Banach space. Then, the following statements are equivalent:*

(i) *F is regular at $(\overline{x}, \overline{y}) \in Gr\ F$;*

(ii) *there exist $\delta > 0$ and $r > 0$ such that*

$$\mathbf{B}_r(\mathbf{0}) \subset F^{(1)}(x, y) \text{ for all } (x, y) \in \overline{\mathbf{B}}_\delta(\overline{x}, \overline{y}) \cap Gr\ F;$$

(iii) *there exist $\delta > 0$ and $\tau > 0$ such that for all $(x, y) \in Gr\ F \cap \overline{\mathbf{B}}_\delta((\overline{x}, \overline{y}))$ and all $\hat{y} \in \overline{\mathbf{B}}_\delta(\overline{y})$, there is a sequence $\{(x_n, y_n)\}_{n\geq 1} \subset Gr\ F \setminus \{(x, y)\}$ converging to (x, y) such that for all n it holds*

$$\|y_n - \hat{y}\| \leq \|y - \hat{y}\| - \tau d((x_n, y_n), (x, y)) .$$

Proof. We have that (iii) implies (i) by Theorem 5.1.

Next, we will show that (i) implies (ii). Let F be regular at $(\overline{x}, \overline{y}) \in \text{Gr } F$. By definition there exist $K > 0$ and $\delta > 0$ such that for all $x \in \mathbf{B}_\delta(\overline{x})$ and all $y \in \mathbf{B}_\delta(\overline{y})$ the following inequality holds:

$$d(x, F^{-1}(y)) \leq K d(y, F(x)) .$$

Fix arbitrary $(x, y) \in \overline{\mathbf{B}}_{\frac{\delta}{2}}((\overline{x}, \overline{y})) \cap \text{Gr } F$, $v \in Y$ with $\|v\| < \dfrac{1}{K} =: r$ and a sequence $t_n \to 0_+$ such that $y_n := y + t_n v \in \mathbf{B}_\delta(\overline{y})$. Then, there exists $\varepsilon > 0$ such that $\|v\| \leq \dfrac{1-\varepsilon}{K}$. Moreover, for every $n \in \mathbb{N}$ there exists $x_n \in F^{-1}(y_n)$ such that $d(x, x_n) \leq d(x, F^{-1}(y_n)) + \varepsilon t_n$. Thus

$$\begin{aligned} d(x, x_n) &\leq d(x, F^{-1}(y_n)) + \varepsilon t_n \leq K d(y_n, F(x)) + \varepsilon t_n \\ &\leq K\|y_n - y\| + \varepsilon t_n \leq (1 - \varepsilon + \varepsilon) t_n = t_n. \end{aligned}$$

Having that $(x_n, y_n) = (x_n, y + t_n v) \in \text{Gr } F$, $v \in F^{(1)}(x, y)$ by definition. We have shown that (ii) holds, if F is regular at $(\overline{x}, \overline{y})$.

It remains to prove that (ii) implies (iii). Assume that (ii) holds.

Let $(x, y) \in \overline{\mathbf{B}}_\delta(\overline{x}, \overline{y}) \cap \text{Gr } F$ and $\hat{y} \in \overline{\mathbf{B}}_\delta(\overline{y})$ be arbitrary. Let us denote $v := \rho \dfrac{\hat{y} - y}{\|\hat{y} - y\|}$ for some $\rho \in (0, r)$. Then, $v \in Y$ with $\|v\| = \rho$ and due to (ii) there exist sequences $t_n \to 0_+$, $v_n \to v$ and $(x_n, y_n) \in \text{Gr } F$ such that $d(x_n, x) \leq t_n$ and $y_n = y + t_n v_n$. Since $\rho t_n < 1$ for n – large enough, we estimate

$$\|y_n - \hat{y}\| = \left\| y - \hat{y} + t_n \rho \, \frac{\hat{y} - y}{\|\hat{y} - y\|} \right\| = \|y - \hat{y}\| - t_n \rho .$$

Moreover, we have that $t_n \geq d(x_n, x) > \dfrac{d(x_n, x)}{\rho + 1}$ and $t_n = \dfrac{\|y_n - y\|}{\|v_n\|} \geq \dfrac{\|y_n - y\|}{\rho + 1}$ for n large enough. Therefore

$$\|y_n - \hat{y}\| \leq \|y - \hat{y}\| - \tau d((x_n, y_n), (x, y)) ,$$

where $\tau := \dfrac{\rho}{2(\rho + 1)}$. The proof is complete. □

Next, we establish the relation between the metric regularity of a set-valued map $F : X \rightrightarrows Y$, X and Y – Banach spaces, and its *graphical (contingent) derivative* $DF(x|y)$. Given $(x, y) \in \text{Gr}F$, define $DF(x|y) : X \rightrightarrows Y$ as the map, whose graph is the (Bouligand) tangent cone $T_{\text{Gr}F}(x, y)$, i.e.

