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The Ever Large Subspace $C_p(Y|X)$: Distinguished, Montel, Covered Nicely?

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$C_p(Y|X)$ denotes the real-valued continuous functions on $Y \subseteq X$ having continuous extensions to a Tychonoff space X, with pointwise topology inherited from $C_p(Y)$. We recently proved $C_p(Y)$ *is distinguished* $\Leftrightarrow$ *it is a large subspace of* $\mathbb{R}^Y$. We prove $C_p(Y|X)$ *is always a large subspace of* $C_p(Y)$. Thus $C_p(Y|X)$ *is always quasibarrelled; always has a feral strong dual; has a quasibarrelled countable enlargement* $\Leftrightarrow$ Y *is infinite; is distinguished* $\Leftrightarrow$ $C_p(Y)$ *is distinguished; is a Montel space* $\Leftrightarrow$ Y *is discrete and* C*-embedded in* X. 'Nice' countable covers for $C_p(Y|X)$ yield potent summary theorems that solve open problems, characterize P-spaces anew, and complete the list of Velichko variations. For example, Summary III: *Assume* Y *is dense in* X. Y *is a* P*-space, or* X *is pseudocompact, or both* $\Leftrightarrow$ $C_p(Y|X)$ *is countably covered by sets that are, respectively, relatively sequentially complete in* $C_p(Y)$*, or bounded, or both.* Putting $Y = X$, one quickly comprehends Velichko variations *à la* Arkhangel'skiĭ.

Keywords: Large subspace, distinguished, Montel, quasibarrelled, $C_p(X)$, C-embedded.

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1. Introduction

In what follows, X is a Tychonoff space with subset Y. X is a *P-space* if G_δ sets are open. We denote by $C_p(X)$ the linear space $C(X)$ of real-valued continuous functions on X equipped with the pointwise topology τ_p. The dual of $C_p(X)$ is denoted by $L(X)$, or by $L_\beta(X)$ when provided with the strong topology. We conventionally view X as a Hamel basis for $L(X)$. By $L_X(Y)$ we denote the subspace of $L_\beta(X)$ spanned by Y. The locally convex space $C_p(Y|X)$ is defined in [1, 0.4] as the range (the image) of $C_p(X)$ in $C_p(Y)$ under the restriction map $f \mapsto f|_Y$ equipped with the relative topology of $C_p(Y)$. A subset S of a locally convex space E is *relatively sequentially complete* if every Cauchy sequence in S converges in E.

Classic and recent characterizations [9, Theorem 2.3] of Montel spaces $C_p(X)$ beg the question: When is $C_p(Y|X)$ Montel? Or when is $C_p(Y|X)$ distinguished, asks the very recently rediscovered world of distinguished $C_p(X)$ spaces [2, 4–13, 16–18, 20, 21, 23]. We shall see that $C_p(Y|X)$ *is distinguished if and only if* $C_p(Y)$

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is distinguished. Also, $C_p(Y|X)$ *is Montel if and only if* $C_p(Y)$ is Montel and $C_p(Y) = C_p(Y|X)$.

A locally convex space E is compact only trivially, when $\dim E = 0$. Yet E is σ-compact (a countable union of compact sets) if, *e.g.*, $\dim E \leq \aleph_0$. Spaces of the form $C_p(X)$ are σ-compact only when $\dim C_p(X) = |X| < \aleph_0$ (Velichko's theorem). But many large-dimensional distinguished spaces $C_p(Y|X)$ are σ-compact. $C_p(\mathbb{N}|\beta\mathbb{N})$ is distinguished, σ-compact, and $\mathfrak{c}$-dimensional. Indeed, setwise, $C_p(\mathbb{N}|\beta\mathbb{N}) = \ell_\infty$, since the discrete $\mathbb{N}$ is C^*-embedded in the Stone-Čech compactification $\beta\mathbb{N}$ (see Theorem 15 and Example 23).

We encountered $C_p(Y|X)$ in writing [9], while extending results by Arkhangel'skiĭ, Buchwalter/Schmets, Velichko, Tkachuk/Shakhmatov, Ferrando/Kąkol. For example,

Theorem 1. [9, Theorem 3.3(α)] *Assume Y is dense in X. If $C_p(Y|X)$ is a countable union of bounded relatively sequentially complete sets, then Y is a P-space and X is pseudocompact.*

In Arkhangel'skiĭ's original [1, I.2.2], $C_p(Y|X)$ *is σ-countably compact.* Thus, with $Y = X$, he deduces *If $C_p(X)$ is σ-countably compact, then X is finite* (since $X = Y$ is a pseudocompact P-space). His clever proof frames Velichko's theorem and fuels multi-authored results we quickly derive, in Theorem 13, from Theorem 1. But Theorem 20 is the grand prize that maximally and properly extends Theorem 1.

Consider another example from [9]:

Theorem 2. [9, Corollary 3.6] *X is a P-space if and only if, for each dense subset Z of X, the space $C_p(Z|X)$ is a countable union of relatively sequentially complete sets.*

Both Theorems 1 and 2 come full blown in Section 5 where, crucially, certain cover sets need be sequentially complete only relative to the superspace $C_p(Y)$ or $C_p(Z)$.

2. Large subspaces

Laurent Schwartz, Grothendieck, Dieudonné, Köthe developed the standard theory of distinguished locally convex (topological vector) spaces. A linear subspace F of a locally convex space E is said to be *large* if each bounded set in E is contained in the closure in E of some bounded set in F. If E is a locally convex space, the (topological) dual E', given its strong topology $\beta(E', E)$, is sometimes denoted E'_β and called the *strong dual* of E. The *bidual* of E, denoted E'', is the dual of E'_β. Of course, E is canonically considered a subspace of E'', and E'' is a subspace of the algebraic dual E'^* of E'. In fact, the bipolar theorem assures E'' is just the union of all $\sigma(E'^*, E')$-closures in E'^* of bounded sets in E. The weak* bidual of E, denoted E''_σ, is E'' endowed with the $\sigma(E'', E')$ topology. If F is a dense (linear) subspace of E, we routinely identify F' and E', so that F'_β is just E'_β with a coarser topology. The topologies coincide precisely when F is large in E. Then the weak* biduals F''_σ and E''_σ also coincide. One says E is *distinguished* if E is a large subspace of E''_σ. The bipolar theorem implies E *is distinguished if and only if* E'_β *is barrelled.*

When E is normed, the celebrated Banach space E'_β is certainly barrelled: E is distinguished. Not necessarily when $E = C_p(X)$: we proved $C_p(X)$ is distinguished if and only if it is large in the product space $\mathbb{R}^X$ [8, 10]. We realize this same simplicity for the more general spaces $C_p(Y|X)$.

Now $C_p(Y|X)$ is a dense linear subspace of $C_p(Y)$, itself a dense subspace of $\mathbb{R}^Y$. Indeed, if $f \in \mathbb{R}^Y$ and σ is a finite subset of Y, there exists $h \in C(X)$ agreeing with f on σ. So $h|_Y$ is a function in $C_p(Y|X)$ which agrees with f on σ, proving density. Thus the three spaces $C_p(Y|X)$, $C_p(Y)$ and $\mathbb{R}^Y$, the latter equipped with the product topology, have the same dual, namely $L(Y)$. The corresponding three strong topologies on $L(Y)$ grow increasingly finer, the finest being $\beta\left(L(Y), \mathbb{R}^Y\right)$, well-known as the strongest locally convex topology on $L(Y)$. We shall denote the bidual of the locally convex space $C_p(Y)$ by $M(Y)$, the bidual of $C_p(Y|X)$ by $M(Y|X)$. Under their weak* topologies, both biduals are considered subspaces of the product space $\mathbb{R}^Y$, and we have $M(Y|X) \subseteq M(Y) \subseteq \mathbb{R}^Y$. The reflexive $\mathbb{R}^Y$ is its own bidual, and is the algebraic dual of $L(Y)$.

