

A Bang-Bang Principle for the Conditional Expectation Vector Measure

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We prove a conditional expectation bang-bang principle. Based on properties of the conditional expectation vector measure, we establish that the conditional expectation of a set-valued mapping coincides with the conditional expectation of the set of selections of its extreme points part. As a by-product, we obtain straightforwardly a purification principle.

Keywords: Conditional expectation, Lyapunov convexity theorem, Young measures, bang-bang principle, purification principle.

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1. Introduction

The classical Lyapunov convexity theorem is a fundamental result for both mathematical economics and the theory of games and optimal control: it is used to provide purification principles and the existence of bang-bang controls. It states that a nonatomic finite dimensional vector measure has a convex and compact range. Generally, when the range space of the vector measure is infinite dimensional, the classical convexity theorem of Lyapunov fails. However, an infinite dimensional version of such a result established by Knowles[29] can be used to establish a purification principle as in Askoura [2].

Furthermore, many works succeeded to recover this property and to establish adequate bang-bang and purification results by using Maharam types and saturated measure spaces; see Greinecker and Podczeck [10], Khan and Sagara [22, 23, 24, 25], Sagara [32] for infinite dimensional Lyapunov convexity theorems, purification processes and applications to the integration of set-valued mappings, equilibrium theory and control systems. We refer to Keisler and Sun [19] and the literature therein for further prevalent studies on this direction.

Soler [33], Dynkin and Evstigneev [8] and He and Sun [14] proved that the conditional expectation vector measure satisfies a Lyapunov convexity theorem by generalizing appropriately the “nonatomicity” notion. More precisely, some condition is assumed on a σ -algebra (to be atomless) over its sub- σ -algebra. The classical

nonatomicity becomes atomless over the trivial σ -algebra. This notion is reformulated as “setwise coarser sub- σ -algebra” in He and Sun [12] where it is successfully used to establish existence of pure strategy equilibria of incomplete information games and some purification process. It is renamed as “nowhere equivalent” σ -algebras and used in He and Sun [13] to handle basic properties such as convexity, closure, and compactness for the distribution of correspondences that are necessary for applied work. He and Sun [13] show their condition is necessary and sufficient for the existence of pure strategy equilibria in some formulations of large games.

Working with this generalized nonatomicity, and the Lyapunov convexity property, we establish a bang-bang principle for the conditional expectation of a set-valued mapping and prove that it generates straightforwardly a purification principle. Such results are very useful in economics as showed by He and Sun [14].

The paper is organized as follows: in Section 2, we revisit and summarize some properties of the conditional expectation vector measure. In Section 3, we prove the bang-bang principle for the conditional expectation vector measure and obtain as a corollary a purification result. We end by an Appendix in Section 4, where we collect some results used in a relevant way and where we give an alternative elementary proof of the Lyapunov property of the conditional expectation vector measure based on Knowles’ characterization of Lyapunov vector measures with infinite dimensional range spaces [29].

2. The conditional expectation vector measure

Consider a probability space $(\Omega, \mathcal{A}, \mu)$, where $\mathcal{A}$ is a σ -algebra. Denote by $\mathcal{A}^+$ the set of elements $E \in \mathcal{A}$ with $\mu(E) > 0$. For a sub- σ -algebra $\mathcal{C} \subset \mathcal{A}$ and $E \in \mathcal{A}$, denote by $\mathcal{C}_E = \{A \cap E : A \in \mathcal{C}\}$ the trace σ -algebra of E in $\mathcal{C}$. We denote by E^c the complement of the set E . Given two sets $E_1, E_2 \in \mathcal{A}$, $E_1 \Delta E_2$ refers to the symmetric difference between E_1 and E_2 . In order to simplify notation, when $\mathcal{C}$ is a sub- σ -algebra of $\mathcal{A}$, we keep μ instead of $\mu_{\mathcal{C}}$ for the restriction of μ to $\mathcal{C}$. Since different σ -algebra are used simultaneously, the L^p -spaces are denoted in a more specific way. For instance $L^1(\Omega, \mathcal{A}, \mu)$, or $L^1(\Omega, \mathcal{C}, \mu)$ to refer to the underlying σ -algebra. Given $f \in L^1(\Omega, \mathcal{A}, \mu)$, we denote by $\mathbb{E}^\mu(f|\mathcal{C})$, or simply $\mathbb{E}(f|\mathcal{C})$ if no confusion may occur, the conditional expectation of f relatively to $\mathcal{C}$ under the measure μ .

The set of real numbers is denoted by $\mathbb{R}$ and that of natural numbers by $\mathbb{N}$.

The following notion is introduced in [13] as a “nowhere equivalence” between σ -algebra. Even if “nowhere equivalence” reflects intuitively the introduced notion, we rename it in order to reflect the influence of the underlying measure μ .

