Journal of Convex Analysis
Volume 33 (2026), No. 1&2, 131–144

Spectral Study about Biharmonic Operator with Landesman-Lazer Condition

Giovany M. Figueiredo, Segundo M. A. Salirrosas, Lorena Soriano
Departamento de Matemática, Universidade de Brasília, Brazil
giovany@unb.br, semaarsa@gmail.com, loresohe199@gmail.com

Received: June 21, 2024
Accepted: November 28, 2024

We study the existence of weak solutions for the following class of problems

$$\begin{cases} \alpha\Delta^2 u + \beta\Delta u = \mu u + \gamma h(x,u) & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geq 1$, $\alpha \geq 0$, $-\infty < \beta < \alpha\lambda_1$, λ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$, $\mu \in (0, \bar{\mu})$, $\bar{\mu} < \mu_2$, μ_2 is the second eigenvalue of the problem $(\alpha\Delta^2 u + \beta\Delta u, H_0^1(\Omega) \cap H^2(\Omega))$, $\gamma \neq 0$ is a real parameter and $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function verifying some conditions, the boundary condition $B(u) = 0$ on $\partial\Omega$ means that $u = \Delta u = 0$ on $\partial\Omega$ when $\alpha > 0$ and $u = 0$ on $\partial\Omega$ when $\alpha = 0$. In this article, we revisit the arguments of Landesman-Lazer, both in the global and local aspects.

Keywords: Biharmonic operator, Landesman-Lazer condition.

2010 Mathematics Subject Classification: Primary 35J60; secondary 35J60, 35J10, 35J20.

1. Introduction

The article [12] due to Landesman and Lazer is often cited in the field of nonlinear differential equations when studying perturbations of linear elliptic boundary value problems when they exhibit resonance. The resonance condition means that the forcing term of the equation has a natural frequency that corresponds to an eigenvalue of the linear problem.

That seminal article [12] has influenced many other papers on this topic. For instance, the authors in [2] present results on the existence and multiplicity of solutions for quasilinear problems in bounded domains involving the p-Laplacian operator under local versions of the Landesman-Lazer condition. The main results do not require any growth restriction at infinity on the nonlinear term which may change sign. The existence of solutions is established by combining variational methods, truncation arguments and approximation techniques based on a compactness result for the inverse of the p-Laplacian operator.

The purpose in [14] is to study the existence, multiplicity and non existence of solutions for nonlinear elliptic problems depending by a parameter under Landesman-Lazer type hypotheses. In order to establish the existence of solution the authors

G.M.F. was supported by CNPq, Capes and Fapdf - Brazil. L.S. was partially supported by CNPq grant. S.M.A.S. was supported by CNPq, Conselho Nacional de Desenvolvimento Científico e Tecnológico - Brazil (150632/2022-3).

ISSN 0944-6532 / $ 2.50

combine the Lyapunov-Schmidt Reduction Method and the term gradient freeze technique with truncation and approximation arguments via bootstrap methods. There is no growth restriction at infinity on the nonlinear term and it may change sign.

In [11] the authors study the resonance problem of a class of singular quasilinear elliptic equations with respect to its higher near-eigenvalues. Under a generalized Landesman–Lazer condition, it is proved that the resonance problem admits at least one nontrivial solution in weighted Sobolev spaces. The proof is based upon applying the Galerkin-type technique, the Brouwer's fixed-point theorem and a compact embedding theorem of weighted Sobolev spaces by Shapiro.

In [4] the authors show that the Ahmad-Lazer-Paul condition for resonant problems is more general than the Landesman-Lazer one, discussing some relations with other existence conditions, as well. As a consequence, such a relation holds, for example, when considering resonant boundary value problems associated with linear elliptic operators, the p-Laplacian and, in the scalar case, with an asymmetric oscillator.

In this paper we deal with the existence of weak solution for the problem

$$(P) \qquad \begin{cases} \alpha\Delta^2 u + \beta\Delta u = \mu u + \gamma h(x,u) & \text{in} \quad \Omega, \\ B(u) = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$, $\alpha \geq 0$, $-\infty < \beta < \alpha\lambda_1$, where λ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$, $\mu \in (0, \bar{\mu})$, $\bar{\mu} < \mu_2$, μ_2 is the second eigenvalue of $(\alpha\Delta^2 + \beta\Delta, H_0^1(\Omega) \cap H^2(\Omega))$, $\gamma \neq 0$ is a real parameter and $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function verifying some conditions, the boundary condition $B(u) = 0$ on $\partial\Omega$ means that $u = \Delta u = 0$ on $\partial\Omega$ when $\alpha > 0$ and $u = 0$ on $\partial\Omega$ when $\alpha = 0$.

The eigenvalue problem associated to problem (P) is given by

$$(E) \qquad \begin{cases} \alpha\Delta^2 u + \beta\Delta u = \mu u & \text{in} \quad \Omega, \\ B(u) = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

Supposing that ϕ_i is an eigenfunction associated to the eigenvalue λ_i of $(-\Delta, H_0^1(\Omega))$, Alves, Santos and Gonçalves [1, Section 4] showed that ϕ_i is also an eigenfunction of $(\alpha\Delta^2 + \beta\Delta, H_0^1(\Omega) \cap H^2(\Omega))$ and the eigenvalues are of the form $\mu_i = \lambda_i(\alpha\lambda_i - \beta)$.

To state our main results, we assume that there exist real numbers t_1 and t_2, with $t_1 < t_2$, such that either

(H_0^+) $\displaystyle\int_\Omega h(x, t_1\phi_1)\phi_1 dx > 0 > \int_\Omega h(x, t_2\phi_1)\phi_1 dx,$

or

(H_0^-) $\displaystyle\int_\Omega h(x, t_1\phi_1)\phi_1 dx < 0 < \int_\Omega h(x, t_2\phi_1)\phi_1 dx$

and

(H_1) There exists $f \in L^q(\Omega)$, $q > \frac{N}{4}$ if $N > 4$ and $q > 1$ if $1 \leq N \leq 4$, such that $|h(x,s)| \leq f(x)$, for every $s \in \mathbb{R}$, a.e. in Ω.

Theorem 1.1. *Suppose h satisfies (H_0^+) and (H_1). Then there exist positive constants γ_1 and ν^* such that, for every $\gamma \in (0, \gamma_1)$ and $|\mu - \mu_1| < \gamma\nu^*$, problem (P) has a weak solution $u_\gamma = t\phi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle \phi_1 \rangle^\perp$.*

We also have the same results as Theorem 1.1 using assumption (H_0^-), but in this case we need to suppose more regularity on the function h.

(H_2) There exists $g \in L^q(\Omega)$, $q > \frac{N}{4}$ if $N > 4$ and $q > 1$ if $1 \le N \le 4$, such that $|h(x, s_2) - h(x, s_1)| \le g(x)|s_2 - s_1|$, for every $s_1, s_2 \in \mathbb{R}$, a.e. in Ω.

