Journal of Convex Analysis
Volume 33 (2026), No. 1&2, 325–333

Approximation of Convex Sets by Homothetic Copies of Polyhedra

Valeriu Soltan
Dept. of Mathematical Sciences, George Mason University, Fairfax, U.S.A.
vsoltan@gmu.edu

Received: December 28, 2024
Accepted: March 14, 2025

We characterize the closed convex sets $K \subset \mathbb{R}^n$ which satisfy the following approximation condition: for any given point $v \in \operatorname{rint} K$ and a scalar $\varepsilon > 0$, there is a polyhedron $P \subset \mathbb{R}^n$ such that $v \in \operatorname{rint} P$ and $P \subset K \subset v + (1+\varepsilon)(P - v)$.

Keywords: Convex set, polyhedron, approximation, neighborhood, homothetic, M-decomposable set.

2020 Mathematics Subject Classification: 52A20, 90C25.

1. Introduction

Various results on polyhedral approximation of convex bodies form an established topic of classical convex geometry (see, e. g., the surveys of Gruber [5] and Bronshteĭn [2]), where the measures of proximity are usually given in terms of some metrics (like Hausdorff distance) on the space of convex bodies in $\mathbb{R}^n$. While this approach is efficient in dealing with compact convex sets, metric estimates alone are seldom suitable to measure the similarity between *unbounded* convex sets. Due to this difficulty, additional assumptions on the nature of polyhedral approximation have to be made. One of such assumptions is that of geometric inclusion, where a given convex set $K \subset \mathbb{R}^n$ is approximated by a polyhedron which either contains K or is contained in K. In this regard, the following historic reference is due to Minkowski (see [7, p. 139] for $n = 3$ and [1, p. 39] for $n \geq 3$).

Proposition 1.1. (Minkowski) *Given a convex body $K \subset \mathbb{R}^n$ and a scalar $\varepsilon > 0$, the assertions below hold.*

(1) *There is an n-dimensional polytope $P \subset \mathbb{R}^n$ satisfying the inclusions $P \subset K \subset B_\varepsilon(P)$, where $B_\varepsilon(P)$ denotes the closed ε-neighborhood of P.*

(2) *For any point $v \in \operatorname{int} K$, there is an n-dimensional polytope $P \subset \mathbb{R}^n$ such that $v \in \operatorname{int} P$ and $P \subset K \subset v + (1+\varepsilon)(P - v)$.* □

We observe that the set $v + (1+\varepsilon)(P - v)$ is the homothetic copy of P with center v and coefficient $1 + \varepsilon$ (see Figure 1).

It is easy to see that Proposition 1.1 does not hold for the case of unbounded convex sets. For instance, the solid parabola $K = \{(x, y) : y \geqslant x^2\}$ in the plane $\mathbb{R}^2$ cannot be approximated by a convex polygonal region. So, it is natural to ask for the

ISSN 0944-6532 / $ 2.50

description of those convex sets in $\mathbb{R}^n$ which can be approximated by polyhedra in the spirit of Minkowski's assertions. Such a description, regarding the first part of Proposition 1.1, is given by Ney and Robinson [8]. We need some auxiliary definitions to formulate their result.

Following [3, 12], we say that a closed convex set $K \subset \mathbb{R}^n$ is *self-bounded* provided there is a bounded set $X \subset \mathbb{R}^n$ such that $K \subset X + \operatorname{rec} K$, where $\operatorname{rec} K$ denotes the recession cone of K. Clearly, a compact convex set K is self-bounded because $\operatorname{rec} K = \{o\}$. (In fact, both papers [3, 12] define the self-boundedness of K by the stronger condition $K \subset y + \operatorname{rec} K$, where y is a suitable point in $\mathbb{R}^n$.)

Next, the closed ε-neighborhood $B_\varepsilon(X)$, $\varepsilon > 0$, of a nonempty set $X \subset \mathbb{R}^n$ is given by $B_\varepsilon(X) = X + B_\varepsilon(o)$, where $B_\varepsilon(o) \subset \mathbb{R}^n$ is the (closed) ball of radius ε centered at the zero vector o.