Definition 2.1. [13] A sub- σ -algebra $\mathcal{C} \subset \mathcal{A}$ is said to be μ -coarser than $\mathcal{A}$ iff for every $E \in \mathcal{A}^+$, there exists $E_0 \in \mathcal{A}$, $E_0 \subset E$ such that

$$\mu(F \Delta E_0) > 0, \forall F \in \mathcal{C}_E.$$

It is worth mentioning that when $\mathcal{C}_E$ is μ -complete, the condition “there exists $E_0 \in \mathcal{A}$, $E_0 \subset E$ such that $\mu(F \Delta E_0) > 0, \forall F \in \mathcal{C}_E$ ” is equivalent to the following: there exists $E_0 \in \mathcal{A}^+$ such that $E_0 \subset E$ and $E_0 \notin \mathcal{C}_E$. So, in this case, Definition

2.1 expresses the absence of $\mathcal{C}$ -atoms in $\mathcal{A}$ in the sense of Hanen and Nevau [11] (p. 443).

This notion generalizes the nonatomicity notion. We see straightforwardly that μ is nonatomic iff $\{\emptyset, \Omega\}$ is μ -coarser than $\mathcal{A}$. In case of nonatomicity of μ , every finite sub- σ -algebra of $\mathcal{A}$ is μ -coarser than $\mathcal{A}$. It is easy to construct an infinitely generated μ -coarser sub- σ -algebra. For instance consider the unit interval $[0, 1]$ endowed with its Borel σ -algebra $\mathfrak{B}([0, 1])$ and the Lebesgue measure λ . Obviously, $\sigma(\{\frac{1}{n+1}, \frac{1}{n}\}, n \in \mathbb{N}\})$ is λ -coarser than $\mathfrak{B}([0, 1])$. It is clear as well that the existence of μ -coarser sub- σ -algebras in $\mathcal{A}$ implies necessarily that μ is nonatomic. Hence, the previous notion is a strengthening of the nonatomicity notion. While the simple nonatomicity of μ does not suffice to handle some situations, the μ -“coarseness” notion can be used successfully. We will see below that this is the case for the Lyapunov convexity property for the conditional expectation vector measure.

An equivalent concept, as proved in [13] (Lemma 1, p 616), is the following notion:

Definition 2.2. [8, 16, 33] $\mathcal{A}$ is said to be *atomless* over a sub- σ -algebra $\mathcal{C} \subset \mathcal{A}$ iff for every $E \in \mathcal{A}^+$, there exists $E_0 \in \mathcal{A}$, $E_0 \subset E$ such that: on some set of $\mathcal{A}^+$

$$0 < \mu(E_0|\mathcal{C}) < \mu(E|\mathcal{C}),$$

where the conditional probability $\mu(D|\mathcal{C})$ corresponds to $\mathbb{E}^\mu(\chi_D|\mathcal{C})$ and χ_D is the indicator function of the set D in the sense of measure theory.

Let X be a Banach space. We comply with [6] and call a vector measure an additive function $G : \mathcal{A} \rightarrow X$. A vector measure $G : \mathcal{A} \rightarrow X$ is said to be μ -continuous, and this is denoted by $G \ll \mu$, iff $\lim_{\mu(E) \rightarrow 0} G(E) = 0$.

Definition 2.3. An X -valued countably additive vector measure G is said to be *Lyapunov* if for every $E \in \mathcal{A}$, $G(\mathcal{A}_E) = \{G(A \cap E) : A \in \mathcal{A}\}$ is convex and weakly compact.

It is known that ([27], Theorem 2, p. 49) the range of a Lyapunov vector measure can be described by:

$$G(\mathcal{A}_E) = \left\{ \int_E f dG : f \text{ is measurable defined from } E \text{ to } [0, 1] \right\}.$$

Conversely, every measure satisfying this property is Lyapunov, since for every $E \in \mathcal{A}^+$, the operator $g \mapsto \int_E g dG$ defined from $L^\infty(E, \mathcal{A}_E, \mu)$ into X is continuous for the weak* topology on $L^\infty(E, \mathcal{A}_E, \mu)$ and the weak topology on X [6].

Let $\mathcal{C} \subset \mathcal{A}$ be a sub- σ -algebra. It is clear that the vector measure

$$G_{\mathcal{C}} : \mathcal{A} \rightarrow L^1(\Omega, \mathcal{C}, \mu) \text{ defined by } E \mapsto \mathbb{E}(\chi_E|\mathcal{C})$$

is bounded, countably additive and since $\mathbb{E}(\chi_A|\mathcal{C}) = 0$ for every μ -null set A , $G_{\mathcal{C}} \ll \mu$ ([6], Theorem 1, p.10). Furthermore, we can check easily that for every $g \in L^\infty(\Omega, \mathcal{A}, \mu)$, $\int_E g dG_{\mathcal{C}} = \mathbb{E}(g\chi_E|\mathcal{C})$. This result follows straightforwardly from the norm-continuity of $\mathbb{E}(\cdot|\mathcal{C}) : L^\infty(\Omega, \mathcal{A}, \mu) \rightarrow L^1(\Omega, \mathcal{C}, \mu)$ and the denseness of the set of simple functions in $L^\infty(\Omega, \mathcal{A}, \mu)$.

