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A Generalization of the Stampacchia Lemma and Applications

Hongya Gao
College of Mathematics and Information Science, Hebei University, Baoding, China

Jiaxiang Zhang
Department of Mathematics, Beijing Jiaotong University, Beijing, China

Hongyan Ma*
College of Mathematics and Information Science, Hebei University, Baoding, China
mahongyan@hbu.edu.cn

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We present a generalization of Stampacchia Lemma and give applications to regularity property of weak and entropy solutions of degenerate elliptic equations of the form

$$\begin{cases} -\operatorname{div}(a(x,u(x))Du(x)) = f(x), & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\dfrac{\alpha}{(1+|u|)^\theta} \le a(x,s) \le \beta$ with $0 < \alpha \le \beta < \infty$ and $0 \le \theta < 1$. The method to derive regularity seems to be simpler than the classical ones.

Keywords: Stampacchia Lemma, generalization, degenerate elliptic equation, regularity.

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1. Introduction

This paper gives a generalization of the classical Stampacchia Lemma and some applications to regularity properties of weak and entropy solutions to some degenerate elliptic equations.

We first recall the classical Stampacchia Lemma, which can be found, for example, in Lemma 4.1 of [17].

Lemma 1.1. *Let c, α, β be positive constants and $k_0 \in \mathbb{R}$. Let the function $\varphi : [k_0, +\infty) \to [0, +\infty)$ be nonincreasing and such that*

$$\varphi(h) \le \frac{c}{(h-k)^\alpha}[\varphi(k)]^\beta \tag{1}$$

for every h, k with $h > k \ge k_0$. It results that:

* Corresponding author.

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(i) *if* $\beta > 1$ *then* $\varphi(k_0 + d) = 0$, *where*

$$d^\alpha = c[\varphi(k_0)]^{\beta-1} 2^{\frac{\alpha\beta}{\beta-1}}.$$

(ii) *if* $\beta = 1$ *then for any* $k \geq k_0$ *we have* $\varphi(k) \leq \varphi(k_0) e^{1-(ce)^{-\frac{1}{\alpha}}(k-k_0)}$.

(iii) *if* $0 < \beta < 1$ *and* $k_0 > 0$ *then for any* $k \geq k_0$,

$$\varphi(k) \leq 2^{\frac{\alpha}{(1-\beta)^2}} \left\{ c^{\frac{1}{1-\beta}} + (2k_0)^{\frac{\alpha}{1-\beta}} \varphi(k_0) \right\} \left(\frac{1}{k}\right)^{\frac{\alpha}{1-\beta}}.$$

The Stampacchia Lemma is an efficient tool in dealing with summability properties of elliptic partial differential equations and systems as well as minima of variational integrals. Till now, this lemma is used repeatedly by many mathematicians, see, for example, [3, 5, 7, 10, 12, 13, 15, 16].

Due to the importance of Stampacchia's Lemma, some generalizations have been given in order to deal with more complicated situations, see [6, 8, 9, 14]. We emphasize that, in [9], Gao, Leonetti and Wang gave some remarks on Stampacchia's Lemma, which compared the conditions of Stampacchia's Lemma and a special case of it; in [6], the authors presented two generalizations of Stampacchia's Lemma and gave some applications to elliptic systems; in [14], Kovalevsky and Voitovich dealt with an analogous condition, but contains an additional factor on its right hand side; in [8], Gao, Huang and Ren gave an alternative, seems to be more elementary, proof of the results due to Kovalevsky and Voitovich [14].

For the convenience of the reader, we list the results due to Kovalevsky and Voitovich [14]. The following version can be found in [8].