Proposition 1.2. ([8]) *Given a closed convex set $K \subset \mathbb{R}^n$, the following conditions are equivalent.*

(a) *K is self-bounded and its recession cone* $\operatorname{rec} K$ *is polyhedral.*

(b) *For any scalar $\varepsilon > 0$, there is a polyhedron $P \subset \mathbb{R}^n$ such that $P \subset K \subset B_\varepsilon(P)$.* □

Our goal here is to establish an affine analog of Proposition 1.2 in the spirit of the second assertion of Minkowski.

Theorem 1.3. *Given a closed convex set $K \subset \mathbb{R}^n$, the following conditions are equivalent.*

(a) *K is self-bounded and its recession cone* $\operatorname{rec} K$ *is polyhedral.*

(b) *For any point $v \in \operatorname{rint} K$ and a scalar $\varepsilon > 0$, there is a polyhedron $P \subset \mathbb{R}^n$ such that $v \in \operatorname{rint} P$ and $P \subset K \subset v + (1+\varepsilon)(P - v)$.*

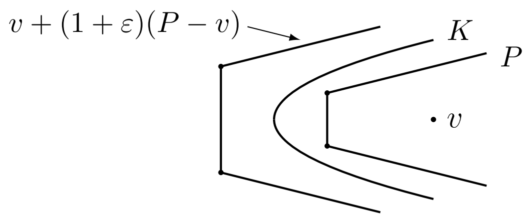

Figure 1: Illustration for Theorem 1.3.

The proof of Theorem 1.3 is based on Proposition 1.2 and the following result of proper interest.

Theorem 1.4. *Given a closed convex set $K \subset \mathbb{R}^n$ and a point $v \in \operatorname{rint} K$, let a ball $B_\rho(v) \subset \mathbb{R}^n$ of radius $\rho > 0$ centered at v satisfy the inclusion* $\operatorname{aff} K \cap B_\rho(v) \subset K$. *Then, for any choice of $\varepsilon > 0$, one has*

$$\operatorname{aff} K \cap B_{\varepsilon\rho}(K) \subset v + (1+\varepsilon)(K - v). \tag{1}$$

Theorem 1.4 shows that, generally, metric neighborhoods $B_\delta(P)$ approximate a self-bounded set K better than the homothetic copies $v + (1+\varepsilon)(P - v)$. On the other hand, the polyhedral sets $v+(1+\varepsilon)(P-v)$ are more suitable for the description from the computational point of view. Also, we observe that the polyhedron P can be of smaller dimension than K in Proposition 1.2 and should be of the same dimension as K in Theorem 1.3.

Generally, the opposite to (1) inclusion does not hold. For instance, Example 3.2 below deals with a convex set $K \subset \mathbb{R}^2$ such that no metric neighborhood $B_\delta(K)$, $\delta > 0$, contains a given homothetic copy $v + (1+\varepsilon)(K-v)$ of K.

Our next theorem describes a family of convex sets in $\mathbb{R}^n$ for which the opposite to (1) inclusion holds. We recall (see, e. g., [4] and [11]) that a closed convex set $K \subset \mathbb{R}^n$ is Motzkin decomposable (M-decomposable, for short) if it can be expressed as $K = C + \operatorname{rec} K$, where C is a compact convex set. Clearly, polyhedra are M-decomposable sets, and the latter are self-bounded.

Theorem 1.5. *Let $K = C + \operatorname{rec} K$ be an M-decomposable set in $\mathbb{R}^n$, $\delta = \operatorname{diam} C$, and $v \in C$. Then, for any choice of $\varepsilon > 0$, one has*

$$K \subset v + (1+\varepsilon)(K-v) \subset \operatorname{aff} K \cap B_{\varepsilon\delta}(K).$$

Corollary 1.6. *Let a polyhedron $P \subset \mathbb{R}^n$ be expressed as $P = Q + \operatorname{rec} P$, where Q is a polytope. Also, let $\delta = \operatorname{diam} Q$ and $v \in Q$. Then, for any choice of $\varepsilon > 0$, one has*

$$P \subset v + (1+\varepsilon)(P-v) \subset \operatorname{aff} P \cap B_{\varepsilon\delta}(P).$$

In view of the above results, we pose the following problems.

