

Asymptotics of the p -Capacity in the Critical Regime

Clément Cosco

*Université Paris Dauphine, France
clement.cosco@gmail.com*

Shuta Nakajima

*Meiji University, Japan
njima@meiji.ac.jp*

Florian Schweiger

*Weizmann Institute of Science, Rehovot, Israel;
current affiliation: Université de Genève, Switzerland
florian.schweiger@unige.ch*

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We are interested in the asymptotics of the p -capacity between the origin and the set nB , where B is the boundary of the unit ball of the lattice $\mathbb{Z}^d$. The p -capacity is defined as the minimum of the Dirichlet energy associated with a discrete version of the p -Laplacian. This variational problem has arisen in particular in the study of large deviations for first passage percolation. For $p < d$, the p -capacity converges to some positive constant, while for $p > d$ the capacity vanishes polynomially fast. The present paper deals with the case $p = d$, for which we prove that the p -capacity vanishes as $c_d(\log n)^{-d+1}$ with an explicit constant c_d . Our proof relies on Thomson's principle for the p -capacity.

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1. Introduction

1.1. Setting

Fix $p \geq 1$. Let $G = (V, E)$ be a graph and $A, B \subset V$. Define the p -capacity between two sets A and B as:

$$\begin{aligned} \text{Cap}_{G,p}(A; B) \\ = \inf_{f: V \rightarrow \mathbb{R}} \left\{ \sum_{e=\langle x,y \rangle \in E} |f(x) - f(y)|^p \mid f(x) \geq 1 \ \forall x \in B, f(x) = 0 \ \forall x \in A \right\}. \end{aligned} \quad (1)$$

Let $D_M = \{x \in \mathbb{Z}^d : |x|_\infty \leq M\}$ be the ball of radius M , $\partial D_M = \{y \in \mathbb{Z}^d : |y|_\infty = M\}$ its boundary, and E_M the set of all edges whose both ends are in D_M . We consider the graph $G_n = (D_n, E_n)$ and define:

$$\text{Cap}_{d,p}(n) = \text{Cap}_{G_n,p}(\{0\}, \partial D_n).$$

In [2], the first two authors have shown the following:

Theorem 1.1. (Theorem 2.2 in [2]) For $d \geq 1$ and $p > 0$, the function

$$\kappa_{d,p}(n) = \begin{cases} \text{Cap}_{d,p}(n) & \text{if } p < d, \\ (\log n)^{d-1} \text{Cap}_{d,p} & \text{if } p = d, \\ n^{p-d} \text{Cap}_{d,p} & \text{if } p > d. \end{cases}$$

is bounded away from 0 and ∞ .

In this article, we give a precise estimate in the critical case $p = d$.

1.2. Main result

Theorem 1.2. Assume that $p = d$. As $n \rightarrow \infty$,

$$\text{Cap}_{d,d}(n) \sim \frac{\text{vol}_{d-1}(S_q^{d-1})}{(\log n)^{d-1}}, \quad (2)$$

where $q = \frac{d}{d-1}$, $S_q^{d-1} = \{x \in \mathbb{R}^d : |x|_q = 1\}$ and $\text{vol}_{d-1}(S_q^{d-1})$ denotes the $(d-1)$ -dimensional volume of S_q^{d-1} .

Remark 1.3. One can replace the norm in the definition of D_M by any norm $|\cdot|_r$ with $r > 0$ and find the same asymptotics as in (2) with the same proof. We keep the norm $|\cdot|_\infty$ for simplicity.

1.3. Background and related works

Asymptotics of the p -capacity have appeared naturally in the study of upper tail large deviations in the context of first passage percolation on $\mathbb{Z}^d$. Let us briefly describe the model. Let $(\tau_e)_{e \in E}$ be i.i.d. weights defined on the edges of $\mathbb{Z}^d$, and

$$T(0, x) = \inf_{\gamma: 0 \rightarrow x} \sum_{e \in \gamma} \tau_e,$$

the infimum of the total weight of all paths from 0 to $x \in \mathbb{Z}^d$, that we call the *passage time*. The passage time satisfies the following law of large numbers:

$$\mu(x) = \lim_{n \rightarrow \infty} \frac{1}{n} T(0, nx) \quad \text{a.s.},$$

where μ is a deterministic function. It is shown in [2] that if the weight distribution satisfies $\mathbb{P}(\tau_e > t) \asymp e^{-\alpha t^p}$ with $1 < p < d$, then for all $\xi > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n^p} \log \mathbb{P}(T(0, nx) \geq (\mu(x) + \xi)n) = -\alpha 2^{1-p} \lambda_{d,p} \xi, \quad (3)$$

where

$$\lambda_{d,p} = \lim_{n \rightarrow \infty} \text{Cap}_{d,p}(n) > 0.$$

An interesting behavior appears in the critical case $p = d$. In the same paper, it is proved that for all $\xi > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n^d \text{Cap}_{d,d}(\ell_n)} \log \mathbb{P}(T(0, nx) \geq (\mu + \xi)n) = -\alpha 2^{1-d} \xi^d. \quad (4)$$

with $\ell_n \sim n / \log n$.

Thus, if we combine (4) with Theorem 1.2, we obtain that

$$\lim_{n \rightarrow \infty} \frac{(\log n)^{d-1}}{n^d} \log \mathbb{P}(T(0, nx) \geq (\mu + \xi)n) = -\alpha 2^{1-d} \xi^d \text{vol}_{d-1}(S_q^{d-1}), \quad (5)$$

and we see that a logarithmic correction appears in the order of magnitude of the large deviations (compare with (3)).

