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Optimal Control of Initial-Boundary Value Problems for Hyperbolic-Type Polyhedral Discrete and Differential Inclusions

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This paper is devoted to the optimization of the discrete and hyperbolic differential inclusions for polyhedral control problems with initial-boundary conditions. We formulate necessary and sufficient optimality conditions in the form of an Euler-Lagrange inclusion and transversality condition for the stated discrete problem. Then, to obtain optimality conditions for a discrete approximation problem, we approximate the discrete problem using the difference approximations of partial derivatives and grid functions on a uniform grid. The idea of that discretization method is to combine the discrete problem with the differential problem. Thus, the derivation of sufficient optimality conditions for the continuous problem is implemented by passing formally to the limit as the discrete steps tend to zero in the discrete-approximation problem. Finally, we conduct a numerical example to illustrate the efficiency of our results.

Keywords: Hyperbolic differential inclusions, boundary-value problem, optimality conditions.

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1. Introduction

This paper focuses on formulating optimality conditions for optimal control problems constrained by partial discrete and differential inclusions. The history of such conditions for optimal control problems governed by a partial differential equation *(PDE)* and/or inclusion *(PDI)* research spans more than fifty years; many significant results on the *PDE* and *PDI* have been reported in the literature; see e.g., [11, 16, 20, 22, 25] and the references therein. *PDE* and *PDI* have recently emerged in the mathematical modeling of various fields in engineering problems, such as fluid mechanics, heat transfer, and mathematical physics. As in the theory of ordinary differential inclusions, they also appear in a variety of purely mathematical considerations including calculus of variations [1, 2, 3, 5, 6, 9, 10, 12, 18, 19, 24].

There are numerous types of *PDEs* and methods for dealing with many of the individual equations that arise as a result of the various sources. Order two elliptic, parabolic, and hyperbolic *PDEs* have been extensively investigated since the early

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twentieth century. The work [28] provides an overview of optimal control problems constrained by linear and semi-linear elliptic and parabolic equations. In that book, the related operator is assumed to be bounded and coercive for problems involving linear *PDEs*. The corresponding operator fails to be coercive in the case of a Laplace equation with Neumann boundary conditions.

Hyperbolic differential inclusions are a significant class of optimal control problems described by *PDIs* and set-valued mappings. To be more specific, the Cauchy problem can be solved locally along any non-characteristic hypersurface for any arbitrary initial data. Since many mechanical equations are hyperbolic, the study of hyperbolic equations is of great current interest. The wave equation is a model of hyperbolic equation that frequently appears in classical physics, such as mechanical or light waves. Hyperbolic *PDEs* are also utilized to model a wide range of significant phenomena such as Aerodynamic flows, atmospheric fluxes [26]. In the paper [27], the authors investigate the optimal planning of drone navigation channels and provide a mathematical model of multi-channel optimal planning based on the optimization of a combination of waves utilizing hyperbolic *PDEs*. The paper [14] proves the equivalence of systems described by a single first-order hyperbolic *PDE* and integral delay equations. The suggested methodology can be used to analyze discontinuous solutions for nonlinear systems with measurable inputs acting on the boundary and/or differential equation given by a single first-order hyperbolic *PDE*.

Existing results for hyperbolic differential inclusions have received remarkable attention in recent years [4]. The authors investigated existence results for the optimization problem with classical initial conditions in separable Banach spaces when the set-valued mapping has convex or non-convex values. In the paper [13], variational problems in two independent variables with hyperbolic *PDEs* as side-conditions are considered to obtain a result analogous to Pontryagin's maximum principle. Differences in the theory of the corresponding *PDEs* or *PDIs* should be considered when solving optimal control problems. The paper [8] is concerned with the boundary value problems of nonlinear *PDIs* driven by a negative Laplacian, as well as the set-valued term containing the gradient. The authors demonstrated the existence of solutions to inclusions with convex and non-convex valued perturbations.

Many authors investigate the optimality conditions for optimal control problems with hyperbolic differential inclusions. The paper [7] is devoted to an optimal control problem given by hyperbolic differential inclusions with boundary conditions and endpoint constraints. Moreover, the author provides an approach involving second-order optimality conditions. The paper [15] deals with optimal control problems for hyperbolic systems with deviating arguments that appear in integral form both in the state equations and in the boundary conditions. Also, the author formulates necessary and sufficient conditions of optimality for the Neumann problem. The paper [21] is about discrete approximation on a uniform grid of the first boundary value problem for hyperbolic differential inclusions. Based on the concepts of locally adjoint mappings, the optimality conditions are derived for discrete and differential problems in the form of Euler-Lagrange inclusions.

Given the above presentation, we are motivated to formulate optimality conditions for the first mixed initial boundary value problems under consideration, which employ discrete and differential inclusions. The present paper is concerned with one of the most complicated and significant field-control theories of partial discrete and differential inclusions. The problems presented, as well as the corresponding conditions of optimality, are new. The structure of this paper is as follows.

In Section 2, we briefly present some basic notations and preliminaries for the set-valued mappings and cone of tangent directions required for the paper; the definitions and concepts can be found in the books [17] and [23].

Section 3 is devoted to the optimization of the discrete polyhedral problem *PDH)* and, in particular, to new discrete results that play a decisive role in deriving optimality conditions for the partial derivatives of the inclusion. Moreover, the specificity of the polyhedrality of the problem posed allows one to skillfully use Theorem 2.3 on minimization of a function for the intersection of a finite number of the set for the general case. Thus, the standard condition for the presence of an interior point of convex analysis is superfluous.

In Section 4, our main interest is to use finite difference approximations for constructing sufficient optimality conditions for hyperbolic *PDIs*. This discrete-approximate connection is of principal importance for the proposed method in our problems. Without such an approach, one can hardly mention any optimal condition for the differential problem. With this aim, we formulate the optimality conditions for discrete-approximate problem *(PA)* by comparing the results for discrete inclusions.

In Section 5, we consider a polyhedral control problem described by hyperbolic *PDIs* with initial-boundary conditions. The basic idea is to establish sufficient conditions for the optimality of the differential problem. We prove the sufficient conditions of optimality in terms of Euler-Lagrange inclusions and transversality conditions, thanks to passing the formal limit in the optimality conditions for problems with discrete-approximate inclusion.

The application of the given results is demonstrated in Section 6 by solving a numerical example. Thus, the investigated example illustrates that the conditions of the proved theorems and well-known results of classical optimal control theory coincide for concrete problems.

2. Preliminary problem statements

In this section, we introduce some basic definitions and facts which are essential tools in the later sections; see [17] and [23] for details. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space equipped with the norm $||\cdot||$ induced by the inner product $\langle x, v\rangle$, $x, v \in \mathbb{R}^n$. $F : \mathbb{R}^{4n} \rightrightarrows \mathbb{R}^n$ is a set-valued mapping from $\mathbb{R}^{4n} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ into the set of subsets of $\mathbb{R}^n$. F is said to be *convex* if its graph

$$gphF = \{(x, v_1, v_2, v_3, v_4) : v_4 \in F(x, v_1, v_2, v_3)\}$$

is a convex subset of $\mathbb{R}^{5n}$. A set-valued mapping F is convex-closed if its $gphF$ is a convex-closed set in $\mathbb{R}^{5n}$. It is convex-valued if $F(x, v_1, v_2, v_3)$ is a convex set for each $(x, v_1, v_2, v_3) \in domF$, where $domF = \{(x, v_1, v_2, v_3) : F(x, v_1, v_2, v_3) \neq \emptyset\}$.

The convex cone is one of the significant concepts in convex analysis and extremal theory. The investigation of its properties is connected with the calculation of the dual cone. The set of vectors $x^* \in \mathbb{R}^n$ for which $\langle x, x^*\rangle \geq 0$ holds for all $x \in K$ is called the dual cone to the cone K and is denoted by K^*. Given a set $A \subset \mathbb{R}^n$ and a point $z_0 \in A$, a vector $\overline{z} \in K_A(z_0)$ is a tangent direction to A at z_0 if there is a function $\gamma : (0, +\infty) \to \mathbb{R}^n$ with $\gamma(t) \to 0$ as $t \downarrow 0$ and such that $z_0 + t\overline{z} + \gamma(t) \in A$ for all $t > 0$ sufficiently small. The set $K_A(z_0)$ of all such vectors is clearly a cone; it is called the cone of tangent directions to A at z_0. It should be pointed out that the cone $K_A(z_0)$ is not uniquely defined. Since $t\overline{z},\ t > 0$ is a vector of tangent direction, if $\overline{z}$ is the same, then it is clear that such vectors form a cone.