Problem 1.7. *Let $K \subset \mathbb{R}^n$ be a closed convex set. Is it true that K is self-bounded if and only if any of the conditions below holds?*

(a) *For any scalar $\varepsilon > 0$, there is an M-decomposable set $F \subset \mathbb{R}^n$ such that $F \subset K \subset B_\varepsilon(F)$.*

(b) *For any point $v \in \operatorname{rint} K$ and a scalar $\varepsilon > 0$, there is an M-decomposable set $F \subset \mathbb{R}^n$ such that $v \in \operatorname{rint} F$ and $F \subset K \subset v + (1+\varepsilon)(F-v)$.*

Problem 1.8. *Let $K \subset \mathbb{R}^n$ be a closed convex set. Is it true that the conditions below are equivalent?*

(a) *K is self-bounded.*

(b) *For any point $v \in K$ and a scalar $\varepsilon > 0$, there is a scalar $\mu > 0$ such that*

$$v + (1+\varepsilon)(K-v) \subset \operatorname{aff} K \cap B_\mu(K).$$

2. Preliminaries

This section describes the necessary notation, terminology, and results on sets in the n-dimensional Euclidean space $\mathbb{R}^n$ (see, e. g., [10] for details). The elements of $\mathbb{R}^n$ are called vectors (or points), and o stands for the zero vector of $\mathbb{R}^n$. Given distinct points $u, v \in \mathbb{R}^n$, we denote by $[u,v]$, (u,v), and $l(u,v)$ the closed segment, the open segment with endpoints u and v, and the line through u and v.

An r-dimensional *plane* L in $\mathbb{R}^n$, where $0 \le r \le n$, is a translate, $L = a + S$, $a \in \mathbb{R}^n$, of a suitable r-dimensional vector subspace S of $\mathbb{R}^n$. A nonempty set $L \subset \mathbb{R}^n$ is a plane if and only if $(1-\lambda)u + \lambda v \in L$ whenever $u, v \in L$ and $\lambda \in \mathbb{R}$. The *affine span* of a set $X \subset \mathbb{R}^n$, denoted aff X, is the intersection of all planes containing X, and dim X is defined as the dimension of the plane aff X.

It what follows, $B_\rho(v)$ denotes the (closed) ball of radius $\rho >$ centered at v. For points $u, v \in \mathbb{R}^n$ and scalars $\rho, \mu > 0$, one has

$$B_{\rho+\mu}(u+v) = B_\rho(u) + B_\mu(v).$$

The *inf*-distance between nonempty sets X and Y in $\mathbb{R}^n$ is defined by

$$\delta(X,Y) = \inf\{\|x-y\| : x \in X,\, y \in Y\}.$$

The interior and closure of a set $X \subset \mathbb{R}^n$ are denoted $\operatorname{int} X$ and $cl\, X$, respectively. A *convex body* in $\mathbb{R}^n$ is a compact convex set with nonempty interior.

A point v of a convex set $K \subset \mathbb{R}^n$ is said to be relatively interior to K provided there is a scalar $\rho > 0$ such that $\operatorname{aff} K \cap B_\rho(v) \subset K$. The set of relatively interior points of K is called the *relative interior* of K and is denoted $\operatorname{rint} K$.

A nonempty set $D \subset \mathbb{R}^n$ is called a *cone* (with apex o) provided $\lambda x \in D$ whenever $\lambda \geq 0$ and $x \in D$. Obviously, this definition implies the inclusion $o \in D$ (although a stronger condition $\lambda > 0$ can be beneficial; see, e. g., [6]). Clearly, $\mu D = D$ whenever $\mu > 0$.

The *recession cone* of a nonempty convex set $K \subset \mathbb{R}^n$ is defined by

$$\operatorname{rec} K = \{e \in \mathbb{R}^n : x + \lambda e \in K \text{ whenever } x \in K \text{ and } \lambda \geq 0\}.$$

The recession cone of a closed convex set $K \subset \mathbb{R}^n$ is closed and convex. Furthermore, $\operatorname{rec} K = \{o\}$ if and only if K is bounded.