When $p = 2$, the capacity admits a representation in terms of probabilities of the hitting time of random walks, see [9, pp. 25 and 36]. Moreover, the properties of the capacity have been extensively studied in this case – in particular, the asymptotics given in Theorem 1.2 are well-known for $p = 2$, see e.g. [7, Proposition 6.3.2]. The standard way to deal with these estimates for $p = 2$ is to rely on Fourier analysis, but this does not adapt well when p is different from 2; thus, we follow another route here.

1.4. Idea of the proof and organization of the paper

The main idea is to compare the discrete p -capacity in (1) to its continuous analogue¹

$$\text{Cap}_{d,p}^c(A; B) = \inf \left\{ \int_{\mathbb{R}^d} \sum_{i=1}^d |\partial_i f|^p \mid f \in W^{1,p}(\mathbb{R}^d), f(x) = 1 \ \forall x \in B, f(x) = 0 \ \forall x \in A \right\},$$

for $A, B \subset \mathbb{R}^d$. Under some general conditions on A and B , one can for example show by using the tools of Γ -convergence that

$$\lim_{n \rightarrow \infty} n^{p-d} \text{Cap}_{\mathbb{Z}^d,p}(nA \cap \mathbb{Z}^d; nB \cap \mathbb{Z}^d) = \text{Cap}_{d,p}^c(A; B).$$

In fact, it is possible to deduce this from general results in [1]. But these tools are not sharp enough to obtain the precise asymptotics in (2), so one needs to be able to compare the continuous capacity to the discrete one with a higher degree of precision.

Our strategy to prove Theorem 1.2 is as follows. We begin by considering the continuous analogue of the variational problem that defines $\text{Cap}_{d,d}(n)$. The corresponding continuous capacity is easily computed thanks to a symmetry argument that reduces the question to a 1-dimensional problem. Moreover, we can give an explicit expression to the continuous minimizer. Then, we can obtain an upper bound on $\text{Cap}_{d,d}(n)$ by choosing discrete approximation of the continuous optimizer. In order to obtain a lower bound, we rely on Thomson's variational formulation of the p -capacity (Theorem 2.13), whose proof follows from the electrical network considerations that we present in Section 2. The problem is then reduced to constructing a discrete flow that is a good approximation to the optimal continuous flow. To do so, we average locally the optimal continuous flow in a careful way to keep the flow property on the lattice. The proof of Theorem 1.2 is presented in Section 3.

¹Note that there are other ways to define the continuous p -capacity; for example the standard way is to choose the operator in the energy as $(\sum_{i=1}^d |\partial_i f|^2)^{p/2}$. Our choice of operator is adapted to the form of the discrete capacity in (1).

2. p -Electrical network and Thomson principle

The p -capacity on a graph can be represented as the effective capacity of an associated p -electrical network. If $p \neq 2$, the network has the property of satisfying a *nonlinear* Ohm law, in the sense that if h denotes the potential of the network and I its current, one has

$$I = cU(\Delta h),$$

where c is the conductance of the network and $U(u) = |u|^{p-1} \times \text{sign}(u) = |u|^{p-2}u$ is nonlinear when $p \neq 2$. The study of nonlinear networks was initiated in [4, 10], and there has been later on much focus on the case of infinite graphs, see e.g. [5] or [12, Appendix 2] and references therein.

In this section, we explain the relation between the p -capacity and its associated p -electrical network, and use it to show Thomson's principle (Theorem 2.14). It is also a good opportunity to introduce the key notions of p -harmonic functions, flows and strength of flows that will play a central role in the proof of Theorem 1.2.

Although different formulations of Thomson's principle have been shown to hold for general types of functions U and infinite graphs [6, 3], we restrict ourselves to the case of finite graphs and for the function $U(u) = |u|^{p-1} \times \text{sign}(u) = |u|^{p-2}u$. Our motivation is to present a simple approach, inspired from [11, Section 8] (which deals with the linear case $p = 2$), that avoids the technicalities arising when considering a more general setting.

2.1. p -Harmonic functions and flows

Let V be a finite set of vertices. For $x, y \in V$, we let $\langle x, y \rangle = \{x, y\}$ denote the *undirected edge* with x, y at both ends. We consider a finite graph $G = (V, E)$ with E being a collection of undirected edges, where edges may be repeated (we allow multiple edges). In the following, we will also need to consider *directed edges* $\vec{x}\vec{y}$ that are given by an ordered couples $(x, y) \in V^2$.

We assume the graph G to be *connected*, in the sense that for all $x, y \in V$, there exists a path $x = x_0, x_1, \dots, x_n = y$ such that $\{x_{i-1}, x_i\} \in E$ for all $i \leq n$.

An electric network on G is defined by the choice of some $a, b \in V$ which are called the *source* and the *sink* of the network, and by a *conductance* $c_{xy} = c_{yx} \in \mathbb{R}_+$ defined on all edges $\langle x, y \rangle \in E$. We also define $r_{xy} = c_{xy}^{-1}$ to be the *resistance* of $\langle x, y \rangle$.

Throughout the section, we fix $p > 1$ if not stated otherwise.

Definition 2.1. A function $h: V \rightarrow \mathbb{R}$ is said to be *p -harmonic at a vertex $x \in V$* if

$$\sum_{y \sim x} c_{xy} |h(y) - h(x)|^{p-2} (h(y) - h(x)) = 0. \quad (6)$$

We say that h is *p -harmonic* (or a *potential*) if it is p -harmonic for all $x \in V \setminus \{a, b\}$.

