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The Lions Concentration-Compactness Principle for the Dirichlet Problem for Partial Differential Equations with Variable Exponent Laplace Operator

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We develop the P. Lions concentration-compactness principle for a sequence of Radon measures on R^n, the P. Lions principle is extended to variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$, $\Omega \subseteq R^n$, $n \geq 3$. Employing this $L^{p(\cdot)}$-extension of the concentration-compactness principle, we establish almost exact conditions under which the Dirichlet problem $u|_{\partial\Omega} = 0$ for variable exponent Laplace equation

$$-div\left(|\nabla u|^{p(x)-2}\nabla u\right) + \lambda |u|^{p(x)-2} u = a(x)|u|^{s(x)-2} u + f(x,\ u)$$

has a weak solution in variable exponent Sobolev space $W_1^{p(\cdot)}(\Omega)$, with critically grown coefficients.

Keywords: Concentration-compactness, variable exponent Lebesgue spaces, variable exponent Sobolev space, mountain pass theorem, concentration-compactness principle, Radon measure, Levy distribution.

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1. Introduction (discussion and definitions)

In 1954 [10], Levy introduced the concentration function of the Borel measure, and in 1984 [11], Lions proved the crucial lemma, which describes the convergence properties of a sequence of positive Lebesgue integrable functions on Euclidean space R^n, particularly, the Lions lemma states that each sequence of strictly positive Lebesgue integrable functions on R^n under the condition that all functions of the sequence have an identical fixed norm, then there exists a subsequence, which fulfills at least one of the three possibilities: the subsequence is compact, the subsequence vanishes for large indices, the subsequence satisfies dichotomy conditions. In 2009 [8], Fu considered some applications of Lions concentration-compactness principle to the solvability problem for the equation with variable Laplacian with a positive parameter controlling critical growth. In 2020 [1], Bashir, Guirao, Siddique and Saeed considered some developments in concentration-compactness principles for the variable exponent Laplacian, especially investigating the asymptotic behavior of some extremals; in the same year [14], Zhang and Huang analyzed the sliding mode control problem for fractional order equations [12, 13, 14].

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Now, we recall some pertinent definitions. Let Ω be a bounded connected domain in R^n. Let $P(\Omega)$ be a subspace of $L^1(\Omega)$ such that $p(\cdot) \in P(\Omega)$, $p(\cdot) : \Omega \to (1, \infty)$. We denote $p_m(\tilde{\Omega}) = ess\ \inf_{x\in\tilde{\Omega}} p(x)$ and $p_S(\tilde{\Omega}) = ess\ \sup_{x\in\tilde{\Omega}} p(x)$ for fixed $\tilde{\Omega} \subset \Omega$. For measurable functions f, we define the Luxemburg norm by

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0\ :\ \rho_p\left(\frac{f}{\lambda}\right) \le 1\right\},$$

where the functional $\rho_p\ :\ L^1(\Omega) \to R$ is given by

$$\rho_p(f) = \int_\Omega |f(x)|^{p(x)}\, dx \quad \text{for } \Omega \subseteq R^n.$$

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of all measurable functions f such that its Luxemburg norm is finite, $\|f\|_{L^{p(\cdot)}} < \infty$.

For variable exponent Lebesgue spaces $L^{p(\cdot)}$ with $1 < p_m \le p_S < \infty$, we can prove a variant of the Holder inequality in the form

$$\int |f(x)\, g(x)|\, dx \le \left(1 + \frac{1}{p_m} - \frac{1}{p_S}\right) \|f\|_{L^{p(\cdot)}} \|g\|_{L^{q(\cdot)}}$$

for all $f \in L^{p(\cdot)}$, $g \in L^{q(\cdot)}$, where the dual exponent function is given by $q(x) = \frac{p(x)}{p(x)-1}$. The Holder constant equals $k_{Hol} = 1 + \frac{1}{p_m} - \frac{1}{p_S}$.

We denote the characteristic function of a measurable set $\Omega \subset R^n,\ n > 2$ by χ_Ω.

We define $p_m := ess \inf\limits_{x\in\Omega} \{p(x)\} > 1$ and $p_S := ess \sup\limits_{x\in\Omega} \{p(x)\} < \infty$.

A function $\varphi : \Omega \to R$ is called *log-Holder continuous* if φ satisfies the estimation

$$|\varphi(x) - \varphi(y)| \le \frac{c}{\log\left(e + |x-y|^{-1}\right)}$$

for all $x,\ y \in \Omega$, with some $c > 0$. A function $\varphi\ :\ \Omega \to R$ satisfies the *log-Holder decay condition* if the inequality

$$|\varphi(x) - \varphi_\infty| \le \frac{c_1}{\log(e + |x|)}$$

holds for all $x \in \Omega$ and some $c_1 > 0$, and some real constants φ_∞.

Definition 1.1. The class $P^{\log}(\Omega)$ is a set of all functions $p(\cdot)$ such that $p(\cdot)^{-1}$ is log-Holder continuous and the function $p(\cdot)$ satisfies the log-Holder decay condition.

In this article, we study the elliptic partial differential equation with variable exponential Laplacian operator in the form

$$-\Delta_{p(x)}u + \lambda |u|^{p(x)-2} u = a(x) |u|^{s(x)-2} u + f(x, u) \tag{1}$$

where $p(\cdot),\ s(\cdot) \in P^{\log}(\Omega)$, $1 < p(x) < s(x) \le p^*(x)$, $\Delta_{p(x)}u \equiv div\big(|\nabla u|^{p(x)-2} \nabla u\big)$. We establish the experience of a weak solution to the problem $u|_{\partial\Omega} = 0$ for the variable Laplace equation under rather general conditions on its coefficients.

To prove the existence of a weak solution for the Dirichlet problem for the variable Laplace equation, we use a synthesis of the concentration-compactness principle for Radon measures and the mountain-pass theorem.

The notion of critical nonlinearity or critical frequency was studied by Byeon and Wang for the nonlinear Schrodinger equation for the description of a certain frequency in the neighborhood in which there exists a standing wave, in 2003 [2]. Applying the comparison principle, Byeon and Wang show the existence of localized solutions to the nonlinear Schrodinger equation with critical frequency [2]. In [3], Chen, Radulescu and Zhang studied the existence of positive solutions to the fractional Kirchhoff-type problem by variational methods; in [4, 5, 7, 8], Chung, Fan and Zhang, Fan and Zhao, and Fu investigated the Dirichlet problem for the variable Laplace equation; some results on concentration compactness can be found in [6]. In 2020 [1], Bashir, Guirao and Siddique studied the variational problem and investigated asymptotic properties of low energy extremals in variable exponent Lebesgue spaces for the variable Laplace operator.

This work contributes to existing studies, and our results can be applied to study mathematical models describing the properties of plasma and electrorheological fluids [1, 12] since the variable exponential growth problems often arise in nonlinear elastic, electrorheological fluids. We consider the equation with, although highly interesting, a specific right side, which can be generalized.

2. The concentration compactness principle for Radon measures

Let $\Omega \subset R^n,\ n > 2$ be a smooth domain in the R^n.