A polyhedral convex set in $\mathbb{R}^n$ is a set that can be expressed as the intersection of some finite family of closed half-spaces, that is, as the set of solutions to some finite system of inequalities of the form $\langle x, x_k^* \rangle \leq \beta_k,\ k = 1, \dots, l$. In particular, if the finite system of inequalities is homogeneous, the set of solutions to this finite system of inequalities is called the polyhedral cone. A polyhedral set is closed.

It is known that if $K_1, K_2, \dots, K_m$ are convex cones in $\mathbb{R}^n$, then $K_1 + K_2 + \dots + K_m$ is also a convex cone and

$$(K_1 + K_2 + \dots + K_m)^* = K_1^* \cap K_2^* \cap \dots \cap K_m^*.$$

Moreover, if the cones $K_1, \dots, K_m$ are closed, then

$$\overline{K_1^* + K_2^* + \dots + K_m^*} = (K_1 \cap K_2 \cap \dots \cap K_m)^*.$$

For polyhedral cones, $K_1^* + K_2^* + \dots + K_m^*$ is also a polyhedral cone, this sum of cones is closed and the bar above it can be removed.

Lemma 2.1. [17] *If $K_1, K_2, \dots, K_m$ are polyhedral cones in $\mathbb{R}^n$, then*

$$K_1^* + K_2^* + \dots + K_m^* = (K_1 \cap K_2 \cap \dots \cap K_m)^*.$$

Theorem 2.2. [17, Farkas's Theorem 1.13] *If K is a polyhedral cone in $\mathbb{R}^n$, then the dual cone K^* is polyhedral and consists of the elements of the form*

$$x^* = \sum_{i=1}^{m} \gamma_i x_i^* \ ,\ \gamma_i \geq 0.$$

A subgradient of a convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at x_0 where f is finite, denoted x^*, is defined by the system of inequalities $f(x) - f(x_0) \geq \langle x^*, x - x_0 \rangle\ \forall\ x$. The set of all subgradients at x_0 denoted $\partial f(x_0)$ is referred to as the subdifferential of f at x_0. If the subdifferential at x_0 consists of only one element, it is equal to the gradient of f at x_0, denoted $\nabla f(x_0)$.

Suppose that x_0 is a point minimizing a function f defined on a set A. It is well known that the conditions that just characterize (somehow) the point x_0 are called the necessary conditions for an extremum. As a rule, if the minimized function and the set A are convex, then the above conditions are sufficient for an extremum. Later on, we use that detailed information about A gives us different necessary conditions for various problems. Thus, in Theorem 3.2 [17], we see that if $x_1 \in A$ is a point at which f is continuous, then in order for x_0 to be such a point it is necessary and sufficient that $\partial f(x_0) \cap K_A^*(x_0) \neq \emptyset$, where $K_A^*(x_0)$ is the dual cone to the $K_A(x_0)$ cone of tangent directions at a point $x_0 \in A$. Then, by Theorem 3.2 [17], in order for x_0 to be a point minimizing f over A, it is necessary and sufficient that $0 \in \partial f(x_0) - K_A^*(x_0)$. Moreover, this result is extended to the case where A is the intersection of finitely many sets as follows.

Theorem 2.3. [17] *Let f be a proper convex function and suppose that $A = \bigcap_{i=1}^{m} A_i$ where A_i are convex sets. Let $x_1 \in A$ be a point of continuity of f. Then, in order for x_0 to be a point minimizing f over A, it is necessary that there exist points $x_i^* \in K_{A_i}^*(x_0)$, $i = 1, 2, \dots, m$; $x^* \in \partial f(x_0)$ and a number $\lambda \in \{0, 1\}$ such that*

$$\lambda x^* = x_1^* + x_2^* + \cdots + x_m^*.$$

Here, if $\lambda = 0$ then at least one of the points $x_1^*, x_2^*, \dots, x_m^*$ is nonzero. If $\lambda = 1$, then these conditions are also sufficient for the minimization of f.

For a convex set-valued mapping F at a point $w^0 = (x^0, v_1^0, v_2^0, v_3^0, v_4^0) \in gphF$ setting $\gamma(\lambda) \equiv 0$, we have

$$\begin{aligned} K_{gphF}(w^0) &= cone\big[gphF - (w^0)\big] \\ &= \big\{(\overline{x}, \overline{v_1}, \overline{v_2}, \overline{v_3}, \overline{v_4}) : \overline{x} = \lambda(x - x^0),\ \overline{v_i} = \lambda(v_i - v_i^0),\ i = 1, 2, 3, 4\big\}, \end{aligned}$$

for all $(x, v_1, v_2, v_3, v_4) \in gphF$. With a convex set-valued mapping F we associate the set-valued mapping F^* defined by

$$F^*\big(v_4^*; w^0\big) := \big\{(x^*, v_1^*, v_2^*, v_3^*) : (x^*, v_1^*, v_2^*, v_3^*, -v_4^*) \in K_{gphF}^*(w^0)\big\}$$

is a locally adjoint set-valued mapping (LAM) to F at a point $w^0 \in gphF$, where $K_{gphF}^*(w^0)$ is the dual to the cone of tangent vectors $K_{gphF}(w^0)$.

In the first part of the paper, we consider the following polyhedral optimization problem for discrete hyperbolic inclusions

$$\text{minimize} \sum_{\substack{t=1,\dots,T \\ x=1,\dots,L-1}} f(u_{x,t}, x, t), \tag{1}$$

$$(PDH) \qquad \begin{aligned} & u_{x,t+1} \in F(u_{x,t-1},\ u_{x-1,t},\ u_{x,t},\ u_{x+1,t}), && (2) \\ & u_{x,0} = \alpha_{0x}, \quad u_{x,1} = \alpha_{1x}, \ \ x = 1, \dots, L-1, && \\ & u_{0,t} = \beta_{0t}, \quad u_{L,t} = \beta_{Lt}, \ \ t = 1, \dots, T-1, && (3) \\ & F(x, v_1, v_2, v_3) = \{v_4 : A_0 x + A_1 v_1 + A_2 v_2 + A_3 v_3 - B v_4 \le d\}, && \end{aligned}$$

where $f(\cdot, x, t) : \mathbb{R}^n \to \mathbb{R}$ is a real-valued polyhedral function, that is, its epigraph epi $f(\cdot, x, t)$ is polyhedral set in $\mathbb{R}^{n+1}$ and $F : \mathbb{R}^{4n} \rightrightarrows \mathbb{R}^n$ is a polyhedral set-valued mapping, A_0, A_1, A_2, A_3 and B, $m \times n$ dimensional matrices, d is a m-dimensional column-vector, L, T are fixed natural numbers, $\alpha_{0x}, \alpha_{1x}, \beta_{0t}, \beta_{1t},\ x = 1, \dots, L-1$, $t = 1, \dots, T-1$ are fixed vectors.

We call that problem (PDH) optimal control of boundary-value problem for a discrete inclusion of hyperbolic-type. A set of points $\{u_{x,t}\}_R = \{u_{x,t} : \ (x, t) \in R\}$ where $R = \{(x, t) : x = 0, \dots, L,\ t = 0, \dots, T,\ (x, t) \ne (0, 0), (L, 0), (0, T), (L, T)\}$ is called a feasible solution for the stated problem if it satisfies the polyhedral inclusions (2) and boundary conditions in (3). One can easily check that, for any fixed L and T, the boundary condition in (3) possesses a feasible solution and the number of points to be determined is equal to the number of discrete inclusions. In this sense, the name "discrete hyperbolic inclusion" is justified.

In the second part of the paper, we investigate the following polyhedral problem for hyperbolic differential inclusion

$$\text{minimize } J[u(\cdot,\cdot)] = \iint\limits_D f(u(x,t),x,t)dxdt + \int_0^L f_0(u(x,T),x)dx, \qquad (4)$$

$$\textit{(PCH)} \quad \Box\, u(x,t) \in F(u(x,t),x,t),\ (x,t) \in D := [0,L]\times[0,T], \qquad (5)$$

$$u(x,0) = \varphi_0(x)\ , \quad \frac{\partial u}{\partial t}(x,0) = \varphi_1(x),$$

$$u(0,t) = \psi_0(t)\ , \quad u(L,t) = \psi_1(t), \qquad (6)$$

where $\Box$ is the operator defined as $\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$. Moreover, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a polyhedral set-valued mapping given by

$$F(y) = \{z :\ Py - Qz \le d\},$$

where P, Q are $m \times n$ dimensional matrices, the functions $f(\cdot,x,t) : \mathbb{R}^n \to \mathbb{R}$ and $f_0(\cdot,x) : \mathbb{R}^n \to \mathbb{R}$ are polyhedral and $\varphi_0(x), \varphi_1(x), \psi_0(t), \psi_1(t)$ are proper continuous functions, L and T are fixed real positive numbers.