An immediate property of p -harmonic functions is the maximum principle:

Theorem 2.2. (Maximum Principle) *If h is p -harmonic, then*

$$\max_{x \in V} h(x) = \max_{x \in \{a, b\}} h(x).$$

Proof. By (6), if there exists $x \notin \{a, b\}$ that is a global maximum then for all $y \sim x$ we have $h(x) = h(y)$. Since G is connected, we find by repeating this observation that h is constant. Therefore the maximum is always attained in $\{a, b\}$. $\square$

Remark 2.3. The last argument also entails that if h is p -harmonic at every $x \in V$, then h must be constant since V is finite.

The maximum principle implies in particular that the only p -harmonic function satisfying $h(a) = h(b) = 0$ is the constant function $h = 0$. In fact, one can show that there is an unique p -harmonic function up to affine transformations. In order to see that, one can rely on the following fundamental comparison principle of p -harmonic functions:

Theorem 2.4. (Comparison principle. Theorem 3.4 in [5]) *If h and g are two p -harmonic functions such that $h(a) \leq g(a)$ and $h(b) \leq g(b)$, then $h(x) \leq g(x)$ for all $x \in V$.*

Corollary 2.5. *Let h be a p -harmonic function such that $h(a) = 0$ and $h(b) = 1$. Then, for every p -harmonic function g , there exist $\alpha, \beta \in \mathbb{R}$ such that $g = \alpha h + \beta$.*

Proof. Let $\beta = g(a)$ and $\alpha = g(b) - g(a)$. Then, the function $\tilde{g} := \alpha h + \beta$ is also p -harmonic and satisfies $g(a) = \tilde{g}(a)$ and $g(b) = \tilde{g}(b)$. By the comparison principle, it must be that $g = \tilde{g}$. $\square$

Definition 2.6. A *flow from a to b* – or simply *flow* – is a function $\theta(\vec{xy})$ on directed edges that is anti-symmetric, in the sense that for all $x, y \in V$, $\theta(\vec{xy}) = -\theta(\vec{yx})$, and which satisfies the *node law*:

$$\forall x \in G \setminus \{a, b\}, \quad \sum_{y \sim x} \theta(\vec{xy}) = 0. \quad (7)$$

If h is a p -harmonic function, one can define a flow I that we call the *current flow* associated with h by the relation:

$$I(\vec{xy}) = c_{xy} |h(y) - h(x)|^{p-2} (h(y) - h(x)). \quad (8)$$

We say that a flow θ satisfies the *cycle law* if for all $x_0, \dots, x_n \in V$ such that $x_n = x_0$, and $\vec{e}_i = \overrightarrow{x_{i-1}x_i}$,

$$\sum_{i=1}^n r_{\vec{e}_i}^{\frac{1}{p-1}} \frac{\theta(\vec{e}_i)}{|\theta(\vec{e}_i)|^{\frac{p-2}{p-1}}} = 0. \quad (9)$$

Note that we set the summand in the above sum to be 0 whenever $\theta(\vec{e}_i) = 0$. By definition, any current flow I satisfies the cycle law. Conversely, any flow that satisfies the cycle law is a current flow:

Theorem 2.7. *Let θ be a flow satisfying the cycle law (9). Then, there exists a p -harmonic function h such that*

$$\theta(\vec{xy}) = c_{xy} |h(y) - h(x)|^{p-2} (h(y) - h(x)). \quad (10)$$

Moreover, p -harmonic functions satisfying (10) are unique up to an additive constant.

Proof. For any $x \in V$, let $a = x_0, x_1, \dots, x_k = x$ be any path from a to x and define

$$h(x) = h(a) + \sum_{i=1}^k r_{\vec{e}_i}^{\frac{1}{p-1}} \frac{\theta(\vec{e}_i)}{|\theta(\vec{e}_i)|^{\frac{p-2}{p-1}}}. \quad (11)$$

Note that by the cycle law, h is well defined. For all $x \sim y$, we have

$$h(y) - h(x) = r_{xy}^{\frac{1}{p-1}} \frac{\theta(\vec{xy})}{|\theta(\vec{xy})|^{\frac{p-2}{p-1}}}, \quad (12)$$

so that (10) is verified and, since $c_{xy}r_{xy} = 1$ when $c_{xy} \neq 0$,

$$\begin{aligned} & \sum_{y \sim x} c_{xy} |h(y) - h(x)|^{p-2} (h(y) - h(x)) \\ &= \sum_{y \sim x} c_{xy} r_{xy}^{\frac{1}{p-1}} r_{xy}^{\frac{p-2}{p-1}} \frac{|\theta(\vec{xy})|^{p-2} \theta(\vec{xy})}{|\theta(\vec{xy})|^{\frac{p-2}{p-1}(p-2)} |\theta(\vec{xy})|^{\frac{p-2}{p-1}}} = \sum_{y \sim x} \theta(\vec{xy}) = 0, \end{aligned}$$

which proves the first part of the theorem. Suppose now that another p -harmonic function h_1 satisfies (10). The function h_1 therefore satisfies (12) and in turn (11), which entails that $h_1 = h + h_1(a) - h(a)$. $\square$

Define the *strength* of a flow by $\|\theta\| := \sum_{x \sim a} \theta(\vec{ax})$. Note that since θ is anti-symmetric, one has $\sum_{y \in V} \sum_{x \sim y} \theta(\vec{xy}) = 0$, so that by the node law we find that $\|\theta\| = \sum_{x \sim b} \theta(\vec{xb})$.

Theorem 2.8. *Let θ and η be two flows that satisfy the cycle law. If $\|\theta\| = \|\eta\|$, then $\theta = \eta$.*

Proof. Let h be a p -harmonic function satisfying (10), and let g be a p -harmonic function associated with η via the same relation. If h is constant, then $\|\theta\| = 0$ so that $\|\eta\| = 0$. Therefore g is p -harmonic at every $x \in V$, and by Remark 2.3 one finds that g is constant and thus, $\theta = \eta = 0$.

