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Subgradient Evolution of Value Functions of Discrete-Time Convex Bolza Problems

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We investigate how the subgradients of the value function of a discrete-time convex Bolza problem evolve over time. In particular, we develop a discrete-time version of the method of characteristics introduced by Rockafellar and Wolenski in the 2000s, by showing that the time-evolution of the subgradients of the value functions can be associated with trajectories of a discrete-time Hamiltonian system. To do so, we first prove that the value function has a dual counterpart, which corresponds to the conjugate of the value function of a suitable dual problem. We finally discuss about the qualification conditions required for our results, showing in particular that classical problems, such as the Linear Quadratic Regulator, satisfy these hypotheses.

Keywords: Optimal control, Discrete-time systems, Convex Bolza problems, Linear Quadratic Regulator, Mixed constraints

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1. Introduction

This paper is concerned with Bolza problems associated with a discrete-time dynamical system, that is, we aim at studying the optimization problem

$$\text{Minimize} \sum_{t=\tau+1}^{T} L_t(x_{t-1}, \Delta x_t) + g(x_T)$$

over all $\{x_t\}_{t=\tau}^{T} \subset \mathbb{R}^n$ such that $x_\tau = \xi$. The variable x_t represents the state of the system at time t and Δx_t quantifies the variation of the state between two consecutive time instants, that is, $\Delta x_t = x_t - x_{t-1}$. Here $\xi \in \mathbb{R}^n$ is the initial state, which for our purposes is considered as a variable. The initial time τ and the final horizon T are integers, assumed to be fixed all along this manuscript and to be such

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that $\tau < T$. In this work, we focus in particular on *convex* Bolza problems, which means that the Lagrangians $L_{\tau+1}, \ldots, L_T$ as well as the terminal cost g are convex functions convex; see Assumption (A) below.

Our goal is to study $\vartheta_\tau(\xi)$, the so-called *value function* of the problem, that is, the optimal value of the problem as a function of the initial state ξ. The convexity of $L_{\tau+1}, \ldots, L_T$ and g implies the convexity of ϑ_τ, which paves the way to study convex Bolza problems from a duality viewpoint. In particular, in this paper we are interested in providing an explicit expression for the conjugate of ϑ_τ.

We are also interested in understanding the evolution over time of the subgradients of ϑ_τ, similarly as done in [12, 22] for continuous-time systems. Let us point out that a global *method of characteristics* for (continuous-time) convex Bolza problems was introduced by Rockafellar and Wolenski in [22, Theorem 2.4]. Their method of characteristics describes the time-evolution of the subgradients of the value functions through following trajectories of a Hamiltonian system. The paper [22] covers the case where the Lagrangian is coercive and no state constraints are allowed. We refer to [12] for an extension to problems with state constraints (which is a major issue in continuous-time settings) and with possibly non-coercive Lagrangian. For discrete-time optimal control problems, a similar result can be found in [4, Proposition 5.13], although in that case the value function is assumed to be smooth, the minimizer to be uniquely determined and all the data of the problem is assumed to be smooth as well. Neither state nor control constraints are allowed.

One of the main advantage when working under convexity is that, a convex Bolza problem can be paired with a dual problem, which turns out also to be a convex Bolza problem. A sample result that can be obtained from the theory is that coextremals are minimizers of the dual problem; see for example [21]. Note that discrete-time convex Bolza problems can be viewed as finite-dimensional convex optimization problems. In this respect, the duality theory presented in Section 2 stems from traditional duality methods applied to this setting. To the best of our knowledge, no duality approach has been reported in the literature for studying the value function of discrete-time convex Bolza problems or the evolution of its subgradients. We refer to [3, Chapter 1] and [4, Chapter 5] for a general account on the value function of discrete-time optimal control problems, and in particular to its relation with the so-called *dynamic programming equation*. We refer also to [25] for a general study on the subgradients of value functions of convex Bolza problems.

Notice that convex Bolza problems with non-finite Lagrangians allow us to recover classical control problems such as the Linear Quadratic Regulator or the linear-convex problems with mixed constraints. This fact has been discussed in [13, Example 2] and [2, 10, 11, 19]) for continuous-time systems, but the arguments can be readily adapted to a discrete-time setting. Let us also mention that convex Bolza problems have been intensively studied in the context of continuous-time systems. Indeed, in the 1970s, in a series of papers [13, 15, 16, 17, 18], Rockafellar set the foundations of a duality theory for this type of problems; [21] is a transposition of the theory to discrete-time problems. The value function, in the case of no pathwise constraints, was studied via the duality theory in [5, 6, 7, 8, 9, 11, 12, 20, 22, 23].

The main contributions of this paper are the following. On the one hand, we transpose the method of characteristics to discrete-time systems, by showing that the time-evolution of the subgradients of the value functions can be associated with trajectories of a discrete-time Hamiltonian system. To do so, we first need to prove that the value function turns out to be the conjugate of the value function of the dual problem. One the other hand, these results require some qualification condition on the primal as well on the dual problem. Qualification conditions over the dual problems may be rather cumbersome to verify, thus we also show that, under appropriate assumptions on the data of the primal problem, the qualification conditions over the dual problems are satisfied immediately.

1.1. Notation and essentials

Throughout this paper we use the following notation: $|\cdot|$ is the Euclidean norm and $a \cdot b$ stands for the Euclidean inner product of $a, b \in \mathbb{R}^n$. We will frequently use the identification $\mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$, with the corresponding inner product. We set $[\![p : q]\!] := \{p, p+1, \ldots, q\}$, that is, it is the collection of all integers between p and q (inclusive), assuming always that $p \leq q$; if $p = q$, we set $[\![p : q]\!] = \{p\}$.

Suppose $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is a function. The *effective domain* of φ is the set

$$\operatorname{dom}(\varphi) := \{x \in \mathbb{R}^n \mid \varphi(x) < +\infty\}.$$

The function φ is said to be *proper* if $\operatorname{dom}(\varphi) \neq \emptyset$ and $\varphi(x) > -\infty$ for all $x \in \mathbb{R}^n$; *convex* if $\operatorname{epi}(\varphi) := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid \varphi(x) \leq r\}$ is a convex set, and *lower semicontinuous (l.s.c. for short)* if $\operatorname{epi}(\varphi)$ is a closed set.

We also use standard nomenclature of convex analysis: $\operatorname{ri}(S)$ stands for the *relative interior* of a convex set $S \subset \mathbb{R}^n$. The *indicator* and *support functions* of a set $S \subset \mathbb{R}^n$ are denoted by δ_S and σ_S, respectively. The set S_∞ stands for the *recession cone* of a convex set $S \subset \mathbb{R}^n$. The *(convex) normal cone* to S at $x \in S$ is the set

$$\mathcal{N}_S(x) := \{z \in \mathbb{R}^n \mid z \cdot (s - x) \leq 0, \ \forall s \in S\}.$$

The *conjugate* of $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the mapping $\varphi^* : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ defined via the formula

$$\varphi^*(y) := \sup\{x \cdot y - \varphi(x) \mid x \in \mathbb{R}^n\}, \qquad \forall y \in \mathbb{R}^n,$$

and its *subdifferential* at $x \in \operatorname{dom}(\varphi)$ is the set

$$\partial\varphi(x) := \{y \in \mathbb{R}^n \mid \varphi(x) + y \cdot (z - x) \leq \varphi(z), \ \forall z \in \mathbb{R}^n\}.$$

These mathematical objects are related via the Fenchel-Young equality:

$$y \in \partial\varphi(x) \quad \Longleftrightarrow \quad \varphi(x) + \varphi^*(y) = x \cdot y. \tag{1}$$

A function $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is called *concave-convex* if $h_y(\cdot) = -h(\cdot, y)$ and $h_x(\cdot) = h(x, \cdot)$ are convex functions. The (concave-convex) subdifferential of h is the set

$$\partial h(x, y) := [-\partial h_y(x)] \times \partial h_x(y), \qquad \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{2}$$

2. Fully convex Bolza problems

The optimization model we analyze corresponds to a Bolza problem associated with a running costs $L_{\tau+1}, \ldots, L_T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ (the Lagrangians) and a given terminal cost $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. In particular, we are concerned with the value function of this discrete-time Bolza problem, which is the mapping that to any given $\xi \in \mathbb{R}^n$ and $\mathbf{s} \in [\![\tau : T-1]\!]$, assigns the optimal value of the problem

$$\vartheta_{\mathbf{s}}(\xi) := \inf \left\{ \sum_{t=\mathbf{s}+1}^{T} L_t(x_{t-1}, \Delta x_t) + g(x_T) \mid \{x_t\}_{t=\mathbf{s}}^{T} \subset \mathbb{R}^n,\ x_{\mathbf{s}} = \xi \right\}, \tag{P}$$

where $\Delta x_t = x_t - x_{t-1}$. Unless otherwise stated, we will always assume that the following assumption is in force:

$$L_{\tau+1}, \dots, L_T \text{ and } g \text{ are proper, convex and l.s.c. functions.} \tag{A}$$

As mentioned in the Introduction, one of the key points provided by the convexity is that the value function turns out to be convex with respect to the state variable.