We write $L_\beta(Y) \succeq L_\beta(Y|X) \succeq L_X(Y)$ to record the evident facts that the strong dual $L_\beta(Y|X)$ of $C_p(Y|X)$ has a topology [coarser] $\langle$finer$\rangle$ than that [of $L_\beta(Y)$] $\langle$of $L_X(Y)\rangle$. We prove, below, that $L_\beta(Y) = L_\beta(Y|X)$. Generally, $L_\beta(Y|X) \neq L_X(Y)$, even when Y is assumed to be dense in X. For counterexample, take $X = \mathbb{R}$ and $Y = \mathbb{Q}$, the dense subset of rationals. Since $\mathbb{Q}$ is countable, $C_p(\mathbb{Q})$ is distinguished and $L_\beta(\mathbb{Q}) = L_\beta(\mathbb{Q}|\mathbb{R})$ is a copy of φ, an $\aleph_0$-dimensional space that has the strongest locally convex topology [6]. But our proof of [12, Theorem 20] explicitly shows $L_\mathbb{R}(\mathbb{Q})$ is *not* a copy of φ. We complete the record by writing

$$L_\beta(Y) = L_\beta(Y|X) \succeq L_X(Y).$$

Thus $M(Y) = M(Y|X)$. There is only one admissible topology for $M(Y)$; indeed, $L_\beta(Y)$ is feral (see (†) and (‡) below), which means that $\beta(M(Y), L(Y)) = \sigma(M(Y), L(Y))$.

Large subspaces preserve quasibarrelledness [22, 8.3.24], so Theorem 4 will prove $C_p(Y|X)$ is quasibarrelled. Or, in addition to the Buchwalther/Schmets classic that

(†) [3] $C_p(X)$ *is always quasibarrelled,*

one could use our simple

(‡) [12, Theorem 11] *A locally convex space E has its weak topology and is [barrelled] $\langle$quasibarrelled$\rangle$ if and only if the [weak*] $\langle$strong$\rangle$ dual is feral* (*i.e.*, all bounded sets are finite-dimensional).

Proof of quasibarrelled $C_p(Y|X)$. Since $C_p(X)$ has its weak topology and is quasibarrelled, $L_\beta(X)$ is feral. Hence its subspace $L_X(Y)$ is feral, as must be $L_\beta(Y|X)$, due to its finer topology. Therefore $C_p(Y|X)$ is quasibarrelled (and has its weak topology). □

Now (†) is the special case $Y = X$. The strong dual $L_\beta(Y|X) = L_\beta(Y)$ is always feral, robustly so: *Every linearly independent sequence in $L_\beta(Y|X)$ admits a subsequence whose span is a complemented copy of φ* [12, Theorem 10]. The weak* dual $L_p(Y|X)$ never contains φ, but when feral, is emphatically so:

Fact. *For $Y \subseteq X$, the following four assertions are equivalent:* (i) $C_p(Y|X)$ *is barrelled;* (ii) $L_p(Y|X)$ *is feral;* (iii) *a set $Z \subseteq Y$ functionally bounded in X [i.e., $h(Z)$ is bounded in $\mathbb{R}$ for all $h \in C(X)$] only if Z is finite;* (iv) *every linearly independent sequence in $L_p(Y|X)$ admits a subsequence whose linear span E is flat [i.e., $E' = E^*$].*

Proof. (Sketch) Surely, (i) $\Leftrightarrow$ (ii) by (‡). Trivially, (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (ii). We show

[(iii) $\Rightarrow$ (iv)]. Suppose W is a linearly independent sequence in $L_p(Y|X)$, where Y serves as a Hamel basis and $L_p(Y|X)' = \{\langle f, \cdot \rangle : f \in C(Y|X)\}$, with the property $\langle f, ay + bz\rangle := af(y) + bf(z)$. For each $w \in W$, let Y_w denote the smallest (finite, nonempty) subset of Y whose span contains w. The set $Z := \bigcup_{w \in W} Y_w$ is denumerable. By (iii), there exists $h \in C(X)$ with $h(Z)$ unbounded in $\mathbb{R}$. Inductively choose $z_n \in Z$ and $w_n \in W$ such that

$$z_n \in Y_{w_n} \text{ and } |h(z_{n+1})| \geq 4 + \max\left\{|h(z)| : z \in \bigcup\nolimits_{1 \leq i \leq n} Y_{w_i}\right\} \text{ for each } n \in \mathbb{N}.$$

Let c_n be the (nonzero) coefficient of z_n in the expansion of w_n. The open neighborhoods

$$\mathcal{O}_n := \{x \in X : |h(x) - h(z_n)| < 1\} \setminus \left\{z \in \bigcup\nolimits_{1 \leq i \leq n} Y_{w_i} : z \neq z_n\right\}$$

of z_n are locally finite. In fact, the triangle inequality implies each x-neighborhood $N_x := \{u \in X : |h(u) - h(x)| < 1\}$ meets at most one of the sets $\mathcal{O}_n$. If each g_i is in $C(X)$ and vanishes off $\mathcal{O}_i$, the formal sum $g := \sum_{i=1}^{\infty} g_i$ is continuous at each $x \in X$, since it agrees with the finite sum $\sum_{i=1}^{k_x} g_i$ on N_x, where k_x is the first positive integer such that N_x misses $\mathcal{O}_i$ for all $i > k_x$. Thus $g \in C(X)$, and $f := g|_Y \in C(Y|X)$.

As g_n vanishes on $\{z \in Y_{w_n} : z \neq z_n\}$, each $\langle g_n, w_n\rangle = c_n g_n(z_n)$. If $i > n$, then $|h(z_i)| \geq 4 + |h(z_n)|$, so $z_n \neq z_i$ and $\mathcal{O}_i$ misses all of Y_{w_n}. Thus $\langle g_i, w_n\rangle = 0$ for $i > n$. Hence

$$\langle f, w_n\rangle = \sum\nolimits_{1 \leq i \leq n} \langle g_i, w_n\rangle = \sum\nolimits_{1 \leq i < n} \langle g_i, w_n\rangle + c_n g_n(z_n).$$

Since X is a Tychonoff space and $c_n \neq 0$, we may inductively choose each g_n such that $\langle f, w_n\rangle$ is an arbitrary real number, and thus $\langle f, \cdot \rangle$ agrees with an arbitrary linear form on the span E of the basis $\{w_n : n \in \mathbb{N}\}$. Therefore $E' = E^*$. □

The case $Y = X$ is classic: *$C_p(X)$ is barrelled if and only if all functionally bounded sets in X are finite* [3, Theorem 1.10].

Of course, $C_p(Y)$ is barrelled if its dense subspace $C_p(Y|X)$ is barrelled. The converse fails, even assuming Y is dense in X: The irrationals $\mathbb{P}$ are dense and relatively discrete in Michael's line $\mathbb{M}$, thus $C_p(\mathbb{P}) = \mathbb{R}^{\mathbb{P}}$ is barrelled. Yet the large subspace $C_p(\mathbb{P}|\mathbb{M})$ is *not* barrelled, since $\mathbb{P}$ contains an infinite set $\left\{\sqrt{2}/n : n \in \mathbb{N}\right\}$ which is functionally bounded in $\mathbb{M}$ (see [12, Example 15]).