If h is non-constant, then by the uniqueness property there exist $\alpha, \beta \in \mathbb{R}$ such that $g = \alpha h + \beta$. Hence $\eta = \alpha \theta$, and since $\|\theta\| \neq 0$ (otherwise h would be constant), we obtain from $\|\eta\| = \|\theta\|$ that $\eta = \theta$. $\square$

2.2. The effective resistance

Theorem 2.9. *Let h be a non-constant p -harmonic function and I be its current flow. The ratio*

$$R_p(a, b) := \frac{|h(b) - h(a)|^{p-2} (h(b) - h(a))}{\|I\|}, \quad (13)$$

is well defined, non-zero and does not depend on the choice of h .

Proof. First observe that if h is a non-constant p -harmonic function, then $\|I\|$ must be non-zero, otherwise h would be p -harmonic at every point $x \in V$ and by Remark 2.3, h would be constant. Moreover, $h(a)$ has to be different from $h(b)$ by the maximum principle.

Now let h_1, h_2 be two non-constant p -harmonic functions and I_1, I_2 be their associated current flows. Up to exchanging h_1 with $-h_1$ or h_2 with $-h_2$, one can assume that both $\|I_1\|$ and $\|I_2\|$ are positive. Denote by $\bar{h}_1 = h_1/\|I_1\|^{\frac{1}{p-1}}$ and $\bar{h}_2 = h_2/\|I_2\|^{\frac{1}{p-1}}$. Further let θ_1, θ_2 be the current flows associated with $\bar{h}_1, \bar{h}_2$. We have

$$\|\theta_1\| = \|I_1\|^{-1} \sum_{x \sim a} c_{ax} |h_1(x) - h_1(a)|^{p-2} (h_1(x) - h_1(a)) = 1,$$

and the same holds for θ_2 , so that by Theorem 2.8 we have $\theta_1 = \theta_2$. Hence by Theorem 2.7, $\bar{h}_1$ and $\bar{h}_2$ only differ from an additive constant, so in particular the ratios (13) with h_1 and h_2 are equal. $\square$

We call $R_p(a, b)$ the p -effective resistance and its inverse $C_p(a, b) := R_p(a, b)^{-1}$ the p -effective conductance or p -capacity. The following Dirichlet and Thomson principles offer effective ways of estimating the p -effective resistance and the p -capacity.

Theorem 2.10. (Dirichlet's principle) *One has:*

$$C_p(a, b) = \inf_{h: V \rightarrow \mathbb{R}} \left\{ \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} c_{xy} |h(y) - h(x)|^p, \ h(a) = 0, \ h(b) = 1 \right\}.$$

Moreover, the infimum is attained at a p -harmonic function h .

Remark 2.11. This entails that $C_p(a, b) = \text{Cap}_{G,p}(\{a\}, \{b\})$ when $c_{xy} \equiv 1$.

Remark 2.12. The theorem shows in particular that non-constant p -harmonic functions always exist and that the p -capacity is always positive.

Proof. The fact that it is p -harmonic is straightforward by differentiating at all points of $V \setminus \{a, b\}$. Now let h be the p -harmonic function minimizing the above infimum. We have

$$\begin{aligned} & \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} c_{xy} |h(y) - h(x)|^p \\ &= \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} c_{xy} (h(y) - h(x)) |h(y) - h(x)|^{p-2} (h(y) - h(x)) \\ &= \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} c_{xy} h(y) |h(y) - h(x)|^{p-2} (h(y) - h(x)) \\ & \quad - \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} c_{xy} h(x) |h(y) - h(x)|^{p-2} (h(y) - h(x)) \\ &= - \sum_{x \in V} \sum_{y \sim x} c_{xy} h(x) |h(y) - h(x)|^{p-2} (h(y) - h(x)). \end{aligned}$$

By p -harmonicity of h , the last sum further equals $-h(a)\|I\| - h(b)(-\|I\|)$ where I is the current flow associated with h . Since $h(b) = 1$ and $h(a) = 0$, this equals $C_p(a, b)$. $\square$

We say that a flow θ is a *unit flow* if $\|\theta\| = 1$. Dirichlet's principle can be restated as a problem involving flows as follows:

Theorem 2.13. (Thomson's principle) *When $p > 1$, the following holds:*

$$R_p(a, b) = \inf \left\{ \sum_{e \in E} r_e^{\frac{1}{p-1}} |\theta(e)|^{\frac{p}{p-1}}, \theta \text{ is a unit flow} \right\}^{p-1}.$$

Moreover, the infimum is attained at a current flow.

Proof. We first show that any minimizer θ of the above problem must satisfy the cycle law. Let $\vec{e}_1, \dots, \vec{e}_n$ form a cycle and set $\gamma(\vec{e}_i) = 1$ for all $1 \leq i \leq n$, and $\gamma(e) = 0$ on all other edges. Note that γ defines a flow.