Proposition 2.1. *The value function $\vartheta_{\mathbf{s}}$ is convex for any $\mathbf{s} \in [\![\tau : T-1]\!]$ fixed.*

Proof. Take $\mathbf{s} \in [\![\tau : T-1]\!]$. It is enough to see that the function

$$\mathcal{L}_{\mathbf{s}}\left(\{x_t\}_{t=\mathbf{s}}^{T}, \xi\right) := \begin{cases} \displaystyle\sum_{t=\mathbf{s}+1}^{T} L_t(x_{t-1}, \Delta x_t) + g(x_T), & \text{if } x_{\mathbf{s}} = \xi, \\ +\infty & \text{otherwise,} \end{cases}$$

is convex and that $\vartheta_{\mathbf{s}}(\xi) = \inf \left\{ \mathcal{L}_{\mathbf{s}}\left(\{x_t\}_{t=\tau}^{T}, \xi\right) \mid \{x_t\}_{t=\mathbf{s}}^{T} \subset \mathbb{R}^n \right\}$. □

2.1. Dual Bolza problem and weak duality

The convex setting we are studying in this paper prompts for a duality theory which can be obtained via the standard approach with perturbation functions (cf. [21]). Inspired by the study done in [22] for continuous-time convex Bolza problems, we consider a dual problem to (P), whose value function (the *dual value function* in the sequel), is given, for any $\eta \in \mathbb{R}^n$ and $\mathbf{s} \in [\![\tau : T-1]\!]$, by the expression

$$\omega_{\mathbf{s}}(\eta) := \inf \left\{ \sum_{t=\mathbf{s}+1}^{T} K_t(p_t, \Delta p_t) + f(p_T) \mid \{p_t\}_{t=\mathbf{s}}^{T} \subset \mathbb{R}^n,\ p_{\mathbf{s}} = -\eta \right\}. \tag{D}$$

Here $\Delta p_t = p_t - p_{t-1}$, $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the *dual terminal cost* given by

$$f(b) := g^*(-b), \qquad \forall b \in \mathbb{R}^n,$$

and $K_{\tau+1}, \dots, K_T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ are the *dual Lagrangians* given by

$$K_t(p, w) := \sup_{x, v \in \mathbb{R}^n} \{x \cdot w + v \cdot p - L_t(x, v)\}, \qquad \forall p, w \in \mathbb{R}^n,\ \forall t \in [\![\tau+1 : T]\!].$$

In the sequel we will refer to $\vartheta_{\mathbf{s}}$ given by (P) as the *primal value function*.

Remark 2.2. Although problem (D) has not been written exactly as the primal problem (P), it is not difficult to see that it can be brought to that form if we consider the Lagrangian $\tilde{L}_t(p, w) = K_t(p + w, w)$. In this case, it is also evident that, since $\tilde{L}_t^*(v, x) = K_t^*(v, x - v) = L_t(x - v, v)$, the dual problem to (D) (defined in the way proposed above) is exactly (P).

Notice that, as for the continuous-time case, we can extend the definition of both value functions up to time $\mathbf{s} = T$ as follows:

$$\vartheta_T(\xi) = g(\xi) \quad \text{and} \quad \omega_T(\eta) = f(-\eta), \qquad \forall \xi, \eta \in \mathbb{R}^n.$$

From the definition of the conjugate, it follows that

$$\vartheta_T(\xi) + \omega_T(\eta) \geq \xi \cdot \eta, \qquad \forall \xi, \eta \in \mathbb{R}^n.$$

Notice that in particular, we have that $\omega_T \geq \vartheta_T^*$ and $\vartheta_T \geq \omega_T^*$. This weak duality relation, becomes strong (with equality) whenever the transversality condition

$$-\eta \in \partial g(\xi)$$

holds; this is a straightforward consequence of (1).

The next proposition shows that this weak duality relation propagates backward in time. Latter on, we will show that the weak duality stated below becomes strong under certain qualification conditions.

Proposition 2.3. *For any* $\mathbf{s} \in [\![\tau : T-1]\!]$*, we have that*

$$\vartheta_{\mathbf{s}}(\xi) + \omega_{\mathbf{s}}(\eta) \geq \xi \cdot \eta, \qquad \forall \xi, \eta \in \mathbb{R}^n.$$

Moreover, we also have that $\omega_{\mathbf{s}} \geq \vartheta_{\mathbf{s}}^*$ *and* $\vartheta_{\mathbf{s}} \geq \omega_{\mathbf{s}}^*$.

Proof. Take $\mathbf{s} \in [\![\tau : T-1]\!]$. It is enough to check the inequality just for the case $\vartheta_{\mathbf{s}}(\xi), \omega_{\mathbf{s}}(\eta) < +\infty$, otherwise the conclusion is direct (considering the convention $+\infty \pm \infty = +\infty$). Note that we are not assuming a priori that $\vartheta_{\mathbf{s}}(\xi), \omega_{\mathbf{s}}(\eta) > -\infty$.

Let $\{x_t\}_{t=\mathbf{s}}^T \subset \mathbb{R}^n$ and $\{p_t\}_{t=\mathbf{s}}^T \subset \mathbb{R}^n$ be feasible trajectories for the optimization problems (P) and (D), respectively. By the definition of the conjugate we have that

$$L_t(x_{t-1}, \Delta x_t) + K_t(p_t, \Delta p_t) \geq x_{t-1} \cdot \Delta p_t + \Delta x_t \cdot p_t = x_t \cdot p_t - x_{t-1} \cdot p_{t-1}.$$

Thus, $\displaystyle\sum_{t=\mathbf{s}+1}^{T} L_t(x_{t-1}, \Delta x_t) + \sum_{t=\mathbf{s}+1}^{T} K_t(p_t, \Delta p_t) \geq x_T \cdot p_T - x_{\mathbf{s}} \cdot p_{\mathbf{s}} = x_T \cdot p_T + \xi \cdot \eta.$

Since $f(b) = g^*(-b)$, the definition of the conjugate of g leads to

$$g(x_T) + f(p_T) \geq -x_T \cdot p_T.$$

It follows that

$$\sum_{t=\mathbf{s}+1}^{T} L_t(x_{t-1}, \Delta x_t) + g(x_T) + \sum_{t=\mathbf{s}+1}^{T} K_t(p_t, \Delta p_t) + f(p_T) \geq \xi \cdot \eta.$$

Taking infimum over $\{x_t\}_{t=\mathbf{s}}^T \subset \mathbb{R}^n$ and $\{p_t\}_{t=\mathbf{s}}^T \subset \mathbb{R}^n$ we get the desired inequality. In particular, we must have (a posteriori) that $\vartheta_{\mathbf{s}}(\xi) > -\infty$ and $\omega_{\mathbf{s}}(\eta) > -\infty$. $\square$

2.2. Strong duality

Let us now focus on proving a strong duality result, which in our case means that the primal and dual value functions are conjugate to each other. For such purpose we introduce the following qualification condition:

$$\begin{cases} \exists\{\bar{x}_t\}_{t=\tau}^T \subset \mathbb{R}^n, & \text{such that } \bar{x}_T \in \operatorname{ri}(\operatorname{dom}(g)), \\ \bar{x}_{t-1} \in \operatorname{ri}(\mathbb{X}(t)) & \text{and} \quad \Delta\bar{x}_t \in \operatorname{ri}(\Gamma_L(t, \bar{x}_{t-1})), \quad \forall t \in [\![\tau+1 : T]\!], \end{cases} \tag{H}$$

where for $x \in \mathbb{R}^n$ and $t \in [\![\tau+1 : T]\!]$ we have

$$\Gamma_L(t,x) := \{v \in \mathbb{R}^n \mid L_t(x,v) \in \mathbb{R}\}, \quad \text{and} \quad \mathbb{X}(t) := \{x \in \mathbb{R}^n \mid \Gamma_L(t,x) \neq \emptyset\}.$$

Remark 2.4. Notice that the minimization in (P) can be restrained to trajectories whose initial state $x_{\mathbf{s}} = \xi$ is brought to the target $\operatorname{dom}(g)$ at time $t = T$.This means that the set $\operatorname{dom}(g)$ can be understood as a terminal constraint, implicitly encoded in the formulation of the problem.

Similarly, by allowing each L_t to take infinite values, we are handling implicitly constraints over the state of system x_t and the variation Δx_t. Indeed, any feasible trajectory of the Bolza problem (P) must satisfy:

$$x_{t-1} \in \mathbb{X}(t) \quad \text{and} \quad \Delta x_t \in \Gamma_L(t, x_{t-1}), \quad \forall t \in [\![\tau+1 : T]\!].$$

In other words, the set-valued maps Γ_L and $\mathbb{X}$ correspond respectively to the dynamics of the system and to the (time-dependent) state constraint.

