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The Generalized Problem of Bolza

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Dedicated to Ralph Tyrell Rockafellar on the occasion of his 90th birthday.

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The paper is a survey of available results concerning the generalized Bolza problem under fairly general assumptions on its component functions that can be extended-real-valued. Specifically, we consider the problem of existence of solutions, necessary conditions and characterization of the value function.

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1. Introduction

The paper is a survey of available results about the generalized Bolza problem:

$$\text{minimize} \quad J(x(\cdot)) = \ell(x(0), x(T)) + \int_0^T L(t, x(t), \dot{x}(t))dt$$

on the space $W^{1,1}$ of absolutely continuous $\mathbb{R}^n$-valued functions on $[0,T]$. The word "generalized" refers to the assumptions on the functions ℓ and L (defined respectively on $\mathbb{R}^n \times \mathbb{R}^n$ and $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n$) which are substantially weaker than in the classical theory. Basically, we shall assume both functions ℓ and L extended real valued with values in $(-\infty, \infty]$ and such that

(A1) ℓ is lower semicontinuous and bounded from below;

(A2) $L(t, \cdot, \cdot)$ is lower semicontinuous for all t and L is measurable with respect to the σ-algebra generated by products of Lebesgue measurable subsets of $[0,T]$ and Borel subsets of $\mathbb{R}^n \times \mathbb{R}^n$.

Under the assumptions the function $t \to L(t, x(t), y(t))$ is measurable for measurable $x(\cdot)$ and $y(\cdot)$. It seems that the term "generalized Bolza problem" appeared for the first time in Rockafellar's 1970 paper [31] and firmly entered the mathematical lexicon after Clarke's 1976 paper [6]. As to the problem itself, it was also considered (in a slightly less general form) a bit earlier by Ioffe and Tikhomirov [19] and Olech [28] mainly in connection with the problem of existence of solutions in optimal control problems. It was shown in the mentioned publications that problems of optimal control can be naturally reformulated as a problem of minimization of functionals

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similar to J. (Actually, the fact that optimal control problems can be reduced to variational problems of Bolza or Lagrange type was already recognized at the very early stage of development of the optimal control theory.) We can specifically refer to Olech's paper for a detailed description of connection between these two types of problems.

Here we touch upon three main topics: existence of solutions, necessary optimality conditions and Hamilton-Jacobi theory. The problem of existence was basically solved at the very early stage of developments in the works of Ioffe and Tikhomirov [19], Olech [28] and Rockafellar [32, 33]. Afterwards and up to now necessary optimality conditions remained the main subject of studies, see e.g. [6]–[10], [12], [16]–[18], [20], [22]–[25], [31, 34, 35, 36]. Still, as we shall see, there are some questions to be answered. Finally, extensions of the Hamilton-Jacobi theory to problems of optimal control was among the main subject of the concluding chapters of the 2000 book by Vinter [35] and the recent book by Bettiol and Vinter [3]. But in both cases it was done under the assumption that the sets of possible velocities of the system are bounded. The assumption played a key role in the discussion, and no adjustment of the corresponding arguments to problems of calculus of variations seem to be possible.

2. Existence of solutions

As we have mentioned in the introduction, conditions that guarantee the existence of solutions in the problem under fairly general assumptions were first found by Ioffe and Tikhomirov in [19], Olech in [28] and Rockafellar in [32, 33] (for problems with $L(t,x,y)$ convex in (x,y) in [32]). The conditions obtained by Ioffe-Tikhomirov and Rockafellar in [33] are equivalent, as we shall see a bit later (surprisingly this fact seems to have never been observed earlier) and imply the result of Olech [29] (as follows from [13]).

But we shall begin with an obvious remark that to prove existence of a solution in the problem we typically need to be sure that[1]

(a) ℓ is a lower semicontinuous function bounded from below and

(b) the function
$$I(x(\cdot)) = \int_0^T L(t,x(t),\dot{x}(t))dt$$
is lower semicontinuous in the weak $W^{1,1}$-topology and the level sets $\{x(\cdot) \in W^{1,1} : I(x(\cdot)) \le c < \infty\}$ are compact in this topology.

Verification of the first property, as a rule, does not create any problems. But analysis of the second property is far less trivial and attracted a lot of attention (see e.g. [2, 5, 21, 30]). It should be observed that in most of the studies of the lower semicontinuity property for $I(x(\cdot))$ a more general functional was considered, namely
$$K(x(\cdot),y(\cdot)) = \int_0^T L(t,x(t),y(t))dt$$

[1] We can though refer to [9] for a proof of existence based on somewhat different principles.

(without any connection between $x(\cdot)$ and $y(\cdot)$) and the problem was to find conditions that guarantee lower semicontinuity of the functional with respect to the uniform convergence of $x(\cdot)$ and weak convergence of $y(\cdot)$ in L^1. A necessary and sufficient condition for lower semicontinuity of K with respect to a wide range of convergences was found in [13]. The following is an immediate corollary of the main result of [13] for our case.