A (convex) *polyhedron* is the intersection of finitely many closed halfspaces, and a *polytope* is a bounded polyhedron. Any polyhedron $P \subset \mathbb{R}^n$ can be expressed as $P = Q + \operatorname{rec} P$, where Q is a polytope and $\operatorname{rec} P$ is a polyhedral cone with apex o.

Lemma 2.1. *For a closed convex set $K \subset \mathbb{R}^n$ the following assertions hold.*

(a) $\operatorname{rec}(u + \mu K) = \operatorname{rec} K$ *whenever $u \in \mathbb{R}^n$ and $\mu > 0$.*

(b) $X + \operatorname{rec} K \subset K$ *for any subset X of K.*

(c) $\operatorname{rec} M \subset \operatorname{rec} K$ *for any convex subset M of K.*

Lemma 2.2. ([9]) *For closed convex sets K_1 and K_2 and a nonempty bounded set X in $\mathbb{R}^n$, one has $K_1 \subset K_2 \Leftrightarrow K_1 + X \subset K_2 + X$.*

3. Proof of Theorem 1.4

Lemma 3.1. *Let $K \subset \mathbb{R}^n$ be a convex set and v be a point in K. For scalars $0 \leq \lambda \leq \mu$, one has*

$$v + \lambda(K - v) \subset v + \mu(K - v) \subset \operatorname{aff} K. \qquad (2)$$

Proof. Clearly, both sets $v+\lambda(K-v)$ and $v+\mu(K-v)$ contain v. Choose any point $x \in v+\lambda(K-v)$. Then $x = v+\lambda(u-v)$ for a suitable point $u \in K$. Since

$$v+\mu(u-v) \in v+\mu(K-v) \ \text{ and } \ 0 < \frac{\lambda}{\mu} \le 1,$$

the convexity of $v+\mu(K-v)$ gives

$$\begin{aligned} x = v+\lambda(u-v) &= \Big(1-\frac{\lambda}{\mu}\Big)v + \frac{\lambda}{\mu}\Big(v+\mu(u-v)\Big) \\ &\in [v, v+\mu(u-v)] \subset v+\mu(K-v). \end{aligned}$$

Consecutively, $v+\lambda(K-v) \subset v+\mu(K-v)$.

For the second inclusion in (2), choose any point $x \in v+\mu(K-v)$. Then we get $x = v+\mu(u-v)$ for a suitable point $u \in K$. Since x is an affine combination of u and v, we have $x \in \operatorname{aff}\{u,v\} \subset \operatorname{aff} K$. □

Proof of Theorem 1.4. Choose any point $x \in \operatorname{aff} K \cap B_{\varepsilon\rho}(K)$. Since we have $K \subset \operatorname{aff} K \cap B_{\varepsilon\rho}(K)$, we consider separately the two possible cases $x \in K$ and $x \in \operatorname{aff} K \cap B_{\varepsilon\rho}(K) \setminus K$:

Case 1. Let $x \in K$. Then, according to Lemma 3.1,

$$x \in K = v+(1+0)(K-v) \subset v+(1+\varepsilon)(K-v).$$

Case 2. Let $x \in \operatorname{aff} K \cap B_{\varepsilon\rho}(K)\setminus K$. Then $x \notin \operatorname{aff} K \setminus B_\rho(v)$, due to the assumption $\operatorname{aff} K \cap B_\rho(v) \subset K$. Consequently, $\|x-v\| > \rho$. The inclusion $x \in B_{\varepsilon\rho}(K)$ implies the existence of a point $u \in K$ such that $\|x-u\| \le \varepsilon\rho$. Clearly, $x \ne u$, due to $x \notin K$.

In view of the inclusion $\operatorname{aff} K \cap B_\rho(v) \subset K$, we consider separately the possible cases $u \in \operatorname{aff} K \cap B_\rho(v)$ and $u \in K \setminus (\operatorname{aff} K \cap B_\rho(v))$.