Denoting by $\mathcal{E}(\theta)$ the energy $\sum_{e \in E} r_e^{\frac{1}{p-1}} |\theta(e)|^{\frac{p}{p-1}}$, for all $\varepsilon \in \mathbb{R}$ we have

$$\begin{aligned} 0 \leq \mathcal{E}(\theta + \varepsilon\gamma) - \mathcal{E}(\theta) &= \sum_{i=1}^n r_{\vec{e}_i}^{\frac{1}{p-1}} |(\theta + \varepsilon)(\vec{e}_i)|^{\frac{p}{p-1}} - \sum_{i=1}^n r_{\vec{e}_i}^{\frac{1}{p-1}} |\theta(\vec{e}_i)|^{\frac{p}{p-1}} \\ &= \frac{p}{p-1} \varepsilon \sum_{i=1}^n r_{\vec{e}_i}^{\frac{1}{p-1}} |\theta(\vec{e}_i)|^{\frac{p}{p-1}} + |\varepsilon|^{\frac{p}{p-1}} \sum_{i=1}^n r_{\vec{e}_i}^{\frac{1}{p-1}} \mathbf{1}_{\theta(\vec{e}_i)=0} + O(\varepsilon^2). \end{aligned}$$

By letting $\varepsilon \rightarrow 0$ from below and from above, we see that necessarily

$$\sum_{i=1}^n r_{\vec{e}_i}^{\frac{1}{p-1}} |\theta(\vec{e}_i)|^{\frac{p}{p-1}} = 0,$$

therefore θ is a current flow.

We will now prove that $\mathcal{E}(\theta)^{p-1} = R_p(a, b)$ for the minimizer θ . Let h be the p -harmonic function associated with θ such that $h(a) = 0$. We have

$$\sum_{e \in E} r_e^{\frac{1}{p-1}} |\theta(e)|^{\frac{p}{p-1}} = \frac{1}{2} \sum_{x \in V} \sum_{x \sim y} c_{xy} |h(y) - h(x)|^p.$$

In the last sum, if we replace h by $h/h(b)$ which satisfies $h(b) = 1$ and $h(a) = 0$, we see from Dirichlet's principle that the sum equals $|h(b)|^p C_p(a, b)$. Since $\|\theta\| = 1$, we have $|h(b)|^{p-1} = R_p(a, b)$. Therefore,

$$(|h(b)|^p C_p(a, b))^{p-1} = (|h(b)| R_p(a, b) C_p(a, b))^{p-1} = |h(b)|^{p-1} = R_p(a, b). \quad \square$$

Let $\text{Cap}_G(A; B)$ denote the capacity between two sets of vertices defined as in (1), except that we allow here the presence of a resistance term r_e in the definition of the energy $\sum_{e=\langle x, y \rangle \in E} r_{xy} |f(x) - f(y)|^p$. Thomson's principle can be easily extended to the case where the source and the sink are sets of vertices:

Theorem 2.14. (Thomson's principle. Set formulation)

$$\text{Cap}_{G,p}(A; B)^{-1} = \inf \left\{ \sum_{e \in E} r_e^{\frac{1}{p-1}} |\theta(e)|^{\frac{p}{p-1}}, \theta \text{ is a unit flow from } A \text{ to } B \right\}^{p-1}, \quad (14)$$

where a flow from A to B is defined as in Definition 2.6 with a, b replaced by A, B . Moreover, the infimum is attained at a current flow.

Proof. Let $A, B \subset G$ such that $A \cap B = \emptyset$. Consider the reduced network where all points $x \in A$ are merged to a single vertex a and all the edges that ended in A now end at a with the same conductances. Do the same for B and merge them to a single vertex b . Choose the potential on A and on B to be constant. Theorem 2.13 then implies (14). $\square$

Of independent interest, we observe that Thomson's principle still holds for the case $p = 1$. It can be stated as follows:

Corollary 2.15. *When $p = 1$, we have:*

$$\text{Cap}_{G,1}(A; B)^{-1} = \inf \{ |r\theta|_\infty, \theta \text{ is a unit flow from } A \text{ to } B \}. \quad (15)$$

Proof. First note that the additional restriction $f(x) \in [0, 1]$ in (1) does not change the value of the capacity. Moreover, on this restriction, $\sum r_{xy} |f(x) - f(y)|^p$ is decreasing in p . Hence,

$$\text{Cap}_{G,1}(A; B) \geq \text{Cap}^{1+\varepsilon}(A; B).$$

We take a sequence $\varepsilon_k \rightarrow 0$ so that

$$\text{Cap}_{G,1}(A; B) \geq \liminf_{\varepsilon \rightarrow 0} \text{Cap}_{G,1+\varepsilon_k}(A; B) = \lim_{k \rightarrow \infty} \text{Cap}_{G,1+\varepsilon_k}(A; B).$$

Let h_ε be a harmonic function for $\text{Cap}_{G,1+\varepsilon}(A; B)$ (i.e. h_ε is harmonic on $V \setminus (A \cup B)$). Then there exists a subsequence ε'_k of ε_k and a function $h: V \rightarrow \mathbb{R}$ such that $h_{\varepsilon'_k}(x) \rightarrow h^*(x)$ for all $x \in V$. This yields

$$\begin{aligned} \text{Cap}_{G,1}(A; B) &\leq \sum_{\langle x, y \rangle \in E} r_{xy} |h^*(x) - h^*(y)| \\ &= \lim_{k \rightarrow \infty} \sum_{\langle x, y \rangle \in E} r_{xy} |h_{\varepsilon'_k}(x) - h_{\varepsilon'_k}(y)| \\ &= \liminf_{\varepsilon \rightarrow 0} \text{Cap}_{G,1+\varepsilon}(A; B). \end{aligned}$$

Hence, $\lim_{\varepsilon \rightarrow 0} \text{Cap}_{G,1+\varepsilon}(A; B) = \text{Cap}_{G,1}(A; B)$.