Taking this into account, it follows that the qualification condition (H) can be understood as a *strict feasibility assumption* over the dynamical system. □

We are now in a position to establish a strong duality relation between the primal and dual value functions.

Proposition 2.5. *Under the Hypothesis* (H)*, it follows that* $\vartheta_{\mathbf{s}}^* = \omega_{\mathbf{s}}$*, for each* $\mathbf{s} \in [\![\tau : T-1]\!]$*. Moreover, the infimum in the definition of* $\omega_{\mathbf{s}}(\eta)$ *is attained for any* $\eta \in \operatorname{dom}(\omega_{\mathbf{s}})$*.*

Proof. Take $\mathbf{s} \in [\![\tau : T-1]\!]$. Note that from (H), we get $\vartheta_{\mathbf{s}}(\bar{x}_{\mathbf{s}}) < +\infty$. In particular, it follows that

$$\vartheta_{\mathbf{s}}^*(\eta) \geq \bar{x}_{\mathbf{s}} \cdot \eta - \vartheta_{\mathbf{s}}(\bar{x}_{\mathbf{s}}) > -\infty, \qquad \forall \eta \in \mathbb{R}^n.$$

Let us focus on proving first $\omega_{\mathbf{s}} = \vartheta_{\mathbf{s}}^*$. Let us define $\mathbf{E}_{\mathbf{s}} = \prod_{t=\mathbf{s}}^T \mathbb{R}^n$, and observe that for $\eta \in \operatorname{dom}(\vartheta_{\mathbf{s}}^*)$ we have

$$-\vartheta_{\mathbf{s}}^*(\eta) = \inf_{\xi \in \mathbb{R}^n} \{\vartheta_{\mathbf{s}}(\xi) - \xi \cdot \eta\} = \inf_{\mathbf{x} \in \mathbf{E}_{\mathbf{s}}} \left\{ \sum_{t=\mathbf{s}+1}^{T} L_t(x_{t-1}, \Delta x_t) + g(x_T) - x_{\mathbf{s}} \cdot \eta \right\},$$

where we have considered $\mathbf{x} = \{x_t\}_{t=\mathbf{s}}^T$. If we define $\ell(a,b) = g(b) - a \cdot \eta$, it is not difficult to see that $\operatorname{ri}(\operatorname{dom}(\ell)) = \mathbb{R}^n \times \operatorname{ri}(\operatorname{dom}(g))$, and so, from the arguments used to prove [21, Theorem 2], it follows that (H) implies that $\partial\phi(0) \neq \emptyset$, where for any $\mathbf{y} = \{y_t\}_{t=\mathbf{s}}^T \in \mathbf{E}_{\mathbf{s}}$ the perturbation function ϕ is

$$\phi(\mathbf{y}) = \inf_{\mathbf{x}\in\mathbf{E_s}} \left\{ \sum_{t=\mathbf{s}+1}^{T} L_t(x_{t-1}, \Delta x_t + y_t) + \ell(x_{\mathbf{s}} + y_{\mathbf{s}}, x_T) \right\}. \tag{3}$$

From [21, Theorem 3] we get that

$$\vartheta^*_{\mathbf{s}}(\eta) = \min_{\mathbf{p}\in\mathbf{E_s}} \left\{ \sum_{t=\mathbf{s}+1}^{T} K_t(p_t, \Delta p_t) + \ell^*(p_{\mathbf{s}}, -p_T) \mid \mathbf{p} = \{p_t\}_{t=\mathbf{s}}^T \right\},$$

and the minimum is attained at some $\{p^*_t\}_{t=\mathbf{s}}^T \in \partial\phi(0) \subset \mathbf{E_s}$. Therefore, since $\ell^*(a,b) = g^*(b)$ if $a = -\eta$ and $\ell^*(a,b) = +\infty$ otherwise, we get that $\omega_{\mathbf{s}} = \vartheta^*_{\mathbf{s}}$; for the case $\vartheta^*_{\mathbf{s}}(\eta) = +\infty$ the equality follows from Proposition 2.3. □

In a symmetric way, if now a qualification condition is imposed over the dual problem, a similar result can be obtained. To be more precise, let us consider the following qualification condition

$$\begin{cases} \exists\{\bar{p}_t\}_{t=\tau}^T \subset \mathbb{R}^n, & \text{such that } \bar{p}_T \in \mathrm{ri}\,(\mathrm{dom}(f))\,, \\ \bar{p}_{t-1} \in \mathrm{ri}\,(\mathbf{P}(t)) & \text{and} \quad \Delta\bar{p}_t \in \mathrm{ri}\,(\Gamma_K(t, \bar{p}_{t-1}))\,, \quad \forall t \in [\![\tau+1 : T]\!], \end{cases} \tag{H'}$$

where for $p \in \mathbb{R}^n$ and $t \in [\![\tau+1 : T]\!]$ we have

$$\Gamma_K(t,p) := \{w \in \mathbb{R}^n \mid \ K_t(p+w, w) \in \mathbb{R}\}, \quad \text{and} \quad \mathbf{P}(t) := \{p \in \mathbb{R}^n \mid \ \Gamma_K(t,p) \neq \emptyset\}.$$

Remark 2.6. Similarly as for the primal problem, the set-valued maps Γ_K and $\mathbf{P}$ can be interpreted respectively as the dynamics and as the state constraints of the *dual system*. Also, $\mathrm{dom}(f)$ corresponds to the terminal constraint of the dual problem. Consequently, any feasible trajectory of (D) must satisfy:

$$p_{t-1} \in \mathbf{P}(t) \quad \text{and} \quad \Delta p_t \in \Gamma_K(t, p_{t-1}), \quad \forall t \in [\![\tau+1 : T]\!] \quad \text{and} \quad p_T \in \mathrm{dom}(f).$$

In this context, the qualification condition (H') can be understood as a *strict feasibility assumption* over the dual problem. □

By symmetry, it is not difficult to see (in the light of Remark 2.2), that the following statement holds true.

Proposition 2.7. *Under the Hypothesis* (H')*, it follows that* $\omega^*_{\mathbf{s}} = \vartheta_{\mathbf{s}}$*, for each* $\mathbf{s} \in [\![\tau : T-1]\!]$*. Moreover, the infimum in the definition of* $\vartheta_{\mathbf{s}}(\xi)$ *is attained for any* $\xi \in \mathrm{dom}(\vartheta_{\mathbf{s}})$*.*

As a consequence, under Hypotheses (H) and (H'), we have that $\vartheta_{\mathbf{s}}$ and $\omega_{\mathbf{s}}$ are convex proper and l.s.c. functions on $\mathbb{R}^n$ for any $\mathbf{s} \in [\![\tau : T-1]\!]$. Moreover, they are conjugate to each other, that is,

$$\vartheta^*_{\mathbf{s}} = \omega_{\mathbf{s}} \quad \text{and} \quad \vartheta_{\mathbf{s}} = \omega^*_{\mathbf{s}}.$$

3. Discrete-time method of characteristics

We now present and prove a discrete-time method of characteristics, which describes of the evolution of the subgradients of the primal value function by means of a discrete-time Hamiltonian system.

Let $H_{\tau+1},\dots,H_T:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}\cup\{\pm\infty\}$ be the Hamiltonians associated with the primal problem (P), that is

$$H_t(x,p):=\sup_{v\in\mathbb{R}^n}\{p\cdot v-L_t(x,v)\},\qquad \forall x,p\in\mathbb{R}^n,\ \forall t\in[\![\tau+1:T]\!].$$

For each $t\in[\![\tau+1:T]\!]$ fixed, H_t is a concave-convex function, thus its subdifferential is given by (2). In particular, we have that $(-w,v)\in\partial H_t(\bar{x},\bar{p})$ if and only if

$$H_t(x,\bar{p})+w\cdot(x-\bar{x})\le H_t(\bar{x},\bar{p})\le H_t(\bar{x},p)-v\cdot(p-\bar{p}),\qquad \forall x,p\in\mathbb{R}^n.$$

Definition 3.1. Let $\mathbf{s}\in[\![\tau:T-1]\!]$ be given. We say that $\{(x_t,p_t)\}_{t=\mathbf{s}}^T\subset\mathbb{R}^n\times\mathbb{R}^n$ is a *discrete-time Hamiltonian trajectory* on $[\![\mathbf{s}:T]\!]$ provided that

$$(-\Delta p_t,\Delta x_t)\in\partial H_t(x_{t-1},p_t),\qquad \forall t\in[\![\mathbf{s}+1,T]\!].\tag{4}$$

The method of characteristics and main result of this paper is the following.