Lemma 1.2. *Let* c, α, β, k_0 *be positive constants and* $0 \leq \theta < 1$. *Let the function* $\varphi : [k_0, +\infty) \to [0, +\infty)$ *be nonincreasing and such that*

$$\varphi(h) \leq \frac{ck^{\theta\alpha}}{(h-k)^\alpha} [\varphi(k)]^\beta \tag{2}$$

for every h, k *with* $h > k \geq k_0 > 0$. *It turns out that:*

(i) *if* $\beta > 1$ *then there exists* $k^* > k_0$ *such that* $\varphi(k^*) = 0$;

(ii) *if* $\beta = 1$ *then for any* $k \geq k_0$,

$$\varphi(k) \leq \varphi(k_0) e^{1-\left(\frac{k-k_0}{\tau}\right)^{1-\theta}},$$

where
$$\tau = \max\left\{ k_0, (ce2^{\theta\alpha}(1-\theta)^\alpha)^{\frac{1}{(1-\theta)\alpha}} \right\};$$

(iii) *if* $0 < \beta < 1$ *then for any* $k \geq k_0$,

$$\varphi(k) \leq 2^{\frac{\alpha(1-\theta)}{(1-\beta)^2}} \left\{ c^{\frac{1}{1-\beta}} + (2k_0)^{\frac{\alpha(1-\theta)}{1-\beta}} \varphi(k_0) \right\} \left(\frac{1}{k}\right)^{\frac{\alpha(1-\theta)}{1-\beta}}.$$

Compare (2) with (1), we find that (2) contains an additional factor depending on k on the numerator of its right hand side, with the power less than the one on the denominator. This case appears naturally in dealing with degenerate second-order elliptic problems considered in [8] and fourth-order elliptic equations considered in [14].

In the present paper we shall give another generalization of Stampacchia's Lemma, that is, we replace $k^{\theta\alpha}$ in (2) with $h^{\theta\alpha}$, and obtain similar results to the classical ones, see Lemma 2.1 in Section 2. In Section 3, we deal with solutions to degenerate elliptic equations of divergence type with the right function belongs to Marcinkiewicz spaces. Using the generalized Stampacchia Lemma, we derive some regularity results of weak and entropy solutions, which can be regarded as generalizations of the classical results, see [2, 4]. The method used in Section 3 seems to be simpler than the classical ones.

2. A generalization of Stampacchia's Lemma

We now present a generalization of Lemma 1.1.

Lemma 2.1. *Let c, α, β, k_0 be positive constants and $0 \le \theta < 1$. Let the function $\varphi : [k_0, +\infty) \to [0, +\infty)$ be nonincreasing and such that*

$$\varphi(h) \le \frac{ch^{\theta\alpha}}{(h-k)^{\alpha}}[\varphi(k)]^{\beta} \tag{3}$$

for every h, k with $h > k \ge k_0 > 0$. It turns out that:

(i) *if $\beta > 1$ then $\varphi(2L) = 0$, where*

$$L = \max\left\{2k_0, c^{\frac{1}{(1-\theta)\alpha}}[\varphi(k_0)]^{\frac{\beta-1}{(1-\theta)\alpha}} 2^{\frac{1}{(1-\theta)\beta}\left(\beta+\theta+\frac{1}{\beta-1}\right)}\right\} > 0. \tag{4}$$

(ii) *if $\beta = 1$ then for any $k \ge k_0$,*

$$\varphi(k) \le \varphi(k_0) e^{1-\left(\frac{k-k_0}{\tau}\right)^{1-\theta}},$$

where $$\tau = \max\left\{k_0, \left(ce2^{\frac{(2-\theta)\theta\alpha}{1-\theta}}(1-\theta)^{\alpha}\right)^{\frac{1}{(1-\theta)\alpha}}\right\}; \tag{5}$$

(iii) *if $0 < \beta < 1$ then for any $k \ge k_0$,*

$$\varphi(k) \le 2^{\frac{(1-\theta)\alpha}{(1-\beta)^2}}\left\{(c_1 2^{\theta\alpha})^{\frac{1}{1-\beta}} + (2k_0)^{\frac{(1-\theta)\alpha}{1-\beta}}\varphi(k_0)\right\}\left(\frac{1}{k}\right)^{\frac{\alpha(1-\theta)}{1-\beta}}, \tag{6}$$

where

$$c_1 = \max\left\{4^{(1-\theta)\alpha}c2^{\theta\alpha}, c_2^{1-\beta}\right\}, \quad c_2 = 2^{\frac{(1-\theta)\alpha}{(1-\beta)^2}}\left[(c2^{\theta\alpha})^{\frac{1}{1-\beta}} + (2k_0)^{\frac{(1-\theta)\alpha}{1-\beta}}\varphi(k_0)\right]. \tag{7}$$

We remark that the difference between (3) and (2) is that $h^{\theta\alpha}$ in place of $k^{\theta\alpha}$. It is obvious that (3) is weaker than (2) since $k < h$. If $\theta = 0$, then (3) and (2) coincide with (1).