Here the solution is understood as the classical solution. We use the following notation: $\Omega = (0,L)\times(0,T)$, $\Gamma = \{(0,t) :\ t \in (0,T)\} \cup \{(L,t) :\ t \in (0,T)\}$, and $\Gamma_0 = \{(x,0) : x \in [0,L]\}$. A function belonging to the space $C^2(\Omega)\cap C^1(\Omega\cup\Gamma\cup\Gamma_0)$, satisfying in Ω the differential inclusion (5), initial conditions $u(x,0) = \varphi_0(x)$, $u_t(x,0) = \varphi_1(x)$ in Γ_0, and boundary conditions $u(0,t) = \psi_0(t)$, $u(L,t) = \psi_1(t)$ in Γ, is called a classical feasible solution. In fact, according to the classical theory of hyperbolic differential equations, problem (4)–(6) can be called a Bolza type problem for the first mixed problem for hyperbolic differential inclusions. It should be noted that the definition of a solution in one sense or another (classical, generalized, etc.) in no way prevents the implementation of this result for the class of problems under consideration. We label this problem *(PCH)*.

3. The optimality principle for polyhedral partial discrete inclusions

In this section, we formulate necessary and sufficient conditions of optimality for the polyhedral problem *(PDH)* with discrete inclusions. For this purpose, we consider the integer $m = 2n(L-1) + n(T-1)(L+1)$, the L-dimensional vectors $u_t = (u_{0,t}, u_{1,t}, \ldots, u_{L,t}) \in \mathbb{R}^{n(L+1)}$, $t = 1,\ldots,T-1$, $u_0 = (u_{1,0},\ldots,u_{L-1,0}) \in \mathbb{R}^{n(L-1)}$ and $u_T = (u_{1,T},\ldots,u_{L-1,T}) \in \mathbb{R}^{n(L-1)}$, $w = (u_0, u_1, \ldots, u_{T-1}, u_T) \in \mathbb{R}^m$ and define the following convex sets in the space $\mathbb{R}^m$

$$M_{x,t} = \Big\{w = (u_0,\ldots,u_T) :\ (u_{x,t-1},\ u_{x-1,t},\ u_{x,t},\ u_{x+1,t},\ u_{x,t+1}) \in gphF\Big\},$$
$$x = 1,\ldots,L-1,\ t = 1,\ldots,T-1,$$

$$M_0 = \Big\{w = (u_0,\ldots,u_T) :\ u_{x,0} = \alpha_{0x},\ x = 1,\ldots,L-1\Big\},$$

$$M_T = \Big\{w = (u_0,\ldots,u_T) :\ u_{x,T} = \alpha_{Tx},\ \ x = 1,\ldots,L-1\Big\},$$

$$N_1 = \Big\{w = (u_0,\ldots,u_T) :\ u_{0,t} = \beta_{0t},\ \ t = 1,\ldots,T-1\Big\},$$

$$N_2 = \Big\{ w = (u_0, \ldots, u_T) : \ u_{L,t} = \beta_{Lt}, \ \ t = 1, \ldots, T-1 \Big\}.$$

We formulate a convex minimization problem in the space $\mathbb{R}^m$ which is equivalent to the problem *(PDH)* as follows:

$$\text{minimize } g(w) = \sum_{\substack{t=1,\ldots,T \\ x=1,\ldots,L-1}} f(u_{x,t}, x, t) \tag{7}$$

$$w \in C := \Big(\bigcap_{\substack{x=1,\ldots,L-1 \\ t=1,\ldots,T-1}} M_{x,t} \Big) \cap M_0 \cap M_T \cap N_1 \cap N_2,$$

where C is the convex set. This transformation allows us to prove rigorously that if $\widetilde{w} = (\widetilde{u}_0, \ldots, \widetilde{u}_T)$ is an optimal solution to the problem *(PDH)*, then $\widetilde{w}$ is an optimal solution to the problem (7).

Let $K_{M_{x,t}}(\tilde{w}), K_{M_0}(\tilde{w}), K_{M_T}(\tilde{w}), K_{N_1}(\tilde{w}), K_{N_2}(\tilde{w})$ be the cone of tangent directions, where $\tilde{w} \in C$. Moreover, let $K^*_{M_0}(\tilde{w})$ be the dual cone to the cone of tangent directions, i.e. $K^*_{M_0}(\tilde{w}) = \{w^* : \ \langle \overline{w}, w^* \rangle \geq 0, \ \forall \ \overline{w} \in K_{M_0}(\tilde{w})\}$. Then, following Theorem 2.3, there exist vectors $w^*(x,t) \in K^*_{M_{x,t}}(\tilde{w})$, $w^{0*} \in K^*_{M_0}(\tilde{w})$, $w^{T*} \in K^*_{M_T}(\tilde{w})$, $w^{1*} \in K^*_{N_1}(\tilde{w})$ and $w^{2*} \in K^*_{N_2}(\tilde{w})$ not all equal to zero such that

$$w_* \ = \sum_{\substack{x=1,\ldots,L-1 \\ t=1,\ldots,T-1}} w^*(x,t) + w^{0*} + w^{T*} + w^{1*} + w^{2*}, \tag{8}$$

where $w_* \in \partial g(\widetilde{w})$. Note that since the considered functions and mappings are polyhedral in the problem *(PDH)*, by virtue of Lemma 2.1 for the dual cone to the algebraic sum of polyhedral cones, the regularity condition in Theorem 2.3 is superfluous. With respect to the necessary and sufficient conditions of optimality, it is clear that the representation in formula (8) holds with $\lambda = 1$ for the point $w_* \in \partial g(\widetilde{w}) \cap K^*_C(\widetilde{w})$. Then by denoting $w^* = (u^*_0, u^*_1, \ldots, u^*_{T-1}, u^*_T)$, it can be easily seen that

$$\begin{aligned}
K^*_{M_0}(\widetilde{w}) &= \{w^* : u^*_t = 0, \ t = 1, \ldots, T\}, \\
K^*_{M_T}(\widetilde{w}) &= \{w^* : u^*_t = 0, \ t \neq T, \ u^*_{0,T} = u^*_{L,T} = 0\}, \\
K^*_{N_1}(\widetilde{w}) &= \{w^* : u^*_{x,t} = 0, \ t = 1, \ldots, T-1 \ , \ x \neq 0, \ u^*_0 = u^*_T = 0\}, \\
K^*_{N_2}(\widetilde{w}) &= \{w^* : u^*_{x,t} = 0, \ t = 1, \ldots, T-1, \ x \neq L, u^*_0 = u^*_T = 0\}.
\end{aligned} \tag{9}$$

Now, taking into account the definition of a polyhedral discrete inclusion (1), we calculate the dual cone $K^*_{M_{x,t}}(\widetilde{w})$. Then $u_{x,t+1} \in F(u_{x,t-1}, \ u_{x-1,t}, \ u_{x,t}, \ u_{x+1,t})$ means that $A_0 u_{x,t-1} + A_1 u_{x-1,t} + A_2 u_{x,t} + A_3 u_{x+1,t} - B u_{x,t+1} \leq d$, $(x = 1, \ldots, L-1$, $t = 1, \ldots, T-1)$. Let $A^i_0, A^i_1, A^i_2, A^i_3, B^i$ be the i-th row of the matrices A_0, A_1, A_2, A_3, B, respectively and d_i be the i-th component of the vector d.