Definition 2.1. Let $\mathrm{M}_R(\Omega)$ be a set of all finite Radon measures on Ω.
Assume $\{\mu_k,\ k \in N\} \subset \mathrm{M}_R(\Omega)$. The sequence $\{\mu_k\}$ *weakly star converges* to the measure $\mu \in \mathrm{M}_R(\Omega)$ if

$$\lim_{k\to\infty} \int_\Omega \phi(x)\, d\mu_k(x) = \int_\Omega \phi(x)\, d\mu(x) \tag{2}$$

holds for all test functions $\phi \in C^\infty(\Omega) \cap C(clos(\Omega))$. This convergence will be denoted as $*weak-\lim\limits_{k\to\infty} \mu_k = \mu$ or $\mu_k \overset{*weakly}{\longrightarrow} \mu$.

Lemma 2.2. (Measure formulation). *Let $\{\theta_k\}$ be a sequence of nonnegative Radon measures such that*

$$\theta_k(\Omega) = \lambda \tag{3}$$

with a given $\lambda > 0$. Then, there exists a subsequence $\left\{\theta_{k(j)}\right\} \subseteq \{\theta_k\}$ of Radon measures $\theta_{k(j)}$ that will be indexed by k, which satisfies one of the following conditions:

(1) *(compactness) there exists a sequence $\{x_k\} \subset \Omega$ such that, for any $\varepsilon > 0$, there is a radius $r \in (0,\ \infty)$, such that*

$$\theta_k(B(x_k, r)) \geq \lambda - \varepsilon,$$

where $B(x_k,\ r)$ is a ball of radius r and center at x_k;

(2) *(vanishing) the upper limit $\lim\limits_{k\to\infty} \sup\limits_{x\in\Omega} \theta_k(B(x, r)) = 0$ holds for all $r \in (0,\ \infty)$;*

(3) *(dichotomy) there exists $\alpha \in (0,\ \lambda)$ such that for all $\varepsilon > 0$ there are k_0 and a pair of positive Radon measures θ_k^1 and θ_k^2 such that, for all $k \geq k_0$,*

$$\left|\theta_k(\Omega) - \left({\theta_k}^1(\Omega) + \theta_k^2(\Omega)\right)\right| \leq \varepsilon,$$
$$|\theta_k^1(\Omega) - \alpha| \leq \varepsilon,$$
$$\left|{\theta_k}^2(\Omega) - (\lambda - \alpha)\right| \leq \varepsilon,$$
$$dist\left(\sup p\left({\theta_k}^1\right),\ \sup p\left({\theta_k}^2\right)\right) \overset{k\to\infty}{\longrightarrow} \infty.$$

Proof. Levy density functions are defined by $\Theta_k(\tau) = \sup\limits_{x\in\Omega}\theta_k(B(x,\ \tau))$ for all numbers $\tau \in [0,\ \infty)$. The Levy functions comprise a sequence of uniformly bounded, nondecreasing nonnegative functions such that

$$\lim_{\tau\to\infty}\Theta_k(\tau) = \lambda.$$

The Helly theorem implies that there exists a nondecreasing nonnegative function Θ such that

$$\lim_{k\to\infty}\Theta_k(\tau) = \Theta(\tau)$$

for all $\tau \in [0,\ \infty)$. The function Θ has property

$$\lim_{\tau\to\infty}\Theta_k(\tau) = \alpha \in [0,\ \lambda]\,.$$

From $\alpha = \lambda$ follows compactness (1). For each $t > \frac{\lambda}{2}$ there is $r(t)$ so that

$$\Theta_k(r) = \sup_{x\in\Omega}\theta_k(B(x,\ r(t))) > t,$$

therefore, there is $x_k = x_k(t)$ such that $\theta_k(B(x_k,\ r(t))) > t$. Taking $x_k = x_k\left(\frac{\lambda}{2}\right)$, we have

$$|x_k(t) - x_k| \leq r\left(\frac{\lambda}{2}\right) + r(t)$$

for all $t \geq \frac{\lambda}{2}$ so there is a sequence $\{y_k\} \subset \Omega$ such that the inequality

$$\theta_k\left(B\left(y_k,\ r\left(\frac{\lambda}{2}\right) + 2r(t)\right)\right) > t$$

holds for all $t \geq \frac{\lambda}{2}$ and all indices k. The compactness has been proven.

From $\alpha = 0$ follows vanishing (2).

From $\alpha \in (0,\ \lambda)$ follows dichotomy. For any $\varepsilon > 0$, there is $r = r(\varepsilon) > 0$ so that $\Theta(r) > \alpha - \varepsilon$, so, the inequality $|\Theta_k(r) - \alpha| \leq \varepsilon$ holds for all large enough indices k, thus there is a sequence r_k such that $\lim\limits_{k\to\infty} r_k = \infty$ and $\Theta_k(r_k) - \alpha \leq \varepsilon$, then we can choose a sequence $\{x_k\} \subset \Omega$ such that $|\theta_k(B(x_k,\ r)) - \alpha| \leq \varepsilon$.

We define two sequences of measures by

$$\theta_k{}^1 = \theta_k \chi \left(B\left(x_k,\ r\right)\right) \quad \text{and} \quad \theta_k{}^2 = \theta_k \chi \left(\Omega \backslash B\left(x_k,\ r_k\right)\right),$$

where χ is a characteristic function. Thus, we obtain

$$\begin{aligned}&\theta_k\left(\Omega\right) - \left(\theta_k{}^1\left(\Omega\right) + \theta_k{}^2\left(\Omega\right)\right) = \theta_k\left(B\left(x_k,\ r_k\right)\right) - \theta_k\left(B\left(x_k,\ r\right)\right)\\ &\leq \Theta_k\left(r_k\right) - \Theta_k\left(r\right) + 2\varepsilon \leq \alpha + \varepsilon - \left(\alpha - \varepsilon\right) + 2\varepsilon = 4\varepsilon,\end{aligned}$$

and the dichotomy has been proven.

We will assume $p\left(\cdot\right),\ s\left(\cdot\right),\ \gamma\left(\cdot\right) \in P^{\log}\left(\Omega\right)$.

We consider a functional sequence $\{u_k\} \subset L^1,\ \|u_k\| \leq c$ as a sequence of Radon measures $\{\mu_k\}$ given by

$$\mu_k\left(E\right) = \int_E \left|u_k\right| dx;$$

$$\mu_k{}^{p(\cdot)}\left(E\right) = \int_E \left|u_k\right|^{p(x)} dx,$$

and

$$\mu_k{}^{\nabla p(\cdot)}\left(E\right) = \int_E \left|\nabla u_k\right|^{p(x)} dx$$

for all $\{u_k\} \subset W_{1,0}^{p(\cdot)}\left(\Omega\right)$.

Lemma 2.3. *Let* $1 < s\left(x\right) < p^*\left(x\right),\ p^*\left(x\right) = \frac{np(x)}{n-p(x)}$. *Let* $\{u_k\}$ *be a bounded sequence of elements of* $W_{1,0}^{p(\cdot)}\left(\Omega\right)$. *Then, there exists a subsequence* $\left\{u_{k(j)}\right\} \subset \{u_k\}$ *still denoted as* $\{u_k\}$*such that*

$$weak-\lim_{k\to\infty} u_k = u,$$

$$\lim_{k\to\infty} \left\|u_k - u\right\|_{L^{s(\cdot)}} = 0,$$

$$*weak-\lim_{k\to\infty} \mu_k{}^{s(\cdot)} = \eta,$$

$$*weak-\lim_{k\to\infty} \mu_k{}^{\nabla p(\cdot)} = \omega.$$

Proposition 2.4. *Let* $\{u_k, \quad k \in N\} \subset L^{p(\cdot)}\left(\Omega\right)$ *such that* $\left\|u_k\right\|_{L^{p(\cdot)}} \leq c$ *with* $c > 0$ *for all* $k \in N$ *and* $\lim u_k = u$ *almost everywhere, then*

$$\lim_{k\to\infty} \int_\Omega \left|u_k - u\right|^{p(x)} dx + \lim_{k\to\infty} \int_\Omega \left|u\right|^{p(x)} dx - \int_\Omega \left|u_k\right|^{p(x)} dx = 0. \tag{4}$$

The proof of the Proposition is straightforward, based on the Taylor approximation theorem with an application of the Holder inequality and Lebesgue's dominated convergence theorem.