Theorem 1.2. *Suppose h satisfies (H_0^-), (H_1) and (H_2). Then there exist positive constants γ_2 and ν^* such that, for every $\gamma \in (0, \gamma_2)$ and $|\mu - \mu_1| < \gamma\nu^*$, problem (P) has a weak solution $u_\gamma = t\phi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle \phi_1 \rangle^\perp$.*

Due to the global growth restriction on the nonlinearity h given by (H_1) and (H_2), we use variational methods to show Theorems 1.1 and 1.2. For our next results we assume that h satisfies

(H_3) For every $M > 0$, there exists $f_M \in L^q(\Omega)$, $q > \frac{N}{4}$ if $N > 4$ and $q > 1$ if $1 \le N \le 4$, such that $|h(x, s)| \le f_M(x)$, for every $|s| < M$, a.e. in Ω.

Theorem 1.3. *Suppose h satisfies (H_0^+) and (H_3). Then there exist positive constants γ^* and ν^* such that, for every $\gamma \in (0, \gamma^*)$ and $|\mu - \mu_1| < \gamma\nu^*$, problem (P) has a weak solution $u_\gamma = t\phi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle \phi_1 \rangle^\perp$.*

(H_4) For every $M > 0$, there exists $g_M \in L^q(\Omega)$, $q > \frac{N}{4}$ if $N > 4$ and $q > 1$ if $1 \le N \le 4$, such that $|h(x, s_1) - h(x, s_2)| \le g_M(x)|s_1 - s_2|$, for every $|s_1|, |s_2| < M$, a.e. in Ω.

Theorem 1.4. *Suppose h satisfies (H_0^-), (H_3) and (H_4). Then there exist positive constants γ^* and ν^* such that, for every $\gamma \in (0, \gamma^*)$ and $|\mu - \mu_1| < \gamma\nu^*$, problem (P) has a weak solution $u_\gamma = t\phi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle \phi_1 \rangle^\perp$.*

To establish the multiplicity of weak solutions for the problem (P) we assume the following condition:

(H_5) There exists $\kappa \in \mathbb{N}$ and $t_i \in \mathbb{R}$, $t_i < t_{i+1}$, $i = 1, 2, \ldots, \kappa$ such that

$$\left[\int_\Omega h(x, t_i\phi_1)\phi_1 dx\right]\left[\int_\Omega h(x, t_{i+1}\phi_1)\phi_1 dx\right] < 0.$$

Theorem 1.5. *Suppose h satisfies (H_3), (H_4) and (H_5). Then there exist positive constants γ^* and ν^* such that, for every $0 < |\gamma| < \gamma^*$ and $|\mu - \mu_1| < |\gamma|\nu^*$, problem (P) has κ weak solutions $u_i = \widehat{t}_i\phi_1 + v_i$, with $\widehat{t}_i \in (t_i, t_{i+1})$ and $v_i \in \langle \phi_1 \rangle^\perp$, $i = 1, 2, \ldots, \kappa$.*

Below we list what we believe that are the main contributions of our paper:

(1) We are extending the result in [14] since in that paper, the case with $\alpha = 0$ and $\beta = -1$ was considered.

(2) We also complete the results in [2], because we are considering the biharmonic operator, which makes the estimates more delicate.

In recent years, problems involving biharmonic or polyharmonic operators have received a special attention, in particular the problems where the nonlinearity has a critical growth. In an interesting book due to Gazzola, Grunau and G. Sweers [7], the reader can find a lot of results involving this class of operator and an excellent bibliography about this subject. In addition to this book, we would like to cite the following papers Gazzola [5], Gazzola, Grunau and Squassina [6], Ge [8, 9], Pucci and Serrin [13], Yangxin, Rongxin and Yaotian [16] and references therein.

The plan of the paper is the following: In Section 2 we recall variational framework. In Section 3 we give the proof of Theorem 1.1. We show the proof of Theorem 1.2 in Section 4. In Section 5 we prove Theorem 1.3. Finally, in Section 6, we prove Theorem 1.4 and in Section 7, we prove Theorem 1.5.

2. Variational framework

The first eigenvalue of $(\alpha\Delta^2 + \beta\Delta, H_0^1(\Omega) \cap H^2(\Omega))$ is characterized by

$$\mu_1 := \inf_{\substack{u \neq 0 \\ u \in X}} \left(\frac{\alpha \int_\Omega |\Delta u|^2 dx - \beta \int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx} \right),$$

where $X = H_0^1(\Omega) \cap H^2(\Omega)$ is a Hilbert space endowed with the norm

$$\|u\|_X^2 = \alpha \int_\Omega |\Delta u|^2 dx - \beta \int_\Omega |\nabla u|^2 dx.$$

We also have that

$$\mu_1 \int_\Omega |u|^2 dx \leq \alpha \int_\Omega |\Delta u|^2 dx - \beta \int_\Omega |\nabla u|^2 dx, \quad u \in X$$

and
$$\mu_2 \int_\Omega |v|^2 dx \leq \alpha \int_\Omega |\Delta v|^2 dx - \beta \int_\Omega |\nabla v|^2 dx,$$

for
$$v \in \langle \phi_1 \rangle^\perp = \{w \in X : \int_\Omega w\phi_1 dx = 0\}.$$

Since $N \geq 4$, we can use Sobolev embedding to get $X \hookrightarrow L^s(\Omega)$, for $2 \leq s \leq 2_{**}$, where $2_{**} = \frac{2N}{N-4}$ in the case $N > 4$ and $2_{**} = +\infty$ for the case $1 \leq N \leq 4$. From $q > \frac{N}{4}$, we have that $2 < 2q' < 2_{**}$, where $q' = \frac{q}{q-1}$. Thus, we obtain

$$X \hookrightarrow L^{2q'}(\Omega) \hookrightarrow L^{q'}(\Omega). \tag{1}$$

The functional associated to problem (P), $I_\gamma : X \to \mathbb{R}$ is given by

$$\begin{aligned} I_\gamma(u) &= \frac{1}{2} \int_\Omega \left[\alpha|\Delta u|^2 - \beta|\nabla u|^2\right] dx - \frac{\mu}{2} \int_\Omega |u|^2 dx - \gamma \int_\Omega H(x,u) dx \\ &= \frac{1}{2}\|u\|_X^2 - \frac{\mu}{2} \int_\Omega |u|^2 dx - \gamma \int_\Omega H(x,u) dx, \end{aligned}$$

where $H(x,t) = \displaystyle\int_0^t h(x,s) ds$.

Under the hypotheses (H_1) and (1) the functional I_γ is well defined and is of class C^1 and its derivative is given by

$$\langle I'_\gamma(u), w\rangle = \int_\Omega [\alpha\Delta u\Delta w - \beta\nabla u\nabla w]\, dx - \mu\int_\Omega uwdx - \gamma\int_\Omega h(x,u)wdx,$$

for every $w \in X$. Thus, the critical points of I_γ are weak solutions of problem (P).