Theorem 3.2. *Let $\xi,\eta\in\mathbb{R}^n$ be given. Suppose that there is a* discrete-time Hamiltonian trajectory *on $[\![\tau:T]\!]$, say $\{(x_t,p_t)\}_{t=\tau}^T\subset\mathbb{R}^n\times\mathbb{R}^n$, such that $(x_\tau,p_\tau)=(\xi,-\eta)$ and that satisfies the transversality condition*

$$-p_T\in\partial g(x_T).$$

Then, $\eta\in\partial\vartheta_\tau(\xi)$ and $-p_{\mathbf{s}}\in\partial\vartheta_s(x_{\mathbf{s}})$ for any $\mathbf{s}\in[\![\tau:T-1]\!]$. Moreover, the converse holds true if the hypotheses (H) *and* (H') *are in force.*

Proof. Let $l:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ be the convex proper and l.s.c. function given by

$$l(a,b):=g(b)-a\cdot\eta,\qquad \forall a,b\in\mathbb{R}^n$$

Let $\{(x_t,p_t)\}_{t=\tau}^T\subset\mathbb{R}^n\times\mathbb{R}^n$ be given. It is not difficult to see that the transversality condition $-p_T\in\partial g(x_T)$ combined with $p_\tau=-\eta$ is equivalent to

$$(p_\tau,-p_T)\in\partial\ell(x_\tau,x_T).\tag{5}$$

Moreover, since $H_t(x,p)=(L_t(x,\cdot))^*(p)$, from [14, Theorem 37.5], we have that the condition of being a *discrete-time Hamiltonian trajectory* on $[\![\tau:T]\!]$ (see equation (4)) is equivalent to the *discrete-time Euler-Lagrange relation*

$$(\Delta p_t,p_t)\in\partial L_t(x_{t-1},\Delta x_t),\qquad \forall t\in[\![\tau+1:T]\!].\tag{6}$$

Let $\{(x_t,p_t)\}_{t=\tau}^T$ be as on the statement of the theorem. From [21, Theorem 1], we have that $\{x_t\}_{t=\tau}^T$ is an optimal trajectory for

$$\inf_{\mathbf{y}\in\mathbf{E}_\tau}\left\{\sum_{t=\tau+1}^T L_t(y_{t-1},\Delta y_t)+g(y_T)-y_\tau\cdot\eta\right\}=\inf_{y_\tau\in\mathbb{R}^n}\{\vartheta_\tau(y_\tau)-y_\tau\cdot\eta\},\tag{$\mathcal{P}_0$}$$

and that $\{p_t\}_{t=\tau}^T\in\partial\phi(0)$, where we recall that ϕ is the perturbation function given by (3). From [21, Theorem 3], it follows that $\{p_t\}_{t=\tau}^T\subset\mathbb{R}^n$ realizes $\omega_\tau(\eta)$, and moreover, $\mathrm{val}\,(\mathcal{P}_0)=-\omega_\tau(\eta)$. Notice that $\{x_t\}_{t=\tau}^T$ realizes $\vartheta_\tau(\xi)$ as well, and that $\vartheta_\tau(\xi)=\mathrm{val}\,(\mathcal{P}_0)+\xi\cdot\eta$. From Proposition 2.3, we get that ϑ_τ and ω_τ are conjugate to each other, and that the Fenchel-Young equality holds at (ξ, η), and so $\eta \in \partial\vartheta_\tau(\xi)$. To see that $-p_{\mathbf{s}} \in \partial\vartheta_s(x_{\mathbf{s}})$ for any $\mathbf{s} \in [\![\tau : T-1]\!]$ it is enough to observe that if $\{(x_t, p_t)\}_{t=\tau}^T$ is a discrete-time Hamiltonian trajectory on $[\![\tau : T]\!]$, then $\{(x_t, p_t)\}_{t=\mathbf{s}}^T$ is a discrete-time Hamiltonian trajectory on $[\![\mathbf{s} : T]\!]$ as well.

Conversely, let us assume that (H) and (H') hold and that $\eta \in \partial\vartheta_\tau(\xi)$. In particular $\xi \in \mathrm{dom}(\vartheta_\tau)$, and so, by Proposition 2.7, there is an optimal trajectory that realizes $\vartheta_\tau(\xi)$, that is, there is $\{x_t\}_{t=\tau}^T \subset \mathbb{R}^n$ such that $x_\tau = \xi$ and

$$\vartheta_\tau(\xi) = \sum_{t=\tau+1}^{T} L_t(x_{t-1}, \Delta x_t) + g(x_T).$$

Notice that the condition $\eta \in \partial\vartheta_\tau(\xi)$ is equivalent to $\xi \in \partial\omega_\tau(\eta)$ because ϑ_τ and ω_τ are conjugate to each other. Hence, $\eta \in \mathrm{dom}(\omega_\tau)$, and thus Proposition 2.5 implies that there is an optimal trajectory that realizes $\omega_\tau(\eta)$, that is, $\{p_t\}_{t=\tau}^T \subset \mathbb{R}^n$ such that $p_\tau = -\eta$ and

$$\omega_\tau(\eta) = \sum_{t=\tau+1}^{T} K_t(p_t, \Delta p_t) + f(p_T).$$

From the Fenchel-Young equality we have that $\vartheta_\tau(\xi) + \omega_\tau(\eta) = \xi \cdot \eta$. Thus $\{x_t\}_{t=\tau}^T$ realizes the infimum in $(\mathcal{P}_0)$. Moreover, from the proof of Proposition 2.5 we also have that $\{p_t\}_{t=\tau}^T \in \partial\phi(0)$. It follows from [21, Theorem 1] that the transversality condition (5) and the discrete-time Euler-Lagrange relation (6) holds. □

4. Examples

Let us now provide some explicit formulae for two type of optimal control problems that fit into the convex setting we have posed for our analysis; we refer to [13] for other optimal control problems that can modelled as Bolza problems. The dynamics that govern both problems are jointly linear in the state and in the control. In particular, throughout this section, we consider matrices $A_\tau, \ldots, A_{T-1}$ and $B_\tau, \ldots, B_{T-1}$ of dimension $n \times n$ and $n \times m$, respectively, and *drifts* $c_\tau, \ldots, c_{T-1} \in \mathbb{R}^n$. The linear system we consider throughout this section is the following:

$$\begin{cases} \Delta x_{t+1} = A_t x_t + B_t u_t + c_t, & \forall t \in [\![\tau : T-1]\!], \\ u_t \in \mathcal{U}_t, \quad \forall t \in [\![\tau : T-1]\!], \\ x_t \in \mathcal{X}_t, \quad \forall t \in [\![\tau : T]\!]. \end{cases} \tag{S}$$

Here $\mathcal{X}_\tau, \ldots, \mathcal{X}_T \subset \mathbb{R}^n$ and $\mathcal{U}_\tau \ldots \mathcal{U}_{T-1} \subset \mathbb{R}^m$ are convex, closed, and nonempty sets; the first ones are the (pure) state constraints and the second ones are the (pure) control constraints.

4.1. Linear Quadratic Regulator problem with state constraints

The first example we study is a finite-time-horizon version of the so-called Linear Quadratic Regulator problem. We show that this model can be studied in the framework of this paper. Throughout this example, we consider positive semidefinite matrices $Q_\tau, \ldots, Q_T$ and $R_\tau, \ldots, R_{T-1}$ of dimension $n \times n$ and $m \times m$, respectively.

For any initial state $\xi \in \mathbb{R}^n$, we consider the following Linear Quadratic Regulator problem:

$$\begin{cases} \text{Minimize} & \frac{1}{2}\|x_T\|^2_{Q_T} + \frac{1}{2}\left[\sum_{t=\tau}^{T-1}\left(\|x_t\|^2_{Q_t} + \|u_t\|^2_{R_t}\right)\right] \\ \text{over all} & \left(\{x_t\}_{t=\tau}^T, \{u_t\}_{t=\tau}^{T-1}\right) \text{ satisfying (S) with } x_\tau = \xi. \end{cases} \tag{$\mathrm{P_{LQ}}$}$$

Here we use the notation $\|z\|^2_M := z \cdot Mz$ for $z \in \mathbb{R}^d$, and for a $d \times d$ matrix M.