Theorem 2.5. *Let $1 < s(x) \le p^*(x)$, $p^*(x) = \frac{np(x)}{n-p(x)}$, $1 < p_m \le p_S < n$ in Ω. Let $\{u_k\}$ be a bounded sequence of elements of $W_{1,0}^{p(\cdot)}(\Omega)$ such that $\|u_k\|_{W_{1,0}^{p(\cdot)}} \le 1$ and $*weak\text{-}\lim_{k\to\infty} {\mu_k}^{s(\cdot)} = \eta$. Then, there exists a subsequence $\left\{u_{k(j)}\right\} \subset \{u_k\}$ still denoted as $\{u_k\}$ and there exists a sequence $\{x_j\} \subset clos(\Omega)$ such that*

$$*weak\text{-}\lim_{k\to\infty} {\mu_k}^{\nabla p(\cdot)} = \omega = |\nabla u|^{p(x)} + \sum_j \omega_j \delta_{x_j}, \tag{5}$$

$$*weak\text{-}\lim_{k\to\infty} {\mu_k}^{s(\cdot)} = \eta = |u|^{s(x)} + \sum_j \eta_j \delta_{x_j}, \tag{6}$$

with the following restrictions

$$\eta(clos(\Omega)) \le c_1 \max\left\{\omega(clos(\Omega))^{\frac{s_S}{p_m}},\ \omega(clos(\Omega))^{\frac{s_m}{p_S}}\right\} \tag{7}$$

and

$$\eta_j \le c_1 \max\left\{{\omega_j}^{\frac{s_S}{p_m}},\ {\omega_j}^{\frac{s_m}{p_S}}\right\}. \tag{8}$$

Proof. Let $\{u_k\}$ be a bounded sequence in $W_{1,0}^{p(\cdot)}(\Omega)$ and let $\|u_k\|_{W_{1,0}^{p(\cdot)}} \le 1$, then there exists a subsequence $\{u_k\}$ such that there is a weak limit $weak\text{-}\lim_{k\to\infty} u_k = u$ and $*weak\text{-}\lim_{k\to\infty} {\mu_k}^{s(\cdot)} = \eta$.

We consider the ball $B(x_0, r)$ with the center at $x_0 \in \Omega$. We have to show that for all $\gamma \in (0, 1)$ there is $\zeta > 0$ such that the inequality

$$\int_{B(x_0,\rho)} |\nabla(\theta u)|^{p(x)} dx$$

$$\le \left(1 + \frac{\gamma}{2}\right) \int_{B(x_0,\rho)} |\nabla u|^{p(x)} dx + \zeta \max\left\{\|\nabla u\|_{L^{p(\cdot)}}^{p_S},\ \|\nabla u\|_{L^{p(\cdot)}}^{p_m}\right\}$$

holds for all $u \in W_{1,0}^{p(\cdot)}(\Omega)$, where $\theta \in W_{1,0}^{\infty}(\Omega)$ is a cutoff function so that

$$\theta = \begin{cases} 0, & x \in clos(\Omega) \backslash B(x_0, \rho), \\ \dfrac{c_B(\ln(\rho) - \ln(|x|))}{c_B(\ln(r) - \ln(\rho))}, & \text{otherwise}, \\ 1, & x \in B(x_0, r), \end{cases}$$

where $0 < r < \rho < 1$ and $r\rho^{-1} \le \zeta$, constant c_B depends on the area of unit sphere in R^n.

Indeed, denoting $\hat{c}_1 = \left(1 + \frac{\gamma}{2}\right)^{\frac{1}{p_S - 1}} - 1$ and employing Holder and Young inequalities, we calculate

$$\int_{B(0,\rho)} |\nabla(\theta u)|^{p(x)} dx = \int_{B(0,\rho)} |\theta\nabla u + u\nabla\theta|^{p(x)} dx$$

$$\le \left(1 + \frac{\gamma}{2}\right) \int_{B(0,\rho)} |\nabla u|^{p(x)} dx$$

$$+ k_{Hol}\left(1 + \frac{1}{\hat{c}_1}\right)^{p_S - 1} \left\||u|^{p(\cdot)}\right\|_{L^{\frac{n}{n-p(\cdot)}}(B(0,\rho))} \left\||\nabla\theta|^{p(\cdot)}\right\|_{L^{\frac{n}{p(\cdot)}}(B(0,\rho))},$$

applying the idea of Fu, we obtain

$$\int_{B(0,\,\rho)} |\nabla(\theta u)|^{p(x)}\,dx$$
$$\leq \left(1+\frac{\gamma}{2}\right)\int_{B(0,\,\rho)} |\nabla u|^{p(x)}\,dx + \gamma\begin{cases} \|\nabla u\|^{p_m}_{L^{p(\cdot)}}, & \text{if } \|u\|_{L^{p^*(\cdot)}} < 1, \\ \|\nabla u\|^{p_S}_{L^{p(\cdot)}}, & \text{if } \|u\|_{L^{p^*(\cdot)}} \geq 1, \end{cases}$$

since, if $\|u\|_{L^{p^*(\cdot)}(B(0,\,\rho))} < 1$, then

$$\left\| |u|^{p(\cdot)} \right\|_{L^{\frac{n}{n-p(\cdot)}}(B(0,\,\rho))} \leq \|u\|_{L^{p^*(\cdot)}(B(0,\,\rho))}{}^{p_m} \leq \|u\|_{L^{p^*(\cdot)}}{}^{p_m} \leq c\,\|\nabla u\|_{L^{p(\cdot)}}{}^{p_m},$$

and if $\|u\|_{L^{p^*(\cdot)}(B(0,\,\rho))} < 1$, then

$$\left\| |u|^{p(\cdot)} \right\|_{L^{\frac{n}{n-p(\cdot)}}(B(0,\,\rho))} \leq \|u\|_{L^{p^*(\cdot)}(B(0,\,\rho))}{}^{p_S} \leq \|u\|_{L^{p^*(\cdot)}}{}^{p_S} \leq c\,\|\nabla u\|_{L^{p(\cdot)}}{}^{p_S}.$$

We define measure $\mu^{\nabla p(\cdot)}(E) = \int_E |\nabla u|^{p(x)}\,dx$, for all $\phi \in C^\infty(\Omega) \cap C(clos(\Omega))$, we can write

$$\int_\Omega |\nabla u|^{p(x)}\,\phi dx \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k|^{p(x)}\,\phi dx,$$

therefore, $\omega \geq \mu^{\nabla p(\cdot)}$, and since the measure ω is finite, we have that the number of atoms of measure ω must be at most countable. Thus, we have the representation

$$\omega = |\nabla u|^{p(x)} + \sum_j \omega_j \delta_{x_j}.$$

Employing the statement above, for all measurable $E \subseteq clos(\Omega)$, we have

$$\lim_{k\to\infty}\inf \int_E |u_k - u|^{s(x)}\,dx + \int_E |u|^{s(x)}\,dx - \lim_{k\to\infty}\inf \int_E |u_k|^{s(x)}\,dx = 0$$

so
$$\tilde{\eta} + \sum_j \eta_j \delta_{x_j} + |u|^{s(x)} - \eta = 0,$$

where $\tilde{\eta}$ is a diffuse measure.