3. Proof of Theorem 1.1

To prove Theorem 1.1, we use a minimization argument. More specifically, we shall verify that for $\gamma > 0$ sufficiently small the infimum of I_γ on the set

$$C := \left\{u = \tau\phi_1 + v : \tau \in [t_1, t_2], \int_\Omega \phi_1 v dx = 0\right\}$$

is attained on an interior point of C.

Lemma 3.1. *Suppose h satisfies (H_1). Then, for every $\gamma > 0$, the functional I_γ is bounded from below and coercive on C.*

Proof. From (H_1) and Hölder inequality we get

$$\begin{aligned}\left|\int_\Omega H(x,\tau\phi_1+v)dx\right| &\le \int_\Omega f(x)|\tau\phi_1+v|dx \le \|f\|_{L^q(\Omega)}\|\tau\phi_1+v\|_{L^{q'}(\Omega)}\\ &\le |\tau|\ \|f\|_{L^q(\Omega)}\|\phi_1\|_{L^{q'}(\Omega)} + \|f\|_{L^q(\Omega)}\|v\|_{L^{q'}(\Omega)}. \end{aligned}\tag{2}$$

Thus, by (1), there exists $C_{q'} > 0$ such that

$$\begin{aligned} &I_\gamma(\tau\phi_1+v) = \frac{\tau^2}{2}\|\phi_1\|_X^2+\frac{1}{2}\|v\|_X^2-\frac{\mu\tau^2}{2}\|\phi_1\|^2_{L^2(\Omega)}-\frac{\mu}{2}\|v\|^2_{L^2(\Omega)} - \gamma\int_\Omega H(x,\tau\phi_1+v)dx\\ &\ge \frac{\mu_1-\bar\mu}{2\mu_1}\|\phi_1\|_X^2\tau^2 - \gamma|\tau|\|f\|_{L^q(\Omega)}\|\phi_1\|_{L^{q'}(\Omega)} + \frac{\mu_2-\bar\mu}{2\mu_2}\|v\|_X^2 - C_{q'}\|f\|_{L^q(\Omega)}\|v\|_X. \end{aligned}$$

This inequality implies that I_γ is coercive on C.

On the other hand, by using (2) and $\mu < \mu_2$ we have for $\|u\|_X < M$ that

$$\begin{aligned} I_\gamma(u) &= \frac{1}{2}\|u\|_X^2 - \frac{\mu}{2}\|u\|^2_{L^2(\Omega)} - \gamma\int_\Omega H(x,u)dx\\ &\ge \frac{1}{2}\|u\|_X^2 - \frac{\mu_2}{2\mu_1}\|u\|_X^2 - C_{q'}\gamma\|f\|_{L^q(\Omega)}\|u\|_X\\ &\ge -\left(\frac{\mu_2-\mu_1}{2\mu_1}\right)M^2 - C_{q'}\gamma\|f\|_{L^q(\Omega)}M, \end{aligned}$$

showing that I_γ is bounded from below on C. □

Now, we define $m_C = \inf\{I_\gamma(u) : u \in C\}$ and, for every $\tau \in [t_1, t_2]$,

$$m_\tau = \inf\{I_\gamma(u) : u = \tau\phi_1 + v, v \in \langle\phi_1\rangle^\perp\}.$$

It follows from Lemma 3.1 that $m_\tau \ge m_C > -\infty$ for every $\tau \in [t_1, t_2]$.

Lemma 3.2. *For every $\gamma > 0$, there exists $u_\gamma \in C$ such that $I_\gamma(u_\gamma) = m_C$. Moreover, for every $\gamma > 0$ and $\tau \in [t_1, t_2]$, there exists $v_\gamma \in \langle \phi_1 \rangle^\perp$ such that $I_\gamma(\tau\phi_1 + v_\gamma) = m_\tau$.*

Proof. Let $(v_n) \subset \overline{B_\rho(0)} \cap \langle \phi_1 \rangle^\perp$ be a sequence such that

$$I_\gamma(\tau\phi_1 + v_n) \to m_C.$$

Since (v_n) is bounded, then, up to a subsequence, we may assume that, there exists $v \in \overline{B_\rho(0)} \cap \langle \phi_1 \rangle^\perp$ such that $v_n \rightharpoonup v$ weakly in X.

We denote $w_n = \tau\phi_1 + v_n$ and $w = \tau\phi_1 + v$. Thus, we may suppose that

$$\begin{cases} w_n \rightharpoonup w \ \text{ weakly in } \ X; \\ w_n \to w \ \text{ strongly in } \ L^s(\Omega); \\ w_n(x) \to w(x) \ \text{ a.e. } \ x \in \Omega; \\ |w_n(x)| \le \zeta(x), \quad \zeta \in L^s(\Omega); \end{cases} \tag{3}$$

for every $1 \le s < 2_{**}$. From (H_1) and (3), we may apply the Lebesgue Dominated Convergence Theorem to get

$$\int_\Omega H(x, w_n)dx \to \int_\Omega H(x, w)dx.$$

Now, we use weak convergence and the fact that $w_n \to w$ strongly in $L^2(\Omega)$ to get

$$\begin{aligned} m_C \le I_\gamma(\tau\phi_1 + v) &= \frac{1}{2}\|w\|_X^2 - \frac{\mu}{2}\|w\|_{L^2(\Omega)}^2 - \gamma\int_\Omega H(x, w)dx \\ &\le \liminf_{n\to\infty}\left[\frac{1}{2}\|w_n\|_X^2 - \frac{\mu}{2}\|w_n\|_{L^2(\Omega)}^2 - \gamma\int_\Omega H(x, w_n)dx\right] \\ &= \liminf_{n\to\infty} I_\gamma(w_n) = m_C. \end{aligned}$$

This implies that $I_\gamma(\tau\phi_1 + v) = m_C$ with $\|v\|_X < \rho$. An analogous argument implies that for every $\gamma > 0$ and $\tau \in [t_1, t_2]$, we may find $v_\gamma \in \langle \phi_1 \rangle^\perp$ such that $I_\gamma(\tau\phi_1 + v_\gamma) = m_\tau$. The proof is complete. □

Now, we define, for each $\tau \in [t_1, t_2]$, the set

$$S_\tau = \left\{ v \in \langle \phi_1 \rangle^\perp : \|v\|_X \le \rho, I_\gamma(\tau\phi_1 + v) = m_\tau \right\}.$$

Remark 3.3. (i) In view of Lemma 3.2, $S_\tau \neq \emptyset$ for every $\gamma > 0$ and $\tau \in [t_1, t_2]$.