Let $f \in L^1(\Omega, \mathcal{A}, \mu)$ and define the vector measure $F_{\mathcal{C}} : \mathcal{A} \rightarrow L^1(\Omega, \mathcal{C}, \mu)$ by $F_{\mathcal{C}}(E) = \mathbb{E}(f\chi_E|\mathcal{C})$. Similarly, $F_{\mathcal{C}}$ is bounded, countably additive, $F_{\mathcal{C}} \ll \mu$, and for every $g \in L^\infty(\Omega, \mathcal{A}, \mu)$, $\int_E g dF_{\mathcal{C}} = \mathbb{E}(gf\chi_E|\mathcal{C})$. By an abuse of notation we designate below $F_{\mathcal{C}}$ by $fG_{\mathcal{C}}$ and then perform the identification $F_{\mathcal{C}} = fG_{\mathcal{C}}$.

Let us now consider the vector case $f = (f_1, f_2, \dots, f_n) \in L^1(\Omega, \mathcal{A}, \mu)^n$ and consider the vector measure $fG_{\mathcal{C}}$ defined by

$$E \mapsto (f_1 G_{\mathcal{C}}(E), f_2 G_{\mathcal{C}}(E), \dots, f_n G_{\mathcal{C}}(E)).$$

The following theorem is obtained in [33]. A proof of an equivalent statement is given in [14]. For the sake of completeness, we give a new and elementary proof in the Appendix.

Theorem 2.4. [33] *Let $\mathcal{N}$ be the set of sub- σ -algebras that are μ -coarser than $\mathcal{A}$. For all $n \in \mathbb{N}$, the following statement holds true*

$$P(n) : \forall f = (f_1, f_2, \dots, f_n) \in L^1(\Omega, \mathcal{A}, \mu)^n, \forall \mathcal{C} \in \mathcal{N}, fG_{\mathcal{C}} \text{ is Lyapunov.}$$

In connection with some results in the literature, let us reformulate the previous theorem in a slightly more general way. Its proof, in the Appendix, shows that it is a simple consequence of Theorem 2.4.

Theorem 2.5. *Let $\mu = (\mu_1, \dots, \mu_n)$ be a vector of finite measures on $(\Omega, \mathcal{A})$ and $\mathcal{C}$ a sub- σ -algebra of $\mathcal{A}$. Assume that $\mathcal{C}$ is μ_i -coarser than $\mathcal{A}$, for every $i = 1, \dots, n$. Let $(f_1, \dots, f_n) \in \prod_{i=1}^n L^1(\Omega, \mathcal{A}, \mu_i)$.*

Consider the vector measure $F_{\mathcal{C}} : \mathcal{A} \rightarrow \prod_{i=1}^n L^1(\Omega, \mathcal{C}, \mu_i)$ defined by

$$F_{\mathcal{C}}(E) = (\mathbb{E}^{\mu_1}(f_1\chi_E|\mathcal{C}), \mathbb{E}^{\mu_2}(f_2\chi_E|\mathcal{C}), \dots, \mathbb{E}^{\mu_n}(f_n\chi_E|\mathcal{C})),$$

Then, $F_{\mathcal{C}}$ is Lyapunov.

Remark 2.6. Observe that the property of being μ -coarser is in some sense hereditary. Specifically, for every $E \in \mathcal{A}^+$, if $\mathcal{C}$ is μ -coarser than $\mathcal{A}$, then $\mathcal{C}_E$ is μ -coarser than $\mathcal{A}_E$ too. Hence, Theorem 2.4 is not different from Soler's ([33], proposition IV.12), stating simply that the range of $fG_{\mathcal{C}}$ is convex and weakly compact. Note that Dynkin and Evstigneev ([8], Lemma 5.1) showed the scalar version of such a result.

In the same spirit, we can see analogously that by setting $f_i = 1$ for every i in Theorem 2.5, we recover Proposition 1 in [14].¹ But the converse implication is also true. Indeed, it is observed in [14], that Proposition 1 in [14] implies Theorem 2.4, when the functions f_i are positive. The general case can be handled by decomposing each f_i into its positive and negative parts $f_i = f_i^+ - f_i^-$ and considering the vector measure F' defined by

$$F'(E) = (\mathbb{E}(f_1^-\chi_E|\mathcal{C}), \dots, \mathbb{E}(f_n^-\chi_E|\mathcal{C}), \mathbb{E}(f_1^+\chi_E|\mathcal{C}), \dots, \mathbb{E}(f_n^+\chi_E|\mathcal{C}))$$

which is Lyapunov as stated above. Hence by Theorem 4.1 in the Appendix, for every $E \in \mathcal{A}^+$ there exists $g \in L^\infty(\Omega, \mathcal{A}, \mu)$ vanishing outside of E , such that

¹ Consider without loss of generality that Proposition 1 in [14] corresponds exactly to the special case of Theorem 2.5, obtained by setting $f_i = 1$, for every i .

$\int_E g dF' = 0$. Then, $\mathbb{E}(f_i^- \chi_{Eg} | \mathcal{C}) = \mathbb{E}(f_i^+ \chi_{Eg} | \mathcal{C}) = 0$ for every i , which is not else but $\int_E g df G_{\mathcal{C}} = 0$. Consequently, by Theorem 4.1 again, the vector measure $fG_{\mathcal{C}}$ is Lyapunov.