We take a cutoff function $\theta \in W^\infty_{1,0}(\Omega)$, $0 < r < \rho < 1$ as above, and calculate

$$\int_{B(y,\,r)} |u|^{s(x)}\,dx \leq \int_{B(y,\,\rho)} |u\theta|^{s(x)}\,dx$$
$$\leq \hat{c}\max\left\{ \left(\int_{B(y,\,\rho)} |\nabla(u\theta)|^{p(x)}\,dx\right)^{\frac{s_S}{p_m}}, \left(\int_{B(y,\,\rho)} |\nabla(u\theta)|^{p(x)}\,dx\right)^{\frac{s_m}{p_S}} \right\}.$$

We take $0 < r < \rho < 1$ and an arbitrary $\varepsilon > 0$, and get

$$\eta(\{y\}) \leq \eta(clos(B(y,r)))$$
$$\leq \hat{c}\max\left\{ (\omega(clos(B(y,\ \rho))) + \varepsilon)^{\frac{s_S}{p_m}}, (\omega(clos(B(y,\ \rho))) + \varepsilon)^{\frac{s_m}{p_S}} \right\},$$

so, taking the limit as $r \to 0$, $\varepsilon \to 0$ and $\rho \to 0$, we obtain

$$\eta(\{y\}) \leq \hat{c} \max\left\{\omega(\{y\})^{\frac{s_S}{p_m}}, \omega(\{y\})^{\frac{s_m}{p_S}}\right\},$$

therefore, for all atoms $\{x_j\}$ we have

$$\eta(\{x_j\}) \leq \hat{c} \max\left\{\omega(\{x_j\})^{\frac{s_S}{p_m}}, \omega(\{x_j\})^{\frac{s_m}{p_S}}\right\}.$$

Thus, atoms of ω and η coincides.

Since $\tilde{u} = u_k - u \overset{weakly \quad W_{1,0}^{p(\cdot)}(\Omega)}{\longrightarrow} 0$ and $*weak-\lim\limits_{k\to\infty} \tilde{\mu}_k{}^{\nabla p(\cdot)} = \tilde{\omega}$ then by the Sobolev theorem

$$\begin{aligned}
&\int_{B(y,\,\rho)} \theta(B(x,\ r)) |\tilde{u}_k|^{s(x)} dx \\
&\leq \int_{B(y,\,\rho)} \theta(B(x,\ r)) |\tilde{u}_k|^{s(x)} dx \\
&\leq \hat{c} \max\left\{\left(\int_{B(y,\,\rho)} \left|\nabla\left(\tilde{u}_k \theta(B(x,\ r))^{\frac{1}{s(x)}}\right)\right|^{p(x)} dx\right)^{\frac{s_S}{p_m}},\right. \\
&\quad \left.\left(\int_{B(y,\,\rho)} \left|\nabla\left(\tilde{u}_k \theta(B(x,\ r))^{\frac{1}{s(x)}}\right)\right|^{p(x)} dx\right)^{\frac{s_m}{p_S}}\right\},
\end{aligned}$$

Taking the limit as $k \to \infty$ we have

$$\begin{aligned}
&\tilde{\eta}(clos(B(y,\rho))) \leq \int_{clos(B(y,\rho))} \theta(B(x,r))\, d\tilde{\eta} \\
&\leq \hat{c} \max\left\{\left(\int_{clos(B(y,\rho))} \theta(B(x,r))^{\frac{n-p(x)}{n}} d\tilde{\omega}\right)^{\frac{s_S}{p_m}},\right. \\
&\quad \left.\left(\int_{clos(B(y,\rho))} \theta(B(x,r))^{\frac{n-p(x)}{n}} d\tilde{\omega}\right)^{\frac{s_m}{p_S}}\right\},
\end{aligned}$$

where $\theta(B(x,r))$ is cut off on $B(x,r)$ relative to $B(x,\ \rho)$, so that

$$\tilde{\eta}(clos(B(y,\rho))) \leq \hat{c} \max\left\{\tilde{\omega}(clos(B(y,\rho)))^{\frac{s_S}{p_m}}, \tilde{\omega}(clos(B(y,\rho)))^{\frac{s_m}{p_S}}\right\}.$$

Now employing the Radon-Nikodym theorem, we obtain the existence of a function $g \in L^1(clos(\Omega),\ \tilde{\omega})$ such that $\frac{d\tilde{\eta}}{d\tilde{\omega}} = g$, the estimation

$$g(x) \leq \hat{c} \max\left\{\tilde{\omega}(\{x\})^{\frac{s_S(x)}{p_m(x)}-1}, \tilde{\omega}(\{x\})^{\frac{s_m(x)}{p_S(x)}-1}\right\}$$

holds with $p_m(x) = \inf\{p_m(y),\ y \in B(x,\rho) \cap clos(\Omega)\}$ and $p_S(x)$, $s_m(x)$, $s_S(x)$ defined similarly. Since atoms of $\tilde{\eta}$ and η coincide, we deduce $\tilde{\eta}$ is tautologically zero.

Since $$\int_\Omega |u|^{s(x)} dx \leq \hat{c} \max\left\{\|\nabla u\|_{L^{p(\cdot)}(\Omega)}{}^{s_S}, \|\nabla u\|_{L^{p(\cdot)}(\Omega)}{}^{s_m}\right\},$$

finally, we obtain

$$\eta(clos(\Omega)) \leq \hat{c} \max\left\{\omega(clos(\Omega))^{\frac{s_S}{p_m}}, \omega(clos(\Omega))^{\frac{s_m}{p_S}}\right\}.$$

The proof of Theorem 2.5 is complete. □

3. The existence of nontrivial solutions to the Dirichlet problem for variable exponential Laplacian equations

We apply the concentration compactness principle to the solvability of the variable exponential Laplace equation with parameter $\lambda > 0$ given in the form

$$-div\left(|\nabla u|^{p(x)-2}\nabla u\right)+\lambda\,|u|^{p(x)-2}u=a\,(x)\,|u|^{s(x)-2}u+f\,(x,\ u)\,, \tag{9}$$

$$u|_{\partial\Omega}=0 \tag{10}$$

in the space $W_{1,0}^{p(\cdot)}(\Omega)$, $\Omega\subset R^n$, $n>2$ under the restrictions

$$1<p\,(x)<s\,(x)\le p^*\,(x),\ p\,(\cdot),\ s\,(\cdot),\ \gamma\,(\cdot)\in P^{\log}\,(\Omega).$$

Definition 3.1. A function $u\in W_{1,0}^{p(\cdot)}(\Omega)$ is called *weak solution* to the variable exponential Laplace equation if the function u is a critical point of the functional $E\ :u\in W_{1,0}^{p(\cdot)}(\Omega)\to R$ given by

$$E(u)=\int_\Omega\frac{1}{p(x)}\left(|\nabla u|^{p(x)}+\lambda|u|^{p(x)}\right)dx$$
$$-\int_\Omega\frac{1}{s\,(x)}a\,(x)\,|u|^{s(x)}\,dx-\int_\Omega\int_{[0,\ u]}f\,(x,\tau)\,d\tau\,dx,$$

the critical points u are defined by

$$0=\langle\delta E\,(u)\,,\ \phi\rangle=\int_\Omega\left(|\nabla u|^{p(x)-2}\nabla u\nabla\phi+\lambda\,|u|^{p(x)-2}u\phi\right)dx$$
$$-\int_\Omega a\,(x)\,|u|^{s(x)-2}u\phi\,dx-\int_\Omega f\,(x,u)\,\phi\,dx$$

for all $\phi\in W_{1,0}^{p(\cdot)}(\Omega)$.