Theorem 2.1. *Assume that the function $L(t,x,\cdot)$ is convex for all t and x. Assume further that $|K(x(\cdot),y(\cdot))| < \infty$ for some $(x(\cdot),y(\cdot)) \in C \times L^1$. A necessary and sufficient condition for the functional $K(x(\cdot),y(\cdot))$ to be lower semicontinuous with respect to the uniform convergence of $x(\cdot)$ and weak convergence of $y(\cdot)$ in L^1 is that the sequence of functions $t \to L^-(t,x_k(t),y_k(t))$ is weakly precompact in L^1 whenever $x_k(\cdot)$ converge uniformly, $y_k(\cdot)$ weakly converge in L^1 and $K(x_k(\cdot),y_k(\cdot)) \le a < \infty$ for all k.*

Here $L^-(t,x,y) = \min\{0, L(t,x,y)\}$.

Actually, convexity of $L(t,x,\cdot)$ is also necessary for lower semicontinuity of the functional under some reasonably weak additional assumption. This follows from the relaxation theorem whose origins go back to Bogolyubov's 1930 result [4] (and closely connected with relaxation theorems in optimal control theory initiated by Warga in early 60s). To state the theorem we need the following definition.

Definition 2.2. Let $Q(t)$ be a set-valued mapping from $[0,T]$ into $\mathbb{R}^n$ with nonempty convex values, and let Q stand for the graph of $Q(\cdot)$. We say following [1] that L is *pseudo-Lipschitz on Q* if there are nonnegative functions $k(\cdot) \in L^1$ and $\beta > 0$ such that for any t, any $x, x' \in Q(t)$ and any $y \in R^n$ with $-\infty < L(t,x,y) < \infty$ there is a y' such that

$$|y - y'| \le (k(t) + \beta|y|)|x - x'|, \quad |L(t,x,y) - L(t,x',y')| \le (k(t) + \beta|y|)|x - x'|.$$

This is true, in particular, if $L(t,\cdot,y)$ is $(k(t)+\beta|y|)$-Lipschitz on $Q(t)$. If, moreover, there is an $x(\cdot)$ such that for any t the set $Q(t)$ contains the ε-neighborhood of $x(t)$ with the same positive ε, we say that L is pseudo-Lipschitz *near* $x(\cdot)$. □

It is an easy matter to see that the pseudo-Lipschitz property of L implies that the set-valued mapping $(t,x) \to F(t,x) = \operatorname{epi} L(t,x,\cdot)$ satisfies the property that

$$F(t,x) \cap NB \subset F(t,x') + (k(t) + \beta N)|x - x'|B, \quad \forall N > 0 \text{ and } x, x' \in Q(t)$$

The latter was introduced in [24] where mappings satisfying the property were called *integrably sub-Lipschitzean in the large at* (t,x).

In what follows we shall consider two families of functionals associated with J, namely

$$I_t(x(\cdot)) = \int_t^T L(s,x(s),\dot{x}(s))ds \quad \text{and} \quad J_t(x(\cdot)) = \ell(x(t),x(T)) + I_t(x(\cdot))).$$

Theorem 2.3. *Let $\hat{L}(t,x,\cdot)$ be the convexification*[2] *of $L(t,x,\cdot)$ and*

$$\hat{I}_t(x(\cdot)) = \int_t^T \hat{L}(t,x(s),\dot{x}(s))ds.$$

If L is pseudo-Lipschitz near some $x(\cdot) \in W^{1,1}$, then for any t there is a sequence of $x_m(\cdot) \in W^{1,1}[t,T]$, with $x_m(\cdot)$, weakly (in $W^{1,1}$) converging to $x(\cdot)$ and such that $\liminf_{m\to\infty} I_t(x_m(\cdot)) \le \hat{I}_t(x(\cdot))$.

Proof. In the proof we basically follow the lines of the proof of Theorem 4.1 in [14]. The theorem itself cannot be applied because of the extra condition ((A4) in [14]) which is actually not needed here.

Clearly, it is sufficient to prove the result only for $t = 0$. Given an $\varepsilon > 0$, we can find measurable $y_i(\cdot)$, and $\alpha_i(\cdot)$, $i = 1,\dots n+1$ such that for almost every t

$$\alpha_i(t) \ge 0,\ \sum \alpha_i(t) = 1,\ \sum \alpha_i(t)y_i(t) = \dot{x}(t),$$

and

$$\sum \alpha_i(t)L(t,x(t),y_i(t)) \le \hat{L}(t,x(t),\dot{x}(t)) + \varepsilon.$$

Given a small $\varepsilon > 0$, we can also find a set $\Delta \subset [0,T]$ whose measure is not smaller than $T - \varepsilon$ and such that the values of $|y_i(t)|$ and $L(t,x(t),y_i(t))$ are bounded by some ρ almost everywhere on Δ. With Δ chosen, we can modify $y_i(t)$ by setting them all equal to $\dot{x}(t)$ outside of Δ. Finally, there are sequences of functions $\alpha_{im}(\cdot)$ with values in $\{0,1\}$ satisfying $\sum_i \alpha_{im}(t) = 1$ a.e. and weak* converging in L^∞ to functions equal to $\alpha_i(t)$ if $t \in \Delta$ and $(n+1)^{-1}$ if $t \notin \Delta$.