Knowing both $C_p(Y|X)$ and $C_p(Y)$ are quasibarrelled with setwise common dual, we prove that $L_\beta(Y|X) = L_\beta(Y)$, equivalently, that $C_p(Y|X)$ is large in $C_p(Y)$. First, some technical details expanding the terminology of [7, 8, 11, 12]. If $Z \in \{X, Y\}$

and $g \in \mathbb{R}^Z$, we put $P_g := \{f \in \mathbb{R}^Z : |f| \leq |g|\}$, a closed bounded set in the Montel space $\mathbb{R}^Z$, hence compact. Obversely, for each nonempty bounded set $A \subseteq \mathbb{R}^Z$ we define $\phi_A \in \mathbb{R}^Z$ by the rule $\phi_A(z) := \sup_{f \in A} |f(z)|$ for each $z \in Z$. We define $\phi_\emptyset \in \mathbb{R}^Z$ to be identically 0. Clearly, $A \subseteq P_{\phi_A}$; $\phi_A \leq \phi_B$ if $A \subseteq B$; and $\phi_A = \phi_B$ if $B = P_{\phi_A}$.

Lemma 3. *If A is a bounded set in $C_p(Y)$, then $A \subseteq \overline{P_{\phi_A} \cap C_p(Y|X)}^{\mathbb{R}^Y}$.*

Proof. There are three steps.

I. Given $y \in Y$, $h \in C(Y)$, $\varepsilon > 0$, and neighborhood U in X of y, we prove *there exist a neighborhood V in X of y with $V \subseteq U$ and a function $f \in C(X)$ such that $|f(y) - h(y)| < \varepsilon$, f vanishes off V, and $f|_Y \in P_h$.*

In case $|h(y)| < \varepsilon$, take $V = U$ with $f = \mathbf{0} \in C(X)$, and we are done.

In case $|h(y)| \geq \varepsilon$, there is scalar a with $|h(y) - a| = |h(y)| - |a| = \varepsilon/2$. As $|h(y)| > |a|$ and $|h|$ is continuous at y, there is a set V open in X with $y \in V \subseteq U$ and $|h(z)| > |a|$ for all $z \in V \cap Y$. The Tychonoff space X admits $f \in C(X)$ such that $f(y) = a$, $|f(x)| \leq |a|$ for each $x \in X$, and f vanishes off V. Now $|f(y) - h(y)| = \varepsilon/2 < \varepsilon$. If $z \in Y \setminus V$, then $|f(z)| = 0 \leq |h(z)|$; and if $z \in V \cap Y$, then $|f(z)| \leq |a| < |h(z)|$. Thus for all $z \in Y$ we have $|f(z)| \leq |h(z)|$; *i.e.*, $f|_Y \in P_h$. □

II. Given a nonempty finite set $\sigma \subseteq Y$, $h \in C(Y)$, and $\varepsilon > 0$, we prove *there exists $f \in C(X)$ such that $f|_Y \in P_h$ and $|f(y) - h(y)| < \varepsilon$ for each $y \in \sigma$.*

Let $\{U_y : y \in \sigma\}$ be a set of disjoint neighborhoods U_y of the points y in σ. For each $y \in \sigma$, use (I) to obtain a y-neighborhood V_y in X with $V_y \subseteq U_y$ and function $f_y \in C(X)$ such that $|f_y(y) - h(y)| < \varepsilon$, f_y vanishes off V_y, and $f_y|_Y \in P_h$. Then $f := \sum_{y \in \sigma} f_y$ is in $C(X)$; for each $y \in \sigma$, disjointness assures $|f(y) - h(y)| = |f_y(y) - h(y)| < \varepsilon$; and if $z \in Y$, then either $f(z) = 0$ or z is in V_y for exactly one $y \in \sigma$. If the latter, then $|f(z)| = |f_y(z)| = |(f_y|_Y)(z)| \leq |h(z)|$ since $f_y|_Y \in P_h$. In both cases, $|f(z)| \leq |h(z)|$, which proves $f|_Y \in P_h$, □

III. Suppose A is a bounded set in $C_p(Y)$ and $h \in A$. From (II), in $\mathbb{R}^Y$ the basic neighborhood of h determined by a finite $\sigma \subseteq Y$ and $\varepsilon > 0$ contains $f|_Y$ for some $f \in C(X)$ such that $f|_Y \in P_h$. Now $f|_Y \in P_h \cap C_p(Y|X)$, so $h \in \overline{P_h \cap C_p(Y|X)}^{\mathbb{R}^Y}$. Hence

$$A \subseteq \bigcup_{h \in A} \overline{P_h \cap C_p(Y|X)}^{\mathbb{R}^Y} \subseteq \overline{\bigcup_{h \in A} P_h \cap C_p(Y|X)}^{\mathbb{R}^Y} \subseteq \overline{P_{\phi_A} \cap C_p(Y|X)}^{\mathbb{R}^Y}. \qquad \square$$

Theorem 4. *$C_p(Y|X)$ is a large subspace of $C_p(Y)$.*

Proof. If A is bounded in $C_p(Y)$, then $P_{\phi_A} \cap C_p(Y|X)$ is bounded in $C_p(Y|X)$ and, via the lemma,

$$A \subseteq C_p(Y) \cap \overline{P_{\phi_A} \cap C_p(Y|X)}^{\mathbb{R}^Y} = \overline{P_{\phi_A} \cap C_p(Y|X)}^{C_p(Y)}. \qquad \square$$

Lemma 5. *If A is a bounded set in $C_p(Y)$, then $P_{\phi_A} = \overline{P_{\phi_A} \cap C_p(Y|X)}^{\mathbb{R}^Y}$.*

Proof. Closures are taken in $\mathbb{R}^Y$. The closed set P_{ϕ_A} contains the right side. We prove the reverse. If Λ is a finite subset of A, then $\phi_\Lambda \in C(Y)$, and as Y is a Tychonoff space, $P_{\phi_\Lambda} = \overline{P_{\phi_\Lambda} \cap C(Y)}$. Define $B := P_{\phi_\Lambda} \cap C(Y)$, a bounded set in $C_p(Y)$, and $D := P_{\phi_\Lambda}$. Note that $B \subseteq D$ and $\phi_D = \phi_\Lambda$. Via Lemma 3,

$$P_{\phi_\Lambda} \cap C(Y) = B \subseteq \overline{P_{\phi_B} \cap C(Y|X)} \subseteq \overline{P_{\phi_D} \cap C(Y|X)} = \overline{P_{\phi_\Lambda} \cap C(Y|X)} \\ \subseteq \overline{P_{\phi_A} \cap C(Y|X)}.$$

Consequently, $P_{\phi_\Lambda} = \overline{P_{\phi_\Lambda} \cap C(Y)} \subseteq \overline{P_{\phi_A} \cap C(Y|X)}$. Hence

$$\overline{\bigcup \{P_{\phi_\Lambda} : \Lambda \text{ is a finite subset of } A\}} \subseteq \overline{P_{\phi_A} \cap C(Y|X)}.$$

Now the closure of the union is clearly P_{ϕ_A}, so $P_{\phi_A} \subseteq \overline{P_{\phi_A} \cap C(Y|X)}$. □

For $g = \mathbf{1}$, Lemma 5 says that the closure $\overline{P_g \cap C(Y|X)}$ in $\mathbb{R}^Y$ is just the unit ball P_g in the Banach space $\ell_\infty(Y)$. Now $\overline{P_g \cap C(Y|X)} \subseteq M(Y|X)$, since $P_g \cap C(Y|X)$ is a bounded set in $C_p(Y|X)$ and $C_p(Y|X)'^* = \mathbb{R}^Y$. Therefore $M(Y|X)$ contains the linear space $\ell_\infty(Y)$. Alternate proof: Since $C_p(Y|X)$ is large in $C_p(Y)$, the biduals $M(Y|X)$ and $M(Y)$ coincide, and it is known that $\ell_\infty(Y) \subseteq M(Y)$.