(ii) For every $\tau \in [t_1, t_2]$, from Lemma 3.2 we assure that

$$\langle I'_\gamma(\tau\phi_1 + v), z\rangle = 0, \quad \text{for every } \ v \in S_\tau, \quad z \in \langle \phi_1 \rangle^\perp.$$

Lemma 3.4. *Suppose h satisfies (H_1). Then, given $\delta > 0$, there exists $\gamma_1 > 0$ such that $\|v\|_X < \delta$ for every $v \in S_{t_1} \cup S_{t_2}$.*

Proof. Suppose, without loss of generality, that $v \in S_{t_1}$. Using (H_1), the Hölder inequality and (1) we obtain

$$\left|\int_\Omega h(x, t_1\phi_1 + v)v dx\right| \le \int_\Omega f|v|dx \le \|f\|_{L^q(\Omega)}\|v\|_{L^{q'}(\Omega)} \le C_{q'}\|f\|_{L^q(\Omega)}\|v\|_X.$$

Hence,
$$\|v\|_X \le \frac{\mu_2}{\mu_2-\bar{\mu}}\gamma C_{q'}\|f\|_{L^q(\Omega)}.$$

Therefore, taking $0<\gamma<\gamma_1 := \dfrac{\mu_2-\bar{\mu}}{\mu_2}\dfrac{\delta}{C_{q'}\|f\|_{L^q(\Omega)}}$, the proof is complete. □

Proof of Theorem 1.1. Following Lemma 3.2, for every $\gamma>0$, there exists some $u_\gamma = t\phi_1 + v \in C$ such that $I_\gamma(u_\gamma) = m_C$. Since $m_C = m_t$ we have that $v \in S_t$. Therefore, of according to Lemma 3.4, $\|v\|_X < \delta$. This shows that $u_\gamma \in \text{int}(C)$ when $t \in (t_1, t_2)$.

It remains to verify that we can not have $t = t_1$ or $t = t_2$. For this purpose, we define $T_i : \langle\phi_1\rangle^\perp \to \mathbb{R}$ by
$$T_i(v) = \int_\Omega h(x, t_i\phi_1 + v)\phi_1 dx, \quad i = 1, 2.$$

By using (H_1), we can show that T_i is continuous. Thus, from (H_0^+), there exists $\delta > 0$ such that, for every $v \in \langle\phi_1\rangle^\perp$, with $\|v\|_X < \delta$ we get
$$\int_\Omega h(x, t_2\phi_1 + v)\phi_1 dx < \frac{T_2(0)}{2} < 0 < \frac{T_1(0)}{2} < \int_\Omega h(x, t_1\phi_1 + v)\phi_1 dx. \tag{4}$$

Then, applying Lemma 3.4, there exists $\gamma_1 > 0$ such that, for every $\gamma \in (0, \gamma_1)$, $\|v\|_X < \delta$, for every $v \in S_{t_1} \cup S_{t_2}$. Thus, we can use (4) to get, for every $\gamma \in (0, \gamma_1)$
$$\int_\Omega h(x, t_2\phi_1 + v_2)\phi_1 dx < \frac{T_2(0)}{2} < 0 < \frac{T_1(0)}{2} < \int_\Omega h(x, t_1\phi_1 + v_1)\phi_1 dx. \tag{5}$$

for every $v_1 \in S_{t_1}$, $v_2 \in S_{t_2}$. This implies that
$$\frac{\mu-\mu_1}{\mu_1\gamma}\|\phi_1\|_X^2 t_1 + \int_\Omega h(x, t_1\phi_1 + v_1)\phi_1 dx \ge -\frac{|\mu-\mu_1|}{\mu_1\gamma}\|\phi_1\|_X^2 |t_1| + \frac{T_1(0)}{2}$$

for every $v_1 \in S_{t_1}$, and
$$\frac{\mu-\mu_1}{\mu_1\gamma}\|\phi_1\|_X^2 t_2 + \int_\Omega h(x, t_2\phi_1 + v_2)\phi_1 dx \le \frac{|\mu-\mu_1|}{\mu_1\gamma}\|\phi_1\|_X^2 |t_2| + \frac{T_2(0)}{2}$$

for every $v_2 \in S_{t_2}$. The above inequalities guarantee that there exists
$$\nu^* < \frac{\mu_1}{2\|\phi_1\|_X^2}\min\left\{\frac{T_1(0)}{|t_1|+1}, -\frac{T_2(0)}{|t_2|+1}\right\}$$

such that, for every $\gamma \in (0, \gamma_1)$ and $|\mu - \mu_1| < \gamma\nu^*$, we have that
$$\begin{aligned}\langle I'_\gamma(t_1\phi_1 + v_1), \phi_1\rangle &= t_1\|\phi_1\|_X^2 - \mu t_1\|\phi_1\|_{L^2(\Omega)}^2 - \gamma\int_\Omega h(x, t_1\phi_1 + v_1)\phi_1 dx \\ &= -\gamma\left[\frac{\mu-\mu_1}{\mu_1\gamma}\|\phi_1\|_X^2 t_1 + \int_\Omega h(x, t_1\phi_1 + v_1)\phi_1 dx\right] < 0 \end{aligned} \tag{6}$$

for every $v_1 \in S_{t_1}$, and
$$\langle I'_\gamma(t_2\phi_1 + v_2), \phi_1\rangle = -\gamma\left[\frac{\mu-\mu_1}{\mu_1\gamma}\|\phi_1\|_X^2 t_2 + \int_\Omega h(x, t_2\phi_1 + v_2)\phi_1 dx\right] > 0 \tag{7}$$

for every $v_2 \in S_{t_2}$.

Supposing for example that $u_\gamma = t_1\phi_1+v$, it follows from (6) that there is $\tau \in (t_1, t_2)$ such that
$$m_C \leq m_\tau \leq I_\gamma(\tau\phi_1 + v) < I_\gamma(t_1\phi_1 + v) = I_\gamma(u_\gamma) = m_C.$$
This implies that we can not have $t = t_1$. A similar argument also implies that we may not have $t = t_2$. Consequently $u_\gamma \in \operatorname{int}(C)$. The proof of theorem is complete. $\square$

4. Proof of Theorem 1.2

For the proof of Theorem 1.2 we will need the following result whose proof can be found in [3].