In order to prove Lemma 2.1, we need two preliminary lemmas. The following lemma can be found in Lemma 7.1 in [11].

Lemma 2.2. *Let β, B, C, x_i be such that $\beta > 1$, $C > 0$, $B > 1$, $x_i \geq 0$ and*

$$x_{i+1} \leq CB^i x_i^\beta, \quad i = 0, 1, 2, \cdots . \tag{8}$$

If

$$x_0 \leq C^{-\frac{1}{\beta-1}} B^{-\frac{1}{(\beta-1)^2}},$$

then, we have

$$x_i \leq B^{-\frac{i}{\alpha}} x_0, \quad i = 0, 1, 2, \cdots ,$$

so that

$$\lim_{i \to +\infty} x_i = 0.$$

The following lemma comes from Remark 1 and its proof in [9].

Lemma 2.3. *Let $c_3, \tilde{\alpha}, \beta, k_0$ be positive constants. Let $\varphi : [k_0, +\infty) \to [0, +\infty)$ be nonincreasing and such that*

$$\varphi(2k) \leq \frac{c_3}{k^{\tilde{\alpha}}} [\varphi(k)]^\beta, \quad 0 < \beta < 1, \tag{9}$$

for every $k \geq k_0$. Then for every h, k with $h > k \geq k_0$,

$$\varphi(h) \leq \frac{c_4}{(h-k)^{\tilde{\alpha}}} [\varphi(k)]^\beta,$$

where $\quad c_4 = \max\left\{4^{\tilde{\alpha}} c_3, c_5^{1-\beta}\right\}, \quad c_5 = 2^{\frac{\tilde{\alpha}}{(1-\beta)^2}} \left[c_3^{\frac{1}{1-\beta}} + (2k_0)^{\frac{\tilde{\alpha}}{1-\beta}} \varphi(k_0)\right].$

Proof of Lemma 2.1.

(i) For $\beta > 1$ we fix $L > k_0$ and choose levels

$$k_i = 2L(1 - 2^{-i-1}), \quad i = 0, 1, 2, \cdots .$$

It is obvious that $k_0 < L \leq k_i < 2L$ and $\{k_i\}$ be an increasing sequence. We choose in (3)

$$k = k_i, \;\; h = k_{i+1}, \;\; x_i = \varphi(k_i) \;\text{ and }\; x_{i+1} = \varphi(k_{i+1}),$$

and noticing that $h - k = k_{i+1} - k_i = L2^{-i-1}$, we have,

$$x_{i+1} \leq \frac{c\left(2L(1-2^{-i-2})\right)^{\theta\alpha}}{(L2^{-i-1})^\alpha} x_i^\beta, \quad i = 0, 1, 2, \cdots ,$$

from which we derive

$$x_{i+1} \leq \frac{c2^{(1+\theta)\alpha}(2^\alpha)^i}{L^{(1-\theta)\alpha}} x_i^\beta, \quad i = 0, 1, 2, \cdots .$$

Thus (8) holds true with (we keep in mind that $\beta > 1$)

$$C = \frac{c2^{(1+\theta)\alpha}}{L^{(1-\theta)\alpha}} \quad \text{and} \quad B = 2^\alpha.$$

We get from Lemma 2.2

$$\lim_{i\to+\infty} x_i = 0 \tag{10}$$

provided that

$$x_0 = \varphi(k_0) = \varphi(L) \le \left(\frac{c2^{(1+\theta)\alpha}}{L^{(1-\theta)\alpha}}\right)^{-\frac{1}{\beta-1}} (2^\alpha)^{-\frac{1}{(\beta-1)^2}}. \tag{11}$$

Note that (10) implies $\varphi(2L) = 0$.