$$K_{gphF}(\widetilde{v}) = \{\overline{v} : \ \widetilde{v} + \mu \overline{v} \in gphF \text{ for sufficiently small } \mu > 0\},$$

where $\widetilde{v} = (\widetilde{x}, \widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3, \widetilde{v}_4)$. Now for

$$I(\widetilde{v}) = \{i : A^i_0 \widetilde{x} + A^i_1 \widetilde{v}_1 + A^i_2 \widetilde{v}_2 + A^i_3 \widetilde{v}_3 - B^i \widetilde{v}_4 = d_i, \ i = 1, \ldots, m\}$$

the inequality

$$A_0^i(\tilde{x}+\mu\overline{x})+\sum_{j=1}^{3}\left[A_j^i(\tilde{v}_j+\mu\overline{v}_j)\right]-B^i(\tilde{v}_4+\mu\overline{v}_4)$$

$$=d_i+\mu\Big(A_0^i\overline{x}+\sum_{j=1}^{3}A_j^i\overline{v}_j-B^i\overline{v}_4\Big)\le d_i$$

holds if $A_0^i\overline{x}+\sum_{j=1}^{3}A_j^i\overline{v}_j-B^i\overline{v}_4\le 0,\ i\in I(\widetilde{v})$. If $i\notin I(\widetilde{v})$ then the inequality

$$A_0^i(\tilde{x}+\mu\overline{x})+\sum_{j=1}^{3}\left[A_j^i(\tilde{v}_j+\mu\overline{v}_j)\right]-B^i(\tilde{v}_4+\mu\overline{v}_4)$$

$$=\Big(A_0^i\tilde{x}+\sum_{j=1}^{3}A_j^i\tilde{v}_j-B^i\tilde{v}_4\Big)+\mu\Big(A_0^i\overline{x}+\sum_{j=1}^{3}A_j^i\overline{v}_j-B^i\overline{v}_4\Big)<d_i$$

is valid for sufficiently small $\mu>0$ regardless of the choice of $\overline{v}=(\overline{x},\overline{v}_1,\overline{v}_2,\overline{v}_3,\overline{v}_4)$. It follows that the polyhedral cone has the form

$$K_{gphF}(\widetilde{v})=\Big\{\overline{v}:\ A_0^i\overline{x}+\sum_{j=1}^{3}A_j^i\overline{v}_j-B^i\overline{v}_4\le 0,\ i\in I(\widetilde{v})\Big\}.$$

Using Farkas' Theorem 2.2 we easily see that $(x^*,v_1^*,v_2^*,v_3^*,v_4^*)\in K^*_{gphF}(\tilde{v})$ if and only if

$$x^*=-\sum_{i\in I(\tilde{v})}A_0^{i*}\mu_i,\ v_j^*=-\sum_{i\in I(\tilde{v})}A_j^{i*}\mu_i,\ j=1,2,3,\ v_4^*=\sum_{i\in I(\tilde{v})}B_i^*\mu_i,\ \mu_i\ge 0, \tag{10}$$

where $A_0^{i*},A_j^{i*},(j=1,2,3),B_i^*$ are transposed vectors of $A_0^i,A_j^i,(j=1,2,3),B_i$ respectively. Now setting $\mu_i=0,\ i\notin I(\tilde{v})$ and denoting by μ the vector column with components μ_i, we obtain that the polyhedral dual cone is calculated by the formula

$$K^*_{gphF}(\tilde{v})=\Big\{(x^*,v_1^*,v_2^*,v_3^*,v_4^*):\ x^*=-A_0^*\mu,\ v_j^*=-A_j^*\mu,\ j=1,2,3,$$
$$v_4^*=B^*\mu,\ \Big\langle A_0\tilde{x}+\sum_{j=1}^{3}A_j\tilde{v}_j-B\tilde{v}_4-d,\ \mu\Big\rangle=0,\ \mu\ge 0\Big\}.$$

Moreover, for the polyhedral LAM from the definition, we have

$$F^*(v_4^*;(\widetilde{x},\widetilde{v}_1,\widetilde{v}_2,\widetilde{v}_3,\widetilde{v}_4))=\Big\{(-A_0^*\mu,-A_1^*\mu,-A_2^*\mu,-A_3^*\mu):\ \ v_4^*=-B^*\mu,$$
$$\Big\langle A_0\tilde{x}+\sum_{j=1}^{3}A_j\tilde{v}_j-B\tilde{v}_4-d,\ \mu\Big\rangle=0\ ,\ \mu\ge 0\Big\}.$$

It should be noted that $F^*(v_4^*;\widetilde{v})$ does not depend on point $\widetilde{v}$ but depends on the set $I(\widetilde{v})$. Since the number of such sets is finite it follows that the number of different LAM $F^*(v_4^*;\widetilde{v})$ is finite. Then we have proved the following lemma.

Lemma 3.1. *The dual cone of the tangent directions $K^*_{M_{x,t}}(\widetilde{w})$ has the form*

$$K^*_{M_{x,t}}(\widetilde{w}) = \Big\{ w^*(x,t) : u^*_{x,t-1}(x,t) = -A_0^*\mu_{x,t},\ u^*_{x-1,t}(x,t) = -A_1^*\mu_{x,t},$$
$$u^*_{x,t}(x,t) = -A_2^*\mu_{x,t}, \quad u^*_{x+1,t}(x,t) = -A_3^*\mu_{x,t},\ u^*_{x,t+1}(x,t) = B^*\mu_{x,t},$$
$$u^*_{ij}(x,t) = 0,\ (i,j) \neq (x,t-1),(x-1,t),(x,t),(x+1,t),(x,t+1),$$
$$\big\langle A_0\tilde{u}_{x,t-1} + A_1\tilde{u}_{x-1,t} + A_2\tilde{u}_{x,t} + A_3\tilde{u}_{x+1,t} - B\tilde{u}_{x,t+1} - d,\ \mu_{x,t}\big\rangle = 0\ ,\ \mu_{x,t} \geq 0 \Big\},$$
$$x = 1,\dots,L-1,\ t = 1,\dots,T-1.$$

As mentioned at the beginning of this section, we are concerned with establishing necessary and sufficient conditions of optimality for the polyhedral problem *(PDH)*. To this end, we propose the following theorem.

Theorem 3.2. *Let $F : \mathbb{R}^{4n} \rightrightarrows \mathbb{R}^n$ be a polyhedral set-valued mapping and f be proper polyhedral function and continuous at the points of some feasible solution. Then for $\{\tilde{u}_{x,t}\}_R$ to be an optimal solution of the problem (PDH) with polyhedral discrete inclusions and boundary constraints, it is necessary and sufficient that there exist vectors $\mu_{x,t}$, $x = 0,\dots,L$, $t = 0,\dots,T$, not all equal zero satisfying the Euler-Lagrange discrete inclusions*

$$-A_0^*\mu_{x,t+1} - A_1^*\mu_{x+1,t} - A_2^*\mu_{x,t} - A_3^*\mu_{x-1,t} + B^*\mu_{x,t-1} \in \partial f(\tilde{u}_{x,t},x,t),$$
$$x = 1,\dots,L-1,\ t = 1,\dots,T-1,$$
$$\big\langle A_0\tilde{u}_{x,t-1} + A_1\tilde{u}_{x-1,t} + A_2\tilde{u}_{x,t} + A_3\tilde{u}_{x+1,t} - B\tilde{u}_{x,t+1} - d,\ \mu_{x,t}\big\rangle = 0\ ,\ \mu_{x,t} \geq 0,$$

and transversality conditions

$$\mu_{0,t} = \mu_{L,t} = 0,\ t = 1,\dots,T-1, \quad \mu_{x,T} = 0\ ,\ x = 1,\dots,L-1,$$
$$B^*\mu_{x,T-1} \in \partial f(\tilde{u}_{x,T},x,T),\ x = 1,\dots,L-1.$$

Proof. Note that $[w_*]_{x,t}$ denotes the components of the vector w_* for the given pair (x,t). Then by using the formulas in (9), we derive that

$$\Big[\sum_{\substack{x=1,\dots,L-1\\ t=1,\dots,T-1}} w^*(x,t) + w^{0*} + w^{T*} + w^{1*} + w^{2*} \Big]_{x,t} = [w_*]_{x,t}$$
$$= u^*_{x,t}(x,t+1) + u^*_{x,t}(x+1,t) + u^*_{x,t}(x,t) + u^*_{x,t}(x-1,t) + u^*_{x,t}(x,t-1), \quad (11)$$

where $w_* \in \partial g(\widetilde{w})$ and it means that $[w_*]_{x,t} = u^{f*}_{x,t} \in \partial f(\tilde{u}_{x,t},x,t)$, $x = 1,\dots,L-1$, $t = 1,\dots,T$. Then from Lemma 3.1, we have

$$-A_0^*\mu_{x,t+1} - A_1^*\mu_{x+1,t} - A_2^*\mu_{x,t} - A_3^*\mu_{x-1,t} + B^*\mu_{x,t-1} \in \partial f(\tilde{u}_{x,t},x,t),$$
$$x = 1,\dots,L-1,\ t = 1,\dots,T-1, \quad (12)$$
$$\big\langle A_0\tilde{u}_{x,t-1} + A_1\tilde{u}_{x-1,t} + A_2\tilde{u}_{x,t} + A_3\tilde{u}_{x+1,t} - B\tilde{u}_{x,t+1} - d,\ \mu_{x,t}\big\rangle = 0\ ,$$
$$\mu_{x,t} \geq 0\ ,\ t = 1,\dots,T-1,\ x = 1,\dots,L-1. \quad (13)$$