Theorem 4.1. *Let Y and Z be closed subspaces of a real Hilbert X such that $X = Y \oplus Z$. Let $J : X \to \mathbb{R}$ be a functional of class C^1. If there exists an increasing function $\psi : (0,\infty) \to (0,\infty)$ such that $\psi(s) \to \infty$ as $s \to \infty$ and*
$$\langle J'(y + z_1) - J'(y + z_2), z_1 - z_2\rangle \geq \|z_1 - z_2\|\psi(\|z_1 - z_2\|)$$
for every $y \in Y$, $z_1, z_2 \in Z$. Then,

(i) *There exists a continuous function $\Phi : Y \to Z$ such that $J(y + \Phi(y)) = \min\limits_{z\in Z} J(y + z)$. Moreover, $\Phi(y)$ is the only element of Z with the property that $\langle J'(y + \Phi(y)), z\rangle = 0$, for every $z \in Z$.*

(ii) *The function $\widehat{J} : Y \to \mathbb{R}$ defined by $\widehat{J}(y) = J(y + \Phi(y))$ is of class C^1 and*
$$\langle \widehat{J}'(y_1), y_2\rangle = \langle J'(y_1 + \Phi(y_1)), y_2\rangle, \text{ for every } y_1, y_2 \in Y.$$

(iii) *$y \in Y$ is a critical point of $\widehat{J}$ if and only if $y + \Phi(y)$ is a critical point of J.*

Proof of Theorem 1.2. For every $v_1, v_2 \in \langle\phi_1\rangle^\perp$ and for every $\tau \in \mathbb{R}$ we have
$$\begin{aligned}&\langle I'_\gamma(\tau\phi_1 + v_1) - I'_\gamma(\tau\phi_1 + v_2), v_1 - v_2\rangle\\ &= \|v_1-v_2\|^2_X - \mu\int_\Omega (v_1-v_2)^2dx - \gamma\int_\Omega [h(x,\tau\phi_1+v_1) - h(x,\tau\phi_1+v_2)]\,(v_1-v_2)dx\end{aligned}$$
Now, we can use (H_1), (H_2), the Hölder inequality and (1) to obtain $C_{2q'} > 0$ such that
$$\begin{aligned}&\langle I'_\gamma(\tau\phi_1 + v_1) - I'_\gamma(\tau\phi_1 + v_2), v_1 - v_2\rangle\\ &\geq \|v_1 - v_2\|^2_X - \frac{\bar{\mu}}{\mu_2}\|v_1 - v_2\|^2_X - \gamma\int_\Omega g(x)|v_1 - v_2|^2dx\\ &\geq \frac{\mu_2 - \bar{\mu}}{\mu_2}\|v_1 - v_2\|^2_X - \gamma\|g\|_{L^q(\Omega)}\|v_1 - v_2\|^2_{L^{2q'}(\Omega)}\\ &\geq \left(\frac{\mu_2 - \bar{\mu}}{\mu_2} - C^2_{2q'}\gamma\|g\|_{L^q(\Omega)}\right)\|v_1 - v_2\|^2_X.\end{aligned}$$
Thus, taking $\gamma_1 < \dfrac{\mu_2 - \bar{\mu}}{2\mu_2}\dfrac{1}{C^2_{2q'}\|g\|_{L^q(\Omega)}}$, we obtain, for every $\gamma \in (0, \gamma_1)$
$$\langle I'_\gamma(\tau\phi_1 + v_1) - I'_\gamma(\tau\phi_1 + v_2), v_1 - v_2\rangle \geq \frac{\mu_2 - \bar{\mu}}{2\mu_2}\|v_1 - v_2\|^2_X.$$

Therefore, I_γ satisfies the hypothesis of Theorem 4.1, with $Y = \langle \phi_1 \rangle$, $Z = \langle \phi_1 \rangle^\perp$ and $\psi(s) = \frac{\mu_2 - \bar{\mu}}{2\mu_2} s$, for every $s \in (0, \infty)$. Hence, by Theorem 4.1(i), there exists a continuous function $\Phi : \langle \phi_1 \rangle \to \langle \phi_1 \rangle^\perp$ such that $\Phi(\tau\phi_1)$ is the only element of the space $\langle \phi_1 \rangle^\perp$ that satisfies

$$\langle I'_\gamma(\tau\phi_1 + \Phi(\tau\phi_1)), v\rangle = 0, \quad \text{for every } \ \tau \in \mathbb{R}, \quad v \in \langle \phi_1 \rangle^\perp. \tag{8}$$

We also have that, from Theorem 4.1(ii), the functional $\widehat{I}_\gamma : \langle \phi_1 \rangle \to \mathbb{R}$ given by $\widehat{I}_\gamma(\tau\phi_1) = I_\gamma(\tau\phi_1 + \Phi(\tau\phi_1))$ is of class C^1 and

$$\begin{aligned}
\langle \widehat{I}'_\gamma(\tau\phi_1), \phi_1\rangle &= \langle I'_\gamma(\tau\phi_1 + \Phi(\tau\phi_1)), \phi_1\rangle \\
&= \tau\|\phi_1\|_X^2 - \tau\mu\|\phi\|^2_{L^2(\Omega)} - \gamma \int_\Omega h\left(x, \tau\phi_1 + \Phi(\tau\phi_1)\right)\phi_1 dx \\
&= -\gamma\left[\frac{\mu - \mu_1}{\mu_1\gamma}\|\phi_1\|_X^2\tau + \int_\Omega h\left(x, \tau\phi_1 + \Phi(\tau\phi_1)\right)\phi_1 dx\right].
\end{aligned} \tag{9}$$

By using (H_1) and (H_0^-) we obtain a similar inequality than (4). More specifically there exists $\delta > 0$ such that, for every $v \in \langle \phi_1 \rangle^\perp$, with $\|v\|_X < \delta$, we have

$$\begin{aligned}
&\int_\Omega h(x, t_1\phi_1 + v)\phi_1 dx < \frac{1}{2}\int_\Omega h(x, t_1\phi_1)\phi_1 dx \\
&\qquad < 0 < \frac{1}{2}\int_\Omega h(x, t_2\phi_1)\phi_1 dx < \int_\Omega h(x, t_2\phi_1 + v)\phi_1 dx.
\end{aligned} \tag{10}$$

On the other hand, by using (H_1), Hölder inequality, (1) and (8) we get $C_{q'} > 0$ such that

$$\begin{aligned}
\|\Phi(\tau\phi_1)\|_X^2 &= \mu\|\Phi(\tau\phi_1)\|^2_{L^2(\Omega)} + \gamma\int_\Omega h\left(x, \tau\phi_1 + \Phi(\tau\phi_1)\right)\Phi(\tau\phi_1) dx \\
&\le \frac{\bar{\mu}}{\mu_2}\|\Phi(\tau\phi_1)\|_X^2 + \gamma C_{q'}\|f\|_{L^q(\Omega)}\|\Phi(\tau\phi_1)\|_X.
\end{aligned}$$

This implies that $\|\Phi(\tau\phi_1)\|_X \le \frac{\mu_2\gamma}{\mu_2 - \bar{\mu}} C_{q'}\|f\|_{L^q(\Omega)}$.