$$v \in DF(x|y)(u) \Leftrightarrow (u, v) \in T_{\text{Gr}F}(x, y) .$$

Corollary 5.3. (cf. Theorem 1.2 in [9] and Theorem 4.13 and Remark 4.14(b) in [2]]) *Let $F : X \rightrightarrows Y$ and $(\overline{x}, \overline{y}) \in Gr\, F$, where X and Y are Banach spaces. Assume there exist $\delta > 0$ and $K > 0$ such that for any $(x, y) \in GrF$ with $\|x-\overline{x}\| \le \delta$, $0 < \|y - \overline{y}\| \le \delta$ and any $v \in Y$, $\|v\| = 1$, it holds*

$$\inf\left\{\|u\| \;\middle|\; v \in DF(x|y)(u)\right\} \le K .$$

Then F is regular at $(\overline{x}, \overline{y}) \in Gr\, F$. The reverse direction is also true when X is finite-dimensional.

Proof. For the first part, we have that for every $\varepsilon > 0$, every $(x, y) \in \mathrm{Gr}F$ with $\|x - \overline{x}\| \le \delta$, $0 < \|y - \overline{y}\| \le \delta$ and for every $v \in Y$ with $\|v\| = 1$ there is some $(u, v) \in T_{\mathrm{Gr}\ F}(x, y)$ such that $\|u\| < K+\varepsilon$. From the definition of Bouligand tangent cone, there exist sequences $u_n \to u$, $v_n \to v$ and a sequence of positive t_n tending to zero, such that $(x + t_n u_n, y + t_n v_n) \in \mathrm{Gr}\ F$. Let us fix an arbitrary $\lambda \in (0, 1]$. We have that

$$\tau_n := \frac{t_n(K+\varepsilon)}{\lambda} \to 0 \quad \text{and} \quad (x+t_n u_n, y+t_n v_n) = (x+\tau_n\frac{\lambda u_n}{K+\varepsilon}, y+\tau_n\frac{\lambda v_n}{K+\varepsilon}) \in \mathrm{Gr}\ F.$$

Without loss of generality we can assume that $\|u_n\| \le K+\varepsilon$ for n large enough. Taking into account that $d(x, x + \tau_n \frac{\lambda u_n}{K+\varepsilon}) \le \lambda\tau_n \le \tau_n$, we obtain that $\frac{\lambda v}{K+\varepsilon} \in F^{(1)}(x, y)$. Since the unit vector $v \in Y$, $\lambda \in (0, 1]$ and $\varepsilon > 0$ are arbitrary and $\|\frac{\lambda v}{K+\varepsilon}\| = \frac{\lambda}{K+\varepsilon}$, we obtain that $\mathbf{B}_{\frac{1}{K}}(\mathbf{0}) \subset F^{(1)}(x, y)$. Then, F is regular at $(\overline{x}, \overline{y})$ due to Corollary 5.2.

For the converse, let X be finite-dimensional and F be regular at $(\overline{x}, \overline{y})$ with constants δ and K. Let $(x, y) \in \mathrm{Gr}F$ with $\|x - \overline{x}\| \le \delta$ and $0 < \|y - \overline{y}\| \le \delta$, $\varepsilon \in (0, \frac{1}{K})$ and $v \in Y$ with $\|v\| = 1$ be arbitrary. Then, we have that $w := (\frac{1}{K} - \varepsilon)v \in F^{(1)}(x, y)$ due to Corollary 5.2. That is, there exist sequences $t_n \to 0_+$, $w_n \to w$ and $(x_n, y_n) \in \mathrm{Gr}\ F$ such that $\|x_n - x\| \le t_n$ and $y_n = y + t_n w_n$. Moreover, since X is finite-dimensional, we have $x_n = x + t_n p_n$, where $\|p_n\| \le 1$ which implies $p_n \longrightarrow_{n\to\infty} p$ (up to a subsequence, labeled in the same way) and $(p, w) \in T_{\mathrm{Gr}\ F}$. Hence $(\frac{p}{\|w\|}, \frac{w}{\|w\|}) = (\frac{p}{\frac{1}{K}-\varepsilon}, v) \in T_{\mathrm{Gr}\ F}$ and $\|\frac{p}{\frac{1}{K}-\varepsilon}\| \le \frac{1}{\frac{1}{K}-\varepsilon}$. We have obtained that for any $v \in Y$, $\|v\| = 1$, it holds