We assume that the coefficients of the variable exponential Laplace equation satisfy the following conditions:

(1) $1<p\,(x)<s\,(x)\le p^*\,(x)$, $p\,(\cdot)\,,\ s\,(\cdot)\,,\ \gamma\,(\cdot)\in P^{\log}\,(\Omega)$, $1<p_m\le p_S<n$, $p^*\,(x)=\frac{np(x)}{n-p(x)}$, function $0<a_0\le a\,(x)\,,\quad a\in L^{\frac{p^*(\cdot)}{p^*(\cdot)-s^*(\cdot)}}$;

(2) function $f\in C\,(clos\,(\Omega)\times R)$, $0<f\,(x,\ z)$ for all $(x,\ z)\in\left(\tilde{\Omega}\times(0,\ \infty)\right)$ for some $\tilde{\Omega}\subseteq\Omega$, and $f\,(x,\ z)=0$ for all $(x,\ z)\in(\Omega\times(-\infty,0])$;

(3) there exist numbers $\theta>p_S$ and $M>0$ such that the inequality

$$0<\theta F\,(x,z)=\theta\int_{[0,\ z]}f\,(x,\tau)\,d\tau\le zf\,(x,z)$$

holds for all $x\in R^n$ and for all $|z|\ge M$;

(4) there is a positive function c_1 and small constant c_2 such that the inequalities $\gamma_0=\inf\limits_{x\in\Omega}\{s\,(x)-\gamma\,(x)-1\}>0$, and $|f\,(x,z)|\le c_1\,(x)+c_2\,|z|^{\gamma(x)}$ hold for all $x\in\Omega$, $z\in R$, $c_1\in L^{\frac{p^*(\cdot)}{p^*(\cdot)-1}}$.

Next, we are going to establish solvability conditions for the variable exponential Laplace equation by application of the concentration compactness principle.

Theorem 3.2. *Let the coefficients of the variable exponential Laplace satisfy conditions* (1)–(4)*, then the variable exponential Laplace equation has a nontrivial weak solution in the Sobolev space* $W_{1,0}^{p(\cdot)}(\Omega)$.

Proof. (1) We show that there is $\tilde{r} > 0$ such that $E(u) > 0$ for all $\|u\|_{W_{1,0}^{p(\cdot)}} = \tilde{r} < 1$. We calculate

$$\int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + \lambda |u|^{p(x)} \right) dx$$
$$- \int_\Omega \frac{1}{s(x)} a(x) |u|^{s(x)} dx - \int_\Omega \int_{[0,\, u]} f(x, \tau) \, d\tau dx$$
$$\geq \frac{1}{p_S} \int_\Omega \left(|\nabla u|^{p(x)} + \lambda |u|^{p(x)} \right) dx$$
$$- \frac{1}{s_m} \int_\Omega c(x) |u|^{s(x)} dx - \int_\Omega \left(c_1(x) |u| + c_2 |u|^{\gamma(x)+1} \right) dx,$$

where we employed the inequality

$$\int_{[0,\, u]} |f(x, \tau)| \, d\tau \leq \left(c_1(x) |u| + c_2 |u|^{\gamma(x)+1} \right).$$

We estimate

$$\int_\Omega c_1(x) |u| \, dx \leq k_{Hol} \|u\|_{L^{p(\cdot)}} \|c_1(\cdot)\|_{L^{q(\cdot)}} \leq c_1 k_{Hol} \|u\|_{W_{1,0}^{p(\cdot)}},$$
$$c_2 \int_\Omega |u|^{\gamma(x)+1} dx \leq c_2 \left(mes(\Omega) + \|u\|_{W_{1,0}^{p(\cdot)}}{}^{p_m} \right),$$

the last inequality is a consequence of the following proposition.

Proposition 3.3. *Let* $1 < p_m \leq p_S < n$ *and* $\inf_{x \in \Omega} \{p^*(x) - \gamma(x)\} > 0$ *then there exists a compact embedding* $W_1^{p(\cdot)}(\Omega) \to L^{\gamma(\cdot)}(\Omega)$ *where* $p(x) \leq \gamma(x)$ *for almost all* $x \in clos(\Omega)$.

We obtain

$$\frac{1}{p_S} \int_\Omega |\nabla u|^{p(x)} dx - \frac{1}{s_m} \int_\Omega c(x) |u|^{s(x)} dx \geq const(\tilde{r}) > 0,$$

and

$$\frac{1}{s_m} \int_\Omega c(x) |u|^{s(x)} dx \leq \frac{1}{s_m} c_S \int_\Omega |u|^{s(x)} dx,$$

so, we can use the estimate $\|u\|_{L^{p(\cdot)}} \leq const(\Omega) \|\nabla u\|_{L^{p(\cdot)}}$ and formulate the following proposition.

Proposition 3.4. *Let* $p, s \in P^{\log}(\Omega)$, $1 < p_m \leq p_S < n$, *and assume that we have* $p(x) \leq s(x) \leq p^*(x)$ *almost everywhere* $x \in clos(\Omega)$, *then there exists a continuous embedding* $W_1^{p(\cdot)}(\Omega) \to L^{s(\cdot)}(\Omega)$.

Thus, we deduce $\|u\|_{L^{p^*(\cdot)}} \le const\,(\Omega)\,\|\nabla u\|_{L^{p(\cdot)}} < 1$ for small enough norm $\|\nabla u\|_{L^{p(\cdot)}}$. Now, we apply Fu's idea of special covering for compact set $clos\,(\Omega)$ constructed as for any $x \in clos\,(\Omega)$ we define a cube

$$Q\,(x,\ l) = \left\{ y \in clos\,(\Omega) \subset R^n \ : \quad \left|x^j - y^j\right| < \frac{l}{2}, \quad j = 1,\ ...,\ n \right\}$$

and such that the inequalities

$$|p\,(x) - p\,(y)| < \varepsilon \quad \text{and} \quad |s\,(x) - s\,(y)| < \varepsilon$$

hold for all $y \in Q\,(x,\ l)$. We denote

$$p_m\,(x) = \inf_{y \in Q(x,\ l)} p\,(y) \quad \text{and} \quad p_S\,(x) = \sup_{y \in Q(x,\ l)} p\,(y)\,.$$