If $Z \in \{X, Y\}$ and A is a bounded set in $C_p(Z)$, the set

$$A^+ := \bigcup \{P_{\phi_\Lambda} \cap C(Z|X) : \Lambda \text{ is a finite subset of } A\}$$

is bounded in $C_p(Z|X)$ [= either $C_p(Y|X)$ or $C_p(X)$]. This agrees with our notation in [7, 8, 11]. The closure $\overline{A^+}$ in $\mathbb{R}^Z$ is a subset of $M(Z|X)$.

A nice form of Saxon's lemma appears in [7]:

(✠) *If A is bounded in $C_p(X)$, then* $\overline{A^+}^{\mathbb{R}^X} = P_{\phi_A}$.

We generalize, using Lemma 5.

Lemma 6. *If A is bounded in $C_p(Y)$ [with $Y \subseteq X$], then* $\overline{A^+}^{\mathbb{R}^Y} = P_{\phi_A}$.

Proof. $$P_{\phi_A} = \overline{\bigcup \{P_{\phi_\Lambda} : \Lambda \text{ is finite in } A\}} = \overline{\bigcup \left\{\overline{P_{\phi_\Lambda} \cap C(Y|X)} : \Lambda \text{ is finite in } A\right\}} \\ = \overline{\bigcup \{P_{\phi_\Lambda} \cap C(Y|X) : \Lambda \text{ is finite in } A\}} = \overline{A^+}.$$ □

So $\overline{A^+}^{\mathbb{R}^Y}$, unlike A^+, is independent of X, and (✠) is Lemma 6 with $Y = X$. For each bounded A in $\mathbb{R}^Y$ we obviously have $\phi_A = \phi_{\overline{A}}$, and $\phi_B = \phi_A$ if $B = P_{\phi_A}$. If A is bounded in $C_p(Y)$, Lemma 6 yields $\phi_{A^+} = \phi_{\overline{A^+}} = \phi_B = \phi_A$, proving $\phi_{A^+} = \phi_A$.

Theorem 4 and the bipolar theorem yield

Corollary 7. *The strong duals of $C_p(Y|X)$ and $C_p(Y)$ coincide, as do their biduals $M(Y|X)$ and $M(Y)$.*

Corollary 8. *$C_p(Y|X)$ is quasibarrelled, and has a quasibarrelled countable enlargement if Y is infinite.*

Proof. Since $C_p(Y)$ is quasibarrelled [3], so are its large subspaces [22, 8.3.24]. If Y is infinite, one easily finds bounded sequences in $C_p(X)$ with restrictions to Y that are linearly independent, so by definition $C_p(Y|X)$ is nonferal, and nonferal quasibarrelled spaces admit quasibarrelled countable enlargements [24, Corollary 3.4]. □

In fact, if Y is infinite, $C_p(Y|X)$ contains (uniformly) bounded $\mathfrak{c}$-dimensional sets, so

Theorem 9. *Assume Y is infinite. $C_p(Y|X)$ has a barrelled countable enlargement if (and only if) $C_p(Y|X)$ is barrelled [25, Theorem 1].*

3. When $C_p(Y|X)$ is distinguished

We [6, 8, 10] unwittingly began the ongoing revival of distinguished $C_p(X)$ spaces, studied fifty years earlier in more obscure terms by Prof. Denny Gulick [13]. Perhaps we are first to study distinguished $C_p(Y|X)$ spaces.

A set $\mathcal{M}$ in $\mathbb{R}^Y$ is *cofinal* [7] if for each $g \in \mathbb{R}^Y$ there exists some $f \in \mathcal{M}$ with $|g| \leq |f|$.

Theorem 10. *These seven assertions are equivalent:*

(1) *$C_p(Y|X)$ is distinguished.*

(2) *$C_p(Y)$ is distinguished.*

(3) *The set $\{\phi_A : A \text{ is bounded in } C_p(Y|X)\}$ is cofinal in $\mathbb{R}^Y$.*

(4) *For each sequence $Y_1, Y_2, \dots$ of pairwise disjoint subsets of Y there exists a sequence $U_1, U_2, \dots$ of open sets in X such that each $U_n \supseteq Y_n$ and for each $y \in Y$, the set $\{n \in \mathbb{N} : y \in U_n\}$ is finite.*

(5) *$M(Y|X) = \mathbb{R}^Y$.*

(6) *$C_p(Y|X)$ is a large subspace of $\mathbb{R}^Y$.*

(7) *The strong dual of $C_p(Y|X)$ bears the strongest locally convex topology.*

Proof. The equivalence of (1) and (2) follows from Corollary 7.

For $3 \leq i \leq 7$, let (i') denote assertion (i) with X replaced by Y. We already know [7, 8, 11] each (i') is equivalent to (2). This quickly proves that the original assertion (i) is equivalent to (2). For instance, we know (2) ⇔ (3′), where (3′) asserts that $S := \{\phi_A : A \text{ is bounded in } C_p(Y)\}$ is cofinal in $\mathbb{R}^Y$.

Let $T := \{\phi_A : A \text{ is bounded in } C_p(Y|X)\}$. Since $\phi_A = \phi_{A^+}$, we quickly see that $S = T$, proving (3) ⇔ (3′). Thus (3) ⇔ (2). Surely, (4) ⇔ (4′), since the topology of Y is inherited from X. Since $C_p(Y)$ and $C_p(Y|X)$ have the same strong duals and biduals, (5) ⇔ (5′) and (7) ⇔ (7′). Finally, (6) ⇒ (6′) is obvious. The converse holds, since being a large subspace is transitive. □

The special case $Y = X$ began with Gulick [13, Theorem 9], who proved (2) ⇔ (5′). Both we [11] and Kąkol/Leiderman [16] proved (2) ⇔ (4′), *etc.* The general case facilitates other known results (*cf.* [12, Lemma 39, Theorems 41 and 5 (6)]).

4. When $C_p(Y|X)$ is a Montel space

A locally convex space E is a *Montel space* if E is quasibarrelled and every bounded set is relatively compact. So, E is Montel if it contains a large subspace F which is Montel: sets compact in F must be compact in E. Therefore

Theorem 11. *If $C_p(Y|X)$ is a Montel space, so is the superspace $C_p(Y)$.*

Theorem 12. *The following six assertions are equivalent:*

(1) *$C_p(Y|X)$ is a Montel space.*

(2) *$P_{\phi_A} \subseteq C_p(Y|X)$ for each bounded set A in $C_p(Y)$.*

(3) *$P_{\phi_A} \subseteq C_p(Y|X)$ for each bounded set A in $C_p(Y|X)$.*

(4) *$C_p(Y)$ is a Montel space and $C_p(Y) = C_p(Y|X)$.*

(5) *Y is discrete and C-embedded in X;* i.e., *$C_p(Y|X) = C_p(Y) = \mathbb{R}^Y$.*