Let us check condition (11) and determine the value of L. (11) is equivalent to

$$\varphi(L) \le c^{-\frac{1}{\beta-1}} 2^{-\frac{\alpha}{\beta-1}\left(1+\theta+\frac{1}{\beta-1}\right)} L^{\frac{(1-\theta)\alpha}{\beta-1}}. \tag{12}$$

In (3) we take $k = k_0$ and $h = L \ge 2k_0$ (which is equivalent to $L - k_0 \ge \frac{L}{2}$) and we have

$$\varphi(L) \le \frac{cL^{\theta\alpha}}{(L-k_0)^\alpha}[\varphi(k_0)]^\beta \le \frac{c2^\alpha L^{\theta\alpha}}{L^\alpha}[\varphi(k_0)]^\beta = \frac{c2^\alpha[\varphi(k_0)]^\beta}{L^{(1-\theta)\alpha}}.$$

Then condition (12) would be satisfied if

$$L \ge 2k_0 \tag{13}$$

and

$$\frac{c2^\alpha[\varphi(k_0)]^\beta}{L^{(1-\theta)\alpha}} \le c^{-\frac{1}{\beta-1}} 2^{-\frac{\alpha}{\beta-1}\left(1+\theta+\frac{1}{\beta-1}\right)} L^{\frac{(1-\theta)\alpha}{\beta-1}},$$

that is

$$c^{\frac{\beta}{\beta-1}}[\varphi(k_0)]^\beta 2^{\alpha+\frac{\alpha}{\beta-1}\left(1+\theta+\frac{1}{\beta-1}\right)} \le L^{\frac{(1-\theta)\alpha\beta}{\beta-1}}. \tag{14}$$

(4) is a sufficient condition for both (13) and (14).

(ii) Let $\beta = 1$ and τ be as in (5). For any $s \in \mathbb{N}^+$ we let

$$k_s = k_0 + \tau s^{\frac{1}{1-\theta}},$$

then $\{k_s\}$ is an increasing sequence and

$$k_{s+1} - k_s = \tau\left[(s+1)^{\frac{1}{1-\theta}} - s^{\frac{1}{1-\theta}}\right].$$

We use Taylor's formula in order to get

$$k_{s+1} - k_s = \tau\left[\frac{1}{1-\theta}s^{\frac{\theta}{1-\theta}} + \frac{\theta}{(1-\theta)^2}\xi^{\frac{2\theta-1}{1-\theta}}\right] \ge \frac{\tau}{1-\theta}s^{\frac{\theta}{1-\theta}}, \tag{15}$$

where ξ lies in the open interval $(s, s+1)$. In (3) we take $\beta = 1$, $k = k_s$ and $h = k_{s+1}$, we use (15) and we get

$$\varphi(k_{s+1}) \le \frac{c\left[k_0 + \tau(s+1)^{\frac{1}{1-\theta}}\right]^{\theta\alpha}}{\left(\frac{\tau}{1-\theta}\right)^\alpha s^{\frac{\theta\alpha}{1-\theta}}}\varphi(k_s) \le \frac{c\left[k_0 + \tau(2s)^{\frac{1}{1-\theta}}\right]^{\theta\alpha}}{\left(\frac{\tau}{1-\theta}\right)^\alpha s^{\frac{\theta\alpha}{1-\theta}}}\varphi(k_s). \tag{16}$$

(5) ensures

$$k_0 \le \tau < \tau(2s)^{\frac{1}{1-\theta}} \quad \text{and} \quad \frac{c\left(2^{1+\frac{1}{1-\theta}}\tau\right)^{\theta\alpha}}{\left(\frac{\tau}{1-\theta}\right)^{\alpha}} \le \frac{1}{e}.$$

From (16) and the above inequalities, one has

$$\varphi(k_{s+1}) \le \frac{c\left(2\tau(2s)^{\frac{1}{1-\theta}}\right)^{\theta\alpha}}{\left(\frac{\tau}{1-\theta}\right)^{\alpha} s^{\frac{\theta\alpha}{1-\theta}}}\varphi(k_s) = \frac{c\left(2^{1+\frac{1}{1-\theta}}\tau\right)^{\theta\alpha}}{\left(\frac{\tau}{1-\theta}\right)^{\alpha}}\varphi(k_s) \le \frac{1}{e}\varphi(k_s).$$