Let $z_m(\cdot)$ be defined by

$$z_m(0) = x(0); \quad \dot{z}_m(t) = \begin{cases} \sum \alpha_{im}(t)y_i(t), & \text{if } t \in \Delta; \\ \dot{x}(t), & \text{otherwise.} \end{cases}$$

Then $z_m(\cdot)$ converges to $x(\cdot)$ in the weak topology of $W^{1,1}$, hence uniformly.

As L is pseudo-Lipschitz near $x(\cdot)$, the distance from $(y_i(t), L(t,x(t),y_i(t)))$ to the set $\operatorname{epi} L(t,z_m(t),\cdot)$ (as a function of t) goes to zero in the L^1-metric. Applying Theorem 3.1 of [14] (with $z_m(\cdot)$ playing the role of $x(\cdot)$ in the theorem), we conclude that there are $a_m(\cdot) \in L^1$ and $x_m(\cdot) \in W^{1,1}$ with $a_m(t) \ge L(t,x_m(t),\dot{x}_m(t))$ a.e. and such that $a_m(\cdot) - L(\cdot,x(\cdot),\dot{z}_m(\cdot)) \to 0$ and $x_m(\cdot) - z_m(\cdot) \to 0$ in L^1 and $W^{1,1}$ respectively. This completes the proof of the theorem. □

We state next two well known facts which are simple consequences of Theorem 2.3 for the cases we need to consider in connection with necessary optimality conditions and extensions of Hamilton-Jacobi theory to our problem. But first we have to introduce a few more definitions and to state one auxiliary result.

Let $\varphi(t,y)$ be a function on $[0,T] \times \mathbb{R}^n$ with values in $(-\infty,\infty]$ and not identically equal to ∞. Let further $\varphi^*(t,p)$ be the Young–Fenchel congugate of $\varphi(t,\cdot)$:

$$\varphi^*(t,p) = \sup_u(\langle p,u\rangle - \varphi(t,u)).$$

[2] By convexification of $f(\cdot)$ we mean the greatest lower semicontinuous convex function majorized by f, that is the second Young-Fenchel conjugate of f.

We assume that φ is measurable in t for any y and $\varphi^*(t,p)$ is measurable in t for every p. It is said that φ satisfies the *growth condition* if

$$\int_0^T \varphi^*(t,p)dt < \infty, \qquad \forall\ p \in \mathbb{R}^n.$$

We shall call φ *normal integrant* if the function $\varphi(t,\cdot)$ is convex and lower semi-continous for every t, and φ satisfies the growth condition. Here is the auxiliary proposition we need.

Proposition 2.4 ([19], § 9.1, Theorem 1)**.** *Let φ be a normal integrant on the product space $[0,T]\times\mathbb{R}^n$. Then for any $\rho\in\mathbb{R}$ the set*

$$\{x(\cdot)\in L^1 : \int_0^T \varphi(t,x(t))dt \le \rho\}$$

is either empty or weakly compact in L^1.

Actually, we shall need a slightly more precise version of the proposition.

Corollary 2.5. *Let again $\varphi(t,x)$ be a function with the same properties as in the proposition. Let further $\beta\ge 0$, $\rho\in\mathbb{R}$ and $r>0$. Then the set*

$$\{x(\cdot)\in W^{1,1}[0,T] : \int_0^T \big(\varphi(t,\dot{x}(t))) - \beta|x(t)|\big)dt \le \rho, \qquad |x(0)|\le r\}$$

is either empty or relatively compact in the weak topology of $W^{1,1}$.

Indeed, we have

$$\begin{aligned}\int_0^T (\varphi(t,\dot{x}(t)) - \beta|x(t)|)dt \ &\ge\ \int_0^T \Big(\varphi(t,\dot{x}(t)) - \beta(r + \int_0^t |\dot{x}(s)|ds)\Big)dt\\ &\ge\ \int_0^T \Big(\varphi(t,\dot{x}(t)) - \beta|\dot{x}(t)|\Big)dt - \beta r\end{aligned}$$

and the function $\psi(t,y)=\varphi(t,y)-\beta T|y|$ obviously satisfies the growth condition.

(Indeed, $$\psi^*(t,p) = \sup_{\|q\|=1}\varphi^*(t,p+\beta Tq)$$

and the ball of radius βT around p lies in the convex hull of $2n$ points $p\pm\beta T\sqrt{n}e_i$, where e_i are elements of an orthonormal basis in $\mathbb{R}^n$.)

The Young-Fenchel conjugate of $L(t,x,\cdot)$

$$H(t,x,p) = \sup_u(\langle p,u\rangle - L(t,x,u))$$

is called the *Hamiltonian* associated with L.