To formulate the problem as a convex Bolza problem, we set the endpoint cost $g^{LQ} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ as

$$g^{LQ}(a) := \frac{1}{2}\|a\|^2_{Q_T} + \delta_{\mathcal{X}_T}(a), \qquad \forall a \in \mathbb{R}^n,$$

and for $t \in [\![\tau : T-1]\!]$ the Lagrangian $L^{LQ}_{t+1} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ as

$$L^{LQ}_{t+1}(x, v) := \frac{1}{2}\|x\|^2_{Q_t} + \delta_{\mathcal{X}_t}(x) + \inf_{u \in \mathcal{U}_t}\left\{\frac{1}{2}\|u\|^2_{R_t} \mid v = A_t x + B_t u + c_t\right\},$$

and set

$$S_t := \{(x, u, v) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \mid v = A_t x + B_t u + c_t,\ x \in \mathcal{X}_t,\ u \in \mathcal{U}_t\}.$$

Notice that each S_t is closed, convex and nonempty. Moreover, we have:

$$\inf_{\substack{\{x_t\}\subset\mathbb{R}^n \\ x_\tau=\xi}} \sum_{t=\tau+1}^{T}\left[L^{LQ}_t(x_{t-1}, \Delta x_t)\right] + g^{LQ}(x_T) = \inf_{\substack{\{x_t\}\subset\mathbb{R}^n \\ x_\tau=\xi}} \sum_{t=\tau}^{T-1}\left[L^{LQ}_{t+1}(x_t, \Delta x_{t+1})\right] + g^{LQ}(x_T)$$

$$= \inf_{\substack{\{x_t\}\subset\mathbb{R}^n \\ x_\tau=\xi}} \frac{1}{2}\sum_{t=\tau}^{T-1}\left[\|x_t\|^2_{Q_t} + \delta_{\mathcal{X}_t}(x_t) + \inf_{u_t \in \mathbb{R}^m}\{\|u_t\|^2_{R_t} + \delta_{S_t}(x_t, u_t, \Delta x_{t+1})\}\right] + g^{LQ}(x_T)$$

$$= \inf_{\substack{\{x_t\}\subset\mathbb{R}^n \\ \{u_t\}\subset\mathbb{R}^m \\ x_\tau=\xi}} \frac{1}{2}\sum_{t=\tau}^{T-1}\left[\|x_t\|^2_{Q_t} + \|u_t\|^2_{R_t} + \delta_{S_t}(x_t, u_t, \Delta x_{t+1})\right] + \frac{1}{2}\|x_T\|^2_{Q_T} + \delta_{\mathcal{X}_T}(x_T).$$

This implies that the optimal value of the optimization problem ($\mathrm{P_{LQ}}$) coincides with the value function of the Bolza problem defined by $L^{LQ}_{\tau+1}, \dots, L^{LQ}_T$ and g^{LQ}. In other words, the problem ($\mathrm{P_{LQ}}$) admits a convex Bolza formulation as in (P), and the corresponding value function satisfies

$$\vartheta^{LQ}_\tau(\xi) := \inf_{x_\tau=\xi}\left\{\frac{1}{2}\sum_{t=\tau}^{T-1}\left[\|x_t\|^2_{Q_t} + \|u_t\|^2_{R_t}\right] + \frac{1}{2}\|x_T\|^2_{Q_T} \mid \{x_t\} \text{ and } \{u_t\} \text{ satisfy (S)}\right\}.$$

Observe that with the assumptions we have done so far, g^{LQ} is proper, convex and l.s.c. and each L^{LQ}_{t+1} is proper and convex, but not necessarily l.s.c. For the lower semicontinuity of L^{LQ}_{t+1}, we introduce the following qualification condition.

$$\ker(B_t) \cap \ker(R_t) \cap (\mathcal{U}_t)_\infty = \{0\}, \qquad \forall t \in [\![\tau : T-1]\!]. \tag{CQ}$$

Proposition 4.1. *If* (CQ) *is satisfied, then for any* $t \in [\![\tau : T-1]\!]$ *the infimum in the definition of* L_{t+1}^{LQ} *is attained whenever* $L_{t+1}^{LQ}(x,v) < +\infty$*, and moreover,* $L_{\tau+1}^{LQ}, \dots, L_T^{LQ}$ *and* g^{LQ} *satisfy* (A).

Proof. Given that the recession function of $u \mapsto \frac{1}{2}\|u\|_{R_t}^2 + \delta_{\mathcal{U}_t}(u)$ is the indicator function of $\ker(R_t) \cap (\mathcal{U}_t)_\infty$, we can conclude that (CQ) implies the qualification condition as per [1, Corollary 3.5.7]. Consequently, the infimum in the definition of the Lagrangian L_{t+1}^{LQ} is attained whenever there is $u \in \mathcal{U}_t$ such that $v = A_t x + B_t u + c_t$. Moreover, [1, Corollary 3.5.7] also implies that L_{t+1}^{LQ} is a l.s.c. function. In particular, the basic assumption (A) are satisfied by $L_{\tau+1}^{LQ}, \dots, L_T^{LQ}$ and g^{LQ}. □

This statement in particular implies that the evolution of over time of the subgradients of the value function ϑ_τ^{LQ} can be described by means of Theorem 3.2. Recall that in order to have a complete characterization of $\partial\vartheta_\tau^{LQ}(\xi)$, we need to ensure that (H) and (H') are satisfied. We proceed now to present some additional conditions that will allow us to guarantee that (H) and (H') are satisfied.

4.1.1. Sufficient conditions for (H)

First, notice that if
$$\mathcal{X}_t = \mathbb{R}^n, \qquad \forall t \in [\![\tau : T]\!] \tag{$\mathrm{C}_1^{\mathrm{LQ}}$}$$
is satisfied, then (H) holds immediately. Indeed, clearly, $\mathrm{dom}(g^{LQ}) = \mathbb{R}^n$.

Recall that $\Gamma_L(t,x) = \{v \in \mathbb{R}^n \mid L_t(x,v) \in \mathbb{R}\}$, then

$$\Gamma_{L^{LQ}}(t,x) := \left\{ v \in \mathbb{R}^n \mid \|x\|_{Q_{t-1}}^2 + \delta_{\mathcal{X}_{t-1}}(x) + \inf_{u \in \mathcal{U}_{t-1}} \left\{ \|u\|_{R_{t-1}}^2 + \delta_{S_{t-1}}(x,u,v) \right\} \in \mathbb{R} \right\}.$$

From this, if $x \notin \mathcal{X}_{t-1}$, then $\Gamma_{L^{LQ}}(t,x) = \emptyset$. If $x \in \mathcal{X}_{t-1}$, then $v \in \Gamma_{L^{LQ}}(t,x)$ if and only if $\inf\left\{\|u\|_{R_{t-1}}^2 + \delta_{S_{t-1}}(x,u,v) \mid u \in \mathcal{U}_{t-1}\right\} \in \mathbb{R}$. Therefore, if $x \in \mathcal{X}_{t-1}$, then

$$\begin{aligned} v \in \Gamma_{L^{LQ}}(t,x) &\Leftrightarrow \exists u \in \mathcal{U}_{t-1} \text{ s.t. } v = A_{t-1}x + B_{t-1}u + c_{t-1}, \\ &\Leftrightarrow v \in B_{t-1}\mathcal{U}_{t-1} + A_{t-1}x + c_{t-1}. \end{aligned}$$

So, if $x \in \mathcal{X}_{t-1}$, $\Gamma_{L^{LQ}}(t,x) = B_{t-1}\mathcal{U}_{t-1} + A_{t-1}x + c_{t-1}$. As $\mathcal{U}_{t-1}$ is nonempty, we have that $\mathbb{X}(t) = \mathcal{X}_{t-1}$ for all t. Thus, if $\mathcal{X}_t = \mathbb{R}^n$ for every t, (H) holds true because

$$\mathrm{ri}(\Gamma_{L^{LQ}}(t,x)) = \mathrm{ri}(B_{t-1}(\mathcal{U}_{t-1})) + A_{t-1}x + c_{t-1} = B_{t-1}(\mathrm{ri}(\mathcal{U}_{t-1})) + A_{t-1}x + c_{t-1}$$

is always nonempty given that $\mathcal{U}_{t-1}$ is convex.

We now study the case where there are (non trivial) state constraints, that is

$$X = \bigcap_{t=\tau}^{T} \mathcal{X}_t \subsetneq \mathbb{R}^n.$$

Assume that $\bigcap_{t=\tau}^T \mathrm{ri}(\mathcal{X}_t)$ is nonempty, so that we have $\bigcap_{t=\tau}^T \mathrm{ri}(\mathcal{X}_t) = \mathrm{ri}(X)$. Then, to ensure (H), one may look for conditions to have an equilibrium point in the system, that is, finding conditions to guarantee that

$$\exists\, \hat{x} \in \mathrm{ri}(X) \text{ such that } 0 \in B_t(\mathrm{ri}(\mathcal{U}_t)) + A_t\hat{x} + c_t, \qquad \forall t \in [\![\tau : T-1]\!]. \tag{$\mathrm{C}_2^{\mathrm{LQ}}$}$$

This condition implies (H), as we can choose $\bar{x}_t = \hat{x}$ for any $t \in [\![\tau : T]\!]$.

In particular, the following statement holds true.