(6) *$C_p(Y|X)$ is a distinguished Montel space.*

Proof. If A is bounded in $C_p(Y)$, then (1) and Lemma 5 respectively imply the closure of $P_{\phi_A} \cap C_p(Y|X)$ in $C_p(Y|X)$ is compact and its closure in the superspace $\mathbb{R}^Y$ is P_{ϕ_A}, also compact. Hence the two closures coincide, proving (2) holds if (1) does. Trivially, (2) implies (3). If (3) holds and A is bounded in $C_p(Y)$, then A^+ is bounded in $C_p(Y|X)$ and $A \subseteq P_{\phi_A} = P_{\phi_{A^+}} \subseteq C_p(Y|X)$. This proves that $C_p(Y) = C_p(Y|X)$ and that A is relatively compact in $C_p(Y)$; *i.e.*, (4) holds. We also have [(4) $\Rightarrow$ (5)] from classic Buchwalter/Schmets results (see [9, Theorem 2.3]) and the fact that, by definition, Y is C-embedded in X if and only if $C_p(Y|X) = C_p(Y)$. If Y is discrete and C-embedded, then $C_p(Y|X) = C_p(Y) = \mathbb{R}^Y$ is a distinguished Montel space; *i.e.*, [(5) $\Rightarrow$ (6)]. Trivially, (6) $\Rightarrow$ (1), and the circle of implications is complete. $\square$

To deny the converse of Theorem 11, take $X = \mathbb{M}$ and $Y = \mathbb{P}$. While $C_p(\mathbb{P}) = \mathbb{R}^{\mathbb{P}}$ is Montel, the proper subspace $C_p(\mathbb{P}|\mathbb{M})$ contravenes (4). $C_p(\mathbb{P}|\mathbb{M})$ is not barrelled (see above), while Montel spaces are, being reflexive. See also Example 23 and Theorem 24.

Recall the nontrivial

[9, Theorem 2.3] *$C_p(X)$ is Montel $\Leftrightarrow$ its bounded σ-compact sets are relatively compact.*

Certainly, one cannot omit *bounded*, but may replace *σ-compact* with *σ-(relatively) countably compact*, or with *σ-(relatively) sequentially complete.* This foreshadows Theorem 13 and Section 5.

5. When $C_p(Y|X)$ is covered nicely

This section assumes Y is dense in X. Theorem 1 affords nice covers (e), (f) that complete the Velichko variations listed in [15, Theorem 9.13].

Theorem 13. (Summary I) *The following six assertions are equivalent:*

(a) *X is finite.*

(b) *$C_p(X)$ is σ-compact* (Velichko).

(c) *$C_p(X)$ is σ-countably compact* (Tkachuk/Shakhmatov; Arkhangel'skiĭ).

(d) $C_p(X)$ *is* σ*-relatively countably compact* (Ferrando/Kąkol).

(e) $C_p(X)$ *is a countable union of bounded relatively sequentially complete sets* (Ferrando/Kąkol/Saxon [9]).

(f) $C_p(X)$ *is a countable union of bounded sequentially complete sets.*

Proof. Trivially, (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e). Theorem 1 applies with $Y = X$ to show that *condition* (e) *ensures* X *is a pseudocompact* P*-space*; equivalently, X *is finite.* That is to say, (e) $\Rightarrow$ (a). Trivially, (a) $\Rightarrow$ (f) $\Rightarrow$ (e). □

The crucial proof of [(e) $\Rightarrow$ (a)] applies Arkhangel'skiĭ's technique and our Theorem 1. A 'best' application needs a 'best' (most relaxed) cover condition $\mathfrak{C}$ for $C_p(Y|X)$ ensuring the conclusion $\mathfrak{B}$ of Theorem 1. Briefly, one needs $\mathfrak{C} \Leftrightarrow \mathfrak{B}$. We determine $\mathfrak{C}$ to be the condition that $C_p(Y|X)$ *is countably covered by bounded sets relatively sequentially complete in the superspace* $C_p(Y)$, and prove Theorem 20 [$\mathfrak{C} \Leftrightarrow \mathfrak{B}$].

Theorem 1's hypothesis $\mathfrak{A}$ relaxes Arkhangel'skiĭ's hypothesis $\mathcal{A}$ that [$C_p(Y|X)$ *is* σ*-countably compact*]. If $\mathcal{V}$ stands for [$C_p(Y|X)$ *is* σ*-compact*], we have

$$\mathcal{V} \Rightarrow \mathcal{A} \Rightarrow \mathfrak{A} \Rightarrow \mathfrak{B}.$$

Arkhangel'skiĭ noted $\mathcal{A} \not\Leftarrow \mathfrak{B}$. Example 22 and [9, Example 3.4] (reprised below) show

$$\mathcal{V} \not\Leftarrow \mathcal{A} \text{ and } \mathfrak{A} \not\Leftarrow \mathfrak{B}.$$

Condition $\mathfrak{B}'$ (below) verifies [$\mathfrak{A} \Leftrightarrow \mathfrak{B}'$]. The climax [$\mathfrak{C} \Leftrightarrow \mathfrak{B}$] lay unknown.

The proof of Theorem 13 gains context if Theorem 20 replaces Theorem 1. Indeed, half of Theorem 20 [$\mathfrak{C} \Rightarrow \mathfrak{B}$] properly enhances Theorem 1 [$\mathfrak{A} \Rightarrow \mathfrak{B}$]. Yet, it cannot properly enhance Theorem 13 *à la* Arkhangel'skiĭ, as $\mathfrak{C}$ reverts to condition (e) when $Y = X$. Rather, in context, new list (a–f) seems a natural completion of old list (a–d).

Let $C_u^b(X)$ denote the linear space $C^b(X)$ of uniformly bounded functions $f \in C(X)$ endowed with the sup-norm $\|f\| = \sup_{x\in X} |f(x)|$. Now define the linear space $C^b(Y|X) := C(Y|X) \cap C^b(Y)$ and let $C_u^b(Y|X)$ denote $C^b(Y|X)$ endowed with the sup-norm. For Y dense in X, the familiar Banach space $C_u^b(X)$ is clearly norm-isomorphic to $C_u^b(Y|X)$. We let $\|\cdot\|$ denote the norm for both Banach spaces. The unit disc for $C_u^b(Y|X)$ is denoted $D_{Y|X}$. Hence $D_{Y|X} = C(Y|X) \cap D_Y$, where D_Y is the unit disc in the Banach space $\ell_\infty(Y)$ of uniformly bounded real functions on Y. And Y *is* C^**-embedded in* $X \Leftrightarrow C^b(Y|X) = C^b(Y) \Leftrightarrow D_{Y|X} = C(Y) \cap D_Y$. Also, $D_{Y|X}$ is closed in both $C_p(Y|X)$ and its subspace $C_p^b(Y|X)$.

Theorem 14. (Folklore, proof of [1, I.2.2]) *Assume that* Y *is dense in* X*. Then* X *is pseudocompact if and only if* $C_p(Y|X)$ *is a countable union of bounded sets* K_n ($n \in \mathbb{N}$).

Proof. If X is pseudocompact, then $C_p(Y|X) = \bigcup_n n \cdot (C(Y|X) \cap D_Y)$ is a countable union of bounded sets.

Conversely, suppose X is not pseudocompact; equivalently, there is a locally finite sequence $\{U_n : n \in \mathbb{N}\}$ of disjoint nonempty open sets in X. For each n, density allows us to pick $y_n \in Y \cap U_n$. Routinely, as in the proofs of our Fact and [19, Theorem 3.1],

$$\{(g(y_1), g(y_2), \dots) : g \in C(X)\} = \mathbb{R}^{\mathbb{N}}.$$

Thus, given an arbitrary sequence of bounded sets $\{K_n : n \in \mathbb{N}\}$ in $C_p(Y|X)$, we may choose $g \in C(X)$ with, say,

$$g(y_n) = 3 + \sup\{|f(y_n)| : f \in K_n\} \text{ for each } n \in \mathbb{N}.$$

Obviously, $g|_Y \in C_p(Y|X) \setminus K_n$ for each n. Therefore $C_p(Y|X)$ cannot be covered by bounded sets K_n $(n \in \mathbb{N})$ if X is not pseudocompact. $\square$

So, if $C_p(Y|X)$ is σ-compact, X is pseudocompact. Also, $D_Y = \overline{D_{Y|X}}^{\mathbb{R}^Y}$ by Lemma 5 with $P_{\phi_A} = D_Y$. Thus $\ell_\infty(Y) = \bigcup_n n \cdot D_Y$ is a σ-compact subspace of $M(Y|X) = M(Y)$.