$$\inf\left\{\|u\| \;\middle|\; v \in DF(x|y)(u)\right\} \le K .$$

The proof is complete. □

A. Proof of Lemma 4.1

Proof of Lemma 4.1. If $f(x^A, x^B) = 0$, the assertion of the theorem is trivial (with $\hat{x}^A = x^A$ and $\hat{x}^B = x^B$). If $f(x^A, x^B) > 0$, we are going to construct inductively three transfinite sequences indexed by ordinal numbers (cf., for example, § 2 Ordinal numbers of Chapter 1 in [20]). More precisely, we prove that there exist an ordinal number α_0 and transfinite sequences

$$\{x^A_\alpha\}_{1\le\alpha\le\alpha_0} \subset X, \quad \{x^B_\alpha\}_{1\le\alpha\le\alpha_0} \subset X, \quad \{t_\alpha\}_{1\le\alpha\le\alpha_0} \subset [0, +\infty),$$

such that $f(x^A_{\alpha_0}, x^B_{\alpha_0}) = 0$ and for each $\alpha \in [1, \alpha_0]$ we have that the following properties hold true:

(S0) $x_\alpha^A \in \overline{\mathbf{B}}_\delta(\overline{x}) \cap A$ and $x_\alpha^B \in \overline{\mathbf{B}}_\delta(\overline{x}) \cap B$;

(S1) $f(x_\alpha^A, x_\alpha^B) \le f(x^A, x^B) - t_\alpha$ (and hence t_α is bounded by $f(x^A, x^B)$);

(S2) $d(x_\alpha^A, \overline{x}) \le d(x^A, \overline{x}) + t_\alpha M$ and $d(x_\alpha^B, \overline{x}) \le d(x^B, \overline{x}) + t_\alpha M$;

(S3) $d(x_\alpha^A, x_\gamma^A) \le M\,(t_\alpha - t_\gamma)$ and $d(x_\alpha^B, x_\gamma^B) \le M\,(t_\alpha - t_\gamma)$ for each $\gamma \le \alpha$.

We implement our construction using induction on α. The process terminates when $f(x_\alpha^A, x_\alpha^B) = 0$ for some α, and this α is named α_0. We start with

$$x_1^A := x^A \in \overline{\mathbf{B}}_{\frac{\delta}{1+2M}}(\overline{x}) \cap A, \quad x_1^B := x^B \in \overline{\mathbf{B}}_{\frac{\delta}{1+2M}}(\overline{x}) \cap B$$

and $t_1 = 0$. It is straightforward to verify the inductive assumptions (S0)–(S3) for $\alpha = 1$.

Assume that $x_\beta^A \in \overline{\mathbf{B}}_\delta(\overline{x}) \cap A$, $x_\beta^B \in \overline{\mathbf{B}}_\delta(\overline{x}) \cap B$ and t_β are constructed and (S1)–(S3) are true for all ordinals β less than α and the process has not been terminated.

Let us first consider the case when α is a successor ordinal, i.e. $\alpha = \beta + 1$. As $\beta < \alpha_0$ (the process has not been terminated), we have $f(x_\beta^A, x_\beta^B) \ne 0$. Moreover $(S0)$ holds, so we can apply the assumption in the statement of the theorem to obtain $\theta > 0, \hat{x}_\beta^A,\ \hat{x}_\beta^B$, and we define $t_\alpha := t_\beta + \theta$, $x_\alpha^A := \hat{x}_\beta^A$ and $x_\alpha^B := \hat{x}_\beta^B$. Now we have $x_\alpha^A \in A$, $x_\alpha^B \in B$, $d(x_\alpha^A, x_\beta^A) \le M\theta$, $d(x_\alpha^B, x_\beta^B) \le M\theta$ and $f(x_\alpha^A, x_\alpha^B) \le f(x_\beta^A, x_\beta^B) - \theta$. Using the inductive assumption, we have

$$f(x_\alpha^A, x_\alpha^B) \le f(x_\beta^A, x_\beta^B) - \theta \le f(x^A, x^B) - t_\beta - \theta = f(x^A, x^B) - t_\alpha\ .$$

Therefore, (S1) is verified for α.

Now the inequalities $d(x_\alpha^A, x_\beta^A) \le M\theta$, $d(x_\alpha^B, x_\beta^B) \le M\theta$ and the inductive assumption (S2) for β yield

$$d(x_\alpha^A, \overline{x}) \le d(x_\beta^A, \overline{x}) + d(x_\alpha^A, x_\beta^A) \le d(x^A, \overline{x}) + t_\beta M + M\theta = d(x^A, \overline{x}) + t_\alpha M,$$

$$d(x_\alpha^B, \overline{x}) \le d(x_\beta^B, \overline{x}) + d(x_\alpha^A, x_\beta^B) \le d(x^B, \overline{x}) + t_\beta M + M\theta = d(x^B, \overline{x}) + t_\alpha M.$$