Proposition 4.2. *If* (CQ) *is satisfied, then* (H) *is satisfied with* $L^{LQ}_{\tau+1}, \ldots, L^{LQ}_T$ *and* g^{LQ} *provided that either* ($\mathrm{C}_1^{\mathrm{LQ}}$) *or* ($\mathrm{C}_2^{\mathrm{LQ}}$) *holds.*

To verify ($\mathrm{C}_2^{\mathrm{LQ}}$), we can solve the following linear system of equations

$$\begin{bmatrix} A_t & B_t \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = -c_t, \tag{7}$$

for each t, and intersect the solution sets with $\mathrm{ri}(X) \times \mathrm{ri}(\mathcal{U}_t)$. If there is some $\bar{x}$ that belongs to this intersection for every t, then we have (H). In particular, we can check if $0 \in \mathrm{ri}(X)$ and $0 \in B_t(\mathrm{ri}(\mathcal{U}_t)) + c_t$. If $c_t = 0$, the condition can be reduced to check that $0 \in \mathrm{ri}(X)$ and $0 \in \mathrm{ri}(\mathcal{U}_t)$.

4.1.2. Sufficient conditions for (H')

Let us turn our attention to the dual problem. Notice that

$$f^{LQ}(b) = \left(g^{LQ}\right)^* (-b) = \sup_{x \in \mathcal{X}_T} \left\{ -x \cdot b - \frac{1}{2}\|x\|^2_{Q_T} \right\}.$$

When Q_T is positive definite and $\mathcal{X}_T = \mathbb{R}^n$, the supremum is achieved at $x = -Q_T^{-1} b$, and consequently, $f^{LQ}(b) = \frac{1}{2}\|b\|^2_{Q_T^{-1}}$, which means that $\mathrm{dom}(f^{LQ}) = \mathbb{R}^n$. In general, we have that $\overline{\mathrm{dom}\,(f^{LQ})} = -(\ker(Q_T) \cap (\mathcal{X}_T)_\infty)^*$; see [1, Theorem 2.5.4].

On the other hand, we have

$$\begin{aligned}
K^{LQ}_{t+1}(p, w) &= \sup_{x, v \in \mathbb{R}^n} \{x \cdot w + v \cdot p - L_{t+1}(x, v)\} \\
&= \sup_{x, v \in \mathbb{R}^n} \left\{ x \cdot w + v \cdot p - \frac{1}{2} \left(\|x\|^2_{Q_t} + \delta_{\mathcal{X}_t}(x) + \inf_{u \in \mathbb{R}^m} \{ \|u\|^2_{R_t} + \delta_{S_t}(x, u, v) \} \right) \right\} \\
&= \sup_{\substack{x \in \mathcal{X}_t \\ u \in \mathcal{U}_t \\ v \in \mathbb{R}^n}} \left\{ x \cdot w + v \cdot p - \frac{1}{2}\|x\|^2_{Q_t} - \frac{1}{2}\|u\|^2_{R_t} \mid v = A_t x + B_t u + c_t \right\} \\
&= \sup_{\substack{x \in \mathcal{X}_t \\ u \in \mathcal{U}_t}} \left\{ x \cdot (A_t^\top p + w) - \frac{1}{2}\|x\|^2_{Q_t} + u \cdot B_t^\top p - \frac{1}{2}\|u\|^2_{R_t} \right\} + c_t \cdot p \\
&= \sup_{x \in \mathcal{X}_t} \left\{ x \cdot (A_t^\top p + w) - \frac{1}{2}\|x\|^2_{Q_t} \right\} + \sup_{u \in \mathcal{U}_t} \left\{ u \cdot B_t^\top p - \frac{1}{2}\|u\|^2_{R_t} \right\} + c_t \cdot p.
\end{aligned}$$

Define $\Gamma_{K^{LQ}}(t+1, p) := \{ w \in \mathbb{R}^n \mid K^{LQ}_{t+1}(p + w, w) \in \mathbb{R} \}$. Then, $w \in \Gamma_{K^{LQ}}(t+1, p)$ if and only if

$$A_t^\top p + A_t^\top w + w \in \mathrm{dom}(\psi_t^1), \tag{8}$$

$$B_t^\top p + B_t^\top w \in \mathrm{dom}(\psi_t^2), \tag{9}$$

where $\psi_t^1(z) := \sup_{x \in \mathcal{X}_t} \left\{ x \cdot z - \frac{1}{2}\|x\|^2_{Q_t} \right\}$ and $\psi_t^2(z) := \sup_{u \in \mathcal{U}_t} \left\{ u \cdot z - \frac{1}{2}\|u\|^2_{R_t} \right\}$.

Also, $p \in \mathbf{P}(t)$ if and only if there is $w \in \mathbb{R}^n$ such that (8) and (9) hold. From [1, Theorem 2.5.4], we have that

$$\overline{\mathrm{dom}(\psi_t^1)} = (\ker(Q_t) \cap (\mathcal{X}_t)_\infty)^* \quad \text{and} \quad \overline{\mathrm{dom}(\psi_t^2)} = (\ker(R_t) \cap (\mathcal{U}_t)_\infty)^*.$$

Therefore:

- If every $(\ker(Q_t) \cap (\mathcal{X}_t)_\infty)^*$ and $(\ker(R_t) \cap (\mathcal{U}_t)_\infty)^*$ are vector spaces, we can take $\bar{p}_\tau = \ldots, \bar{p}_T = 0$, and (H') will be satisfied. Indeed, if $w \in \Gamma_{K^{LQ}}(t, 0)$, then $-w \in \Gamma_{K^{LQ}}(t, 0)$ as well. Thus, $0 \in \mathrm{ri}(\Gamma_{K^{LQ}}(t, p))$. Also, if $w \in \Gamma_{K^{LQ}}(t, p)$ then, $-w \in \Gamma_{K^{LQ}}(t, -p)$. This means that $0 \in \mathrm{ri}(\mathbf{P}(t))$. This case covers Linear Quadratic Regulator problems that are common in control engineering literature: no state constraints or compact state constraints, and R_t being positive definite or compact control constraints.
- If Q_t is positive definite or $\mathcal{X}_t$ is compact for every t, then given $p \in \mathbb{R}^n$ the first condition (8) holds true for every $p, w \in \mathbb{R}^n$, because $(\ker(Q_t) \cap (\mathcal{X}_t)_\infty)^* = \mathbb{R}^n$. We can choose $w = -p$ to ensure that the second condition (9) also remains true. Thus, we conclude that in this case $\mathbf{P}(t) = \mathbb{R}^n$ for every t and the condition (H') holds true because $\mathrm{ri}(\Gamma_{K^{LQ}}(t, p))$ is always nonempty.
- If R_t is positive definite or $\mathcal{U}_t$ is compact for every t, then given $p \in \mathbb{R}^n$ the second condition (9) is true for every $w \in \mathbb{R}^n$, because $(\ker(R_t) \cap (\mathcal{U}_t)_\infty)^* = \mathbb{R}^m$. If we also have that the matrix $A_t^\top + I_n$ has rank n, then we can choose w such that $(A_t^\top + I_n)w = -A_t^\top p$ and so $\mathbf{P}(t) = \mathbb{R}^n$ for every t, and (H') holds in this case since $\mathrm{ri}(\Gamma_{K^{LQ}}(t, p))$ is always nonempty.

This discussion can be summarized in the following statement.

Corollary 4.3. *If* (CQ) *is satisfied, then* (H') *is satisfied with* $L_{\tau+1}^{LQ}, \ldots, L_T^{LQ}$ *and* g^{LQ} *provided that* $(\ker(Q_t) \cap (\mathcal{X}_t)_\infty)^*$ *and* $(\ker(R_s) \cap (\mathcal{U}_s)_\infty)^*$ *are vector spaces for any* $t \in [\![\tau : T]\!]$ *and* $s \in [\![\tau : T-1]\!]$*, respectively. In the case this condition is not satisfied, one can still ensure that* (H') *is satisfied provided that one of the following conditions holds:*

(1) $(\ker(Q_t) \cap (\mathcal{X}_t)_\infty)^* = \mathbb{R}^n$ *for any* $t \in [\![\tau : T]\!]$*;*

(2) $(\ker(R_t) \cap (\mathcal{U}_t)_\infty)^* = \mathbb{R}^m$ *for any* $t \in [\![\tau : T-1]\!]$ *and* $A_t^\top + I_n$ *has rank* n*.*

4.2. Linear-convex problem with mixed constraints

The second model we study is a linear-convex problem with mixed constraints, that is, constraints that affect jointly the state and control of the system (at the same time). To be more precise, we consider the following mixed constraints:

$$f_t(x_t, u_t) \leq 0, \qquad \forall t \in [\![\tau : T-1]\!], \tag{$\mathrm{M_c}$}$$

where $f_\tau, \ldots, f_{T-1} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are given real-valued convex functions.

Following an analogous procedure as for the Linear Quadratic Regulator problem, we first show that this model can be stated as convex Bolza problem satisfying assumption (A).