Case 2a. Let $u \in \operatorname{aff} K \cap B_\rho(v)$. Then $\|u-v\| \le \rho$ and

$$\|x-v\| \le \|x-u\| + \|u-v\| \le \varepsilon\rho + \rho = (1+\varepsilon)\rho.$$

Let $y = v+(1+\varepsilon)^{-1}(x-v)$. Clearly, $y \in l(x,v) \subset \operatorname{aff} K$. Furthermore, $y \in B_\rho(v)$, due to

$$\|y-v\| = (1+\varepsilon)^{-1}\|x-v\| \le (1+\varepsilon)^{-1}(1+\varepsilon)\rho = \rho.$$

Consequently, $y \in \operatorname{aff} K \cap B_\rho(v) \subset K$, which gives

$$x = v+(1+\varepsilon)(y-v) \in v+(1+\varepsilon)(K-v).$$

Case 2b. Let $u \in K \setminus (\operatorname{aff} K \cap B_\rho(v))$. Then $\|u-v\| > \rho$. Under this assumption, we will consider separately the cases $u \in l(x,v)$ and $u \notin l(x,v)$

Case 2b1. Suppose first that $u \in l(x,v)$. Since the points x, u, v are pairwise distinct, possible placements of u on the line $l(x,v)$ can be characterized by one of the inclusions: $x \in (u,v)$, $u \in (x,v)$, and $v \in (x,u)$. We consider each of these cases separately.

First, let $x \in (u, v)$. Then $x \in K$ by the convexity of K, which contradicts the above assumption $x \notin K$.

Next, let $u \in (x, v)$. Then $u = \eta x + (1 - \eta)v$ for a suitable scalar $0 < \eta < 1$. Consequently, $x = v + \eta^{-1}(u - v)$, where $\eta^{-1} > 1$. Then

$$\eta^{-1} = \frac{\|x - v\|}{\|u - v\|} = \frac{\|x - u\| + \|u - v\|}{\|u - v\|} = 1 + \frac{\|x - u\|}{\|u - v\|} < 1 + \frac{\varepsilon\rho}{\rho} = 1 + \varepsilon.$$

According to Lemma 3.1,

$$x = v + (x - v) = v + \eta^{-1}(u - v) \in v + \eta^{-1}(K - v) \subset v + (1 + \varepsilon)(K - v).$$

Finally, let $v \in (x, u)$. Since $\|x - v\| > \rho$, there is a point $u' \in (x, v)$ such that $\|u' - v\| = \rho$. Then $u' \in \operatorname{aff} K \cap B_\rho(v) \subset K$. Repeating the argument from the previous case, we obtain the existence of a scalar $0 < \mu < 1 + \varepsilon$ such that $x - v = \mu(u' - v)$ and

$$x = v + (x - v) = v + \mu(u' - v) \in v + \mu(K - v) \subset v + (1 + \varepsilon)(K - v).$$

Case 2b2. Suppose that $u \notin l(x, v)$. Let

$$w = v + \rho\,\frac{x - u}{\|x - u\|}.$$

Then $w \in \operatorname{aff}\{x, u, v\} \subset \operatorname{aff} K$. Clearly, $\|v - w\| = \rho$, which gives the inclusion $w \in \operatorname{aff} K \cap B_\rho(v) \subset K$. Furthermore, the segments $[v, w]$ and $[u, x]$ are parallel (see Figure 2). Denote by z the point of intersection of the open segments (x, v) and (u, w). We have $z \in (u, w) \subset K$ due to the convexity of K, and the inclusion $z \in (x, v)$ implies that $z = \gamma x + (1 - \gamma)v$, where $0 < \gamma < 1$. Equivalently, we have $(1 - \gamma)(z - v) = \gamma(x - z)$.

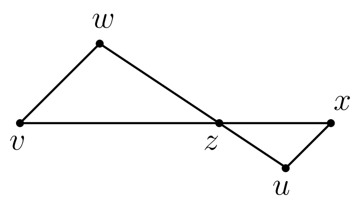

Figure 2: Illustration to Case 2b2.