Thus (S2) is verified for α. Using the estimate $t_\beta \le f(x^A, x^B) \le d(x^A, x^B)$ from (S1) for β, the assumption of the theorem and the above inequalities, we obtain

$$\begin{aligned} d(x_\alpha^A, \overline{x}) &\le d(x^A, \overline{x}) + t_\alpha M \le d(x^A, \overline{x}) + M d(x^A, x^B) \\ &\le d(x^A, \overline{x}) + M(d(x^A, \overline{x}) + d(x^B, \overline{x})) \\ &\le \frac{\delta}{1+2M} + M\frac{2\delta}{1+2M} = \delta \end{aligned}$$

which means that $x_\alpha^A \in \overline{\mathbf{B}}_\delta(\overline{x})$. Similarly $x_\alpha^B \in \overline{\mathbf{B}}_\delta(\overline{x})$. Thus (S0) holds.

Now let $\gamma < \alpha$. Then

$$d(x_\alpha^A, x_\gamma^A) \le d(x_\beta^A, x_\gamma^A) + d(x_\alpha^A, x_\beta^A) \le M\,(t_\beta - t_\gamma) + M\theta = M\,(t_\alpha - t_\gamma)$$

and in the same way

$$d(x_\alpha^B, x_\gamma^B) \le d(x_\beta^B, x_\gamma^B) + d(x_\alpha^B, x_\beta^B) \le M\,(t_\beta - t_\gamma) + M\theta = M\,(t_\alpha - t_\gamma)\ .$$

We have verified the inductive assumptions (S0)-(S3) for the case of a successor ordinal α.

We next consider the case when α is a limit ordinal number. Let $\beta < \alpha$ be arbitrary. Then $\beta + 1 < \alpha$ too. Since the transfinite process has not stopped at $\beta + 1$, then $f(x^A_\beta, x^B_\beta) > 0$, and taking into account (S1) we obtain that $t_\beta < f(x^A, x^B)$. Hence the increasing transfinite sequence $\{t_\beta\}_{1\le\beta<\alpha}$ is bounded, and so it is convergent. We denote $t_\alpha := \lim_{\beta\to\alpha} t_\beta$. Since $d(x^A_\beta, x^A_\gamma) \le (t_\beta - t_\gamma)M$, the transfinite sequence $\{x^A_\beta\}_{1\le\beta<\alpha}$ is fundamental. Hence there exists x^A_α so that $\{x^A_\beta\}_{1\le\beta<\alpha}$ tends to x^A_α as β tends to α with $\beta < \alpha$. In the same way one can prove the existence of x^B_α so that the transfinite sequence $\{x^B_\beta\}_{1\le\beta<\alpha}$ tends to x^B_α as β tends to α. To verify the inductive assumptions (S1)–(S3) for α, one can just take a limit for β tending to α with $\beta < \alpha$ in the same assumptions written for each $\beta < \alpha$ (for (S1) one takes lim inf on the left and uses that this is greater than the value of f at the limit point, since the function is lower semicontinuous). For (S0) one uses that A and B are closed.

We have constructed inductively the transfinite sequences

$$\{x^A_\beta\}_{\beta\le\alpha} \subset A, \ \ \{x^B_\beta\}_{\beta\le\alpha} \subset B \ \text{ and } \ \{t_\beta\}_{\beta\le\alpha} \subset [0, +\infty).$$

The process will terminate when $f(x^A_\alpha, x^B_\alpha) = 0$ for some α. Since

$$f(x^A_\alpha, x^B_\alpha) \le f(x^A, x^B) - t_\alpha$$

and the transfinite sequence t_α is strictly increasing, the equality $f(x^A_\alpha, x^B_\alpha) = 0$ will be satisfied for some $\alpha = \alpha_0$ strictly preceding the first uncountable ordinal number. Indeed, the successor ordinals indexing the so constructed transfinite sequences form a countable set (because to every successor ordinal $\alpha + 1$ corresponds the open interval $(t_\alpha, t_{\alpha+1}) \subset \mathbb{R}$, these intervals are disjoint and the rational numbers are countably many and dense in $\mathbb{R}$). Therefore, α_0 is countably accessible. On the other hand-side, assuming the Axiom of countable choice, ω_1 is not countably accessible. Hence our inductive process terminates before ω_1.

Then we set $\tilde{x}^A := x^A_{\alpha_0} \in A$ and $\tilde{x}^B := x^B_{\alpha_0} \in B$ and because of (S1) we have that $t_{\alpha_0} \le f(x^A, x^B)$. Applying (S3) we obtain

$$d(x^A_1, x^A) \le M(t_{\alpha_0} - t_1) \le Mf(x^A, x^B).$$

Similarly $d(x^B_1, x^B) \le Mf(x^A, x^B)$. This completes the proof. □

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