For a convenient form in the previous formula, we can arrange $\mu_{0,t} = \mu_{L,t} = 0$, $t = 1,\dots,T-1$, $\mu_{x,T} = 0$, $x = 1,\dots,L-1$. Moreover, from the components of the equality (11), we have $u^*_{x,T}(x,T-1) \in \partial f(\tilde{u}_{x,T},x,T)$, $x = 1,\dots,L-1$ or more convenient form $B^*\mu_{x,T-1} \in \partial f(\tilde{u}_{x,T},x,T)$, $x = 1,\dots,L-1$. This ends the proof of the theorem. □

4. Polyhedral partial discrete-approximate inclusions

The discretization method combines the polyhedral problem of discrete inclusions *(PDH)* with the differential problem *(PCH)*. We use difference operators to approximate the problem *(PCH)* and apply Theorem 3.2 to obtain the necessary and sufficient conditions of optimality. We choose steps h and δ on the x, t-axes, respectively and use the grid functions $u_{x,t} = u_{h\delta}(x,t)$ on a uniform grid on D.

$$\Box\, u(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = Eu(x,t) - Gu(x,t),$$

where $Eu(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t)$, $Gu(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$. The following difference operators are easily defined on the three-point models, i.e. each of the operators $Eu(x,t)$, $Gu(x,t)$ is approximated by the $\tilde{E}u(x,t)$, $\tilde{G}u(x,t)$:

$$\tilde{E}u(x,t) := \frac{u(x,t+\delta) - 2u(x,t) + u(x,t-\delta)}{\delta^2}$$

$$\tilde{G}u(x,t) := \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}.$$

As a result, we have the following discrete-approximate problem associated with the problem *(PCH)*:

$$\min\, J_{h\delta}(u(x,t)) = \sum_{\substack{x=0,\dots,L-h \\ t=0,\dots,T-\delta}} \delta h f(u(x,t),x,t) + \sum_{x=0,\dots,L-h} h f_0(u(x,T),x), \tag{14}$$

$$\text{(PA)} \quad \widetilde{E}u(x,t) - \widetilde{G}u(x,t) \in F(u(x,t),x,t), \tag{15}$$

$$u(x,0) = \varphi_0(x)\ ,\ u(x,\delta) = \varphi_0(x) + \delta\varphi_1(x),\ x = h, 2h, \dots, L-h,$$

$$u(0,t) = \psi_0(t)\ ,\ u(L,t) = \psi_1(t),\ t = \delta, 2\delta, \dots, T-\delta. \tag{16}$$

Here $$(x,t) \in R_{\delta h} = \left\{ (x,t) : \begin{array}{c} x = 0, h, \dots, L,\ t = 0, \delta, \dots, T, \\ (x,t) \neq (0,0),\ (0,T),\ (L,0),\ (L,T) \end{array} \right\}.$$

Then we establish the optimality conditions of the discrete-approximate problem *(PA)* associated with the continuous problem *(PCH)* as follows.

Theorem 4.1. *Let $f(\cdot,x,t)$ and $f_0(\cdot,x)$ be proper polyhedral functions and continuous at the points on some feasible solution. Then for the optimality of the solution $\{\tilde{u}(x,t)\}$ in the problem (PA), it is necessary and sufficient that there exist functions $\mu = \mu_{h\delta}$ not all equal to zero, such that*

(i) $Q^*\big[\widetilde{E}\mu(x,t) - \widetilde{G}\mu(x,t)\big] - P^*\mu(x,t) \in \partial f(\tilde{u}(x,t),x,t)$,
$x = h, \dots, L-h,\ t = \delta, \dots, T-\delta,$

(ii) $\big\langle P\widetilde{u}(x,t) - Q\big[\widetilde{E}\widetilde{u}(x,t) - \widetilde{G}\widetilde{u}(x,t)\big] - d,\ \mu(x,t)\big\rangle = 0\ ,\ \mu(x,t) \geq 0,$

(iii) $\mu(0,t) = \mu(L,t) = 0,\ t = \delta, \dots, T-\delta, \quad \mu(x,T) = 0\ ,\ x = h, \dots, L-h,$
$Q^*\Big[\dfrac{\mu(x,T-\delta) - \mu(x,T)}{\delta}\Big] \in \partial f_0(\tilde{u}(x,T),x,T)\ ,\ x = h, \dots, L-h.$

Proof. From the inclusion (15), it follows that

$$Pu(x,t) - Q\Big[\frac{1}{\delta^2}\Big(u(x,t+\delta) - 2u(x,t) + u(x,t-\delta)\Big) - \frac{1}{h^2}\Big(\big(u(x+h,t) \\ -2u(x,t) + u(x-h,t)\Big)\Big] \le d\ ,\ x = h,\dots,L-h,\ t = \delta,\dots,T-\delta. \tag{17}$$

Then the following inequality is equivalent with (4) by rewriting the form of the inclusion (2) in the discrete problem *(PDH)*:

$$-\frac{1}{\delta^2}Qu(x,t-\delta) + \frac{1}{h^2}Qu(x-h,t) + \Big[P + 2(\frac{1}{\delta^2} - \frac{1}{h^2})Q\Big]u(x,t) + \frac{1}{h^2}Qu(x+h,t) \\ -\frac{1}{\delta^2}Qu(x,t+\delta) \le d\ ,\ x = h,\dots,L-h,\ t = \delta,\dots,T-\delta.$$

Now denoting $\overline{A}_0 = -\frac{1}{\delta^2}Q$, $\overline{A}_1 = \frac{1}{h^2}Q$, $\overline{A}_2 = P + 2\big(\frac{1}{\delta^2} - \frac{1}{h^2}\big)Q$, $\overline{A}_3 = \frac{1}{h^2}Q$, and $\overline{B} = \frac{1}{\delta^2}Q$ we have a problem (1)–(3) where

$$\overline{A}_0u_{x,t-1}+\overline{A}_1u_{x-1,t}+\overline{A}_2u_{x,t}+\overline{A}_3u_{x+1,t}-\overline{B}u_{x,t+1} \le d,\ x = 1,\dots,L-1,\ t = 1,\dots,T-1.$$

Therefore, taking into account $(k_0\overline{A}_0+k_1\overline{A}_1)^* = k_0\overline{A}_0^*+k_1\overline{A}_1^*$ (k_0, k_1 are real numbers) and applying Theorem 3.2 we have

$$-\overline{A}_0^*\overline{\mu}_{x,t+1} - \overline{A}_1^*\overline{\mu}_{x+1,t} - \overline{A}_2\overline{\mu}_{x,t} - \overline{A}_3^*\overline{\mu}_{x-1,t} + \overline{B}^*\overline{\mu}_{x,t-1} \in \partial f(\tilde{u}_{x,t}, x, t), \\ x = 1,\dots,L-1,\ t = 1,\dots,T-1.$$

Then for the discrete-approximate problem we deduce that

$$\frac{1}{\delta^2}Q^*\overline{\mu}(x,t+\delta) - \frac{1}{h^2}Q^*\overline{\mu}(x+h,t) - \Big[P^* + 2(\frac{1}{\delta^2} - \frac{1}{h^2})Q^*\Big]\overline{\mu}(x,t) \\ -\frac{1}{h^2}Q^*\overline{\mu}(x-h,t) + \frac{1}{\delta^2}Q^*\overline{\mu}(x,t-\delta) \in \delta h\partial f(\tilde{u}(x,t),x,t), \\ x = h,\dots,L-h,\ t = \delta,\dots,T-\delta. \tag{18}$$

Then divide both sides of the inclusion (4) by δh and denoting $\overline{\mu}(x,t) = \delta h\mu(x,t)$, we write that

$$Q^*\Big[\frac{\mu(x,t+\delta) - 2\mu(x,t) + \mu(x,t-\delta)}{\delta^2}\Big] - Q^*\Big[\frac{\mu(x+h,t) - 2\mu(x,t) - \mu(x-h,t)}{h^2}\Big] \\ -P^*\mu(x,t) \in \partial f(\tilde{u}(x,t),x,t)\ ,\ x = h,\dots,L-h,\ t = \delta,\dots,T-\delta,$$

or, in terms of the difference operators $\widetilde{E}$ and $\widetilde{G}$ we rewrite inclusion (i) of the theorem in a more convenient form as follows

$$Q^*\big[\widetilde{E}\mu(x,t) - \widetilde{G}\mu(x,t)\big] - P^*\mu(x,t) \in \partial f(\tilde{u}(x,t),x,t), \\ x = h,\dots,L-h,\ t = \delta,\dots,T-\delta. \tag{19}$$