Theorem 15. *Assume Y is dense in X. The following three assertions are equivalent:*

(1) *$C_p(Y|X)$ is σ-compact.*

(2) *The Banach disc $D_{Y|X}$ is, in the space $C_p(Y|X)$, absorbing and compact.*

(3) *Setwise, $C(Y|X) = \ell_\infty(Y)$.*

Proof. [(1) ⇒ (2)]. If compact sets K_n $(n \in \mathbb{N})$ cover $C_p(Y|X)$, then X is pseudocompact by Theorem 14. Equivalently, $C_p(Y|X) = C_p^b(Y|X)$, wherein $D_{Y|X}$ is absorbing and closed. The compact K_n are also closed, hence norm-closed, and countably cover the complete metric space $C_u^b(Y|X)$. Baire's theorem ensures some K_m is a norm-neighborhood of some point $h \in C(Y|X)$. Thus $h + \delta \cdot D_{Y|X} \subseteq K_m$ for some $\delta > 0$. That is to say, the closed set $D_{Y|X}$ lies inside the compact set $\delta^{-1} \cdot (-h + K_m)$. Therefore the absorbing disc $D_{Y|X}$ is compact.

[(2) ⇒ (3)]. Since $D_{Y|X}$ is always pointwise dense in D_Y by Lemma 5 and is, by (2), compact, hence closed, we have $D_{Y|X} = D_Y$. Also by (2), $D_{Y|X}$ is absorbing, so that, setwise, $C(Y|X) = \bigcup_n n \cdot D_{Y|X} = \bigcup_n n \cdot D_Y = \ell_\infty(Y)$.

[(3) ⇒ (1)]. By (3), $C(Y|X) = \ell_\infty(Y) = \bigcup_n n \cdot D_Y$. As $n \cdot D_Y$ is compact in $\mathbb{R}^Y$, it is also compact in the subspace $C_p(Y|X)$. Thus $C_p(Y|X)$ is σ-compact. $\square$

When A is specified as a subset of a locally convex space E, we said A *is relatively sequentially complete* if every Cauchy sequence in A converges in E. If every Cauchy sequence in A converges, perhaps not in E, but in a larger locally convex space $F \supseteq E$, we indicate this by saying A is *relatively sequentially complete in F*. The space E is *σ-sequentially complete* if it is a countable union of sequentially complete sets, and is *σ-relatively sequentially complete in F* if it is a countable union of sets that are relatively sequentially complete in F.

Theorem 16. *Assume Y is dense in X. The following three assertions are equivalent:*

(1) *$C_p(Y|X)$ is a countable union of bounded sequentially complete sets.*

(2) *$C_p(Y|X)$ is a countable union of bounded relatively sequentially complete sets.*

(3) *The Banach disc $D_{Y|X}$ is, in $C_p(Y|X)$, absorbing and sequentially complete.*

Proof. Trivially, (1) implies (2).

Assume (2) holds. Then X is, by Theorem 14, pseudocompact, which means that $D_{Y|X}$ absorbs points of $C(Y|X) = C^b(Y|X)$. Let $\{g_n\}_n$ be a pointwise Cauchy sequence in $D_{Y|X}$. The sequence converges to some g in the Montel space $\mathbb{R}^Y$. Baire's theorem provides a relatively sequentially complete K_m whose norm-closure $\overline{K}_m$ is a norm-neighborhood of some point h in the Banach space $C_u^b(Y|X)$. Hence $h + \delta D_{Y|X} \subseteq \overline{K}_m$ for some $\delta > 0$. In particular, each $h + \delta g_n \in \overline{K}_m$. Thus there exists $f_n \in K_m$ such that $\|h + \delta g_n - f_n\| < 1/n$ for each n. The pointwise limit of $\{f_n\}_n$ is clearly $h + \delta g$. Since K_m is relatively sequentially complete, the limit $h + \delta g$ is in $C(Y|X)$. Therefore $g \in C(Y|X)$; the Cauchy sequence $\{g_n\}_n$ converges in $C_p(Y|X)$, and (3) holds.

If (3) holds, the sets $K_n := n \cdot D_{Y|X}$ $(n \in \mathbb{N})$ are bounded and sequentially complete in $C_p(Y|X)$, and cover $C_p(Y|X)$ since $D_{Y|X}$ is absorbing; (1) holds. □

Now [(1) ⇔ (2)] simply says [(e) ⇔ (f)] holds more generally, with $C_p(Y|X)$ in place of $C_p(X)$. Unlike [(b) ⇔ (c)] (Example 22). In [9] we added only (e) to the list (a–d), being unsure of [(1) ⇔ (2)] and [(e) ⇔ (f)].

Assertion (2) is Theorem 1's hypothesis $\mathfrak{A}$. We appropriate (3) as the promised $\mathfrak{B}'$ with $\mathfrak{A} \Leftrightarrow \mathfrak{B}'$. It greatly expedites [9, Example 3.4], showing $\mathfrak{A} \not\Leftarrow \mathfrak{B}$, as follows:

Recall in [9, Example 3.4] that $X := \alpha\mathbb{N}$ is the one-point compactification of $Y := \mathbb{N}$, so X is pseudocompact and the dense Y, being discrete, is surely a P-space. The conclusion $\mathfrak{B}$ of Theorem 1 holds. But not hypothesis $\mathfrak{A}$, as assertion (3) fails: *The disc* $D_{\mathbb{N}|\alpha\mathbb{N}}$ *is not sequentially complete in* $C_p(\mathbb{N}|\alpha\mathbb{N})$.

Proof. Setwise, $C_p(\mathbb{N}|\alpha\mathbb{N})$ is just the Banach space c of convergent real sequences. Fix a real sequence $v \in \ell_\infty \setminus c$ of norm 1. Define $v_n \in c$ so that v_n agrees with v on the first n coordinates and is 0 elsewhere $(n \in \mathbb{N})$. The sequence $\{v_n\}_n \subseteq D_{\mathbb{N}|\alpha\mathbb{N}}$ is Cauchy but does not converge in $C_p(\mathbb{N}|\alpha\mathbb{N})$, since $v \notin c = C_p(\mathbb{N}|\alpha\mathbb{N})$. □

However, v is in the superspace $C_p(\mathbb{N}) = \mathbb{R}^{\mathbb{N}}$. In general,

Theorem 17. *Assume Y is dense in X. Then Y is a P-space if and only if $C_p(Y|X)$ is σ-relatively sequentially complete in the superspace $C_p(Y)$.*

Proof. If Y is a P-space, then $C_p(Y)$ is sequentially complete (Buchwalter/Schmets), so $K_n := C(Y|X)$ is relatively sequentially complete in $C_p(Y)$.

Conversely, suppose $C_p(Y|X)$ is the union of sets K_n $(n \in \mathbb{N})$ that are relatively sequentially complete in $C_p(Y)$. We show that if U is the intersection of a sequence $\{U_n : n \in \mathbb{N}\}$ of neighborhoods of a point y in Y, then U is also a neighborhood of y.