At this stage, Proposition 1 in [14] implies Theorem 2.4. However, Theorem 2.4 is a particular case of Theorem 2.5 which is itself a simple consequence of the former as will be shown in the Appendix. $\square$

3. Bang-bang principle for conditional expectations

By default, topological spaces are endowed with their Borel σ -algebra denoted for a topological space E by $\mathcal{B}(E)$. The convex hull operator is denoted by $\text{co}(\cdot)$ and the topological closure by $\text{cl}(\cdot)$.

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, X a separable Banach space and $T : \Omega \rightrightarrows X$ a set-valued map. Then, T is said to be *measurable* iff $\{\omega \in \Omega : T(\omega) \cap O \neq \emptyset\} \in \mathcal{A}$ for all open sets $O \subset X$. A *measurable selection* of T is a measurable function $f : \Omega \rightarrow X$, such that $f(\omega) \in T(\omega)$, for all $\omega \in \Omega$. If $T : \Omega \rightrightarrows X$ is with convex values, the extreme points of values of T are represented in the following set-valued map $\text{ext}(T) : \Omega \rightrightarrows X$ by $\text{ext}(T)(\omega) = \{y \in T(\omega) : y \text{ is an extreme point of } T(\omega)\}$. We say that T is L^1 -bounded if there is an integrable function $\varphi : \Omega \rightarrow \mathbb{R}$ such that

$$\|x\| \leq \varphi(\omega), \forall \omega \in \Omega, \forall x \in T(\omega).$$

By assuming that T is L^1 -bounded, define the integral of T by the set of Bochner integrals of its measurable selections

$$\int_{\Omega} T d\mu = \left\{ \int_{\Omega} f d\mu : f \text{ is a measurable selection of } T \right\}.$$

Note that since X is separable, the measurability and strong measurability (or μ -measurability) of X -valued functions² are identical ([6], Theorem 2, p.42). We denote by $L_X^1(\Omega, \mathcal{A}, \mu)$ the set of equivalence classes of μ -a.e. equal Bochner integrable functions defined from Ω to X . In the particular case of $X = \mathbb{R}^n$, integration of $\mathbb{R}^n$ -valued function is understood coordinatewise or equivalently in the Bochner sense. The specification of the norm $\|\cdot\|$ used in $\mathbb{R}^n$ is not relevant for the results below.

Definition 3.1. Consider a probability space $(\Omega, \mathcal{A}, \mu)$ and a separable Banach space X . Let $T : \Omega \rightrightarrows X$ be a measurable and L^1 -bounded set-valued map, with compact nonempty values and $\mathcal{C} \subset \mathcal{A}$ a sub- σ -algebra. Denote the set of measurable selections of T by $\mathcal{S}_T$ and denote the set of their conditional expectations by

$$\mathbb{E}(\mathcal{S}_T | \mathcal{C}) = \{\mathbb{E}(f | \mathcal{C}) : f \in \mathcal{S}_T\}.$$

The following theorem is a bang-bang principle for the conditional expectation operator.

² Being a μ -a.e. limit of a sequence of simple functions.

Theorem 3.2. Consider a probability space $(\Omega, \mathcal{A}, \mu)$, where $\mathcal{A}$ is a μ -complete σ -algebra. Let $T : \Omega \rightrightarrows \mathbb{R}^n$ be a measurable, L^1 -bounded set-valued map, with compact convex nonempty values and $\mathcal{C} \subset \mathcal{A}$ a sub- σ -algebra which is μ -coarser than $\mathcal{A}$.

Then,
$$\mathbb{E}(\mathcal{S}_T | \mathcal{C}) = \mathbb{E}(\mathcal{S}_{\text{ext}(T)} | \mathcal{C}), \mu\text{-a.e. on } \Omega.$$

Proof. It is obvious that $\mathbb{E}(\mathcal{S}_{\text{ext}(T)} | \mathcal{C}) \subset \mathbb{E}(\mathcal{S}_T | \mathcal{C}), \mu\text{-a.e. on } \Omega$. Let us consider the converse inclusion. Let $h \in \mathcal{S}_T$. From Theorem 4.4 (in the Appendix), there is $h_i, \alpha_i, i \in I = \{1, \dots, n+1\}$ such that h_i is a measurable selection of $\text{ext}(T)$ for every $i \in I$ and the weight functions $\alpha_i : \Omega \rightarrow [0, 1], i \in I$, are measurable and satisfy $\sum_{i \in I} \alpha_i = 1$. Furthermore, $h = \sum_{i \in I} \alpha_i h_i$. Since T is L^1 -bounded, the functions $h_i, i \in I$, are integrable.