Let us return to J_t. We shall also adopt the following assumption in what follows:

(A3) there are a $\beta\ge 0$ and a function $\varphi(t,y)$ on $[0,T]\times\mathbb{R}^n$ measurable with respect to the sigma-algebra generated by products of Lebesgue measurable subsets of $[0,T]$ and Borel subsets of $\mathbb{R}^n$, satisfying the growth condition and such that $L(t,x,y)\ge\varphi(t,y)-\beta|x|$ for all (t,x,y).

The two theorems below give sufficient conditions for lower semicontinuity of J_t with respect to uniform convergence of $x(\cdot)$. Theorem 2.3 suggests that in order to prove this property we must assume that

(A4) $L(t, x, \cdot)$ is a convex function for all t and all x.

Theorem A ([19], §9.1, Theorem 3). *Assume in addition to* (A1)–(A4) *that there are* $\varepsilon > 0$ *and a normal integrant* φ *on* $[0, T] \times \mathbb{R}^n$ *such that*

$$L(t, x, y) \geq \varphi(t, y), \tag{1}$$

for all t *and* y *and, for every* t*, all* x ε*-close to* $x(t)$*. Then the functional* I_t *is lower semicontinuous in the topology of uniform convergence in the* ε*-neighborhood of* $x(\cdot)$*.*

In what follows it will be often convenient for us to say that L itself *satisfies the growth condition near* $x(\cdot)$ if the condition of Theorem A is satisfied.

Theorem B ([33]). *Suppose that there are an integrand* $\psi(t, p)$ *and* $\varepsilon > 0$ *such that for any* t *and* p *the inequality* $H(t, x, p) \leq \psi(t, p)$ *holds for all* x *in the* ε*-neighborhood of* $x(t)$ *and*

$$\int_0^T \psi(t, p) dt < \infty, \qquad \forall \;\; p. \tag{2}$$

Then the functional I_t *is lower semicontinuous in the topology of uniform convergence in the* ε*-neighborhood of* $x(\cdot)$*.*

We should note that, while Theorem A coincides with the corresponding result in [19], the statement of Theorem B is a bit different from the statement of the semicontinuity theorem in [33]. But it is an easy matter to pass from one of them to the other.

It is also easy to see that *the theorems are equivalent.* Indeed, if Theorem A holds, then

$$H(t, x, p) = \sup_y(\langle p, y\rangle - L(t, x, y)) \leq \sup_y(\langle p, y\rangle - \varphi(t, y)) \leq \varphi^*(t, p)$$

and (2) holds with $\psi(t, p) = \varphi^*(t, p)$ as φ satisfies the growth condition.

On the other hand, under the assumptions of Theorem B, we may assume that $\psi(t, \cdot)$ is convex. (Otherwise, we can replace it by $\psi^{**}(t, p)$. The inequality $H(t, x, p) \leq \psi^{**}(t, p)$ will hold as $H(t, x, \cdot)$ is convex and lower semicontinuous for all t, x.) We have furthemore

$$L(t, x, y) = \sup_p(\langle p, y\rangle - H(t, x, p)) \geq \sup_p(\langle p, y\rangle - \psi(t, p)) = \psi^*(t, y).$$

and ψ^* satisfies the growth condition as its conjugate coincides with ψ.

Thus, if the conditions of Theorem A are satisfied, then the assumption of Theorem B is also satisfied with $\psi(t, p) = \varphi^*(t, p)$, and vice versa, the conditions of Theorem A are satisfied with $\varphi(t, y) = \psi^*(t, y)$, provided the condition of Theorem B holds with the given $\psi(t, p)$ (where, of course, in both cases the stars relate to conjugates of functions of the second argument). Combining Theorem A with Proposition 2.4, we conclude with the following existence result.

Theorem 2.6. *Suppose that $J(x(\cdot)) < \infty$ for some arc $x(\cdot) \in W^{1,1}$. Assume further that there is a normal integrant φ such that (1) holds for all $(t,x,y) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n$. Then J attains its minimal value on $W^{1,1}$.*

Of course, a similar result can be stated with a reference to Theorem B instead of Theorem A.

3. Necessary conditions

As in the case of existence of solutions, necessary conditions for a given trajectory $\overline{x}(\cdot)$ to be a local minimum of J in the $W^{1,1}$-topology, were first obtained in early 70s by Rockafellar [31] for the case of convex ℓ and $L(t,\cdot,\cdot)$. Namely, it was shown (under certain conditions to be specified a bit later) that there exists an absolutely continuous $p(\cdot)$ satisfying the relations:

$$(\dot{p}(t), p(t)) \in \partial L(t,\cdot,\cdot)(\overline{x}(t), \dot{\overline{x}}(t)) \quad a.e.; \tag{3}$$