Thomson's principle states that $\text{Cap}_{G,1+\varepsilon}(A; B) = |r^{\frac{1}{1+\varepsilon}} \theta_\varepsilon|_{\frac{1+\varepsilon}{\varepsilon}}^{-(1+\varepsilon)}$ for a current flow θ_ε . We can take a subsequence ε_k such that θ_{ε_k} converges to a unit flow θ^* . We get:

$$\lim_{k \rightarrow \infty} \left| r^{\frac{1}{1+\varepsilon_k}} \theta_{\varepsilon_k} \right|_{\frac{1+\varepsilon_k}{\varepsilon_k}}^{-(1+\varepsilon_k)} = |r\theta^*|_\infty^{-1} \leq \sup \{ |r\theta|_\infty^{-1} : \theta \text{ is a unit flow} \}.$$

On the other hand, if we let θ_* maximize $|r\theta|_\infty^{-1}$, we obtain from the fact that θ_{ε_k} is a maximizer for the $(1 + \varepsilon_k)$ -capacity

$$\lim_{k \rightarrow \infty} \left| r^{\frac{1}{1+\varepsilon_k}} \theta_{\varepsilon_k} \right|_{\frac{1+\varepsilon_k}{\varepsilon_k}}^{-(1+\varepsilon_k)} \geq \lim_{k \rightarrow \infty} \left| r^{\frac{1}{1+\varepsilon_k}} \theta_* \right|_{\frac{1+\varepsilon_k}{\varepsilon_k}}^{-(1+\varepsilon_k)} = |r\theta_*|_\infty^{-1}.$$

Putting things together, we obtain

$$\begin{aligned} \sup \left\{ |r\theta|_\infty^{-1} : \theta \text{ is a unit flow} \right\} &= \lim_{k \rightarrow \infty} \left| r^{\frac{1}{1+\varepsilon_k}} \theta_{\varepsilon_k} \right|_{\frac{1+\varepsilon_k}{\varepsilon_k}}^{-(1+\varepsilon_k)} \\ &= \lim_{\varepsilon \rightarrow 0} \text{Cap}_{G,1+\varepsilon}(A; B) = \text{Cap}_{G,1}(A; B). \end{aligned} \quad \square$$

3. Proof of Theorem 1.2

3.1. The continuous analogue

As mentioned in the introduction, it turns out that the analogous continuous p -capacity problem can be easily solved. Given $q \geq 1$, we define

$$B_q^d = \{x \in \mathbb{R}^d : |x|_q \leq 1\}, \quad S_q^{d-1} = \{x \in \mathbb{R}^d : |x|_q = 1\}.$$

In all that follows, q will be chosen as $q = \frac{d}{d-1}$. Let,

$$\text{Cap}_d^c(n) = \inf \left\{ \int_{nB_q^d} \sum_{i=1}^d |\partial_i f(x)|^d dx : f(x) = 0 \ \forall x \in B_q^d, f(y) = 1 \ \forall y \in nS_q^{d-1} \right\}.$$

Theorem 3.1. *We have*

$$\text{Cap}_d^c(n) = \frac{\text{vol}_{d-1}(S_q^{d-1})}{(\log n)^{d-1}}, \quad (16)$$

where vol_{d-1} is the $d-1$ -dimensional volume.

Proof. The proof follows from a standard computation. If we take

$$f(x) = \frac{\log |x|_q}{\log n} = \frac{\log \sum_{i=1}^d |x_i|^q}{q \log n},$$

then
$$|\partial_i f(x)| = \frac{|x_i|^{q-1}}{\log n \sum_i |x_i|^q}.$$

Hence,
$$\begin{aligned} \int_{nB_q^d} \sum_{i=1}^d |\partial_i f(x)|^d dx &= \frac{1}{(\log n)^d} \int_{nB_q^d} \sum_{i=1}^d \left(\frac{|x_i|^{q-1}}{\sum_i |x_i|^q} \right)^d dx \\ &= \frac{1}{(\log n)^d} \int_{nB_q^d} \left(\sum_i |x_i|^q \right)^{-(d-1)} dx. \end{aligned} \quad (17)$$

Using polar coordinates,

$$x = r\theta \text{ with } r = |x|_q > 0 \text{ and } \theta = x/|x|_q \in S_q^{d-1}, \quad (18)$$

so that $dx = r^{d-1} dr d\theta$, since $|x|_q^{-(d-1)q} |x|_q^{d-1} = r^{-1}$, (17) is equal to

$$\frac{\text{vol}_{d-1}(S_q^{d-1})}{(\log n)^d} \int_1^n r^{-1} dr = \frac{\text{vol}_{d-1}(S_q^{d-1})}{(\log n)^{d-1}}. \quad (19)$$

This gives an upper bound on the continuous capacity.

Next, we consider the lower bound. We use polar coordinates as in (18) again. We denote $\theta = (\theta_i)_{i=1}^d$. By the Hölder inequality, we have:

$$|\partial_r f(r\theta)|^d = \left| \sum_i \theta_i \partial_i f(x) \right|^d \leq \left(\sum_i |\theta_i|^q \right)^{1/q} \left(\sum_i |\partial_i f(x)|^d \right) = \sum_i |\partial_i f(x)|^d.$$

Hence,

$$\begin{aligned} \int_{nB_q^d} \sum_{i=1}^d |\partial_i f(x)|^d dx &\geq \iint_{[1,n] \times S_q^{d-1}} |\partial_r f(r\theta)|^d r^{d-1} dr d\theta \\ &\geq \text{vol}_{d-1}(S_q^{d-1}) \inf \left\{ \int_1^n |g'(t)|^d t^{d-1} dt : g : [1, n] \rightarrow \mathbb{R}, g(1) = 0, g(n) = 1 \right\} \\ &= \frac{\text{vol}_{d-1}(S_q^{d-1})}{(\log n)^{d-1}}, \end{aligned}$$

where we have changed variables in the first line, and used that the variational problem of dimension 1 in the last line is exactly solvable ($g(t) = \frac{\log t}{\log n}$ achieves the infimum). $\square$