From now, we work with equivalence classes, and μ -a.e. equal functions are confused. We prove thereafter that there exists a finite measurable partition $B_i, i \in I$, of Ω satisfying

$$\mathbb{E}(h | \mathcal{C}) = \sum_{i \in I} \mathbb{E}(\alpha_i h_i | \mathcal{C}) = \sum_{i \in I} \mathbb{E}(\chi_{B_i} h_i | \mathcal{C}), \mu\text{-a.e. on } \Omega. \quad (1)$$

Consider the vector measure G defined on $\mathcal{A}$ by

$$G(E) = \mathbb{E}[(h_1, \dots, h_{n+1}) \chi_E | \mathcal{C}].$$

From Theorem 2.4, G is Lyapunov. Hence, from Lemma 4.3 (in the Appendix) there exists a measurable partition $B_1, \dots, B_{|I|}$ of Ω such that, for every $i \in I$,

$$G(B_i) = \int_{\Omega} \alpha_i dG.$$

That is, for every $i, j \in I$,

$$\mathbb{E}(\chi_{B_i} h_j | \mathcal{C}) = \mathbb{E}(\alpha_i h_j | \mathcal{C}), \mu\text{-a.e. on } \Omega.$$

By summing up the diagonal elements (over the same indices $i = j$), we obtain,

$$\mathbb{E}(h | \mathcal{C}) = \sum_{i \in I} \mathbb{E}(\alpha_i h_i | \mathcal{C}) = \sum_{i \in I} \mathbb{E}(\chi_{B_i} h_i | \mathcal{C}), \mu\text{-a.e. on } \Omega.$$

This proves Equation (1).

Define on Ω the function f by setting $f|_{B_i} = h_i$, for every $i \in I$. We obtain a measurable function $f : \Omega \rightarrow \mathbb{R}^n$ satisfying

$$\mathbb{E}(f | \mathcal{C}) = \mathbb{E}(h | \mathcal{C}), \mu\text{-a.e. on } \Omega.$$

Furthermore, by construction, $f(\omega) \in \text{ext}(T(\omega)), \mu\text{-a.e. on } \Omega$. □

Theorem 5 in [14] establishes that if $M : \Omega \rightrightarrows \mathbb{R}^n$ is a set-valued map satisfying assumptions of Theorem 3.2 except the convexity of its values, then $\mathbb{E}(\mathcal{S}_M | \mathcal{C}) = \mathbb{E}(\mathcal{S}_{\text{co}(M)} | \mathcal{C})$. Since T has convex compact values (in Theorem 3.2), $T = \text{co}(\text{ext}(T))$. So, we are tempted to write $\mathbb{E}(\mathcal{S}_T | \mathcal{C}) = \mathbb{E}(\mathcal{S}_{\text{co}(\text{ext}(T))} | \mathcal{C}) = \mathbb{E}(\mathcal{S}_{\text{ext}(T)} | \mathcal{C})$. But this conclusion is not correct since the set of extreme points is not necessarily closed, and the set-valued map $\text{ext}(T)$ does not satisfy necessarily all the requirements of

Theorem 5 in [14]. Consequently, Theorem 3.2 cannot be deduced straightforwardly from this result. On the contrary, Theorem 5 in [14] can be deduced from Theorem 3.2.

Corollary 3.3. [14] *Apart from the convexity of values of T , suppose that the assumptions of Theorem 3.2 are satisfied. Then,*

$$\mathbb{E}(\mathcal{S}_T|\mathcal{C}) = \mathbb{E}(\mathcal{S}_{\text{co}(T)}|\mathcal{C}).$$

Proof. The set-valued map $\omega \mapsto \text{co}(T(\omega))$ is measurable (Castaing and Valadier [5], Theorem III.40), with convex compact values. It is obviously L^1 -bounded as well. Hence, from Theorem 3.2, $\mathbb{E}(\mathcal{S}_{\text{co}(T)}|\mathcal{C}) = \mathbb{E}(\mathcal{S}_{\text{ext}(\text{co}(T))}|\mathcal{C})$.

But, since T has compact values, we have $\text{ext}(\text{co}(T(\omega))) \subset T(\omega)$, for every $\omega \in \Omega$, and hence $\mathbb{E}(\mathcal{S}_{\text{ext}(\text{co}(T))}|\mathcal{C}) \subset \mathbb{E}(\mathcal{S}_T|\mathcal{C})$. That is, $\mathbb{E}(\mathcal{S}_{\text{co}(T)}|\mathcal{C}) \subset \mathbb{E}(\mathcal{S}_T|\mathcal{C})$. The converse inclusion being obvious, we deduce that $\mathbb{E}(\mathcal{S}_T|\mathcal{C}) = \mathbb{E}(\mathcal{S}_{\text{co}(T)}|\mathcal{C})$. $\square$

Remark 3.4. Consider a probability space $(\Omega, \mathcal{A}, \mu)$, where $\mathcal{A}$ is a μ -complete σ -algebra. Let $T : \Omega \rightrightarrows \mathbb{R}^n$ be a measurable set-valued map, with compact nonempty values and $\mathcal{C} \subset \mathcal{A}$ a sub- σ -algebra.