On the other hand, from formula (13) written in terms of $\overline{A}_i, (i = 0,1,2,3)$ and $\overline{B}$, we see that

$$\big\langle\overline{A}_0u(x,t-\delta) + \overline{A}_1u(x-h,t) + \overline{A}_2u(x,t) + \overline{A}_3u(x+h,t) - \overline{B}u(x,t+\delta) - d\,, \\ \overline{\mu}(x,t)\big\rangle = 0\ ,\ \overline{\mu}(x,t) \ge 0,\ x = h,\dots,L-h,\ t = \delta,\dots,T-\delta,$$

or, equivalently, dividing both sides of the last relation by δh and expressing A_i, $(i = 0, 1, 2, 3)$ and B again by $P, Q, \widetilde{E}, \widetilde{G}$ we derive that

$$\left\langle P\widetilde{u}(x,t) - Q\left[\widetilde{E}\widetilde{u}(x,t) - \widetilde{G}\widetilde{u}(x,t)\right] - d,\ \mu(x,t)\right\rangle = 0\ ,$$
$$\mu(x,t) \geq 0\ ,\ x = h, \ldots, L-h,\ t = \delta, \ldots, T-\delta. \tag{20}$$

Using the optimality conditions of Theorem 3.2 involved in the transversality conditions above, we further have

$$\overline{\mu}(0,t) = \overline{\mu}(L,t) = 0,\ t = \delta, \ldots, T-\delta, \quad \overline{\mu}(x,T) = 0\ ,$$
$$\frac{1}{\delta^2} Q^* \overline{\mu}(x, T-\delta) \in h\partial f_0(\tilde{u}(x,T), x, T)\ ,\ x = h, \ldots, L-h.$$

Again using $\overline{\mu}(x,t) = \delta h \mu(x,t)$, we have

$$\mu(0,t) = \mu(L,t) = 0,\ t = \delta, \ldots, T-\delta, \quad \mu(x,T) = 0\ ,$$
$$Q^* \left[\frac{\mu(x, T-\delta) - \mu(x,T)}{\delta}\right] \in \partial f_0(\tilde{u}(x,T), x, T)\ ,\ x = h, \ldots, L-h. \tag{21}$$

Then, from relations (4), (4), and (4), we obtain the proof of the theorem. □

5. Sufficient conditions of optimality for partial differential inclusions of hyperbolic type

By passing formally to the limit in conditions of Theorem 4.1 as $h \to 0,\ \delta \to 0$, we derive sufficient conditions of optimality for polyhedral problem *(PCH)* with *PDIs* of hyperbolic type:

(a) $Q^* \Box \mu(x,t) - P^* \mu(x,t) \in \partial f(\tilde{u}(x,t), x, t),\ (x,t) \in D,$

(b) $\left\langle P\widetilde{u}(x,t) - Q\Box\widetilde{u}(x,t) - d,\ \mu(x,t)\right\rangle = 0\ ,\ \mu(x,t) \geq 0,$

(c) $\mu(0,t) = \mu(L,t) = 0,\ t \in (0,T),$
$\mu(x,T) = 0\ ,\ -Q^* \dfrac{\partial \mu(x,T)}{\partial t} \in \partial f_0(\tilde{u}(x,T), x, T),\ x \in (0,L).$

It should be noted that the solution is taken in the space of classical solutions only for the sake of simplicity in the Euler-Lagrange inclusions (a). Now we are ready to present our results for differential inclusions as follows.

Theorem 5.1. *Let the functions f and f_0 be continuous and polyhedral functions with respect to u and F be a polyhedral mapping. Then for the optimality of a solution $\widetilde{u}(\cdot,\cdot)$ in problem (PCH) for hyperbolic differential inclusions, it is sufficient that there exists $\mu(x,t)$ such that conditions* (a),(b) *and* (c) *hold.*

Proof. Because of the definition of subdifferential, we have

$$f(u(x,t), x, t) - f(\tilde{u}(x,t), x, t) \geq \left\langle Q^* \Box \mu(x,t) - P^* \mu(x,t),\ u(x,t) - \widetilde{u}(x,t)\right\rangle. \tag{22}$$

By condition (b) of the theorem and the inclusion (5), we write

$$\left\langle Pu(x,t) - Q\Box u(x,t) - d,\ \mu(x,t)\right\rangle \leq \left\langle P\widetilde{u}(x,t) - Q\Box\widetilde{u}(x,t) - d,\ \mu(x,t)\right\rangle$$

in other words,

$$\Big\langle -P^*\mu(x,t)\ ,\ u(x,t)-\widetilde{u}(x,t)\Big\rangle \geq -\Big\langle \Box u(x,t)-\Box\widetilde{u}(x,t),\ Q^*\mu(x,t)\Big\rangle. \tag{23}$$

Using (22) and (23), we further obtain

$$\begin{aligned} f(u(x,t),x,t)-f(\tilde{u}(x,t),x,t) \geq \Big\langle Q^*\Box\mu(x,t)\ ,\ u(x,t)-\widetilde{u}(x,t)\Big\rangle \\ -\Big\langle \Box u(x,t)-\Box\widetilde{u}(x,t),\ Q^*\mu(x,t)\Big\rangle. \end{aligned} \tag{24}$$

Then integrating the inequality (5) over $D=[0,L]\times[0,T]$, we have

$$\begin{aligned} \iint\limits_D \Big[f(u(x,t),x,t)-f(\tilde{u}(x,t),x,t)\Big]dxdt \geq \iint\limits_D \Big[\Big\langle Q^*\Box\mu(x,t),u(x,t)-\widetilde{u}(x,t)\Big\rangle\Big]dxdt \\ -\iint\limits_D \Big[\Big\langle \Box u(x,t)-\Box\widetilde{u}(x,t),\ Q^*\mu(x,t)\Big\rangle\Big]dxdt. \end{aligned} \tag{25}$$

To calculate the right-hand side of the inequality (25), let's transform the following integral:

$$\begin{aligned} \iint\limits_D \Big[\Big\langle Q^*\Box\mu(x,t)\ ,\ u(x,t)-\widetilde{u}(x,t)\Big\rangle\Big]dxdt - \iint\limits_D \Big[\Big\langle \Box u(x,t)-\Box\widetilde{u}(x,t), \\ Q^*\mu(x,t)\Big\rangle\Big]dxdt = \iint\limits_D \Big[\Big\langle Q^*\frac{\partial^2\mu(x,t)}{\partial t^2}\ ,\ u(x,t)-\widetilde{u}(x,t)\Big\rangle\Big]dxdt \\ -\iint\limits_D \Big[\Big\langle \frac{\partial^2\big(u(x,t)-\widetilde{u}(x,t)\big)}{\partial t^2},\ Q^*\mu(x,t)\Big\rangle\Big]dxdt - \iint\limits_D \Big[\Big\langle Q^*\frac{\partial^2\mu(x,t)}{\partial x^2}\ , \\ u(x,t)-\widetilde{u}(x,t)\Big\rangle\Big]dxdt + \iint\limits_D \Big[\Big\langle \frac{\partial^2\big(u(x,t)-\widetilde{u}(x,t)\big)}{\partial x^2},\ Q^*\mu(x,t)\Big\rangle\Big]dxdt. \end{aligned} \tag{26}$$

Then it is easy to see that

$$\begin{aligned} \iint\limits_D \Big[\Big\langle Q^*\frac{\partial^2\mu(x,t)}{\partial t^2}\ ,\ u(x,t)-\widetilde{u}(x,t)\Big\rangle\Big]dxdt - \iint\limits_D \Big[\Big\langle \frac{\partial^2\big(u(x,t)-\widetilde{u}(x,t)\big)}{\partial t^2}, \\ Q^*\mu(x,t)\Big\rangle\Big]dxdt = \iint\limits_D \frac{\partial}{\partial t}\Big[\Big\langle u(x,t)-\widetilde{u}(x,t)\ ,\ Q^*\frac{\partial\mu(x,t)}{\partial t}\Big\rangle\Big]dxdt \\ -\iint\limits_D \Big[\frac{\partial}{\partial t}\Big\langle \frac{\partial\big(u(x,t)-\widetilde{u}(x,t)\big)}{\partial t}\ ,\ Q^*\mu(x,t)\Big\rangle\Big]dxdt \\ = \int_0^L \Big[\Big\langle u(x,T)-\widetilde{u}(x,T)\ ,\ Q^*\frac{\partial\mu(x,T)}{\partial t}\Big\rangle - \Big\langle u(x,0)-\widetilde{u}(x,0)\ ,\ Q^*\frac{\partial\mu(x,0)}{\partial t}\Big\rangle\Big]dx \\ -\int_0^L \Big[\Big\langle \frac{\partial\big(u(x,T)-\widetilde{u}(x,T)\big)}{\partial t}\ ,\ Q^*\mu(x,T)\Big\rangle - \Big\langle \frac{\partial\big(u(x,0)-\widetilde{u}(x,0)\big)}{\partial t}\ ,\ Q^*\mu(x,0)\Big\rangle\Big]dx. \end{aligned}$$