$$\dot{\overline{x}}(t) \in \partial H(t, \overline{x}(t), \cdot)(p(t)), \quad \dot{p}(t) \in \partial(-H)(t,\cdot,p(t))(\overline{x}(t)); \tag{4}$$

$$(p(0), -p(T)) \in \partial\ell(\overline{x}(0), \overline{x}(T)), \tag{5}$$

with ∂ being the subdifferential in the sense of convex analysis. Moreover, it was shown that, under the assumptions, the relations (3) and (4) are equivalent. The conditions on L (other than convexity of $L(t,\cdot,\cdot)$) were very general, actually, the same as were stated in the beginning of the paper: L may assume the value $+\infty$, the set-valued mapping $t \to \operatorname{epi} L(t,\cdot,\cdot)$ is measurable and for almost every t the function $L(t,\cdot,\cdot)$ is lower semicontinuous.

The assumption that $L(t,\cdot,\cdot)$ is convex may be restrictive in many applications. On the other hand, as follows from Theorem 2.3, convexity of $L(t,x,\cdot)$ may be a natural assumption. On the other hand, if $L(t,x,\cdot)$ is not convex, its convexification may result in a lost of some good properties needed for further analysis. So Clarke dropped this condition in [6, 7]. In [6] he proved (3),(5) *with ∂ being the generalized gradient*, introduced by him a year earlier, assuming in addition to the assumptions in the beginning of the paper that

(a) the function L satisfies the growth condition and is *epi-Lipschitz* at $\overline{x}(\cdot)$, that is there are a summable positive valued $k(\cdot)$ and an $\varepsilon > 0$ such that for almost every t the Hausdorff distance between $\operatorname{epi} L(t,x_1,\cdot)$ and $\operatorname{epi} L(t,x_2,\cdot)$ is not greater that $k(t)|x_1 - x_2|$ if both x_1 and x_2 belong to the ε-ball around $\overline{x}(t)$, and

(b) the problem is *calm* in the sense that one of the functions

$$\Phi^0(s) = \inf\{\ell(x(0)+s, x(1)) + \int_0^T L(t,x(t),\dot{x}(t))dt : \ |x(t) - \overline{x}(t)| < \varepsilon \ \text{a.e.}\},$$

$$\Phi^1(s) = \inf\{\ell(x(0)+s, x(1)) + \int_0^T L(t,x(t),\dot{x}(t))dt : \ |x(t) - \overline{x}(t)| < \varepsilon \ \text{a.e.}\}$$

satisfies
$$\liminf_{s\to+0} \frac{\Phi^i(s) - \Phi^i(0)}{s} > -\infty.$$

In [7] the Hamiltonian condition: there is a $q(\cdot)$ such that

$$(-\dot{q}(t), \dot{x}(t)) \in \partial H(t, \cdot, \cdot)(\overline{x}(t), q(t)) \text{ a.e.} \tag{6}$$

was added to (3),(5) under the additional assumption that H satisfies the condition of Theorem B. However, no connection was established in the paper between $q(\cdot)$ in (6) and the $p(\cdot)$ that appears in (3)–(5).

It is also clear that the calmness condition is satisfied if ℓ is Lipschitz as a function of one of its arguments near $(\overline{x}(0), \overline{x}(T))$. But this does not happen in typical situations with constraints on possible values of $x(\cdot)$ at the ends of the interval, e.g. when these values are fixed. As was shown by Clarke in [8], the absence of calmness may lead to the appearance of some degenerate conditions instead of (3)–(5) (cf. e.g. Theorem 3.1 below). As far as the first condition (a) is concerned, it should be said that the Lipschitz asssumption for a set-valued mapping is fairly natural if the mapping has bounded values but it is rather restrictive for maps with unbounded values.

The other problem associated with Clarke's result is that the generalized gradient can be a fairly big set which could make the results not very efficient for application. Therefore the subsequent studies were to a large extent concentrated on the replacement of the calmness condition by something easier to verify and the generalized gradient by something smaller.

The first step was probably made by Frankowska in 1985 in [12] where she proved a theorem in which the relations (3) and (5) were obtained with ∂ being some *asymptotic gradient* introduced in the paper. As the generalized gradient, this was a convex set which, however, can be smaller (and never greater) then the generalized gradient. There was also no need in [12] in a condition similar to calmness as ℓ was assumed Lipschitz near $(\overline{x}(0), \overline{x}(T))$.

The next step was made by Mordukhovich in [26]. Namely, it was shown in [26] that a more precise (than (3)) relation

$$\dot{p}(t) \in \operatorname{conv}\{v : \ (v, p(t)) \in \partial L(t, \cdot, \cdot)(\overline{x}(t), \dot{\overline{x}}(t))\} \tag{7}$$

holds with some $p(\cdot)$ still satisfying (5) and ∂ being the *limiting subdifferential* in both cases. But the conditions on ℓ and L were in [26] stronger that in [6]: it was assumed that both ℓ and L are continuous functions of all its arguments and $L(t, \cdot, \cdot)$ is Lipschitz in a neighborhood of $(\overline{x}(t), \dot{\overline{x}}(t))$ for all t with Lipschitz constant not depending on t.