- Denote and define the set of $\mathcal{C}$ -measurable and integrable μ -a.e. selections of T by

$$\mathcal{S}_T^1(\mathcal{C}) = \{f \in L_{\mathbb{R}^n}^1(\Omega, \mathcal{A}, \mu) : f \text{ is } \mathcal{C}\text{-measurable and } f(\omega) \in T(\omega), \mu\text{-a.e.}\}.$$

Denoted simply by $\mathcal{S}_T^1$ when $\mathcal{C} = \mathcal{A}$.

- The conditional expectation of $\mathcal{S}_T^1$ is defined by

$$\mathbb{E}(\mathcal{S}_T^1|\mathcal{C}) = \{\mathbb{E}(f|\mathcal{C}) : f \in \mathcal{S}_T^1\}.$$

Following Hiai and Umegaki ([15], Theorem 5.1), for a measurable L^1 -bounded set-valued map T , there is a unique set-valued map $\mathbb{E}(T|\mathcal{C})$, call it *conditional expectation* of T satisfying

$$\mathcal{S}_{\mathbb{E}(T|\mathcal{C})}^1(\mathcal{C}) = \text{cl}(\mathbb{E}(\mathcal{S}_T^1|\mathcal{C})),$$

where the closure is taken with respect to the norm in $L_{\mathbb{R}^n}^1(\Omega, \mathcal{A}, \mu)$.

Since T is assumed to be L^1 -bounded in Theorem 3.2, the set of measurable selections $\mathcal{S}_T$ of T coincides with the set of measurable and integrable μ -a.e. selections $\mathcal{S}_T^1$ of T . This remains also true for $\mathcal{S}_{\text{ext}(T)}^1$. Therefore, the previous theorem establishes, under its assumptions, that

$$\mathcal{S}_{\mathbb{E}(T|\mathcal{C})}^1(\mathcal{C}) = \text{cl}(\mathbb{E}(\mathcal{S}_T^1|\mathcal{C})) = \text{cl}(\mathbb{E}(\mathcal{S}_{\text{ext}(T)}^1|\mathcal{C})), \mu\text{-a.e. on } \Omega. \quad (2)$$

But, since T has convex compact values, $\mathcal{S}_T^1$ is clearly convex and norm closed. Then, it is weakly closed. Furthermore, the L^1 -boundedness of T implies the L^1 -boundedness of $\mathcal{S}_T^1$ then its uniform integrability³. Consequently, $\mathcal{S}_T^1$ is weakly

³ Recall that a subset $\mathcal{H} \subset L_{\mathbb{R}^n}^1(\Omega, \mathcal{A}, \mu)$, is said to be uniformly integrable if and only if $\lim_{M \rightarrow +\infty} \int_{\|f\| > M} \|f\| d\mu = 0$, uniformly in $f \in \mathcal{H}$. Equivalently, $\mathcal{H}$ is bounded in $L_{\mathbb{R}^n}^1(\Omega, \mathcal{A}, \mu)$ and $\lim_{\mu(E) \rightarrow 0} \int_E \|f\| d\mu = 0$, uniformly in $f \in \mathcal{H}$.

compact which implies that the convex set $\mathbb{E}(\mathcal{S}_T^1|\mathcal{C})$ is weakly compact, so norm closed, as $\mathbb{E}(\cdot|\mathcal{C})$ is continuous for the weak topologies.

Note that we have $\mathbb{E}(\mathcal{S}_T^1|\mathcal{C}) = \mathbb{E}(\mathcal{S}_{\text{ext}(T)}^1|\mathcal{C})$, so that Equation (2) becomes

$$\mathcal{S}_{\mathbb{E}(T|\mathcal{C})}^1(\mathcal{C}) = \mathbb{E}(\mathcal{S}_T^1|\mathcal{C}) = \mathbb{E}(\mathcal{S}_{\text{ext}(T)}^1|\mathcal{C}), \mu\text{-a.e. on } \Omega. \quad \square$$

If μ is nonatomic, then $\mathcal{C} = \{\emptyset, \Omega\}$ is μ -coarser than $\mathcal{A}$ as already mentioned in section 2. In this particular case, Theorem 3.2 reduces to the following well-known result ([23], Theorem 3.1).

Corollary 3.5. *Consider a probability space $(\Omega, \mathcal{A}, \mu)$, where $\mathcal{A}$ is a μ -complete σ -algebra. Let $T : \Omega \rightrightarrows \mathbb{R}^n$ be a measurable, L^1 -bounded set-valued map, with compact convex nonempty values. Assume that μ is nonatomic. Then,*