The optimal control problem we deal with now is the following:

$$\begin{cases} \text{Minimize } \displaystyle\sum_{t=\tau}^{T-1} \ell_t(x_t, u_t) + g(x_T), \\ \text{over all } \left(\{x_t\}_{t=\tau}^{T}, \{u_t\}_{t=\tau}^{T-1}\right) \text{ satisfying (S) and } (\mathrm{M_c}) \text{ with } x_\tau = \xi. \end{cases} \tag{$\mathrm{P_M}$}$$

Hereinafter $\ell_\tau, \ldots, \ell_{T-1} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are convex functions. We define, for every $t \in [\![\tau : T-1]\!]$, the Lagrangian associated to $(\mathrm{P_M})$ as

$$L_{t+1}^M(x, v) = \inf_{u \in \mathbb{R}^m} \{\ell_t(x, u) + \delta_{\mathcal{U}_t}(u) + \delta_{D_t}(x, u) + \delta_{S_t}(x, u, v)\} + \delta_{\mathcal{X}_t}(x),$$

where $D_t = \{(x, u) : f_t(x, u) \leq 0\}$ represents the mixed constraints. It is not difficult to see that, with these data, $(\mathrm{P_M})$ can be written equivalently as a Bolza problems since

$$\vartheta_\tau^M(\xi) := \inf_{\substack{\{x_t\} \subset \mathbb{R}^n \\ x_\tau = \xi}} \sum_{t=\tau+1}^{T} \left[L_t^M(x_{t-1}, \Delta x_t)\right] + g(x_T) = \inf_{\substack{\{x_t\} \subset \mathbb{R}^n \\ x_\tau = \xi}} \sum_{t=\tau}^{T-1} \left[L_{t+1}^M(x_t, \Delta x_{t+1})\right] + g(x_T)$$

$$= \inf_{\substack{\{x_t\} \subset \mathbb{R}^n \\ \{u_t\} \subset \mathbb{R}^m \\ x_\tau = \xi}} \sum_{t=\tau}^{T-1} [\ell_t(x_t, u_t) + \delta_{\mathcal{U}_t}(u_t) + \delta_{D_t}(x_t, u_t) + \delta_{S_t}(x_t, u_t, \Delta x_{t+1}) + \delta_{\mathcal{X}_t}(x_t)] + g(x_T).$$

In order to ensure that (A) is satisfied by $L_{\tau+1}^M, \ldots, L_T^M$, we make the following assumption

$$\begin{cases} \forall t \in [\![\tau : T-1]\!],\ \exists \psi_t : \mathbb{R}^n \to \mathbb{R} \text{ upper semicontinuous such that} \\ \displaystyle\sup_{u \in \mathcal{U}_t} \{|u| \mid f_t(x, u) \leq 0\} \leq \psi_t(x), \quad \forall x \in \mathcal{X}_t. \end{cases} \tag{$\mathrm{A^M}$}$$

This is the discrete version of [2, Assumption (A5)]. Let us define

$$\Omega_t := \{(x, u) \mid x \in \mathcal{X}_t, u \in \mathcal{U}_t,\ f_t(x, u) \leq 0\}, \qquad \forall t \in [\![\tau : T-1]\!].$$

Notice that these sets are closed. In the sequel we assume that each Ω_t is nonempty, otherwise $(\mathrm{P_M})$ becomes immediately infeasible.

Proposition 4.4. *If* $(\mathrm{A^M})$ *is satisfied, then for any* $t \in [\![\tau : T-1]\!]$ *the infimum in the definition of* L_{t+1}^M *is attained whenever* $L_{t+1}^M(x, v) < +\infty$*, and moreover,* $L_{\tau+1}^M, \ldots, L_T^M$ *are convex, proper and l.s.c. functions.*

Proof. Let us divide the proof into several steps.

(a) *Convexity.* Let $L_{t+1}^M(x, v) = \Phi_t(x, v) + \delta_{\mathcal{X}_t}(x)$, where $\Phi_t(x, v) = \inf_{u \in \mathbb{R}^m} \varphi_t(x, v, u)$ and $\varphi_t(x, v, u) = \ell_t(x, u) + \delta_{\mathcal{U}_t}(u) + \delta_{D_t}(x, u) + \delta_{S_t}(x, u, v)$. Then, φ_t is convex, and so Φ_t is convex too. Thus, L_{t+1}^M is convex.

(b) *Properness.* It is clear that $L_{t+1}^M(x, A_t x + B_t + c_t) < +\infty$ for any $(x, u) \in \Omega_t$. Let us prove by contradiction that $L_{t+1}^M > -\infty$. Suppose there is (x, v) such that

$L_{t+1}^M(x,v) = -\infty$. Then there is a sequence $\{u_k\}$ such that $(x,u_k) \in \Omega_t$ for every $k \in \mathbb{N}$ and $\ell_t(x,u_k) \to -\infty$. By $(\mathrm{A^M})$, $\{u_k\}$ is bounded by $\psi_t(x)$, which contradicts the continuity of ℓ_t (ℓ_t is a real valued convex function). Therefore, L_{t+1}^M is proper.

(c) *Lower semi-continuity.* To prove that L_{t+1}^M is l.s.c., we rely on [24, Chapter 5]. We first show that φ_t is level-bounded in u locally uniformly in (x,v), as defined in [24]. Let $(\bar{x},\bar{v}) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Fix $\varepsilon > 0$ and let $M > 0$ such that $\sup\{\psi_t(x) \mid |x-\bar{x}| \le \varepsilon\} \le M$ (its existence is guaranteed by the upper semi-continuity of ψ_t). If for $u \in \mathbb{R}^m$, we have that $\varphi_t(x,v,u) \le \alpha$, with $|x-\bar{x}| \le \varepsilon$ and $v \in \mathbb{R}^n$, then $u \in \mathcal{U}_t$ and $f_t(x,u) \le 0$. By $(\mathrm{A^M})$, $|u| \le \psi_t(x) \le M$. So, φ_t is level-bounded in u locally uniformly in (x,v).

We prove now that Φ_t is l.s.c. Let $(\bar{x},\bar{v}) \in \mathrm{dom}(\Phi_t)$ and $(x_k,v_k) \to (\bar{x},\bar{v})$. Without loss of generality, we can assume that there is a subsequence with $\Phi_t(x_{k_j},v_{k_j}) < +\infty$ for every $j \in \mathbb{N}$, otherwise the conclusion is straightforward. By continuity of ℓ_t, we can choose $\{(x_{k_j},v_{k_j})\}$ so that $\Phi_t(x_{k_j},v_{k_j}) < M$ for some $M > 0$ large enough. Then, $\{\Phi_t(x_{k_j},v_{k_j})\}$ is bounded from above. As we have already proved that φ_t is level-bounded in u locally uniformly in (x,v), by [24, Proposition 5.4] we obtain that the sequence $\varphi_t(x_{k_j},v_{k_j},\cdot)$ is tight in the sense of [24, Definition 5.3]. With this, we can follow the steps on [24, Theorem 5.6(a)] to prove that

$$\liminf_{k\to\infty} \Phi_t(x_k,v_k) = \liminf_{j\to\infty} \Phi_t(x_{k_j},v_{k_j}) \ge \Phi_t(\bar{x},\bar{v}).$$

To conclude, we show that $\mathrm{dom}\,\Phi_t$ is closed. Let $\{(x_k,v_k)\} \subset \mathrm{dom}\,\Phi_t$ such that $(x_k,v_k) \to (x,v)$. Then for every $k \in \mathbb{N}$ there is $u_k \in \mathcal{U}_t$ such that $f_t(x_k,u_k) \le 0$ and $v_k = A_t x_k + B_t u_k + c_t$. The sequence $\{x_k\}$ is bounded because it is convergent. By $(\mathrm{A^M})$, the sequence $\{u_k\}$ is also bounded. Therefore, up to a subsequence (which we do not relabel), we may assume that $u_k \to u$. By continuity of f_t, we obtain that $f_t(x,u) \le 0$ and $v = A_t x + B_t u + c_t$, concluding that $(x,v) \in \mathrm{dom}\,\Phi_t$. This implies that Φ_t is l.s.c. and so is L_{t+1}^M. □

Proposition 4.4 implies that the evolution of over time of the subgradients of the value function

$$\vartheta_\tau^M(\xi) = \inf_{x_\tau=\xi} \left\{ \sum_{t=\tau}^{T-1} \ell_t(x_t,u_t) + g(x_T) \mid \{x_t\},\{u_t\} \text{ satisfy (S) and } (\mathrm{M_c}) \right\}$$

can be described by means of Theorem 3.2. Similarly to what was done for the Linear Quadratic Regulator problem $(\mathrm{P_{LQ}})$, we now study some additional conditions on the data of the problem that will allow us to guarantee that (H) and (H') are satisfied by the data of $(\mathrm{P_M})$.