A standard induction produces a decreasing sequence $\{V_n : n \in \mathbb{N}\}$ of neighborhoods in X of y and a sequence $\{h_n : n \in \mathbb{N}\} \subseteq C^b(X)$ such that, for each $n \in \mathbb{N}$,

$$V_n \cap Y \subseteq U_n, \quad h_n \text{ vanishes off } V_n, \quad \|h_n\| = h_n(y) = 1, \text{ and}$$
$$|h_1(x) - 1|, |h_2(x) - 1|, \ldots, |h_n(x) - 1| < 1/n \text{ for all } x \in V_{n+1}.$$

Clearly, as in our proof of [9, (7) ⇒ (1)], the pointwise limit h of $\{h_n : n \in \mathbb{N}\}$ is identically 1 on $V := \cap_k V_k$ and vanishes elsewhere. Also clear: $V \cap Y \subseteq U$. Since the K_n cover the Banach space $C_u^b(Y|X)$, Baire's theorem ensures the norm-closure A of some $K_m \cap C^b(Y|X)$ contains a set of the form $g + \delta D_{Y|X}$, where $g \in C^b(Y|X)$ and $\delta > 0$. In particular, each $g + \delta h_n|_Y \in A$, so that for each $j \in \mathbb{N}$ there is some $f_j \in K_m \cap C^b(Y|X)$ with $\|g + \delta h_j|_Y - f_j\| < 1/j$. The pointwise limit of $\{f_j : j \in \mathbb{N}\}$ is $g + \delta h|_Y$. Since K_m is relatively sequentially complete in $C_p(Y)$, we must have $g + \delta h|_Y \in C(Y)$. Therefore $h|_Y$ is in $C(Y)$, vanishes on $Y \setminus V \supseteq Y \setminus U$, and is 1 at y. It follows that U is a neighborhood of y in Y. □

Theorem 18. *X is a P-space if and only if, for each dense subset Z of X, the space $C_p(Z|X)$ is sequentially complete.*

Proof. If each $C_p(Z|X)$ is sequentially complete, so is $C_p(X|X) = C_p(X)$, which means X is a P-space [3].

Conversely, assume X is a P-space with dense subset Z. An arbitrary Cauchy sequence in $C_p(Z|X)$ is of the form $\{f_n|_Z\}_n$, where each f_n is in $C(X)$ and, for each z in Z, the scalar sequence $\{f_n(z)\}_n$ is Cauchy in $\mathbb{R}$. Let x be an arbitrary point in X. Since X is a P-space, there is a neighborhood V_n of x on which f_n has constant value $f_n(x)$ $(n \in \mathbb{N})$. The intersection $V := \cap_n V_n$ is a neighborhood of x. As Z is dense, we may select a point z in $Z \cap V$. Then $\{f_n(z)\}_n$ is Cauchy, since $z \in Z$. And $f_n(x) = f_n(z)$, since $z \in V_n$ for each $n \in \mathbb{N}$. Therefore $\{f_n\}_n$ is Cauchy at each x in X, and converges to some f in $C_p(X)$, a sequentially complete space [3]. So $\{f_n|_Z\}_n$ converges to $f|_Z$ in $C_p(Z|X)$; *i.e.*, $C_p(Z|X)$ is sequentially complete. □

Theorems 2, 17, 18, most of [9, Theorem 3.1], and analogs of Theorem 13 constitute

Theorem 19. (Summary II) *The following seven assertions are equivalent:*

(1) *X is a P-space.*

(2) *$C_p(Z|X)$ is sequentially complete for every dense $Z \subseteq X$.*

(3) *$C_p(Z|X)$ is σ-sequentially complete for every dense $Z \subseteq X$.*

(4) *$C_p(Z|X)$ is σ-relatively sequentially complete for every dense $Z \subseteq X$.*

(5) *$C_p(Z|X)$ is σ-relatively sequentially complete in $C_p(Z)$ for every dense $Z \subseteq X$.*

(6) *$C_p(X)$ is σ-relatively sequentially complete.*

(7) *$C_p(X)$ is σ-sequentially complete.*

Proof. Theorem 18 equates (1) and (2). Trivially, (2) ⇒ (3) ⇒ (4) ⇒ (5). We apply (5) with $Z := X$ to obtain [(5) ⇒ (6)]. To obtain [(6) ⇔ (1)], apply Theorem 17 with $Y := X$. We now have the equivalence of six assertions (1 - 6). Buchwalter/Schmets proves [(1) ⇒ (7)]. And [(7) ⇒ (6)] is trivial. □

Theorem 20. (Criterion $\mathfrak{C}$) *Assume Y is dense in X. Then $C_p(Y|X)$ is a countable union of bounded sets K_n $(n \in \mathbb{N})$ that are relatively sequentially complete in the superspace $C_p(Y)$ if and only if Y is a P-space and X is pseudocompact.*

Proof. Half of the proof is immediate from Theorems 14 and 17.

For the other half, suppose Y is a P-space and X is pseudocompact. By Theorems 14 and 17, there are sequences $\{I_n : n \in \mathbb{N}\}$ and $\{J_n : n \in \mathbb{N}\}$, each a cover of $C_p(Y|X)$, one consisting of bounded sets, the other, of sets relatively sequentially complete in $C_p(Y)$. We cover $C_p(Y|X)$ with sets

$$K_n := \left(\bigcup\nolimits_{1\leq k\leq n} I_k\right) \bigcap \left(\bigcup\nolimits_{1\leq k\leq n} J_k\right)$$

which are obviously both bounded and relatively sequentially complete in $C_p(Y)$. □

Theorem 21. (Summary III) *Assume that Y is dense in X. Then Y is a P-space, or X is pseudocompact, or both $\Leftrightarrow$ $C_p(Y|X)$ is countably covered by sets that are, respectively, relatively sequentially complete in $C_p(Y)$, or bounded, or both.*

Having seen criterion $\mathfrak{C}$ is *properly* more general than hypothesis $\mathfrak{A}$ of Theorem 1 (not just *formally* so), we prove $\mathcal{A}$ properly relaxes $\mathcal{V}$: we exhibit choices for X and Y where $C_p(Y|X)$ is σ-countably compact (even σ-sequentially compact), but is not σ-compact. Our Y must be uncountable, else the conditions of Theorems 15 and 16 coalesce. Indeed, if Y is countable then $\mathbb{R}^Y$ is metrizable, so that the bounded $D_{Y|X}$ is compact $\Leftrightarrow$ $D_{Y|X}$ is sequentially complete.

Example 22. Let Y be the one-point Lindelöfication of an uncountable discrete space. Choose a pseudocompact X with $Y \subseteq X \subseteq \beta Y$. Our X is either the Stone-Čech compactification βY or a noncompact choice; *e.g.*, X may consist of Y and one arbitrarily chosen cluster point of S in βY for each denumerable subset S of Y, *à la* Haydon's third example [14]. In any case, Y is obviously dense in X. We claim $C_p(Y|X)$ is not σ-compact, but is σ-countably compact. □

Proof of claim. By definition, there is a point $p \in Y$ whose neighborhoods are *co-countable* (complements in Y are denumerable or finite). A function f in $\mathbb{R}^Y$ is in $C(Y)$ if and only if the set $\{y \in Y : f(y) = f(p)\}$ is co-countable. So the function with value 1 at p, and 0 elsewhere, is in $\ell_\infty(Y) \setminus C(Y)$, thus in $\ell_\infty(Y) \setminus C(Y|X)$, proving $C(Y|X) \neq \ell_\infty(Y)$. Hence $C_p(Y|X)$ is not σ-compact (Theorem 15).