$$\int_{\Omega} T d\mu = \int_{\Omega} \text{ext}(T) d\mu.$$

Based on the bang-bang principle (Theorem 3.2), we are going to deduce a purification principle analogous to that established by He and Sun ([12], Proposition 2 and [14], Theorem 7). Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, where $\mathcal{A}$ is a μ -complete σ -algebra, and A be a Polish space. Denote by $\mathbb{P}(A)$ the set of Borel probability measures on A . Consider the space of Young measures⁴ $\mathcal{R}(\Omega, A)$ with respect to $(\Omega, \mathcal{A}, \mu)$ and A : the set of functions $\delta : \Omega \rightarrow \mathbb{P}(A)$ such that $\omega \mapsto \delta_{\omega}(B)$ is measurable, for every $B \in \mathfrak{B}(A)$. Endow $\mathcal{R}(\Omega, A)$ with the topology of stable convergence⁵, i.e., the coarsest topology making continuous the functions $\delta \mapsto \int_{\Omega} [\int_A \phi d\delta_{\omega}] d\mu$, where $\phi : \Omega \times A \rightarrow \mathbb{R}$ is a Carathéodory integrand ($\phi(\cdot, a)$ is measurable for every $a \in A$ and $\phi(\omega, \cdot)$ is continuous on A for every $\omega \in \Omega$) which is L^1 -bounded, i.e., there exists a μ -integrable function $h : \Omega \rightarrow \mathbb{R}_+$ such that $|\phi(\omega, x)| \leq h(\omega)$, for every $(\omega, x) \in \Omega \times A$. Following Valadier ([35], Lemma A2), the measurability condition defining Young measures is equivalent to the measurability of the functions $\omega \mapsto \delta_{\omega}(\phi) = \int_A \phi d\delta_{\omega}$, $\phi \in \mathcal{C}_b(A)$, where $\mathcal{C}_b(A)$ is the set of continuous and bounded functions defined on A . We associate to every measurable function $f : \Omega \rightarrow A$, a (degenerate) Dirac-Young measures δ^f , defined at every $\omega \in \Omega$ by $\delta_{\omega}^f(B) = 1$ if $f(\omega) \in B$ and $\delta_{\omega}^f(B) = 0$ otherwise. Denote by $D(\Omega, A)$ the set of Dirac-Young measures.

Given $V : \Omega \times A \rightarrow \mathbb{R}^n$ and $\delta \in \mathcal{R}(\Omega, A)$ denote by $\delta.V$ the function defined on Ω by

$$\delta.V(\omega) = \int_A V(\omega, a) d\delta_{\omega}(a),$$

provided that the integral is well defined. Denote $f.V$ the similarly defined function for a measurable function $f : \Omega \rightarrow A$. That is, $f.V(\omega) = V(\omega, f(\omega))$.

⁴ More generally the space of measures on $\Omega \times A$ having marginals on Ω equal to μ is referred to as the space of Young measures. We define, here, the space of disintegrated Young measures. However, under our assumptions (metrizability of A), all measure on the product having its marginal on Ω equal to μ can be disintegrated, see Theorem 3.2, p.179 in [9] and [34]. Hence, we can consider that we do not modify the definition of this space.

⁵ Since A is metrizable Suslin, we do not make distinction between the different topologies known on $\mathcal{R}(\Omega, A)$: the S, M, N, W defined in [4], because they are identical, see the Portmanteau Theorem in [4].

Corollary 3.6. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, where $\mathcal{A}$ is a μ -complete σ -algebra and A a compact metric space and $n \in \mathbb{N}$. Consider a function $V : \Omega \times A \rightarrow \mathbb{R}^n$ such that $V(\cdot, a)$ is measurable, for every $a \in A$, and $V(\omega, \cdot)$ is continuous on A for every $\omega \in \Omega$, that is, $V = (V_1, \dots, V_n)$ is a vector consisting of n Carathéodory integrands. Assume further that V_i is L^1 -bounded, for every i , and $\mathcal{C}$ is μ -coarser than $\mathcal{A}$. Then, for every $\delta \in \mathcal{R}(\Omega, A)$, there exists a measurable function $f : \Omega \rightarrow A$ such that*

- $\mathbb{E}(\delta.V|\mathcal{C}) = \mathbb{E}(f.V|\mathcal{C})$, and
- $f(\omega) \in \text{supp}(\delta_\omega)$, for μ -a.e. $\omega \in \Omega$.

Proof. In this proof, some arguments dealing mainly with measurability considerations are taken from [2].

Since δ_ω has a compact support and $V(\omega, \cdot)$ is continuous, for every $\omega \in \Omega$:

$$\text{co}(V(\omega, \text{supp}(\delta_\omega))) \text{ is compact for every } \omega \in \Omega. \quad (3)$$

Fix a function $\bar{\delta} : \Omega \rightarrow \mathbb{P}(A)$ representing the μ -a.e. equal equivalence class of δ .

Observe now that the set-valued map $\omega \mapsto \text{supp}(\bar{\delta}_\omega)$ is measurable. Indeed, for every open set U of A , $\{\omega \in \Omega : \text{supp}(\bar{\delta}_\omega) \cap U \neq \emptyset\} = \{\omega \in \Omega : \bar{\delta}_\omega(U) > 0\}$. Hence, by definition of $\mathcal{R}(\Omega, A)$, $\omega \mapsto \text{supp}(\bar{\delta}_\omega)$ is measurable.