Thus, by taking $\gamma_2 < \min\left\{\gamma_1, \frac{\mu_2 - \bar{\mu}}{\mu_2\gamma C_{q'}\|f\|_{L^q(\Omega)}}\delta\right\}$, we can use (10) to get

$$\begin{aligned}
&\frac{\mu - \mu_1}{\gamma\mu_1}\|\phi_1\|_X^2 t_1 + \int_\Omega h\left(x, t_1\phi_1 + \Phi(t_1\phi_1)\right)\phi_1 dx \\
&< \frac{|\mu - \mu_1|}{\gamma\mu_1}\|\phi_1\|_X^2|t_1| + \frac{1}{2}\int_\Omega h(x, t_1\phi_1)\phi_1 dx
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&\frac{\mu - \mu_1}{\gamma\mu_1}\|\phi_1\|_X^2 t_2 - \int_\Omega h\left(x, t_2\phi_1 + \Phi(t_2\phi_1)\right)\phi_1 dx \\
&< \frac{|\mu - \mu_1|}{\gamma\mu_1}\|\phi_1\|_X^2|t_2| - \frac{1}{2}\int_\Omega h(x, t_2\phi_1)\phi_1 dx.
\end{aligned} \tag{12}$$

By using the above inequalities in (9), we have

$$\langle \widehat{I}'_\gamma(t_1\phi_1), \phi_1\rangle > -\gamma \left[\frac{|\mu-\mu_1|}{\gamma\mu_1}\|\phi_1\|_X^2|t_1| + \frac{1}{2}\int_\Omega h(x, t_1\phi_1)\phi_1 dx\right]$$

and $$\langle \widehat{I}'_\gamma(t_2\phi_1), \phi_1\rangle < -\gamma \left[-\frac{|\mu-\mu_1|}{\gamma\mu_1}\|\phi_1\|_X^2|t_2| + \frac{1}{2}\int_\Omega h(x, t_2\phi_1)\phi_1 dx\right].$$

Thus, by taking

$$\nu^* < \frac{\mu_1}{2\|\phi_1\|_X^2}\min\left\{\frac{-1}{|t_1|+1}\int_\Omega h(x,t_1\phi_1)\phi_1 dx,\ \frac{1}{|t_2|+1}\int_\Omega h(x,t_2\phi_1)\phi_1 dx\right\}$$

we obtain $$\langle \widehat{I}'_\gamma(t_1\phi_1), \phi_1\rangle > 0 \quad \text{and} \quad \langle \widehat{I}'_\gamma(t_2\phi_1), \phi_1\rangle < 0$$

for every $\gamma \in (0, \gamma_2)$ and $|\mu - \mu_1| < \gamma\nu^*$. Then, by the Intermediate Value Theorem, there exists $t \in (t_1, t_2)$ such that $\langle \widehat{I}'_\gamma(t\phi_1), \phi_1\rangle = 0$. It follows from Theorem 4.1(iii) that $u_\gamma = t\phi_1 + \Phi(t\phi_1)$ is a critical point of the functional I_γ and that completes the proof. □

5. Proof of Theorem 1.3

For the proof of Theorem 1.3 we define the following function

$$h_R(x, s) = \chi(s)h(x, s), \quad \text{for every } x \in \Omega,\ s \in \mathbb{R},$$

where $R > \max\{|t_1|, |t_2|\}\|\phi_1\|_\infty > 0$ and $\chi \in C^\infty(\mathbb{R}, [0, 1])$ is a function satisfying

$$\chi(s) = \begin{cases} 1, & \text{if } |s| \le R+1, \\ 0, & \text{if } |s| \ge R+2. \end{cases}$$

In view of definition of the function h_R, we deal in the sequel with the modified problem

$$(P_R) \qquad \begin{cases} \alpha\Delta^2 u + \beta\Delta u = \mu u + \gamma h_R(x, u) & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega, \end{cases}$$

and we will look for solutions $u_\gamma = t\phi_1 + v$, $t \in (t_1, t_2)$ and $v \in \langle\phi_1\rangle^\perp$ verifying $\|u_\gamma\|_\infty < R + 1$, because, in this case, $h_R(x, u_\gamma) = h(x, u_\gamma)$ and therefore u_γ is also a solution of problem (P).

Note that $\|t_i\phi_1\|_\infty < R$, $i = 1, 2$. Thus, we can see that h_R satisfies the hypotheses (H_0^+). Moreover, from (H_3), there exists $f_{R+2} \in L^q(\Omega)$ such that $|h(x, s)| \le f_{R+2}(x)$, for every $|s| \le R + 2$, a.e. $x \in \Omega$. It follows, from definition of h_R that

$$|h_R(x, s)| \le f_{R+2}(x), \text{ for every } s \in \mathbb{R}, \text{ a.e. } x \in \Omega. \tag{13}$$

Then, we can use Theorem 1.1 to find γ_1 and ν^* such that, for every $\gamma \in (0, \gamma_1)$ and $|\mu - \mu_1| < \gamma\nu^*$, problem (P_R) has a weak solution $u_\gamma = t_\gamma\phi_1 + v_\gamma$, with $t_\gamma \in (t_1, t_2)$ and $v_\gamma \in \langle\phi_1\rangle^\perp$.

For the next result, we consider the following problem

$$(P_\zeta) \qquad \begin{cases} \alpha\Delta^2 u + \beta\Delta u = \zeta(x) & \text{in} \quad \Omega, \\ B(u) = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\alpha \geq 0$, $-\infty < \beta < \alpha\lambda_1$ and $\zeta \in L^p(\Omega)$, $p \geq 2$. Then the only solution u of (P_ζ) satisfies

$$u \in W^{4,p}(\Omega) \cap H_0^1(\Omega) \quad \text{and} \quad \|u\|_{W^{4,p}(\Omega)} \leq \widehat{K}\|\zeta\|_{L^p(\Omega)} \tag{14}$$

cf. Gupta and Kwong [10] for related results and remarks.

Lemma 5.1. *Suppose h_R satisfies* (13) *and that there exist positive constants γ^* and ν^* such that, for every $\gamma \in (0,\gamma^*)$ and $|\mu-\mu_1| < \gamma\nu^*$, $u_\gamma = t\phi_1 + v$ is a weak solution of problem (P_R), with $t \in (t_1,t_2)$ and $v \in \langle\phi_1\rangle^\perp$. Then there exist $p > \frac{N}{4}$ and $C_p > 0$ such that $\|v\|_{W^{4,p}(\Omega)} \leq C_p\gamma$.*

Proof. Since $u = t\phi_1 + v$ is a weak solution of problem (P_R) we have that v is a weak solution of the problem

$$\begin{cases} \alpha\Delta^2 v + \beta\Delta v = t(\mu-\mu_1)\phi_1 + \mu v + \gamma h_R(x, t\phi_1 + v) & \text{in} \quad \Omega, \\ B(v) = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Then, using $0 < \mu < \bar{\mu} < \mu_2$, (13), Hölder inequality and (1) we obtain

$$\begin{aligned} \|v\|_X^2 &= t(\mu-\mu_1)\int_\Omega \phi_1 v\,dx + \mu\int_\Omega v^2 dx + \gamma\int_\Omega h_R(x,t\phi_1+v)v\,dx \\ &\leq \frac{\mu}{\mu_2}\|v\|_X^2 + \gamma\|f_{R+2}\|_{L^q(\Omega)}\|v\|_{L^{q'}(\Omega)} \\ &\leq \frac{\bar{\mu}}{\mu_2}\|v\|_X^2 + \gamma C_{q'}\|f_{R+2}\|_{L^q(\Omega)}\|v\|_X \end{aligned}$$

or equivalently,

$$\|v\|_X \leq \frac{\mu_2}{\mu_2-\bar{\mu}}\gamma C_{q'}\|f_{R+2}\|_{L^q(\Omega)}. \tag{15}$$