It is interesting to note that no Weierstrass-type condition was stated in any of the mentioned papers, although the function $L(t, x, \cdot)$ was not assumed convex in either of them. In the context of Clarke's result [6] it was added by Jourani in [20]. But for the first time it probably appeared in the 1996 paper by Ioffe and Rockafellar [18] in which the conditions (7) and (5) (with the limiting subdifferential) were established under weaker assumptions on ℓ and L than in [26], namely that ℓ is lower semicontinuous, L is $\mathcal{L} \times \mathcal{B}$-measurable, $L(t, \cdot, \cdot)$ is lower semicontinuous, although still everywhere finite, for almost all t and $L(t, \cdot, y)$ satisfies the following Lipschitz-type assumption: for any $N > 0$ there are $\varepsilon > 0$ and summable positive valued $c(t)$ and $k(t)$ such that the inequalities

$$|L(t,x,y) - L(t,x',y)| \le k(t)|x - x'| \quad \text{and} \quad |L(t,x,y)| \le c(t)$$

if $|y - \dot{\overline{x}}(t)| \le N$, $|x - \overline{x}(t)| \le \varepsilon$, $|x' - \overline{x}(t)| \le \varepsilon$. It was also mentioned in [18] that conditions (7) and (5) are necessary for a weak minimum although, to the best of my knowledge, a direct proof of this fact was given in [17] (and a bit earlier in [16] in case when ℓ is Lipschitz in a neighborhood of $(\overline{x}(0), \overline{x}(T))$).

Around the same time, Loewen and Rockafellar in a series of papers [23]–[25] considered the problem with L being an extended real valued function, convex in the last argument. In [23] they showed that there is a $p(\cdot)$ satisfying both (3) and (6) (as $q(\cdot)$ in the latter) – a result that falls short of verification of whether the relations are actually equivalent. The question was positively answered in [25] even for more precise (than (3) and (6)) relations, namely, (7) and

$$\dot{p}(t) \in \operatorname{conv}\{w : \ (-w, \dot{\overline{x}}(t)) \in \partial H(t,\cdot,\cdot)(\overline{x}(t), p(t))\}. \tag{8}$$

Specifically, the following result was proved in [25] (along with the fact that the relations (7) and (8) are equivalent):

Theorem 3.1. *Assume that for all t the function $L(t,\cdot,\cdot)$ is lower semicontinuous with values in $(-\infty,\infty]$ and $L(t,x,\cdot)$ is a convex function for all t, x in a neighborhhod of the graph of $\overline{x}(\cdot)$ in $[0,T] \times \mathbb{R}^n$. Suppose further that the problem is regular at $\overline{x}(\cdot)$ in the sense that $p(t) \equiv 0$ is the only solution of the relations*

$$\dot{p}(t) \in \operatorname{conv}\{v : \ (v, p(t)) \in \partial^\infty L(t,\cdot,\cdot)(\overline{x}(t), \dot{\overline{x}}(t))\}, \tag{9}$$

$$(p(0), -p(T)) \in \partial^\infty \ell(\overline{x}(0), \overline{x}(T)). \tag{10}$$

If under these assumptions $\overline{x}(\cdot)$ is a local minimum of J in the topology of uniform convergence, then conditions (7), (8) *and* (5) *hold with some $p(\cdot)$, provided there are a summable function $\delta(t)$, an $\varepsilon > 0$ and a $\lambda \ge 0$ such that for almost every t and all (x,v) satisfying $|x - \overline{x}(t)| < \varepsilon$, $|v - \dot{\overline{x}}(t)| < \delta(t)$*

(a) $|w| \le \lambda\delta(t)(1 + |p|)$ *whenever* $(w,p) \in \partial L(t,\cdot,\cdot)(x,v)$ *and* $|L(t,x,v) - L(t,\overline{x}(t),\dot{\overline{x}}(t))| < \delta(t)$;

(b) *if* $L(t,x,v) < \infty$ *and* $x_k \to x$, *then there are* $v_k \to v$ *such that* $L(t,x_k,v_k) \to L(t,x,v)$.

An important point emphasized in [25] is that under the assumptions for almost every t the set-valued mapping $E(x) = \operatorname{epi} L(t,x,\cdot)$ satisfies the Aubin pseudo-Lipschitz property with summable constant $k(t)$ near $(\overline{x}, (\overline{y}, \overline{v}))$, where $\overline{y} = \dot{\overline{x}}(t)$ and $\overline{v} = L(t, \overline{x}(t), \dot{\overline{x}}(t))$. (Recall that, according to Aubin [1], a set-valued mapping $F(x)$ is pseudo-Lipshitz near $(\overline{x}, \overline{y}) \in \operatorname{Graph} F$ with constant K if there are $\varepsilon > 0$ such that for all x, u in a neighborhood of $\overline{x}$