4.2.1. Sufficient conditions for (H)

Let $\Gamma_{L^M}(t+1,x) := \{v \in \mathbb{R}^n \mid L_{t+1}^M(x,v) \in \mathbb{R}\}$. By $(\mathrm{A^M})$, for $x \in \mathcal{X}_t$ fixed we have

$$\inf_{u\in\mathcal{U}_t} \{\ell_t(x,u) \mid f_t(x,u) \le 0\} \ge \inf_{|u|\le\psi_t(x)} \{\ell_t(x,u)\} > -\infty.$$

Therefore, if $x \in \mathcal{X}_t$:

$$\begin{aligned} v \in \Gamma_L(t+1,x) &\Leftrightarrow \exists u \in \mathcal{U}_t \text{ s.t. } v = A_t x + B_t u + c_t \,\wedge\, f_t(x,u) \le 0, \\ &\Leftrightarrow v \in B_t\{u \in \mathcal{U}_t \mid f_t(x,u) \le 0\} + A_t x + c_t. \end{aligned}$$

Since $\{u \in \mathcal{U}_t \mid f_t(x,u) \le 0\}$ is nonempty convex, $\mathrm{ri}(\Gamma_L(t+1,x))$ is also nonempty. In particular,

$$\mathbb{X}(t+1) = \mathcal{X}_t \cap \Xi, \quad \text{where} \;\; \Xi := \{x \in \mathbb{R}^n \mid \exists u \in \mathcal{U}_t \text{ s.t. } f_t(x,u) \le 0\}.$$

Notice that the set Ξ is convex. It follows that $\mathrm{ri}(\mathcal{X}_t) \cap \mathrm{ri}(\Xi) \subseteq \mathrm{ri}(\mathbf{X}(t+1))$ with equality if $\mathrm{ri}(\mathcal{X}_t) \cap \mathrm{ri}(\Xi) \neq \emptyset$. Thus, for ensuring that (H) holds we may also look for conditions that guarantee an equilibrium point in the system.

Proposition 4.5. *Assume that* $(\mathrm{A}^{\mathrm{M}})$ *is satisfied,* $X = \bigcap_{t=\tau}^{T} \mathrm{ri}(\mathcal{X}_t)$ *is nonempty and for any* $t \in [\![\tau : T-1]\!]$

$$\exists\, \hat{x} \in \mathrm{ri}(X), u_t \in \mathrm{ri}(\mathcal{U}_t) \text{ such that } 0 = B_t u_t + A_t \hat{x} + c_t \text{ and } f_t(\hat{x}, u_t) < 0. \qquad (\mathrm{C}^{\mathrm{M}})$$

Then (H) *is satisfied with* $L^M_{\tau+1}, \dots, L^M_T$ *and* g^M.

Proof. We only need to note that u_t in $(\mathrm{C}^{\mathrm{M}})$ belongs to $\mathrm{ri}(\mathcal{U}_t) \cap \mathrm{ri}(\Xi) \subset \mathrm{ri}(\mathcal{U}_t) \cap \Xi)$. Clearly, $\hat{x} \in \mathbf{X}(t+1)$ and $0 \in \Gamma_L(t+1,\hat{x})$ for any $t \in [\![\tau : T-1]\!]$. □

4.2.2. Sufficient conditions for (H')

To conclude, we now focus on the dual problem. First, let us seek conditions to ensure that the dual problem has no state constraints. These conditions will imply (H'). Notice that

$$K^M_{t+1}(p,w) = \sup_{(x,u)\in\Omega_t} \{x \cdot (w + A_t^\top p) + u \cdot B_t^\top p - \ell_t(x,u)\} + c_t \cdot p,$$

and consequently

$$\Gamma_{K^M}(t+1,p) = \Big\{ w \in \mathbb{R}^n \mid \sup_{(x,u)\in\Omega_t} \{x \cdot (w + A_t^\top w + A_t^\top p) + u \cdot (B_t^\top p + B_t^\top w) - \ell_t(x,u)\} \in \mathbb{R} \Big\}.$$

By continuity of ℓ_t, we have that $\mathbf{P}(t+1) = \mathbb{R}^n$ if Ω_t is compact. This is true if for example

- $\mathcal{X}_t$ is compact. In this case, we have that

$$\sup_{(x,u)\in\Omega_t} \{|u| \mid |f_t(x,u) \le 0\} \le \sup_{x\in\mathcal{X}_t} \{\psi_t(x)\} < +\infty$$

  and therefore Ω_t is compact.
- there is a coercive function $h_t : \mathbb{R}^n \to \mathbb{R}$ such that $h_t(x) \le f_t(x,u)$; this is the case for example if $f_t(x,u) = |x|^2 + |u|^2 - 1$. Indeed, this implies that Ω_t is bounded. To see this assume by contradiction that $\{(x_k,u_k)\} \subset \Omega_t$ is such that $|(x_k,u_k)| \to \infty$. Note that $h_t(x_k) \le 0$, and so $\{x_k\}$ is bounded. Therefore, necessarily $|u_k| \to \infty$, however, this contradicts $(\mathrm{A}^{\mathrm{M}})$.

On the other hand, if Ω_t is not compact, (H') also holds if $\mathcal{U}_t$ is compact and $A_t^\top + I_n$ has rank n for every t. To prove this, let $p \in \mathbb{R}^n$ and choose w such that $w + A_t^\top w = A_t^\top p$. Then

$$\begin{aligned}\sup_{(x,u)\in\Omega_t} &\{x \cdot (w + A_t^\top w + A_t^\top p) + u \cdot (B_t^\top p + B_t^\top w) - \ell_t(x,u)\} \\ &= \sup_{(x,u)\in\Omega_t} \{u \cdot 2B_t^\top p - \ell_t(x,u)\} \in \mathbb{R}\end{aligned}$$

It is easy to see that if ℓ_t is coercive for any t, then (H') holds. If the matrix $A_t^\top + I_n$ has rank n for all t, we can refine this argument by asking

$$\begin{cases} \exists\, \kappa_0 \in \mathbb{R},\ h_\tau, \dots, h_{T-1} : \mathbb{R}^m \to \mathbb{R} \text{ such that} \\ \sup\limits_{(x,u)\in\Omega_t} \{|x| \mid \ell_t(x,u) - z\cdot u \le \kappa_0\} \le h_t(z), \quad \forall z \in \mathbb{R}^m. \end{cases} \tag{$\mathrm{B^M}$}$$

In this case, $\mathbf{P}(t) = \mathbb{R}^n$ for every t as well. To prove this, let $p \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$ such that $w + A_t^\top w = -A_t^\top p$. Assume by contradiction that there is $\{(x_k, u_k)\} \subset \Omega_t$ such that

$$x_k\cdot(w + A_t^\top w + A_t^\top p) + u_k\cdot(B_t^\top p + B_t^\top w) - \ell_t(x_k,u_k) = u_k\cdot(B_t^\top p + B_t^\top w) - \ell_t(x_k,u_k) \to \infty.$$

This implies that there is $N_0 \in \mathbb{N}$ such that

$$u_k\cdot(B_t^\top p + B_t^\top w) - \ell_t(x_k,u_k) \ge -\kappa_0, \qquad \forall k \ge N_0.$$

By taking $z = B_t^\top p + B_t^\top w$ in ($\mathrm{B^M}$), then $\sup_{k\ge N_0} |x_k| \le h_t(z)$. Then, by ($\mathrm{A^M}$), there exists some $M \in \mathbb{R}$ such that $\sup_{k\ge N_0} |u_k| \le M$, which is a contradiction with $u_k\cdot(B_t^\top p + B_t^\top w) - \ell_t(x_k,u_k) \to \infty$ as ℓ_t is continuous.

The condition ($\mathrm{B^M}$) corresponds to the discrete version of [2, Assumption (A6)], and it is satisfied, for example, if ℓ_t is coercive in u and has at least linear growth in x.

The previous discussion will be summarized in the following proposition:

Proposition 4.6. *If* ($\mathrm{A^M}$) *is satisfied, then* (H') *is satisfied with* $L^M_{\tau+1}, \dots, L^M_T$ *and* g^M *if* Ω_t *is compact or* ℓ_t *is coercive for every* $t \in [\![\tau : T-1]\!]$*. If none of these conditions is satisfied,* (H') *holds provided that the matrix* $A_t^\top + I_n$ *has rank* n *and either* $\mathcal{U}_t$ *is compact or* ℓ_t *satisfies* ($\mathrm{B^M}$) *for every* $t \in [\![\tau : T-1]\!]$.

5. Conclusion

In this paper we have shown that the evolution over time of the subgradients of the value function of a (deterministic) discrete-time convex Bolza problem can be associated with trajectories of a discrete-time Hamiltonian system. This is the so-called *discrete-time method of characteristics*. We have also demonstrated that the qualification conditions required for our results to hold are satisfied in several cases of interest; e.g. the Linear Quadratic Regulator problem and the linear-convex problem with mixed constraints.

This paper paves the way for future extension to stochastic convex Bolza problems. Indeed, our main result (Theorem 3.2) as well as our intermediate results (Proposition 2.5 and 2.7) are consequences of the research reported by Rockafellar and Wets in [21]. For our paper, we have only used the deterministic part, but we plan to use the stochastic part in a forthcoming paper.

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