3.2. Upper bound

Lemma 3.2. $\limsup_{n \rightarrow \infty} (\log n)^{d-1} \text{Cap}_{d,d}(n) \leq \text{vol}_{d-1}(S_q^{d-1}).$

Proof. Let us set $f(x) = \frac{\log(d|x|_q+1)}{\log n}$. Then for $x \in \partial D_n$, we have $f(x) \geq 1$, and also $f(0) = 0$. Moreover, for $x \in D_n$,

$$|f(x + \mathbf{e}_i) - f(x)| = (1 + o(1)) \frac{|x_i|^{q-1}}{\log n \sum_i |x_i|^q}. \quad (20)$$

Hence, we have

$$\begin{aligned} \sum_{x,y \in E} |f(x) - f(y)|^d &\leq (1 + o(1)) (\log n)^{-d} \sum_{x \in D_n} \left(\frac{|x_i|^{q-1}}{\sum_i |x_i|^q} \right)^d \\ &\leq (1 + 2o(1)) (\log n)^{-d} \int_{nB_q^d} \sum_{i=1}^d \left(\frac{|x_i|^{q-1}}{\sum_i |x_i|^q} \right)^d dx \\ &\leq (1 + 2o(1)) \frac{\text{vol}_{d-1}(S_q^{d-1})}{(\log n)^{d-1}}, \end{aligned} \quad (21)$$

which gives an upper bound. $\square$

Before turning to the proof of the lower bound, let us introduce the notion of p -harmonic functions on $\mathbb{R}^d$.

3.3. A p -harmonic function and its associated flow

We say that a continuous function u is Δ_p -harmonic² on $A \subset \mathbb{R}^d$ if for all $x \in A$ one has $\Delta_p u(x) = 0$, where

$$\Delta_p u = \operatorname{div} \left(\left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right)_{i=1, \dots, d} \right).$$

We observe that when $p = d$, the function

$$g(x) = q^{-1} \log \|x\|_q^q \quad \text{where } q = \frac{d}{d-1}, \quad (22)$$

is Δ_p -harmonic in $\mathbb{R}^d \setminus \{0\}$. Indeed, one has

$$\frac{\partial g}{\partial x_i}(x) = \frac{|x_i|^{\frac{d}{d-1}-1}}{|x|_q^q} \frac{x_i}{|x_i|},$$

so that

$$\Theta_i(x) := \left| \frac{\partial g}{\partial x_i}(x) \right|^{d-2} \frac{\partial g}{\partial x_i}(x) = \frac{x_i}{|x|_q^d}.$$

Therefore,

$$\frac{\partial \Theta_i}{\partial x_i}(x) = \frac{(\sum_{i=1}^d |x_i|^q)^{d-2}}{|x|_q^{2d}} \left(\sum_{i=1}^d |x_i|^q - d|x_i|^q \right),$$

from where we see that

$$\forall x \in \mathbb{R}^d \setminus \{0\}, \quad \operatorname{div} \Theta(x) = 0, \quad (23)$$

with $\Theta = (\Theta_i)_{i=1, \dots, d}$, i.e. g is Δ_p -harmonic. In particular, the vector field Θ defines a continuous *flow* (i.e. a divergence-free vector field) on $\mathbb{R}^d \setminus \{0\}$ associated with the Δ_p -harmonic function g .

3.4. Lower bound via discretization of flow

Lemma 3.3. *We have $\liminf_{n \rightarrow \infty} (\log n)^{d-1} \operatorname{Cap}_{d,d}(n) \geq \operatorname{vol}_{d-1}(S_q^{d-1})$.*

Proof. We will use Thomson's principle (Theorem 2.13). The divergence-free function Θ defined in Section 3.3 is a good candidate for a flow, but one needs first to construct an approximating discrete flow out of it. To do so, we use a construction described by Lyons in [8, Section 6]. Let x, y be two neighboring points in $\mathbb{Z}^d$ and denote by S_{xy} the face of the cube of length one centered at x that is perpendicular to $\vec{xy}$. We then define the flow $\theta(\vec{xy})$ to be the value of the flow of Θ that goes through S_{xy} in the direction from x to y . By the divergence theorem, θ defines a flow on $\mathbb{Z}^d \setminus \{0\}$.

Now let $x \in \mathbb{Z}^d$ and $y = (x_1, \dots, x_i \pm 1, \dots, x_d)$ for some $i \leq d$. The discrete flow θ satisfies

$$\theta(\vec{xy}) = \pm \int_{\Pi_{j \neq i} [x_j - \frac{1}{2}, x_j + \frac{1}{2}]} \frac{x_i \pm \frac{1}{2}}{\left(\sum_{j \neq i} |z_j|^q + |x_i \pm \frac{1}{2}|^q \right)^{\frac{d-1}{d}}} \prod_{j \neq i} dz_j.$$

² Note that the standard definition of a p -harmonic function would be with the operator $\operatorname{div}(|\nabla u|^{p-2} \nabla u)$. This is why we call it Δ_p -harmonic instead.