$$F(x) \cap (\overline{y} + \varepsilon B) \subset F(u) + K|x - u|B.)$$

A similar conclusion was recently obtained in [17] under different set of assumptions. Below we state a simplified version of this result. The assumptions on L in this theorem are stronger in certain respects but on the other hand $L(t, x, \cdot)$ is not assumed convex and the regularity property is substantially weaker: there is no nonzero $b \in \mathbb{R}^n$ such that $(0, b, -b)$ belongs to the limiting normal cone to epi ℓ at $\big(\ell(\overline{x}(0), \overline{x}(T)), \overline{x}(0), \overline{x}(T)\big)$ and for almost every t the vector $(0, 0, b)$ belongs to the limiting normal cone to epi $L(t, \cdot, \cdot)$. Specifically, the following theorem was proved in [17].

Theorem 3.2. *Assume as before that for every t the function $L(t, \cdot, \cdot)$ is lower semicontinuous with values in $(\infty, \infty]$. Assume also that there are positive valued $\varepsilon_0(t)$ and summable $k_0(t)$ such that the function*

$$\sup\{|L(t, \overline{x}(t), y)| : \ |y - \dot{\overline{x}}(t)| \leq \varepsilon_0(t)\}$$

is summable and the function $L(t, \cdot, y)$ is $k_0(t)$-Lipschitz in a neighborhood of $\overline{x}(t)$ (depending on t) whenever $|y - \dot{\overline{x}}(t)| \leq \varepsilon_0(t)$. Suppose further that the regularity property introduced above holds at $\overline{x}(\cdot)$.

Let finally $Q(t)$ be a measurable set-valued mapping into $\mathbb{R}^n$ such that for any $u \in Q(t)$ the function $L(t, \cdot, u)$ is continuous in a neighborhood of $\overline{x}(t)$ (depending on t and u).

If under these conditions $\overline{x}(\cdot)$ is a local minimum in the $W^{1,1}$-topology, then there is an absolutely continuous $\mathbb{R}^n$-valued function $p(\cdot)$ satisfying (7), (5) and such that

$$L(t, \overline{x}(t), u) - L(t, \overline{x}(t), \dot{\overline{x}}(t)) - \langle p(t), u - \dot{\overline{x}}(t)\rangle \geq 0, \qquad \forall\, u \in Q(t). \tag{11}$$

The two theorems seem to be independent. An interesting question is whether it is possible to find a more general statement that would contain the theorems as consequences.

4. The value function

In this section we shall consider a family of simplified versions of the problem, namely, the problems of minimization of pure integral functional

$$I_t(x(\cdot)) = \int_t^T L(s, x(s), \dot{x}(s))ds$$

on the set $X(t, x)$ of absolutely continuous functions $x(\cdot)$ on $[t, T]$ such that $x(t) = x$. Let $V(t, x)$ be the lower bound of the functional in the problem. If $X(t, x) = \emptyset$, we set $V(t, x) = \infty$. This function will be the subject of our main interest in this section. The concluding result describes some properties of the value function. The question whether these properties do characterize the value function remains open.

Theorem 4.1. *Assume, as in Theorem* 2.6*, that there is a normal integrant φ such that* (1) *holds for all $(t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$. Then the value function $V(t, x)$ is lower semicontinuous.*

Proof. To begin with we recall that by Corollary 2.5 *for any $\beta > 0$, $\rho \in \mathbb{R}$ and $r \geq 0$ the set*

$$\{x(\cdot) \in W^{1,1} : \int_0^T (\varphi(t, \dot{x}(t)) - \beta|x(t)|)dt \leq \rho, \quad |x(0)| \leq r\}$$

is either empty or relatively compact in the weak topology of $W^{1,1}$.

Now let $(t_i, x_i) \to (t, x)$. We have to show that $\liminf_{i\to\infty} V(t_i, x_i) \geq V(t, x)$. If $V(t_i, x_i) \to \infty$, this is obviously true. So we may assume that $V(t_i, x_i) \leq C < \infty$ for all i. By Theorem 2.6 for any i there is a minimizer of I_{t_i} on $X(t_i, x_i)$, that is such an $x_i(\cdot)$ that $x_i(t_i) = x_i$ and $I_{t_i}(x_i(\cdot)) = V(t_i, x_i)$. We can extend every $x_i(\cdot)$ to the entire segment $[0, T]$ by setting $x_i(t) = x_i(t_i)$ for $t \in [0, t_i]$. As $\varphi(t, 0)$ is a summable function according to our assumption, integrals of $\varphi(t, x_i(t))$ are uniformly bounded. Hence by Corollary 2.5 the sequence $(x_i(\cdot))$ is precompact in the weak topology of $W^{1,1}$ and we may assume that it weakly converges to some $x(\cdot)$. We can also be sure that there is an $r > 0$ such that for every i the enequality $|x_i(t)| \leq r < \infty$ holds for all t.