Then, since $u(x,t)$ and $\widetilde{u}(x,t)$ are feasible on the initial conditions $u(x,0)=\widetilde{u}(x,0)=\varphi_0(x)$ and $u_t(x,0)=\widetilde{u}_t(x,0)=\varphi_1(x)$, we deduce from the last relation that

$$\iint\limits_D \Big[\big\langle Q^*\frac{\partial^2\mu(x,t)}{\partial t^2}\ ,\ u(x,t)-\widetilde{u}(x,t)\big\rangle\Big]dxdt-\iint\limits_D\Big[\big\langle\frac{\partial^2\big(u(x,t)-\widetilde{u}(x,t)\big)}{\partial t^2},$$
$$Q^*\mu(x,t)\big\rangle\Big]dxdt=\int_0^L\Big[\big\langle u(x,T)-\widetilde{u}(x,T)\ ,\ Q^*\frac{\partial\mu(x,T)}{\partial t}\big\rangle\Big]dx. \tag{27}$$

Moreover, it is clear to see that

$$-\iint\limits_D \Big[\big\langle Q^*\frac{\partial^2\mu(x,t)}{\partial x^2}\ ,\ u(x,t)-\widetilde{u}(x,t)\big\rangle\Big]dxdt+\iint\limits_D\Big[\big\langle\frac{\partial^2\big(u(x,t)-\widetilde{u}(x,t)\big)}{\partial x^2},$$
$$Q^*\mu(x,t)\big\rangle\Big]dxdt=-\iint\limits_D\Big[\frac{\partial}{\partial x}\big\langle u(x,t)-\widetilde{u}(x,t)\ ,\ Q^*\frac{\partial\mu(x,t)}{\partial x}\big\rangle\Big]dxdt$$
$$+\iint\limits_D\Big[\frac{\partial}{\partial x}\big\langle\frac{\partial\big(u(x,t)-\widetilde{u}(x,t)\big)}{\partial x},\ Q^*\mu(x,t)\big\rangle\Big]dxdt$$
$$=-\int_0^T\Big[\big\langle u(L,t)-\widetilde{u}(L,t)\ ,\ Q^*\frac{\partial\mu(L,t)}{\partial x}\big\rangle-\big\langle u(0,t)-\widetilde{u}(0,t)\ ,\ Q^*\frac{\partial\mu(0,t)}{\partial x}\big\rangle\Big]dt$$
$$+\int_0^T\Big[\big\langle\frac{\partial\big(u(L,t)-\widetilde{u}(L,t)\big)}{\partial x}\ ,\ Q^*\mu(L,t)\big\rangle-\big\langle\frac{\partial\big(u(0,t)-\widetilde{u}(0,t)\big)}{\partial x}\ ,\ Q^*\mu(0,t)\big\rangle\Big]dt. \tag{28}$$

Again in view of the boundary conditions $u(0,t)=\widetilde{u}(0,t)=\psi_0(t)$ and $u(L,t)=\widetilde{u}(L,t)=\psi_1(t)$, the difference between the integrals in formula (28) is equal to zero. Considering equations (26), (27) and (28), we obtain the following formula

$$\iint\limits_D\Big[\big\langle Q^*\Box\mu(x,t)\ ,\ u(x,t)-\widetilde{u}(x,t)\big\rangle\Big]dxdt-\iint\limits_D\Big[\big\langle\Box u(x,t)-\Box\widetilde{u}(x,t),$$
$$Q^*\mu(x,t)\big\rangle\Big]dxdt=\int_0^L\Big[\big\langle u(x,T)-\widetilde{u}(x,T)\ ,\ Q^*\frac{\partial\mu(x,T)}{\partial t}\big\rangle\Big]dx. \tag{29}$$

Therefore, from the inequality (25), we conclude that

$$\iint\limits_D\Big[f(u(x,t),x,t)-f(\tilde{u}(x,t),x,t)\Big]dxdt$$
$$\geq\int_0^L\Big[\big\langle u(x,T)-\widetilde{u}(x,T)\ ,\ Q^*\frac{\partial\mu(x,T)}{\partial t}\big\rangle\Big]dx. \tag{30}$$

Moreover, by the transversality condition of the theorem, we have

$$f_0(u(x,T),x,T)-f_0(\tilde{u}(x,T),x,T)\geq\big\langle-Q^*\frac{\partial\mu(x,T)}{\partial t}\ ,\ u(x,T)-\widetilde{u}(x,T)\big\rangle. \tag{31}$$

Then, integrating inequality (31) on the interval $[0, L]$, we obtain:

$$\int_0^L \big[f_0(u(x,T),x,T) - f_0(\tilde{u}(x,T),x,T)\big]dx$$
$$\geq -\int_0^L \big\langle Q^* \frac{\partial \mu(x,T)}{\partial t}\ ,\ u(x,T) - \widetilde{u}(x,T)\big\rangle dx. \tag{32}$$

Then, combining (30) with (32), we get

$$\iint\limits_D \big[f(u(x,t),x,t) - f(\tilde{u}(x,t),x,t)\big]dxdt$$
$$+\int_0^L \big[f_0(u(x,T),x,T) - f_0(\tilde{u}(x,T),x,T)\big]dx \geq 0. \tag{33}$$

Consequently, for any feasible solution $u(\cdot,\cdot)$, $J(u(x,t)) \geq J(\widetilde{u}(x,t))$; i.e. $\widetilde{u}(\cdot,\cdot)$ is an optimal solution. This completes the proof. $\square$

6. Application

The purpose of this section is to provide a bit of insight into why our main result is effective and applicable. One of the convenient ways to apply Theorem 3.2 is to formulate its optimality conditions for optimal control of a boundary value problem for discrete inclusions. For example, consider the following discrete model:

$$\text{minimize} \sum_{\substack{x=1,\dots,9\\ t=1,\dots,10}} (x+2t)u_{x,t}, \tag{34}$$

$$\begin{gathered}
u_{x,t+1} \in\ F(u_{x,t-1},\ u_{x-1,t},\ u_{x,t},\ u_{x+1,t}),\quad x=1,\dots,9,\ t=1,\dots,9\\
u_{1,0}=0,\ \ u_{2,0}=4,\ \ u_{3,0}=10,\ u_{4,0}=12,\ \ u_{5,0}=-16,\ \ u_{6,0}=20,\\
u_{7,0}=0,\ \ u_{8,0}=24,\ \ u_{9,0}=0,\ \ u_{0,1}=2,\ u_{0,2}=0,\ \ u_{0,3}=-4,\ \ u_{0,4}=8,\\
u_{0,5}=12,\ \ u_{0,6}=-3,\ \ u_{0,7}=5,\ \ u_{0,8}=0,\ u_{0,9}=1,\ \ u_{10,1}=7,\ \ u_{10,2}=3,\\
u_{10,3}=-2,\ \ u_{10,4}=0,\ \ u_{10,5}=1,\ \ u_{10,6}=14,\ \ u_{10,7}=0,\ \ u_{10,8}=6,\ \ u_{10,9}=0,\\
u_{1,10}=-34,\ \ u_{2,10}=0,\ \ u_{3,10}=16,\ \ u_{4,10}=0,\ \ u_{5,10}=-5,\ \ u_{6,10}=12,\\
u_{7,10}=0,\ \ u_{8,10}=-1,\ \ u_{9,10}=0.
\end{gathered} \tag{35}$$

Here $f(u_{x,t},x,t) \equiv (x+2t)u_{x,t}$, $F(x,v_1,v_2,v_3) = \{v_4 : -x - v_1 + v_2 - 2v_3 - 4v_4 \leq 0\}$, $A_0 \equiv -1$, $A_1 \equiv -1$, $A_2 \equiv 1$, $A_3 \equiv -2$, $B \equiv 4$, $d = 0$ and $T = 10$, $L = 10$ in the problem *(PDH)*. Since f is differentiable function, we have $\partial f(\tilde{u}_{x,t},x,t) \equiv x+2t$, $x = 1,\dots,9,\ t = 1,\dots,10$. Then taking into account Theorem 3.2, we write

$$\begin{gathered}
\mu_{x,t+1} + \mu_{x+1,t} - \mu_{x,t} + 2\mu_{x-1,t} + 4\mu_{x,t-1} = x+2t,\ x=1,\dots,9,\ t=1,\dots,9,\\
\mu_{0,t} = \mu_{10,t} = 0,\ t=1,\dots,9,\quad \mu_{x,10} = 0\ ,\ 4\mu_{x,9} = x+20,\ x=1,\dots,9.
\end{gathered} \tag{36}$$

By the formula (36), we find $\mu_{x,t}$, $x = 1,\dots,9,\ t = 0,\dots,9$ in the following Table 1.