Now, we define $\zeta(x) = t(\mu-\mu_1)\phi_1(x) + \mu v(x) + \gamma h_R(x, t\phi_1(x) + v(x))$, $x \in \Omega$. Therefore, for $p_1 = \min\{q, 2_{**}\}$, we can use (13), (15), Sobolev embedding and the fact that $L^q(\Omega) \hookrightarrow L^{p_1}(\Omega)$ to get $K_1 > 0$ such that

$$\begin{aligned} \|\zeta\|_{L^{p_1}(\Omega)} &\leq |\mu-\mu_1|\max\{|t_1|,|t_2|\}\|\phi_1\|_{L^{p_1}(\Omega)} + \mu\|v\|_{L^{p_1}(\Omega)} + \gamma\|f_{R+2}\|_{L^{p_1}(\Omega)} \\ &\leq \gamma\nu^*\max\{|t_1|,|t_2|\}\|\phi_1\|_\infty|\Omega|^{1/p_1} + \bar{\mu}C_{p_1}\|v\|_X + \gamma\|f_{R+2}\|_{L^{p_1}(\Omega)} \\ &\leq K_1\gamma, \end{aligned}$$

where

$$K_1 = \nu^*\max\{|t_1|,|t_2|\}\|\phi_1\|_\infty|\Omega|^{1/p_1} + \bar{\mu}C_{p_1}C_{q'}\|f_{R+2}\|_{L^q(\Omega)}\frac{\mu_2}{\mu_2-\bar{\mu}} + \|f_{R+2}\|_{L^{p_1}(\Omega)}.$$

Thus, by using (14), we find $\widehat{K}_1 > 0$ such that

$$\|v\|_{W^{4,p_1}(\Omega)} \leq \widehat{K}_1\|\zeta\|_{L^{p_1}(\Omega)} \leq \widehat{K}_1 K_1\gamma. \tag{16}$$

Now, we consider three cases.

Case 1. If $4p_1 > N$, the lemma follows from (16), by taking $p = p_1$.

Case 2. If $4p_1 = N$, we have $W^{4,p_1}(\Omega) \hookrightarrow L^q(\Omega)$, because $p_1 \leq q$. Then $v \in L^q(\Omega)$ and there exists $K_2 > 0$ such that

$$\begin{aligned} \|\zeta\|_{L^q(\Omega)} &\leq |\mu - \mu_1| \max\{|t_1|, |t_2|\} \|\phi_1\|_{L^q(\Omega)} + \mu \|v\|_{L^q(\Omega)} + \gamma \|f_{R+2}\|_{L^q(\Omega)} \\ &\leq \gamma \nu^* \max\{|t_1|, |t_2|\} \|\phi_1\|_\infty |\Omega|^{1/q} + \bar{\mu} C_q \|v\|_{W^{4,p_1}(\Omega)} + \gamma \|f_{R+2}\|_{L^q(\Omega)} \\ &\leq K_2 \gamma, \end{aligned} \tag{17}$$

where $\quad K_2 = \nu^* \max\{|t_1|, |t_2|\} \|\phi_1\|_\infty |\Omega|^{1/q} + \bar{\mu} C_q \widehat{K}_1 K_1 + \|f_{R+2}\|_{L^q(\Omega)}$.

Thus, by using (14), we find $\widehat{K}_2 > 0$ such that

$$\|v\|_{W^{4,q}(\Omega)} \leq \widehat{K}_2 \|\zeta\|_{L^q(\Omega)} \leq \widehat{K}_2 K_2 \gamma. \tag{18}$$

This proves the lemma in this case, by taking $p = q$.

Case 3. If $4p_1 < N$, we have $W^{4,p_1}(\Omega) \hookrightarrow L^{p_2}(\Omega)$, where $p_2 = \frac{Np_1}{N-4p_1}$. Thus, if $\min\{p_2, q\} = q$, we get $W^{4,p_1}(\Omega) \hookrightarrow L^q(\Omega)$. Therefore the lemma follows using the same arguments as in (17) and (18).

If $\min\{p_2, q\} = p_2$, we can argue as in (17) and using (16) to obtain constants $\widehat{K}_3, K_3 > 0$ such that $\|v\|_{W^{4,p_2}(\Omega)} \leq \widehat{K}_3 K_3 \gamma$. If $4p_2 \geq N$, we can argue as in the Cases 1 or 2, and the lemma is proved. Otherwise, $W^{4,p_2}(\Omega) \hookrightarrow L^{p_3}(\Omega)$, where $p_3 = \frac{Np_2}{N-4p_2}$. Moreover

$$\frac{p_3}{p_2} = \frac{p_2}{p_1} \frac{N - 4p_1}{N - 4p_2} > 1.$$

If $\min\{p_3, q\} = q$, $W^{4,p_2}(\Omega) \hookrightarrow L^q(\Omega)$ and consequently the lemma follows from (17) and (18).

If $\min\{p_3, q\} = p_3$, we can argue as in (17) and using (16) to obtain constants $\widehat{K}_4, K_4 > 0$ such that $\|v\|_{W^{4,p_3}(\Omega)} \leq \widehat{K}_4 K_4 \gamma$. If $4p_3 \geq N$, we can argue as in the Cases 1 or 2, the lemma is proved. Otherwise, we can repeat the above arguments k times, for $k \in \mathbb{N}$ sufficiently large, and obtain p_k such that $\min\{p_k, q\} = q > \frac{N}{4}$. Thus, the lemma follows from (17) and (18), by taking $p = q$.

To ensure the existence of p_k, note that $p_j = \frac{Np_{j-1}}{N-4p_{j-1}}$, with $j = 2, 3, \cdots$ verifies

$$\frac{p_4}{p_3} = \frac{p_3}{p_2} \frac{N - 4p_2}{N - 4p_3} > \frac{p_3}{p_2}$$

and we can see that
$$\frac{p_j}{p_{j-1}} > \frac{p_{j-1}}{p_{j-2}} > \ldots > \frac{p_3}{p_2}.$$

Thus
$$\frac{p_j}{p_2} = \frac{p_j}{p_{j-1}} \frac{p_{j-1}}{p_{j-2}} \cdots \frac{p_4}{p_3} \frac{p_3}{p_2} > \left(\frac{p_3}{p_2}\right)^{j-2} > 1.$$

Therefore $\lim\limits_{j \to \infty} p_j = \infty$. Then, there exists $j_0 \in \mathbb{N}$ such that, for every $j > j_0$ we have $p_j > q$. The lemma is proved. $\square$

Proof of Theorem 1.3. Since h_R satisfies (H_0^+), from (13) and Theorem 1.1, there exist positive constants γ_1 and ν^* such that, for every $\gamma \in (0, \gamma_1)$ and $|\mu - \mu_1| < \gamma\nu^*$, problem (P_R) has a weak solution $u_\gamma = t\phi_1 + v$ with $t \in (t_1, t_2)$ and $v \in \langle \phi_1 \rangle^\perp$.