Suppose first that $t_i \leq t$ for all i. Then

$$\liminf J_{t_i}(x_i(\cdot)) \geq \liminf \int_{t_i}^{t} L(s, x_i(s), \dot{x}_i(s))ds + \liminf J_t(x_i(\cdot)).$$

The first limit must be nonnegative by (3) and uniform boundedness of $x_i(\cdot)$. Theorem 2 now implies that $\liminf J_{t_i}(x_i(\cdot)) \geq J_t(x(\cdot))$.

Let now $t_i > t$ for all i. By Theorem 2 for any $\tau > t$

$$\liminf I_\tau(x_i(\cdot)) = \liminf \int_\tau^T L(s, x_i(s), \dot{x}_i(s))ds \geq \int_\tau^T L(s, x(s), \dot{x}(s))ds = I_\tau(x(\cdot)).$$

It remains to apply Fatou's lemma (keeping in mind the pseudo-Lipschitz property of L and boundedness of $x(\cdot)$). $\square$

We need one more concept to state the next result. Let $f(x)$ be a function on a Banach space X which is finite at a certain x. The *Dini-Hadamard subdifferential* $\partial^- f(x)$ of f at x is the collection of $x^* \in X^*$ such that

$$f(x + h) - f(x) \geq \langle x^*, h\rangle + o(\|h\|)$$

for all h of a neighbvorhood of zero (that may depend on x^*). If f is a convex function, then $\partial^- f(x)$ coincides with the subdifferential of f at x in the sense of convex analysis.

Recall next that, as immediately follows from (A3)

$$H(t, x, p) \leq \varphi^*(t, p) + \beta|x|.$$

Note also that in our case $H(t, x, \cdot)$ is everywhere finite (or to be more precise, for almost every t it is everywhere finite for every x). This is immediate from (A3). Here is the main rsult of the section.

Theorem 4.2. *Assume* (A1) *and* (A2), *and let* $(\alpha, p) \in \partial^- V(t, x)$.

(a) *If* $L(\cdot, \cdot, u)$ *is upper semicontinuous at* (t, x) *from the left in the sense that the upper limit of* $L(s, w, u)$ *when* $(s, w) \to (t, x)$ *with* $s < t$ *is not greater that* $L(t, x, u)$, *then*
$$-\alpha + H(t, x, -p) \leq 0.$$

(b) *If* $H(\cdot, \cdot, p)$ *is upper semicontinuous at* (t, x) *from the right, then*
$$-\alpha + H(t, x, -p) \geq 0.$$

Proof. (a) We have for any $t > 0$, any $\lambda > 0$ and any u

$$V(t-\lambda, x-\lambda u) \leq V(t,x) + \int_{t-\lambda}^{t} L(s, x(s), \dot{x}(s))ds$$

whenever $x(t-\lambda) = x - \lambda u$ and $x(t) = x$. In particular

$$V(t-\lambda, x-\lambda u) \leq V(t,x) + \int_{t-\lambda}^{t} L(s, su + x - tu, u)ds.$$

Hence for any $\lambda > 0$

$$-\alpha + \langle -p, u\rangle \leq \frac{1}{\lambda}\int_{t-\lambda}^{t} L(s, su + x - tu, u)ds + r(\lambda),$$

where $r(\lambda) \to 0$ when $\lambda \to 0$. Thus, $-\alpha \leq L(t,x,u) - \langle -p, u\rangle$ for any u by the assumed upper semicontinuity of $L(\cdot,\cdot,u)$, so that

$$-\alpha \leq \inf_u (L(t,x,u) - \langle -p, u\rangle) = -H(t,x,-p).$$

(b) Let now $0 \leq t < T$. Take an $\overline{x}(\cdot)$ defined on $[t,T]$ and such that $\overline{x}(t) = x$ and $V(t,x) = J_t(\overline{x}(\cdot))$. Set $\overline{u}(s) = \overline{x}(t+s) - \overline{x}(t)$. We have for any sufficiently small positive λ

$$V(t,x) = V(t+\lambda, x + \overline{u}(\lambda)) + \int_0^{\lambda} L(t+s, x+\overline{u}(s), \dot{\overline{u}}(s))ds$$

Thus, if $(\alpha, p) \in \partial^- V(t,x)$, then

$$0 \geq \lambda(\alpha + \langle p, \overline{u}(\lambda)\rangle) + \int_0^{\lambda} L(t+s, x+\overline{u}(s), \dot{\overline{u}}(s))ds + o(\lambda)$$

It follows that for any sufficiently small $\lambda > 0$

$$0 \leq -\alpha + \frac{1}{\lambda}\int_t^{t+\lambda} \sup_u [\langle -p, u\rangle - L(s, \overline{x}(s), u)]ds \leq -\alpha + \frac{1}{\lambda}\int_t^{t+\lambda} H(s, \overline{x}(s), -p)ds.$$

and the result follows. □

It is not yet clear whether the converse implication is valid, that is whether the value function is fully defined by the conditions (a), (b) along with the equality $V(T,x) \equiv 0$.

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