The triangles $\Delta(v, w, z)$ and $\Delta(x, u, z)$ are similar. Furthermore, $\|v - w\| = \rho$ and $\|x - u\| \le \varepsilon\rho$. So,

$$\frac{1}{\varepsilon} = \frac{\rho}{\varepsilon\rho} \le \frac{\|v - w\|}{\|x - u\|} = \frac{\|z - v\|}{\|x - z\|} = \frac{\gamma}{1 - \gamma},$$

which gives $\gamma^{-1} \le 1 + \varepsilon$. Consequently, Lemma 3.1 gives

$$x = v + \gamma^{-1}(z - v) \in v + \gamma^{-1}(K - v) \subset v + (1 + \varepsilon)(K - v). \qquad \square$$

The example below shows that the opposite to inclusion (1) does not hold.

Example 3.2. Let $K = \{(x, y) : y \geqslant x^2\}$ and $v = (0, 1)$. We are going to show that no ρ-neighborhood $B_\rho(K)$ of K contains its homothetic copy

$$K' = v + 2(K - v) = \{(x, y) : y \geq \tfrac{1}{2}x^2 - 1\}.$$

For this, choose a point $u_1 = (x_1, x_1^2) \in \operatorname{bd} K$, with $x_1 > 0$. The line l tangent to K at u_1 is given by the equation $y - x_1^2 = 2x_1(x - x_1)$, and the orthogonal to l line l^* is given by $y - x_1^2 = -\frac{1}{2x_1}(x - x_1)$. The line l^* meets $\operatorname{bd} K'$ (given by $y = \frac{1}{2}x^2 - 1$) at two distinct points. An easy computation shows that the nearest to u_1 point in $l^* \cap \operatorname{bd} K'$ is $u_2 = (x_2, x_2^2)$, where

$$x_2 = \frac{\sqrt{8x_1^4 + 12x_1^2 + 1} - 1}{2x_1}.$$

Clearly, u_1 is the nearest to u_2 point in K; so, the inf-distance $\delta(u_2, K)$ equals $\|u_1 - u_2\|$. It is easy to see that

$$\begin{aligned} \|u_1 - u_2\| &= \sqrt{(x_1 - x_2)^2 + (x_1^4 - x_2^4)^2} \geqslant |x_1 - x_2| \\ &= \frac{\sqrt{8x_1^4 + 12x_1^2 + 1} - 2x_1^2 - 1}{2x_1} \geqslant \frac{x_1}{3}. \end{aligned}$$

Therefore, $\delta(u_2, K) \to \infty$ as $x_1 \to \infty$. So, no closed ρ-neighborhood $B_\rho(K)$ of K contains K'.

4. Proof of Theorem 1.3

Lemma 4.1. *Let $L \subset \mathbb{R}^n$ be a plane and $B_\rho(v) \subset \mathbb{R}^n$ be a ball centered at a point $v \in L$. Then there is a polytope $T \subset L$ such that $\operatorname{aff} T = L$ and*

$$L \cap B_{\rho/2}(v) \subset T \subset L \cap B_\rho(v). \tag{3}$$

Proof. Since the case $\dim L = 0$ trivially holds (then $L = \{v\}$), we let L be of a positive dimension m. Let $S = L - v$. Then S is a subspace of dimension m and $S \cap B_\rho(o)$ is the ball in S of radius ρ centered at o. We identify S with $\mathbb{R}^m$ and $S \cap B_\rho(o)$ with the ball $V_\rho(o) \subset \mathbb{R}^m$ By Proposition 1.1, there is an m-dimensional polytope $T' \subset \mathbb{R}^m$ satisfying the inclusions $T' \subset V_\rho(o) \subset V_{\rho/2}(T')$. Since

$$V_\rho(o) = V_{\rho/2}(o) + V_{\rho/2}(o) \quad \text{and} \quad V_{\rho/2}(T') = T' + V_{\rho/2}(o),$$

Lemma 2.2 gives $V_{\rho/2}(o) \subset T'$. Thus

$$\begin{aligned} L \cap B_{\rho/2}(v) = v + S \cap B_{\rho/2}(o) &= v + V_{\rho/2}(o) \subset v + T' \\ &\subset v + V_\rho(o) = v + S \cap B_\rho(o) = L \cap B_\rho(v). \end{aligned}$$

Hence the polytope $T = v + T'$ satisfies the inclusions (3). □

Proof of Theorem 1.3. Since the case $\dim K = 0$ is trivial, we may assume that $\dim K > 0$.