Moreover, by the definition of the set-valued mapping F, it should be noted that all $\mu_{x,t}$, $x = 1,\dots,9$, $t = 0,\dots,9$ in Table 1 being positive coincides with the results in the optimal control theory. This kind of reasoning does not compute the minimum value, of course, but it gives us a strong clue about whether the discrete problem is well-posed.

$\mu_{x,t}$										
x \ t	0	1	2	3	4	5	6	7	8	9
1	0,547	0,999	1,457	1,904	2,347	2,857	3,273	3,531	4,687	5,250
2	0,17	0,356	0,556	0,729	0,872	1,195	1,312	0,75	2,313	5,5
3	0,783	1,12	1,491	1,822	2,068	2,68	3,234	2,125	2,438	5,75
4	0,553	0,786	1,076	1,334	1,355	1,781	2,766	2,25	2,563	6
5	0,984	1,26	1,616	2,03	2,101	2,234	2,922	2,375	2,688	6,25
6	0,912	1,134	1,394	1,796	2,036	2,346	3,078	2,5	2,813	6,5
7	1,286	1,574	1,829	2,125	2,304	2,655	3,352	2,625	2,938	6,75
8	1,056	1,333	1,62	1,915	2,076	2,395	2,949	2,281	3,063	7
9	1,99	2,342	2,716	3,108	3,413	3,829	4,387	4,172	5,062	7,25

Table 1: The values of variables $\mu_{x,t}$, $x = 1, \ldots, 9$, $t = 0, \ldots, 9$.

On the other hand, from Theorem 3.2, we have

$$\left\langle -\tilde{u}_{x,t-1} + \tilde{u}_{x-1,t} + \tilde{u}_{x,t} - 2\tilde{u}_{x+1,t} - 4\tilde{u}_{x,t+1},\ \mu_{x,t} \right\rangle = 0\ ,\ \mu_{x,t} \geq 0, \tag{37}$$

$x = 1, \ldots, 9$, $t = 1, \ldots, 9$ and since $\mu_{x,t} \neq 0$, $x = 1, \ldots, 9$, $t = 1, \ldots, 9$ in Table 1, we derive that

$$-\tilde{u}_{x,t-1} + \tilde{u}_{x-1,t} + \tilde{u}_{x,t} - 2\tilde{u}_{x+1,t} - 4\tilde{u}_{x,t+1} = 0,\ x = 1, \ldots, 9,\ t = 1, \ldots, 9. \tag{38}$$

Then, taking into account boundary conditions (36), the found solutions of equalities (38) $\widetilde{u}_{x,t}$, $x = 1, \ldots, 9$, $t = 1, \ldots, 9$ are given in Table 2.

$\tilde{u}_{x,t}$									
x \ t	1	2	3	4	5	6	7	8	9
1	5768,808	6192,379	1990,713	-538,404	-713,068	-423,086	-34,223	198,506	102,338
2	-9501,353	-3769,641	-1022,024	157,576	752,839	214,938	-205,081	-88,311	19,416
3	-97,799	1813,715	63,299	-646,676	224,290	352,754	-16,276	-79,7	2,694
4	1069,347	2713,98	929,157	-882,356	-646,444	-10,685	77,425	7,055	-0,511
5	-4850,386	-1942,854	840,651	710,471	27,181	-13,347	38,083	5,687	-5,130
6	933,843	-1584,526	-493,767	321,727	8,271	-91,088	-24,37	-9,466	4,847
7	6051,167	699,777	-518,4	35,970	11,858	5,735	33,249	-5,085	-14,279
8	1159,108	-846,632	-434,145	92,606	-27,661	-24,015	36,113	14,123	-7,020
9	-764,765	-484,468	280,232	300,711	-18,032	-73,271	-14,806	5,588	-1,432

Table 2: The values of optimal trajectory $\widetilde{u}_{x,t}$, $x = 1, \ldots, 9$, $t = 1, \ldots, 9$.

Here note that the set of points $\{\widetilde{u}_{x,t}\}_R = \{\widetilde{u}_{x,t} :\ (x, t) \in R\}$, where

$$R = \{(x, t) : x = 0, \ldots, 10,\ t = 0, \ldots, 10,\ (x, t) \neq (0, 0), (10, 0), (0, 10), (10, 10)\}$$

is a feasible solution for the stated problem which satisfies boundary conditions in (36). Moreover, taking into account details in Table 2, we find the values of the cost functional (34) in the Table 3.

$(x+2t)\tilde{u}_{x,t}$									
x \ t	1	2	3	4	5	6	7	8	9
1	17306,424	30961,895	13934,99	-4845,636	-7843,748	-5500,118	-513,345	3374,602	1944,422
2	-38005,412	-22617,846	-8176,192	1575,76	9034,068	3009,132	-3281,27	-1589,59	388,32
3	-488,995	12696,005	569,691	-7113,436	2915,77	5291,31	-276,692	-1514,3	56,574
4	6416,082	21711,84	9291,57	-10588,272	-9050,216	-170,96	1393,65	141,1	-11,242
5	-33952,702	-17485,686	9357,161	9236,123	407,715	-226,899	723,577	119,427	-117,99
6	7470,744	-15845,26	-5925,204	4504,178	132,336	-1639,584	-487,4	-208,252	116,328
7	54460,503	7697,547	-6739,2	539,55	201,586	108,965	698,229	-116,955	-356,975
8	11591,08	-10159,584	-6078,03	1481,696	-497,898	-480,3	794,486	338,952	-182,52
9	-8412,415	-6298,084	4203,48	5112,087	-342,608	-1538,691	-340,538	139,7	-38,664

Table 3: The values of $(x+2t)\widetilde{u}_{x,t}$, $x = 1,\dots,9,\ t = 1,\dots,9$.

Finally, the sum of the numbers in each column we denote by S_j, $j = 1,\dots,9$, where

$$\begin{array}{lll} S_1 = 16385,31, & S_2 = 660,827, & S_3 = 10438,266, \\ S_4 = -97,95, & S_5 = -5043,02, & S_6 = -1147,145 \\ S_7 = -1289,303, & S_8 = 684,684, & S_9 = 1798,253. \end{array}$$

Then from the boundary conditions (36) for $T = 10$, we have

$$S_{10} = \sum_{x=1,\dots,9} (x+20)\widetilde{u}_{x,10} = -187. \tag{39}$$

On the basis of the equality (39), we can conclude that the total minimum value is

$$\sum_{\substack{x=1,\dots,9\\ t=1,\dots,10}} (x+2t)u_{x,t} = \sum_{j=1}^{10} S_j = 22202,92.$$

7. Conclusion

This paper investigates the optimal control of boundary-value problems for hyperbolic-type discrete and differential inclusions to an attractive class of *PDIs* in variational analysis. The discrete polyhedral problem *(PDH)* typically allows us to reduce solving a convex system of equations with as many unknowns as equations. Then, it is possible to formulate optimality conditions for the discrete problem *(PDH)*. Theorem 4.1 on discrete-approximate gives us strong reason to believe that the discrete and differential problems are related. It assists in the interpretation of discrete and differential problems by allowing us to reformulate optimality conditions for discrete inclusions. Then, we formulate sufficient conditions of optimality for differential problem *(PCH)* by passing formally to the limit in the optimality conditions for discrete approximate problem. Finally, an example is given to demonstrate the efficiency of our key results. To summarize, we list the following contributions of the paper: one can deduce that the necessary and sufficient conditions of optimality for such partial discrete and differential problems. It is also easily seen that the suggested technique can be utilized to solve different optimization problems with higher-order partial discrete and differential inclusions. In practice, the wave equation, which is a significant second-order linear *PDE* for the definition of waves, can be studied as a particular case of the hyperbolic *PDI* in the problem *(PCH)*. The wave equation and its modifications play fundamental roles in continuum mechanics, quantum mechanics, plasma physics, general relativity, geophysics, and many other scientific and technical disciplines.

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