Since X is pseudocompact, $C_p(Y|X) = \bigcup_{n=1}^{\infty} n \cdot D_{Y|X}$. Since Y is always C^*-embedded in βY, thus also in $X \subseteq \beta Y$, we have $D_{Y|X} = C(Y) \cap D_Y$. Hence

$$D_{Y|X} = \{f \in D_Y : f \text{ has constant value } f(p) \text{ on some co-countable subset of } Y\}.$$

We complete the proof by showing $D_{Y|X}$, thus each $n \cdot D_{Y|X}$, is countably compact.

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $D_{Y|X}$. For each $n \in \mathbb{N}$, choose a countable set H_n in $Y \setminus \{p\}$ such that $f_n(y) = f_n(p)$ for all $y \in Y \setminus H_n$. Then $H := \bigcup_{n=1}^{\infty} H_n$ is countable, and each f_n has constant value $f_n(p)$ on $Y \setminus H$. Each f_n is in the set

$$\mathcal{D} := \{f \in D_Y : f \text{ is constant on } Y \setminus H\},$$

and $\mathcal{D} \subseteq D_{Y|X}$. Easily, $\mathcal{D}$ is bounded and sequentially complete, and its (pointwise) topology is metrizable, determined by the countably many points in $H \bigcup \{p\}$. In fact, $\mathcal{D}$ may be identified as a closed, bounded subset of the familiar Fréchet-Montel space $\mathbb{R}^{\mathbb{N}}$. Therefore $\mathcal{D}$ is sequentially compact and contains the limit of some subsequence of $\{f_n\}_{n=1}^{\infty}$, as does the superset $D_{Y|X}$. Hence $D_{Y|X}$ is also sequentially (and countably) compact. □

Example 23. Many distinguished spaces $C_p(Y|X)$ are σ-compact. Indeed, if Y is discrete, dense and C^*-embedded in pseudocompact X, e.g., X could be any pseudocompact set between Y and βY then, setwise, $C(Y|X) = \ell_\infty(Y)$, so that, by Theorem 15, $C_p(Y|X)$ is σ-compact. Since $C_p(Y) = \mathbb{R}^Y$ is distinguished, so is $C_p(Y|X)$ by Theorem 10. □

Note: In Example 22, the C^*-embedded Y is not discrete. In [9, Example 3.4], the discrete Y is not C^*-embedded. Example 23 fails when *Montel* replaces *distinguished*:

Theorem 24. $C_p(Y|X)$ is a σ-compact Montel space (if and) only if Y is finite.

Proof. By hypothesis and Theorem 12, $C_p(Y|X) = C_p(Y)$ is σ-compact. Hence Y is finite (Velichko). □

Concerned here with σ-compact spaces $C_p(Y|X)$ and $C_p(X)$, one also considers σ-compact X *a la* Kąkol/Leiderman who, having proved we may conveniently write [*X is a Δ-space*] to mean [*$C_p(X)$ is distinguished*], also proved

[17, Corollary 2.3] *The continuous image of a σ-compact Δ-space is, itself, a Δ-space.*

They then posed two problems:

[17, Problems 2.7 and 2.8] *Suppose X is a Lindelöf subspace of a compact Δ-space Z and W is a continuous image of X.*

(i) *Is X necessarily σ-compact?*

(ii) *Is W necessarily a Δ-space?*

We cannot say. They knew, of course, that a positive answer for (i) would imply the same for (ii) via [17, Corollary 2.3], quoted above.

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References

[1] A. V. Arkhangel'skiĭ: *Topological Function Spaces*, Kluwer, Dordrecht (1992).

[2] T. Banakh, J. Kakol, J. P. Schürz: *ω^ω-base and infinite-dimensional compact sets in locally convex spaces*, Rev. Mat. Complut. 35/2 (2022) 599–614.

[3] H. Buchwalter, J. Schmets: *Sur quelques propriétés de l'espace $C_s(T)$*, J. Math. Pures Appl. 52 (1973) 337–352.

[4] J. C. Ferrando: *Descriptive topology for analysts*, RACSAM 114/2 (2020), art.no. 107, 34p.

[5] J. C. Ferrando: *Approximation by pointwise bounded sets of continuous functions*, Topology Appl. 317 (2022), art.no. 108191.

[6] J. C. Ferrando, J. Kakol: *Metrizable bounded sets in $C(X)$ spaces and distinguished $C_p(X)$ spaces*, J. Convex Analysis 26 (2019) 1337–1346.

[7] J. C. Ferrando, J. Kakol: *Distinguished vector-valued function spaces and injective tensor products*, Bull. Belg. Math. Soc. Simon Stevin 28 (2021) 709–721.

[8] J. C. Ferrando, J. Kakol, A. Leiderman, S. A. Saxon: *Distinguished $C_p(X)$ spaces*, RACSAM 115/1 (2021), art.no. 27, 18p.

[9] J. C. Ferrando, J. Kakol, J., S. A. Saxon: *Characterizing P-spaces X in terms of $C_p(X)$*, J. Convex Analysis 22 (2015) 905–915.

[10] J. C. Ferrando, J. Kakol, S. A. Saxon: *Examples of nondistinguished function spaces $C_p(X)$*, J. Convex Analysis 26 (2019) 1347–1348.

[11] J. C. Ferrando, S. A. Saxon: *If not distinguished, is $C_p(X)$ even close?*, Proc. Amer. Math. Soc. 149 (2021) 2583–2596.

[12] J. C. Ferrando, S. A. Saxon: *Distinguished $C_p(X)$ spaces and the strongest locally convex topology*, RACSAM 117/4 (2023), art.no. 166, 20p.

[13] D. Gulick: *Duality theory for the topology of simple convergence*, J. Math. Pures Appl. 52 (1973) 453–472.

[14] A. Haydon: *Three examples in the theory of spaces of continuous functions*, C. R. Acad. Sci. Sér. A 276 (1973) 685–687.

[15] J. Kakol, W. Kubiś, M. López-Pellicer: *Descriptive Topology in Selected Topics of Functional Analysis*, Springer, Berlin (2011).

[16] J. Kakol, A. Leiderman: *A characterization of X for which spaces $C_p(X)$ are distinguished and its applications*, Proc. Amer. Math. Soc., Ser. B 8 (2021) 86–99.

[17] J. Kąkol, A. Leiderman: *Basic properties of X for which the space $C_p(X)$ is distinguished*, Proc. Amer. Math. Soc., Ser. B 8 (2021) 267–280.

[18] J. Kąkol, A. Leiderman: *On linear continuous operators between distinguished spaces $C_p(X)$*, RACSAM 115/4 (2021), art.no. 199, 11p.

[19] J. Kąkol, S. A. Saxon, A. R. Todd: *Pseudocompact spaces X and df-spaces $C_c(X)$*, Proc. Amer. Math. Soc. 132 (2004) 1703–1712.

[20] A. Leiderman, V. V. Tkachuk: *Pseudocompact Δ-spaces are often scattered*, Monatshefte f. Math. 197 (2022) 493–503.

[21] S. López-Alfonso, M. López-Pellicer, S. Moll-López: *Distinguished property in tensor products and weak* dual spaces*, Axioms 2021, 10(3), 151.

[22] P. Pérez Carreras, J. Bonet: *Barrelled Locally Convex Spaces*, North-Holland Mathematics Studies 131, North-Holland, Amsterdam (1987).

[23] R. Pol: Private communication (2022).

[24] I. Tweddle, S. A. Saxon: *Bornological countable enlargements*, Proc. Edinb. Math. Soc. 46 (2003) 35–44.

[25] I. Tweddle, F. E. Yeomans: *On the stability of barrelled topologies II*, Glasgow Math. J. 21 (1980) 91–95.