By recursion we obtain

$$\varphi(k_s) \le \frac{1}{e^s}\varphi(k_0), \quad s \in \mathbb{N}^+.$$

For any $k \ge k_0$, there exists $s \in \mathbb{N}^+$ such that

$$k_0 + \tau(s-1)^{\frac{1}{1-\theta}} \le k < k_0 + \tau s^{\frac{1}{1-\theta}}.$$

Thus, considering $\varphi(k)$ is nonincreasing, one has

$$\varphi(k) \le \varphi\left(k_0 + \tau(s-1)^{\frac{1}{1-\theta}}\right) = \varphi(k_{s-1}) \le e^{1-s}\varphi(k_0) \le \varphi(k_0)e^{1-\left(\frac{k-k_0}{\tau}\right)^{1-\theta}},$$

as desired.

(iii) We take $h = 2k$ in (3) and we get for every $k \ge k_0 > 0$,

$$\varphi(2k) \le \frac{c2^{\theta\alpha}}{k^{(1-\theta)\alpha}}[\varphi(k)]^{\beta}.$$

Thus (9) holds true with $c_3 = c2^{\theta\alpha}$ and $\tilde{\alpha} = (1-\theta)\alpha$. Lemma 2.3 tells us that for every h, k with $h > k \ge k_0$,

$$\varphi(h) \le \frac{c_1}{(h-k)^{(1-\theta)\alpha}}[\varphi(k)]^{\beta},$$

where c_1 be as in (7). We use Lemma 1.1 (iii) and we get (6).

3. Applications

In this section we are interested in the following degenerate elliptic problem

$$\begin{cases} -\operatorname{div}(a(x,u(x))Du(x)) = f(x), & \text{in } \Omega; \\ u(x) = 0, & \text{on } \partial\Omega. \end{cases} \tag{17}$$

Here Ω is a bounded, open subset of $\mathbb{R}^n$, $n > 2$, and $a(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (that is, measurable with respect to x for every $s \in \mathbb{R}$, and continuous with respect to s for almost every $x \in \Omega$) satisfying the following conditions:

$$\frac{\alpha}{(1+|s|)^{\theta}} \leq a(x,s) \leq \beta, \tag{18}$$

for some positive number θ such that

$$0 \leq \theta < 1,$$

for almost every $x \in \Omega$ and for every $s \in \mathbb{R}$, where α and β are positive constants. As far as the source f is concerned, we assume

$$f \in L^m_{weak}(\Omega), \quad m > 1, \tag{19}$$

where $L^m_{weak}(\Omega)$ is the Marcinkiewicz space (see Definition 3.8 in [3]), which consists of all measurable functions f on Ω with the following property: there exists a constant γ such that

$$|\{x \in \Omega : |f| > \lambda\}| \leq \frac{\gamma}{\lambda^m}, \quad \forall \lambda > 0, \tag{20}$$

where $|E|$ is the Lebesgue measure of the set E. The norm of $f \in L^m_{weak}(\Omega)$ is defined by

$$\|f\|^m_{L^m_{weak}(\Omega)} = \inf\{\gamma > 0, (20) \text{ hlods}\}.$$

A Hölder inequality holds true for $f \in L^m_{weak}(\Omega)$ and $m > 1$: there exists some $B = B(\|f\|_{L^m_{weak}(\Omega)}, m) > 0$ such that for every measurable subset $E \subset \Omega$,

$$\int_E |f| dx \leq B|E|^{1-\frac{1}{m}}, \tag{21}$$

see Proposition 3.13 in [3].

For $m > (2^*)' = \left(\frac{2n}{n-2}\right)' = \frac{2n}{n+2}$ we consider weak solutions of (17).