By the reverse triangular inequality, the above denominator is bounded from below by $(|x|_q - |a|_q)^d$ where $a = (1/2, \dots, 1/2)$. Since $|a|_q \leq d$, we obtain that for all x such that $|x|_q > d$,

$$|\theta(x, y)| \leq \frac{|x_i \pm \frac{1}{2}|}{(|x|_q - d)^d},$$

and hence, by the triangular inequality,

$$\frac{1}{2} \sum_{y \sim x} |\theta(x, y)|^{\frac{d}{d-1}} \leq \frac{(|x|_q + |a|_q)^q}{(|x|_q - d)^{dq}} \leq \frac{(|x|_q + d)^q}{(|x|_q - d)^{dq}}.$$

We now fix $\varepsilon > 0$ and choose $R = R_\varepsilon > 0$ big enough such that for all $x \in \mathbb{Z}^d$ with $|x|_q > R$ and any neighboring point $x' \in \prod_{i \leq d} [x_i - 1/2, x_i + 1/2]$, we have (recall that $q = \frac{d}{d-1}$),

$$\frac{(|x|_q + d)^q}{(|x|_q - d)^{dq}} \leq (1 + \varepsilon) |x'|_q^{-d}.$$

This implies in particular that for all $|x|_q > R$,

$$\begin{aligned} \frac{(|x|_q + d)^q}{(|x|_q - d)^{dq}} &= \int_{\prod_{i \leq d} [x_i - 1/2, x_i + 1/2]} \frac{(|x|_q + d)^q}{(|x|_q - d)^{dq}} dx' \\ &\leq (1 + \varepsilon) \int_{\prod_{i \leq d} [x_i - 1/2, x_i + 1/2]} |x'|_q^{-d} dx'. \end{aligned}$$

Then we let $c > 0$ be such that $c^{-1}|x|_\infty \leq |x|_q \leq c|x|_\infty$, so that

$$\begin{aligned} &\frac{1}{2} \sum_{x \in \mathbb{Z}^d: 0 \leq |x|_\infty \leq n} \sum_{y \sim x} |\theta(x, y)|^{\frac{d}{d-1}} \\ &\leq \frac{1}{2} \sum_{x \in \mathbb{Z}^d: 0 \leq |x|_q \leq R} \sum_{y \sim x} |\theta(x, y)|^{\frac{d}{d-1}} + \sum_{x \in \mathbb{Z}^d: R < |x|_q \leq cn} \frac{(|x|_q + d)^q}{(|x|_q - d)^{dq}}. \end{aligned} \quad (24)$$

The last sum of (24) is less than

$$(1 + \varepsilon) \int_{1 \leq |x'|_q \leq cn+2} |x'|_q^{-d} dx' \leq (1 + \varepsilon) \text{vol}_{d-1}(S_q^{d-1}) \log(cn + 2),$$

by the same computation as in (19), while the first sum of (24) is bounded from above by CR^d for some $C = C(d) > 0$. This can be seen by using that

$$|\Theta_i(x)| \leq \frac{|x|_\infty}{|x|_q^d} \leq \frac{c}{|x|_q^{\frac{d-1}{d}}},$$

which implies that $|\Theta|_\infty$ is uniformly bounded from above on all faces S_{xy} and that $|\theta(x, y)|$ is uniformly bounded from above. Therefore,

$$\limsup_{n \rightarrow \infty} (\log n)^{-1} \frac{1}{2} \sum_{x \in \mathbb{Z}^d: 0 \leq |x|_\infty \leq n} \sum_{y \sim x} |\theta(x, y)|^{\frac{d}{d-1}} \leq (1 + \varepsilon) \text{vol}_{d-1}(S_q^{d-1}).$$

Since ε is arbitrary, we obtain from Thomson's principle (Theorem 2.14) that

$$\liminf_{n \rightarrow \infty} (\log n)^{d-1} \text{Cap}_{d,d}(n) \geq \|\theta\|^{q(d-1)} \text{vol}_{d-1}(S_q^{d-1})^{1-d}. \quad (25)$$

We are now left with computing $\|\theta\|$. We know that $\text{div } \Theta = 0$ on $\mathbb{R}^d \setminus \{0\}$, and so from the divergence theorem we obtain that

$$\|\theta\| = \int_{\partial D_{n-1/2}} \Theta \cdot d\vec{S} = \int_{nS_q^{d-1}} \Theta \cdot d\vec{S},$$

and from the same theorem applied to $g\Theta$ with g from (22), that also

$$\int_{(nB_q^d) \setminus B_q^d} \sum_{i=1}^d |\partial_i g(x)|^d + g \Delta_p g dx = \int_{\partial((nB_q^d) \setminus B_q^d)} (g\Theta) \cdot d\vec{S}.$$

Therefore, combining that $g = 0$ on S_q^{d-1} and $g = \log n$ on nS_q^{d-1} with the fact that g is Δ_p -harmonic, we find that

$$\int_{(nB_q^d) \setminus B_q^d} \sum_{i=1}^d |\partial_i g(x)|^d = (\log n) \int_{nS_q^{d-1}} \Theta \cdot d\vec{S}. \quad (26)$$

Furthermore, we know from (19), where $f = g/\log n$, that

$$\int_{(nB_q^d) \setminus B_q^d} \sum_{i=1}^d |\partial_i g(x)|^d dx = (\log n) \text{vol}_{d-1}(S_q^{d-1}),$$

and thus
$$\|\theta\| = \int_{nS_q^{d-1}} \Theta \cdot d\vec{S} = \text{vol}_{d-1}(S_q^{d-1}).$$

We thus obtain from (25) that

$$\liminf_{n \rightarrow \infty} (\log n)^{d-1} \text{Cap}_{d,d}(n) \geq \text{vol}_{d-1}(S_q^{d-1}). \quad \square$$

Remark 3.4. Equation (26) expresses that the continuous Dirichlet energy is equal to the strength of the continuous flow associated with the harmonic function g with boundary conditions $g = 0$ on S_q^{d-1} and $g = \log n$ on nS_q^{d-1} . This is the continuous analogue of what is shown in the proof of Theorem 2.10 (Dirichlet's principle). In particular, the integration by part that we used to obtain (26) may be compared to the manipulation of sums that is performed in the proof of Theorem 2.10.

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