Assuming ε small enough, we have $p_S\,(x) < p_m^*\,(x)$. We take a finite subsystem $\{Q\,(x_i, l_i)\,,\, i = 1, ..., m\}$ covering $clos(\Omega)$, from the system $\{Q\,(x, l)\,,\, x \in clos(\Omega)\}$. We split $\cup_i\, Q\,(x_i, l_i)$ into finite system of finite open cubes $\{Q(x_k, l_k), k = 1, ..., m_1\}$ such that for any $Q\,(x_k, l_k)$, $k = 1, ..., m_1$ there is at least one $Q\,(x_i, l_i)$, $i = 1, ..., m$ with the property $Q\,(x_k, l_k) \subset Q\,(x_i, l_i)$. So, we estimate

$$\int_{Q(x_k,\ l_k)} |u|^{p^*(x)}\, dx \le \hat{c}\, \|u\|_{W_1^{p(\cdot)}(Q(x_k,\ l_k))}{}^{p_m{}^*(k)}, \quad k = 1,\ ...,\ m_1$$

and $$\int_{Q(x_k,\ l_k)} |\nabla u|^{p(x)}\, dx \ge (1 + \hat{c}_1)^{-1}\, \|u\|_{W_1^{p(\cdot)}(Q(x_k,\ l_k))}{}^{p_S(k)}, \quad k = 1,\ ...,\ m_1.$$

So, we have

$$\frac{1}{p_S} \int_\Omega |\nabla u|^{p(x)}\, dx - \frac{1}{p_m} \hat{c}_1 \int_\Omega |u|^{p^*(x)}\, dx$$
$$\ge \breve{c}_1\, \|u\|_{W_1^{p(\cdot)}(Q(x_k,\ l_k))}{}^{p_S(k)} - \breve{c}_2\, \|u\|_{W_1^{p(\cdot)}(Q(x_k,\ l_k))}{}^{p_m{}^*(k)}, \quad k = 1,\ ...,\ m_1,$$

since $p_m{}^*\,(k) > p_S\,(k)$ we obtain

$$\breve{c}_1\, \|u\|_{W_1^{p(\cdot)}(Q(x_k,\ l_k))}{}^{p_S(k)} - \breve{c}_2\, \|u\|_{W_1^{p(\cdot)}(Q(x_k,\ l_k))}{}^{p_m{}^*(k)} \ge const\,(\tilde{r}\,(k)) > 0$$

for $\tilde{r}\,(k) > 0$. Let $0 < \tilde{r} \le \tilde{r}\,(k) < (1 + \hat{c}_1)^{-1}$ and $\tilde{r} = \|u\|_{W_1^{p(\cdot)}(Q(x_k,\ l_k))}$ then we have $\|u\|_{L^{p(\cdot)}} \le \hat{c}_1\, \|\nabla u\|_{L^{p(\cdot)}}$. We put $\tilde{r}_m = \min_k \{l_k\}$ and obtain

$$E(u) \ge \sum_{k=1,\dots,m_1} \left(\frac{1}{p_S} \int_\Omega |\nabla u|^{p(x)}\, dx - \frac{1}{p_m} \hat{c}_1 \int_\Omega |u|^{p^*(x)}\, dx \right)$$
$$\ge \breve{c}_1 \left(\frac{\tilde{r}_m}{m_1} \right)^{p_m} \left(1 + \frac{\breve{c}_2}{\breve{c}_1} \tilde{r}_m{}^{\frac{p_m{}^2}{n - p_m}} \right) > 0.$$

Second, we show that there is a function v, $\|v\|_{W_1^{p(\cdot)}} > r$ such that $E\,(v) < 0$ for some $0 < r$. Assume $\tilde{x} \in \tilde{\Omega}$, the constant $\rho \in \left(0,\ \frac{1}{2}\right)$ such that

$$p_S\,(\tilde{x}, B\,(\tilde{x}, 2\rho)) < p_m^*\,(\tilde{x}, B\,(\tilde{x}, 2\rho)) \quad \text{for all } B\,(\tilde{x}, 2\rho) \subset \tilde{\Omega}.$$

We take $\phi \in C_0^\infty(B(\tilde{x},2\rho))$ so that $\phi(x) \in [0,1]$ and $|\nabla\phi| \le \rho^{-1}$, and obtain

$$\begin{aligned}
E(\zeta\phi) &= \int_{B(\tilde{x},2\rho)} \frac{\zeta^{p(x)}}{p(x)} \left(|\nabla\phi|^{p(x)} + \lambda|\phi|^{p(x)}\right) dx \\
&\quad - \int_{B(\tilde{x},2\rho)} \frac{\zeta^{s(x)}}{s(x)} a(x)|\phi|^{s(x)}\, dx - \int_{B(\tilde{x},\,2\rho)} \int_{[0,\,\zeta\phi]} f(x,\,\tau)\, d\tau dx \\
&\le \int_{B(\tilde{x},2\rho)} \frac{\zeta^{p(x)}}{p(x)} \left(|\nabla\phi|^{p(x)} + \lambda|\phi|^{p(x)}\right) dx \\
&\quad - \int_{B(\tilde{x},2\rho)} \frac{\zeta^{s(x)}}{s(x)} a(x)|\phi|^{s(x)}\, dx \\
&\quad - \int_{B(\tilde{x},2\rho)} \left(c_1(x)\zeta|\phi| + c_2\zeta^{\gamma(x)+1}|\phi|^{\gamma(x)+1}\right) dx
\end{aligned}$$

for all $\zeta > 1$. Since $s_m > p_S$ we can choose a large enough number ζ so that

$$\begin{aligned}
E(\zeta\phi) &\le \frac{\zeta^{p_S}}{p_m} \int_{B(\tilde{x},\,2\rho)} \left(|\nabla\phi|^{p(x)} + \lambda|\phi|^{p(x)}\right) dx \\
&\quad - \frac{\zeta^{s_m}}{s_S} a_m \int_{B(\tilde{x},\,2\rho)} |\phi|^{s(x)}\, dx \\
&\quad - \int_{B(\tilde{x},\,2\rho)} \left(c_1(x)\zeta|\phi| + c_2\zeta^{\gamma(x)+1}|\phi|^{\gamma(x)+1}\right) dx \overset{\zeta\to\infty}{\longrightarrow} -\infty
\end{aligned}$$

so $E(\zeta\phi) < 0$.

Third, we prove that assuming $E(u_k) \overset{k\to\infty}{\longrightarrow} c$ and $\delta E(u_k) \overset{k\to\infty}{\longrightarrow} 0$ in the topology $W_{-1,0}^{q(\cdot)}$, $q(x) = \frac{p(x)}{p(x)-1}$, then sequence $\{u_k\}$ is bounded in $W_{1,\,0}^{p(\cdot)}$. We write

$$c + 1 \ge E(u_k) = E(u_k) - \frac{1}{s_m}\langle \delta E(u_k),\, u_k\rangle + \frac{1}{s_m}\langle \delta E(u_k),\, u_k\rangle$$

and
$$\begin{aligned}
E(u_k) &= \int_\Omega \frac{1}{p(x)} \left(|\nabla u_k|^{p(x)} + \lambda|u_k|^{p(x)}\right) dx \\
&\quad - \int_\Omega \frac{1}{s(x)} a(x)|u_k|^{s(x)}\, dx - \int_\Omega \int_{[0,\,u_k]} f(x,\tau)\, d\tau dx \\
&\ge \frac{1}{p_S} \int_\Omega \left(|\nabla u_k|^{p(x)} + \lambda|u_k|^{p(x)}\right) dx \\
&\quad - \int_\Omega \frac{1}{s(x)} a(x)|u_k|^{s(x)}\, dx - \int_\Omega \frac{1}{\theta} f(x,\,u_k)\, u_k dx \\
&\ge \frac{1}{p_S} \int_\Omega \left(|\nabla u_k|^{p(x)} + \lambda|u_k|^{p(x)}\right) dx - \int_\Omega \frac{1}{s(x)} a(x)|u_k|^{s(x)}\, dx \\
&\quad - \int_\Omega \frac{1}{\theta} \left(c_1(x)|u_k| + c_2|u_k|^{\gamma(x)+1}\right) dx \\
&\quad - \frac{1}{s_m}\langle \delta E(u_k),\, u_k\rangle + \frac{1}{s_m}\langle \delta E(u_k),\, u_k\rangle .
\end{aligned}$$