Definition 3.1. A function $u \in W^{1,2}_0(\Omega)$ is a solution to (17) if for any $v \in W^{1,2}_0(\Omega)$,

$$\begin{cases} -\operatorname{div}(a(x,u(x))Du(x)) = f(x), & \text{in } \Omega; \\ u(x) = 0, & \text{on } \partial\Omega. \end{cases} \tag{22}$$

We note that, because of assumption (18), the differential operator

$$-\operatorname{div}(a(x,v)Dv)$$

is not coercive on $W^{1,2}_0(\Omega)$, even if it is well defined between $W^{1,2}_0(\Omega)$ and its dual, see the last chapter in the monograph [3]. We mention that, for $f \in L^m(\Omega)$, Boccardo, Dall'Aglio and Orsina derived in [4] some existence and regularity results of (17), see also the last chapter in [3].

In the first part of this section, we consider degenerate elliptic equations with the right hand side belongs to Marcinkiewicz space $L^m_{weak}(\Omega)$ with $m > (2^*)'$ by using Lemma 2.1. This method seems to be simpler than the classical ones.

Theorem 3.2. *Let $f \in L^m_{weak}(\Omega)$, $m > (2^*)'$, and $u \in W^{1,2}_0(\Omega)$ be a solution to* (17). *Then*

(i) $m > \frac{n}{2} \Rightarrow \exists L = L(n, \alpha, \|f\|_{L^m_{weak}(\Omega)}, m, |\Omega|) > 0$ *such that* $|u| \le 2L$, *a.e.* Ω*;*

(ii) $m = \frac{n}{2} \Rightarrow \exists \lambda = \lambda(n, \alpha, \|f\|_{L^m_{weak}(\Omega)}, m) > 0$ *such that* $e^{\lambda |u|^{1-\theta}} \in L^1(\Omega)$*;*

(iii) $(2^*)' < m < \frac{n}{2} \Rightarrow u \in L^{m^{**}(1-\theta)}_{weak}(\Omega)$, $m^{**} = \frac{Nm}{N-2m}$.

In [18, 19], Stampacchia considered the following linear Dirichlet problem

$$\begin{cases} -\mathrm{div}(a(x)Du(x)) = f(x), & \text{in } \Omega; \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{23}$$

and obtained the following results

Theorem 3.3. *If f belongs to the Marcinkiewicz space $L^m_{weak}(\Omega)$, $(2^*)' < m < \frac{n}{2}$, then the weak solutions u of (23) belong to $L^{m^{**}}_{weak}(\Omega)$; if $f \in L^m_{weak}(\Omega)$, $m > \frac{n}{2}$, then $u \in L^\infty(\Omega)$.*

We compare the results in Theorem 3.2 with the ones in Theorem 3.3: if $\theta = 0$, then Theorem 3.2(i), (iii) coincide with the results in Theorem 3.3. The result Theorem 3.2(ii) seems to be new.

If we weaken the suitability hypotheses on f, then it is possible to give a meaning to solution for problem (17), using the concept of entropy solutions which has been introduced in [1].

Definition 3.4. Let $f \in L^m_{weak}(\Omega)$, $1 < m < (2^*)'$. A measurable function u is an entropy solution of (17) if $T_\ell(u)$ belongs to $W^{1,2}_0(\Omega)$ for every $\ell > 0$ and if

$$\int_\Omega a(x,u) Du DT_\ell(u - \varphi) dx \le \int_\Omega f T_\ell(u - \varphi) dx, \tag{24}$$

for every $\ell > 0$ and for every $\varphi \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega)$.

In (24), $T_\ell(v) = \max\{-v, \min\{v, \ell\}\}$ is the truncation of v at level $\ell > 0$.

We have the following result for entropy solutions.