(a)⇒(b) Let $K \subset \mathbb{R}^n$ be a self-bounded closed convex set with polyhedral recession cone $\operatorname{rec} K$. Choose a point $v \in \operatorname{rint} K$ and a ball $B_\rho(v) \subset \mathbb{R}^n$ satisfying the condition $\operatorname{aff} K \cap B_\rho(v) \subset K$. According to Lemma 4.1, there is a polytope T such that $\operatorname{aff} T = \operatorname{aff} K$ and

$$\operatorname{aff} K \cap B_{\rho/2}(v) \subset T \subset \operatorname{aff} K \cap B_\rho(v) \subset K.$$

Let $\varepsilon > 0$ and $\delta = \varepsilon\rho/2$. By Proposition 1.2, there is a polyhedron P such that $P \subset K \subset B_\delta(P)$. Since $K \subset \operatorname{aff} K$, we have $K \subset \operatorname{aff} K \cap B_\delta(P)$. Furthermore, $\operatorname{rec} P = \operatorname{rec} K$, as proved in [8].

Express P as $P = Q + \operatorname{rec} P$, where Q is a suitable polytope. Put $Q' = \operatorname{Conv}(Q \cup T)$. Clearly, Q' is a polytope and $T \subset Q' \subset K$ by the convexity of K. Furthermore, $\operatorname{aff} Q' = \operatorname{aff} K$ and

$$\operatorname{aff} K \cap B_{\rho/2}(v) \subset T \subset Q'.$$

Consider the polyhedron $P' = Q' + \operatorname{rec} P$. It easy to see that $\operatorname{aff} P' = \operatorname{aff} Q' = \operatorname{aff} K$ and $\operatorname{rec} P' = \operatorname{rec} P$. Thus $\operatorname{rec} P' = \operatorname{rec} K$. Consequently,

$$P = Q + \operatorname{rec} P \subset Q' + \operatorname{rec} P' = P' = Q' + \operatorname{rec} K \subset K.$$

Furthermore,

$$K \subset \operatorname{aff} K \cap B_\delta(P) \subset \operatorname{aff} K \cap B_\delta(P') \quad \text{and} \quad \operatorname{aff} P' \cap B_{\rho/2}(v) \subset P'.$$

Consequently, $v \in \operatorname{rint} P'$. Theorem 1.4 implies the inclusion

$$P' \subset K \subset \operatorname{aff} K \cap B_\delta(P') = \operatorname{aff} P' \cap B_{(\rho/2)\varepsilon}(P') \subset v + (1+\varepsilon)(P' - v).$$

Summing up, the polyhedron P' satisfies condition (b) of Theorem 1.3.

(b)⇒(a) Choose a point $v \in \operatorname{rint} K$ and scalar $\varepsilon > 0$. By the assumption, there is a polyhedron $P \subset \mathbb{R}^n$ such that $v \in \operatorname{rint} P$ and $P \subset K \subset v + (1+\varepsilon)(P - v)$. Express P as $P = Q + \operatorname{rec} P$, where Q is a suitable polytope and $\operatorname{rec} P$ is a polyhedral cone. Since all three sets P, K, and $v + (1+\varepsilon)(P - v)$ are closed and convex, Lemma 2.1 gives

$$\operatorname{rec} P \subset \operatorname{rec} K \subset \operatorname{rec}(v + (1+\varepsilon)(P - v)) = \operatorname{rec} P.$$

Hence $\operatorname{rec} K = \operatorname{rec} P$, which shows that the cone $\operatorname{rec} K$ is polyhedral. Furthermore,

$$\begin{aligned} K &\subset v + (1+\varepsilon)(P - v) = v + (1+\varepsilon)(Q + \operatorname{rec} K - v) \\ &= v + (1+\varepsilon)(Q - v) + (1+\varepsilon)\operatorname{rec} K = v + (1+\varepsilon)(Q - v) + \operatorname{rec} K. \end{aligned}$$