Assuming $0 < \varpi < 1$ and applying the Young inequality, we have

$$\int_{\Omega} |u_k|^{\gamma(x)+1} dx \leq \varpi \int_{\Omega} |u_k|^{p(x)} dx + \int_{\Omega} \varpi^{\frac{\gamma(x)+1}{\gamma(x)+1-p(x)}} dx,$$

we can assume that $\|u_k\|_{W_{1,0}^{p(\cdot)}} > 1$ and since $\|\delta E(u_k)\|_{W_{-1,0}^{q(\cdot)}} \leq const$ then

$$c+1 \geq \left(\frac{1}{p_S} - \frac{1}{s_m}\right) \|u_k\|_{W_{1,0}^{p(\cdot)}}{}^{p_m} - \frac{1}{s_m} const \|u_k\|_{W_{1,0}^{p(\cdot)}}.$$

Therefore, the sequence $\left\{\|u_k\|_{W_{1,0}^{p(\cdot)}}\right\}$ is bounded.

Fourth, we must show that the conditions of the compactness concentration theorem are satisfied.

Thus, we have the existence of the $W_{1,0}^{p(\cdot)}$-bounded sequence $\{u_k\}$ and a sequence $\{x_j\} \subset clos(\Omega)$, that satisfies the following conditions:

$$weak-\lim_{k\to\infty} u_k = u,$$

$$\lim_{k\to\infty} \|u_k - u\|_{L^{s(\cdot)}} = 0,$$

$$*weak-\lim_{k\to\infty} \mu_k{}^{s(\cdot)} = \eta = |u|^{s(x)} + \sum_j \eta_j \delta_{x_j},$$

$$*weak-\lim_{k\to\infty} \mu_k{}^{\nabla p(\cdot)} = \omega = |\nabla u|^{p(x)} + \sum_j \omega_j \delta_{x_j},$$

$$\eta_j \leq c_1 \max\left\{\omega_j{}^{\frac{s_S}{p_m}},\ \omega_j{}^{\frac{s_m}{p_S}}\right\}.$$

We consider the delta system $\phi_{j,\varepsilon}(x) = \phi\left(\frac{x_j - x}{\varepsilon}\right)$, $\phi \in C_0^{\infty}(\Omega)$, $\varepsilon > 0$ where $\phi \in [0,1]$,

$$\phi(x) = \begin{cases} 1, & \|x\| < 1, \\ 0, & \|x\| > 2. \end{cases}$$

We have

$$\int_{\Omega} \left(|\nabla u_k|^{p(x)-2} \nabla u_k \nabla(\phi_{j,\varepsilon} u_k) + \lambda |u_k|^{p(x)-2} u_k \phi_{j,\varepsilon}\right) dx$$
$$\leq a_S \int_{\Omega} |u_k|^{s(x)} \phi_{j,\varepsilon} dx + c_{1,S} \int_{\Omega} \left(|u_k| + \frac{c_2}{c_{1,S}} |u_k|^{\gamma(x)+1}\right) \phi_{j,\varepsilon} dx,$$

so, from $\varepsilon \to 0$ follows $\omega(\{x_j\}) \leq \hat{c}\,\eta(\{x_j\})$ so $\omega_j \leq \hat{c}\,\eta_j$. Since $\hat{c}\eta_j^{\frac{1}{s(x_j)}} \leq \eta_j{}^{\frac{1}{p(x_j)}}$ we have $\omega(\{x_j\}) = \eta(\{x_j\}) = 0$ for all $x_j \in clos(\Omega)$.

Next, we have $\frac{|u_k - u|^{s(x)}}{2^{s_m}} \leq |u_k|^{s(x)} + |u|^{s(x)}$ and

$$2\int_K |u|^{s(x)} dx \leq \liminf \int_K \left(|u|^{s(x)} + |u_k|^{s(x)}\right) dx - \int_K \frac{|u_k - u|^{s(x)}}{2^{s_m}} dx,$$

where the subset K is a compact. So,

$$\limsup \int_K \frac{|u_k - u|^{s(x)}}{2^{s_m}} dx \leq 0$$

and $\lim_{k\to\infty} \|u_k - u\|_{L^{s(\cdot)}} = 0$.

To conclude our proof, we must show that there exists a subsequence $\left\{u_{k(j)}\right\} \subset \{u_k\}$ such that the limit

$$\lim_k \nabla u_k(x) = \nabla u(x)$$

holds for almost all $x \in clos(\Omega)$. For any compact $K \subseteq \Omega$, we write

$$\begin{aligned}
&\langle \delta E(u) - \delta E(u_k),\ u - u_k \rangle = \\
&= \int_K \left(|\nabla u|^{p(x)-2} \nabla u - |\nabla u_k|^{p(x)-2} \nabla u_k \right) \nabla (u - u_k)\, dx \\
&\quad + \lambda \int_K \left(|u|^{p(x)-2} u - |u_k|^{p(x)-2} u_k \right) (u - u_k)\, dx \\
&\quad - \int_K a(x) \left(|u|^{s(x)-2} u - |u_k|^{s(x)-2} u_k \right) (u - u_k)\, dx \\
&\quad + \int_K f(x,\ u_k)(u - u_k)\, dx - \int_K f(x,\ u)(u - u_k)\, dx.
\end{aligned}$$

We further estimate

$$\begin{aligned}
&\left| \int_K a(x) \left(|u|^{s(x)-2} u - |u_k|^{s(x)-2} u_k \right) (u - u_k)\, dx \right| \\
&\leq \|a\|_{L^{\frac{p^*(\cdot)}{p^*(\cdot)-s^*(\cdot)}}} \left(\hat{c}_1 \|u\|_{L^{p^*(\cdot)}}^{s_S - 1} \|u - u_k\|_{L^{p^*(\cdot)}} + \hat{c}_2 \|u_k\|_{L^{p^*(\cdot)}}^{s_S - 1} \|u - u_k\|_{L^{p^*(\cdot)}} \right),
\end{aligned}$$

$$\begin{aligned}
&\int_K |f(x,\ u)| (u - u_k)\, dx \leq \int_K \left(c_1(x) + c_2 |u|^{\gamma(x)} \right) (u - u_k)\, dx \\
&\leq k_{Hol} \|u - u_k\|_{L^{p^*(\cdot)}} \|c_1\|_{L^{\frac{p^*(\cdot)}{p^*(\cdot)-1}}} \\
&\quad + c_2\, k_{Hol} \|u - u_k\|_{L^{p^*(\cdot)}} \begin{cases} \|u\|_{L^{p^*(\cdot)}}{}^{p^*{}_m - 1}, & \|u\|_{L^{p^*(\cdot)}} < 1, \\ \|u\|_{L^{p^*(\cdot)}}{}^{p^*{}_S - 1}, & \|u\|_{L^{p^*(\cdot)}} < 1, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\int_K |f(x,\ u_k)| (u - u_k)\, dx \leq \int_K \left(c_1(x) + c_2 |u_k|^{\gamma(x)} \right) (u - u_k)\, dx \\
&\leq k_{Hol} \|u - u_k\|_{L^{p^*(\cdot)}} \|c_1\|_{L^{\frac{p^*(\cdot)}{p^*(\cdot)-1}}} \\
&\quad + c_2\, k_{Hol} \|u - u_k\|_{L^{p^*(\cdot)}} \begin{cases} \|u_k\|_{L^{p^*(\cdot)}}{}^{p^*{}_m - 1}, & \|u\|_{L^{p^*(\cdot)}} < 1, \\ \|u_k\|_{L^{p^*(\cdot)}}{}^{p^*{}_S - 1}, & \|u\|_{L^{p^*(\cdot)}} < 1. \end{cases}
\end{aligned}$$