Theorem 3.5. *Let u be an entropy solution of* (17) *and $f \in L^m_{weak}(\Omega)$, where $1 < m < (2^*)'$. Then $u \in L^{m^{**}(1-\theta)}_{weak}(\Omega)$.*

In [2] the author considered distributional solutions to

$$\begin{cases} -\mathrm{div}(a(x, u(x), Du(x)) = f(x), & \text{in } \Omega; \\ u(x) = 0, & \text{on } \partial\Omega. \end{cases} \tag{25}$$

Under some coercivity and controllable growth conditions of the operator $a(x, s, \xi)$, the author derived (see [2, Theorem 2.1]) the following

Theorem 3.6. *If f belongs to the Marcinkiewicz space $L^m_{weak}(\Omega)$, $1 < m < (2^*)'$, then there exists a distributional solution u of (25) belonging to $L^{m^{**}}_{weak}(\Omega)$ and we have $|Du| \in L^{m^*}_{weak}(\Omega)$.*

The difference between Theorems 3.5 and 3.6 is that in Theorem 3.5 we consider entropy solutions, while in Theorem 3.6 the author considered distributional solutions. In case of $\theta = 0$, the result in Theorem 3.5 coincide with the first result in Theorem 3.6.

We now use Lemma 2.1 to prove Theorems 3.2 and 3.5.

Proof of Theorem 3.2. Denote $G_k(u) = u - T_k(u)$. For $h > k > 0$, we take

$$v = T_{h-k}(G_k(u))$$

as a test function in (22) and we have, by (18), that

$$\alpha \int_{B_{k,h}} \frac{|Du|^2}{(1+|u|)^\theta} dx \le \int_{A_k} f T_{h-k}(G_k(u)) dx, \tag{26}$$

where $\qquad B_{k,h} = \{x \in \Omega : k < |u| \le h\}, \quad A_k = \{x \in \Omega : |u| > k\}.$

We now estimate both sides of (26). The left hand side can be estimated as

$$\begin{aligned}
\alpha \int_{B_{k,h}} \frac{|Du|^2}{(1+|u|)^\theta} dx &\ge \frac{\alpha}{(1+h)^\theta} \int_{B_{k,h}} |Du|^2 dx \\
&= \frac{\alpha}{(1+h)^\theta} \int_\Omega |DT_{h-k}(G_k(u))|^2 dx \\
&\ge \frac{\alpha \mathcal{S}^2}{(1+h)^\theta} \left(\int_\Omega |T_{h-k}(G_k(u))|^{2^*} \right)^{\frac{2}{2^*}} \\
&\ge \frac{\alpha \mathcal{S}^2}{(1+h)^\theta} \left(\int_{A_h} |T_{h-k}(G_k(u))|^{2^*} \right)^{\frac{2}{2^*}} \\
&= \frac{\alpha \mathcal{S}^2}{(1+h)^\theta} (h-k)^2 |A_h|^{\frac{2}{2^*}},
\end{aligned} \tag{27}$$

where we have used the Sobolev inequality

$$u \in W^{1,p}_0(\Omega) \Longrightarrow \mathcal{S}\|u\|_{L^{p^*}(\Omega)} \le \|Du\|_{L^p(\Omega)}, \quad p \ge 1, \ \mathcal{S} = \mathcal{S}(n,p).$$

Using Hölder inequality (21), the right hand side can be estimated as

$$\int_{A_k} f T_{h-k}(G_k(u)) dx \le (h-k) \int_{A_k} |f| dx \le (h-k) B |A_k|^{\frac{1}{m'}}, \tag{28}$$

where B is a constant depending on $\|f\|_{L^m_{weak}(\Omega)}$ and m.

(26) together with (27) and (28) implies, for $h \geq 1$,

$$|A_h| \leq \frac{c_6(1+h)^{\frac{\theta 2^*}{2}}}{(h-k)^{\frac{2^*}{2}}}|A_k|^{\frac{2^*}{2m'}} \leq \frac{c_7 h^{\frac{\theta 2^*}{2}}}{(h-k)^{\frac{2^*}{2}}}|A_k|^{\frac{2^*}{2m'}},$$

where c_6 is a constant depending on $n, \alpha, \|f\|_{L^m_{weak}(\Omega)}, m$ and $c_7 = c_6 2^{\frac{\theta 2^*}{2}}$. Thus (3) holds true with

$$\varphi(k) = |A_k|,\ c = c_7,\ \alpha = \frac{2^*}{2},\ \beta = \frac{2^*}{2m'} \ \text{ and } \ k_0 = 1.$$

We use Lemma 2.1 and we have:

(i) If $m > \frac{n}{2}$, then $\beta > 1$. We use Lemma 2.1 (i) and we have $|A_{2L}| = 0$ for some constant L depending on $n, \alpha, \|f\|_{L^m_{weak}(\Omega)}, m$ and $|\Omega|$, from which we derive $|u| \leq 2L$ a.e. Ω;

(ii) If $m = \frac{n}{2}$, then $\beta = 1$. We use Lemma 2.1 (ii) and we derive that there exists a constant τ depending on $n, \alpha, \|f\|_{L^m_{weak}(\Omega)}, m$ and θ, such that

$$|\{|u| > k\}| \leq |\{|u| > 1\}| e^{1-\left(\frac{k-1}{\tau}\right)^{1-\theta}} \leq |\Omega| e^{1-\left(\frac{k-1}{\tau}\right)^{1-\theta}}.$$

We let $2^{2-\theta}\lambda = \tau^{\theta-1}$ and $k \geq 2$ ($\Leftrightarrow k-1 \geq \frac{k}{2}$) and we have

$$|\{|u| > k\}| \leq |\Omega| e^{1-2^{2-\theta}\lambda(k-1)^{1-\theta}} \leq |\Omega| e^{1-2^{2-\theta}\lambda\left(\frac{k}{2}\right)^{1-\theta}} = c_7 e^{-2\lambda k^{1-\theta}},\ k \geq 2,$$

where $c_7 = |\Omega| e$. Hence

$$|\{e^{\lambda|u|^{1-\theta}} > e^{\lambda k^{1-\theta}}\}| = |\{|u| > k\}| \leq c_7 e^{-2\lambda k^{1-\theta}}.$$

Let $\tilde{k} = e^{\lambda k^{1-\theta}}$, then

$$|\{e^{\lambda|u|^{1-\theta}} > \tilde{k}\}| \leq \frac{c_7}{\tilde{k}^2}, \quad \text{for } \ \tilde{k} \geq e^{\lambda 2^{1-\theta}}.$$

Let $\left[e^{\lambda 2^{1-\theta}}\right] = k_1$. We now use Lemma 3.11 in [3] which states that the sufficient and necessary condition for $g \in L^r(\Omega)$, $r \geq 1$, is

$$\sum_{k=0}^{\infty} k^{r-1} |\{|g| > k\}| < +\infty.$$

We use the above lemma for $g = e^{\lambda|u|^{1-\theta}}$ and $r = 1$. Since

$$\sum_{\tilde{k}=0}^{\infty} |\{e^{\lambda|u|^{1-\theta}} > \tilde{k}\}| = \left(\sum_{\tilde{k}=0}^{k_1} + \sum_{\tilde{k}=k_1+1}^{\infty}\right) \leq (k_1+1)|\Omega| + c_7 \sum_{\tilde{k}=k_1+1}^{\infty} \frac{1}{\tilde{k}^2} < +\infty,$$

then $e^{\lambda|u|^{1-\theta}} \in L^1(\Omega)$;

(iii) If $(2^*)' < m < \frac{n}{2}$, then $\beta < 1$. We use Lemma 2.1 (iii) and we have for all $k \geq 1$,

$$|\{|u| > k\}| \leq c_8 \left(\frac{1}{k}\right)^{\frac{(1-\theta)\alpha}{1-\beta}} = c_8 \left(\frac{1}{k}\right)^{m^{**}(1-\theta)},$$

where c_8 is a constant depending on $n, \alpha, \|f\|_{L^m_{weak}(\Omega)}, m$ and $|\Omega|$.

Thus $$|\{|u| > \lambda\}| \le \max\{c_8, |\Omega|\} \left(\frac{1}{\lambda}\right)^{m^{**}(1-\theta)}, \ \forall \lambda > 0,$$

that is $u \in L^{m^{**}(1-\theta)}_{weak}(\Omega)$, as desired. □

Proof of Theorem 3.5. Let $h > k > 0$. In (24) we take

$$\varphi = T_k(u) \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega) \text{ and } \ell = h - k.$$

By (18) we derive (26). The result follows from the proof of Theorem 3.2 (iii). □

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