Because $v + (1+\varepsilon)(Q - v)$ is a polytope, it follows that K is self-bounded. □

5. Proof of Theorem 1.5

Lemma 5.1. *Let $C \subset \mathbb{R}^n$ be a bounded convex set of diameter $\delta > 0$ and v be a point in C. For any choice of $\varepsilon > 0$, one has*

$$C \subset v + (1+\varepsilon)(C - v) \subset \operatorname{aff} C \cap B_{\varepsilon\delta}(C).$$

Proof. Choose a point $x \in v + (1+\varepsilon)(C - v)$. Then $x = v + (1+\varepsilon)(u - v)$ for a suitable point $u \in C$. Hence $x - u = \varepsilon(u - v)$, which gives

$$\|x - u\| = \varepsilon \, \|u - v\| \le \varepsilon \operatorname{diam} C = \varepsilon\delta.$$

So, $x \in B_{\varepsilon\delta}(v) \subset B_{\varepsilon\delta}(C)$. Since x is an affine combination of the points $u, v \in C$, we obtain $x \in \operatorname{aff} C \cap B_{\varepsilon\delta}(C)$. $\square$

Proof of Theorem 1.5. Because $v + (1+\varepsilon)(K - v) \subset \operatorname{aff} K$ (see Lemma 3.1), it suffices to prove the inclusion

$$v + (1+\varepsilon)(K - v) \subset B_{\varepsilon\delta}(K).$$

By Lemma 5.1, $v + (1+\varepsilon)(C - v) \subset B_{\varepsilon\delta}(C)$. Therefore,

$$\begin{aligned} v + (1+\varepsilon)(K - v) &= v + (1+\varepsilon)(C + \operatorname{rec} K - v) \\ &= v + (1+\varepsilon)(C - v) + (1+\varepsilon)\operatorname{rec} K \\ &= v + (1+\varepsilon)(C - v) + \operatorname{rec} K \subset B_{\varepsilon\delta}(C) + \operatorname{rec} K \\ &= B_{\varepsilon\delta}(o) + C + \operatorname{rec} K = B_{\varepsilon\delta}(o) + K = B_{\varepsilon\delta}(K). \qquad \square \end{aligned}$$

Acknowledgements. The author thanks the referee for helpful comments on an earlier draft of the paper.

References

[1] T. Bonnesen, W. Fenchel: *Theorie der konvexen Körper*, Springer, Berlin (1934).

[2] E. M. Bronshteĭn: *Approximation of convex sets by polyhedra*, J. Math. Sci. (N.Y.) 153 (2008) 727–762.

[3] D. Dörfler: *On the approximation of unbounded convex sets by polyhedra*, J. Optim. Theory Appl. 194 (2022) 265–287.

[4] M. Goberna, E. González, J. E. Martínez-Legaz, M. I. Todorov: *Motzkin decomposition of closed convex sets*, J. Math. Anal. Appl. 364 (2010) 209–221.

[5] P. M. Gruber: *Aspects of approximation of convex bodies*, in: *Handbook of Convex Geometry, Vol. A*, North-Holland, Amsterdam (1993) 319–345.

[6] J. Lawrence, V. Soltan: *On unions and intersections of nested families of cones*, Beitr. Algebra Geom. 57 (2016) 655–665.

[7] H. Minkowski: *Gesammelte Abhandlungen. Band 2*, Teubner, Leipzig (1911).

[8] P. E. Ney, S. M. Robinson: *Polyhedral approximation of convex sets with an application to large deviation probability theory*, J. Convex Analysis 2 (1995) 229–240.

[9] H. Rådström: *An embedding theorem for spaces of convex sets*, Proc. Amer. Math. Soc. 3 (1952) 165–169.

[10] V. Soltan: *Lectures on Convex Sets*, 2nd ed., World Scientific, Hackensack (2020).

[11] V. Soltan: *On M-decomposable sets*, J. Math. Anal. Appl. 485/2 (2020), art.no. 123816, 15p.

[12] F. Ulus, *Tractability of convex vector optimization problems in the sense of polyhedral approximation*, J. Global Optim. 72 (2018) 731–742.