Since the sequence $\{u_k\}$ is bounded in $W_{1,\,0}^{p(\cdot)}$-space, and

$$\langle \delta E(u) - \delta E(u_k),\ u - u_k \rangle \overset{k\to\infty}{\longrightarrow} 0,$$

we have $$\int_K \left(|\nabla u|^{p(x)-2} \nabla u - |\nabla u_k|^{p(x)-2} \nabla u_k \right) \nabla (u - u_k)\, dx \overset{k\to\infty}{\longrightarrow} 0\,,$$

so
$$\int_K |\nabla u - \nabla u_k|^{p(x)}\,dx$$
$$= \int_{\{x\in K\,:\,p(x)<2\}} |\nabla u - \nabla u_k|^{p(x)}\,dx + \int_{\{x\in K\,:\,p(x)\geq 2\}} |\nabla u - \nabla u_k|^{p(x)}\,dx$$
$$\leq \int_{\{x\in K\,:\,p(x)<2\}} \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla u_k|^{p(x)-2}\nabla u_k\right)(\nabla u - \nabla u_k)^{\frac{p(x)}{2}}$$
$$\left(|\nabla u|^{p(x)} - |\nabla u_k|^{p(x)}\right)^{\frac{2-p(x)}{2}}dx$$
$$+\int_{\{x\in K\,:\,p(x)\geq 2\}} \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla u_k|^{p(x)-2}\nabla u_k\right)\nabla(u-u_k)\,dx,$$
therefore, $\lim_k \|\nabla u - \nabla u_k\|_{L^{p(\cdot)}(K)} = 0$ for each compact set $K \subseteq \Omega$. Thus, there exists the subsequence $\left\{u_{k(j)}\right\} \subset \{u_k\}$, still denoted as $\{u_k\}$, such that the limit
$$\lim_k \nabla u_k(x) = \nabla u(x)$$
holds for almost all $x \in \Omega$. Applying the Vitali convergence theorem, we obtain
$$\lim_k \textstyle\int_\Omega |u_k|^{p(x)-2} u_k\phi dx = \int_\Omega |u|^{p(x)-2} u\phi dx,$$
$$\lim_k \textstyle\int_\Omega |u_k|^{s(x)-2} u_k\phi dx = \int_\Omega |u|^{s(x)-2} u\phi dx,$$
$$\lim_k \textstyle\int_\Omega |u_k|^{p^*(x)-2} u_k\phi dx = \int_\Omega |u|^{p^*(x)-2} u\phi dx,$$
$$\lim_k \textstyle\int_\Omega |\nabla u_k|^{p(x)-2} \nabla u_k\nabla\phi dx = \int_\Omega |\nabla u|^{p(x)-2} \nabla u\nabla\phi dx,$$
$$\lim_k \textstyle\int_\Omega f(x,\ u_k)\,\phi dx = \int_\Omega f(x,\ u_k)\,\phi dx$$
for all $\phi \in C_0^\infty(\Omega)$. Thus, we conclude
$$\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla\phi + \lambda |u|^{p(x)-2} u\phi\right)dx$$
$$-\int_\Omega a(x)\,|u|^{s(x)-2} u\phi\,dx - \int_\Omega f(x,\ u)\,\phi\,dx = 0$$
for all $\phi \in C_0^\infty(\Omega)$, and $u \in W_{1,0}^{p(\cdot)}(\Omega)$ is a weak solution to the variable Laplacian equation.

4. Conclusions

In this article, the Lions concentration-compactness principle has been employed to study the Dirichlet problem for an elliptic partial differential equation with variable exponent Laplace operator. We formulate the conditions under which the Dirichlet problem for an elliptic equation with variable exponent Laplace operator has at least one weak solution in appropriate Sobolev space. Since weak solutions coincide with critical points of the energy functional, we applied the mountain pass theorem to show the existence of a Palais-Smale sequence to which we employed the concentration-compactness principle and established the solvability of the Dirichlet problem. We assumed that the right part of the elliptic equation does not contain convective terms, to deal with the variable Laplace equation with convective terms, the problem requires future additional investigations.

References

[1] Z. Bashir, J. L. G. Guirao, A. Siddique, T. Saeed: *Generalized concentration-compactness principles for variable exponent Lebesgue spaces with asymptotic analysis of low energy extremals*, Mathematics 8/10 (2020), art.no. 1849.

[2] J. Y. Byeon, Z. Q. Wang, *Standing waves with a critical frequency for nonlinear Schrodinger equations II*, Calc. Var. Partial Diff. Equations 18 (2003) 207–219.

[3] W. Chen, V. Radulescu, B. Zhang: *Fractional Choquard-Kirchhoff problems with critical nonlinearity and Hardy potential*, Anal. Math. Phys. 11/3 (2021), art.no. 132, 25p.

[4] N. T. Chung: *Eigenvalue problems for fractional $p(x,y)$-Laplacian equations with indefinite weight*, Taiwanese J. Math. 23 (2019) 1153–1173.

[5] X. L. Fan, Q. H. Zhang: *Existence of solutions for $p(x)$-Laplacian Dirichlet problems*, Nonlinear Anal. 52 (2003) 1843–1852.

[6] X. L. Fan, Q. H. Zhang: *Existence of solutions for $p(x)$-Laplacian Dirichlet problem*, Nonlinear Anal. 52 (2003) 1843–1852.

[7] X. L. Fan, D. Zhao: *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Analysis Appl. 263/2 (2001) 424–446.

[8] Y. Q. Fu: *The principle of concentration compactness in $L^{p(x)}$ spaces and its application*, Nonlinear Anal. 71/5-6 (2009) 1876–1892.

[9] J. M. Kim, Y. H. Kim, J. Lee: *Existence and multiplicity of solutions for equations of $p(x)$-Laplace type in $\mathbb{R}^N$ without (AR)-condition*, Diff. Int. Equations 31 (2018) 435–464.

[10] P. Levy: *Théorie de l'Addition des Variables Aléatoires*, Gauthier-Villars, Paris (1954).

[11] P. L. Lions: *The concentration-compactness principle in the calculus of variations. The limit case. I*, Rev. Mat. Iberoamericana 1/2 (1985) 145–201.

[12] B. L. Zhang, N. Pan, J. Cao: *Weak solutions for parabolic equations with $p(x)$-growth*, Elect. J. Diff. Equations 2016 (2016), art.no. 209, 15p.

[13] C. Zhang, X. Zhang, *Renormalized solutions for the fractional $p(x)$-Laplacian equation with L^1 date*, Nonlinear Anal. Theory Methods Appl., Ser. A 190 (2020), art.no. 111610, 15 p.

[14] X. Zhang, W. Huang: *Adaptive neural network sliding mode control for nonlinear singular fractional order systems with mismatched uncertainties*, Fractal Fractional 4 (2020) 50p.