By Lemma 5.1 we can find p sufficiently large such that $p > \frac{N}{4}$, $W^{4,p}(\Omega) \hookrightarrow C^{0,\lambda}(\overline{\Omega})$ and such that $v \in C^{0,\lambda}(\overline{\Omega})$. Thus, there exists $C > 0$ such that

$$\|v\|_\infty \leq C\|v\|_{W^{4,p}(\Omega)} \leq CC_p\gamma.$$

Consequently, $\|v\|_\infty \to 0$, as $\gamma \to 0$. This implies that there exists $\gamma^* \in (0, \gamma_1)$ such that $\|v\|_\infty < 1$, for every $\gamma \in (0, \gamma^*)$. Thus, for every $\gamma \in (0, \gamma^*)$ and $|\mu - \mu_1| < \gamma\nu^*$ we have

$$\|u_\gamma\|_\infty = \|t\phi_1 + v\|_\infty \leq \|t\phi_1\|_\infty + \|v\|_\infty < R + 1.$$

Hence $\chi(u_\gamma) = 1$ and $h_R(x, u_\gamma) = h(x, u_\gamma)$, for every $\gamma \in (0, \gamma^*)$ and $|\mu - \mu_1| < \gamma\nu^*$. Consequently, u_γ is a solution of problem (P). □

6. Proof of Theorem 1.4

Proof of Theorem 1.4. As in the proof of Theorem 1.3 we consider

$$h_R(x, s) = \chi(s)h(x, s), \quad \text{for every } x \in \Omega,\ s \in \mathbb{R}$$

and the associated truncated problem

$$(P_R) \qquad \begin{cases} \alpha\Delta^2 u + \beta\Delta u = \mu u + \gamma h_R(x, u) & \text{in} \quad \Omega, \\ B(u) = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We can verify that h_R satisfies the hypothesis (H_0^-) and (H_1) with $f = f_{R+2}$. Furthermore, from the definition of h, (H_3) and (H_4), we obtain that h_R satisfies (H_2) with $g = \|\chi'\|_\infty f_{R+2} + g_{R+2} \in L^q(\Omega)$, where f_{R+2} and g_{R+2} are given by (H_3) and (H_4), respectively. Thus, we can apply Theorem 1.2, finding positive constants γ_2 and ν^* such that, for every $\gamma \in (0, \gamma_2)$ and $|\mu - \mu_2| < \gamma\nu^*$, problem (P_R) has a weak solution $u_\gamma = t\phi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle \phi_1 \rangle^\perp$. Next, using Lemma 5.1 and arguing as in the proof of Theorem 1.3, we obtain $\gamma^* \in (0, \gamma_2)$ such that u_γ is a solution of problem (P) for every $\gamma \in (0, \gamma^*)$ and $|\mu - \mu_1| < \gamma\nu^*$. The proof is complete. □

7. Proof of Theorem 1.5

Proof of Theorem 1.5. By taking $-h$ whenever $\gamma < 0$, we can apply Theorem 1.3 or 1.4, for every $i \in \{1, 2, \dots, \kappa\}$, to find positive constants γ_i^* and ν_i^* such that, for every $0 < |\gamma| < \gamma_i^*$ and $|\mu - \mu_1| < |\gamma|\nu_i^*$, the problem (P) has a weak solution $u_i = \widehat{t_i}\phi_1 + v_i$, with $\widehat{t_i} \in (t_i, t_{i+1})$ and $v_i \in \langle \phi_1 \rangle^\perp$. Now we can take $0 < \gamma^* < \min\{\gamma_1^*, \dots, \gamma_\kappa^*\}$ and $0 < \nu^* < \min\{\nu_1^*, \dots, \nu_\kappa^*\}$ to conclude the proof of the theorem. □

References

[1] C. O. Alves, J. A. Santos, J. V. A. Gonçalves: *On multiple solutions for multivalued elliptic equations under Navier boundary conditions*, J. Convex Analysis 18/3 (2011) 627–644.

[2] D. Arcoya, M. C. M. Rezende, E. A. B. Silva: *Quasilinear problems under local Landesman–Lazer condition*, Calc. Var. Partial Diff. Equations 58/6 (2019), art.no. 210, 27p.

[3] A. Castro: *Reduction methods via minimax*, in: *Differential Equations*, Proc. São Paulo 1981, Lecture Notes in Mathematics 957, Springer, Berlin (1982) 1–20.

[4] A. Fonda, M. Garrione: *Nonlinear resonance: a comparison between Landesman-Lazer and Ahmad-Lazer-Paul conditions*, Adv. Nonlinear Studies 11 (2011) 391–404.

[5] F. Gazzola: *Critical growth problems for polyharmonic operators*, Proc. R. Soc. Edinburgh, Sect. A Math. 128 (1998) 251–263.

[6] F. Gazzola, H. C. Grunau, M. Squassina: *Existence and nonexistence results for critical growth biharmonic elliptic equations*, Calc. Var. Partial Diff. Equations 18 (2003) 117–143.

[7] F. Gazzola, H. C. Grunau, G. Sweers: *Polyharmonic Boundary Value Problems*, Lectures Notes in Mathematics 1991, Springer, Berlin (2010).

[8] Y. Ge: *Brezis-Niremberg problem and Coron problem for polyharmonic operators*, Progr. Nonlinear Differential Equations Appl. 63 (2005) 291–297.

[9] Y. Ge: *A critical elliptic problem for polyharmonic operators*, J. Funct. Anal. 260 (2011) 2247–2282.

[10] C. P. Gupta, Y. C. Kwong: *Biharmonic eigenvalue problems and L^p estimates*, Internat. J. Math. Math. Sci. 13 (1990) 469–480.

[11] G. Jia, F. Li, Z. Ding: *Resonance problem of a class of singular quasilinear elliptic equations*, Applicable Analysis 94/10 (2015) 2095–2109.

[12] E. M. Landesman, A. C. Lazer: *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, J. Math. Mech. 19 (1970) 609–623.

[13] P. Pucci, G. Serrin: *Critical exponents and critical dimensions for polyharmonic operators*, J. Math. Pures Appl. 69 (1990) 55–83.

[14] M. C. M. Rezende, P. M. Sánchez-Aguilar, E. A. B. Silva: *A Landesman-Lazer local condition for semilinear elliptic problems*, Bull. Braz. Math. Soc., New Series 50 (2019) 889–911.

[15] M. Willem: *Minimax Theorems*, Birkhäuser, Boston (1996).

[16] Y. Yangxin, W. Rongxin, S. Yaotian: *Nontrivial solution for a class of semilinear biharmonic equation involving critical exponents*, Acta Math. Scientia 27B (2007) 509–514.