Consider the function $\bar{p} : \Omega \rightarrow \mathbb{R}^n$ defined for every $\omega \in \Omega$ by

$$\bar{p}(\omega) = \int_A V(\omega, a) d\bar{\delta}_\omega(a).$$

Let us check that $\bar{p}$ is measurable. For every fixed $\omega \in \Omega$, $V(\omega, \cdot)$ is continuous from the compact metric space A into $\mathbb{R}^n$, hence, $\int_A V(\omega, a) d\bar{\delta}_\omega(a)$ is well defined. The function $\omega \mapsto V(\omega, \cdot)$ is $\mathcal{A}\text{-}\mathfrak{B}(\mathcal{C}(A))$ measurable (Aliprantis and Border [1], Theorem 4.55, p. 155), where $\mathcal{C}(A)$ is the set of continuous functions defined on A . Then, since $\bar{\delta}$ is $\mathcal{A}$ -weak* measurable, the function $\omega \mapsto \int_A V(\omega, a) d\bar{\delta}_\omega(a) = \bar{p}(\omega)$ is measurable as a composition of measurable functions : $\omega \mapsto (V(\omega, \cdot), \bar{\delta}_\omega)$ from Ω to $\mathcal{C}(A) \times \mathbb{P}(A)$ obviously measurable and $(\varphi, q) \mapsto \int_A \varphi dq$ from $\mathcal{C}(A) \times \mathbb{P}(A)$ to $\mathbb{R}$ which is continuous for the product of the norm by the weak* topology, then measurable. Hence $\bar{p}$ is measurable. From the mean value theorem for the integral of vector valued functions (for instance Diestel and Uhl [6], Corollary 8, p. 48) and using (3),

$$\bar{p}(\omega) \in \text{co}(V(\omega, \text{supp}(\bar{\delta}_\omega))), \text{ for every } \omega \in \Omega. \quad (4)$$

Remark now that $\omega \mapsto V(\omega, \text{supp}(\bar{\delta}_\omega))$ is measurable. Indeed, since $\omega \mapsto \text{supp}(\bar{\delta}_\omega)$ is measurable with compact values, it possesses a Castaing representation. That is, for every $\omega \in \Omega$, $\text{supp}(\bar{\delta}_\omega) = \text{cl}(\{\sigma_k(\omega) : k \in \mathbb{N}\})$, where σ_n are measurable functions from Ω to A , see (Castaing and Valadier [5], Theorem III.7, p. 66). But, from the continuity of $V(\omega, \cdot)$, $\omega \in \Omega$, $V(\omega, \text{supp}(\bar{\delta}_\omega)) = \text{cl}(V(\omega, \{\sigma_k(\omega) : k \in \mathbb{N}\}))$ for every $\omega \in \Omega$. Clearly, $\omega \mapsto V(\omega, \{\sigma_k(\omega) : k \in \mathbb{N}\})$ is measurable as a countable union of measurable functions. Hence, $\omega \mapsto V(\omega, \text{supp}(\bar{\delta}_\omega))$ is measurable (Castaing and Valadier [5], Theorem III.4). Therefore, $\omega \mapsto \text{co}(V(\omega, \text{supp}(\bar{\delta}_\omega)))$ is measurable, Castaing and Valadier ([5], Theorem III.40).

Since $V(\cdot, \text{supp}(\bar{\delta}))$ has compact values, it results from Theorem 3.2 that there is a measurable selection h of $\text{ext}(\text{co}(V(\cdot, \text{supp}(\bar{\delta})))) \subset V(\cdot, \text{supp}(\bar{\delta}))$ such that

$$\mathbb{E}(\delta.V|\mathcal{C}) = \mathbb{E}(\bar{p}(\cdot)|\mathcal{C}) = \mathbb{E}(h|\mathcal{C}), \mu\text{-a.e. on } \Omega.$$

An application of the implicit measurable selection theorem (Castaing and Valadier [5], Theorem III.38) allows to find a measurable selection⁶ f of $\omega \mapsto \text{supp}(\bar{\delta}_\omega)$, defined on Ω , such that $h(\omega) = V(\omega, f(\omega))$ for every $\omega \in \Omega$. Therefore,

$$\mathbb{E}(\delta.V|\mathcal{C}) = \mathbb{E}(f.V|\mathcal{C}), \mu\text{-a.e. on } \Omega.$$

Furthermore, by construction, $f(\omega) \in \text{supp}(\delta_\omega)$, μ -a.e. on Ω . $\square$

Remark 3.7. Corollary 3.6 is obtained by He and Sun ([14], Theorem 7). A different version is obtained in ([12], Proposition 2, p. 134), where a $\mathcal{C}$ -measurability restriction for $\delta \in \mathcal{R}(\Omega, A)$ is assumed but the space A is a general Polish Space. It is obtained in a more general setting in [18]. $\square$

An immediate density result can be inferred from the previous corollary. Endow $\mathcal{R}(\Omega, A)$ by the following topology $\mathcal{T}$ described by: a net $(\delta_\lambda)_{\lambda \in \Lambda}$ converges to δ iff for every L^1 -bounded Carathéodory integrand φ on $\Omega \times A$,

$$(\mathcal{T}) \quad \mathbb{E}(\delta^\lambda.\varphi|\mathcal{C}) \text{ converges in } L^1(\Omega, \mathcal{C}, \mu) \text{ to } \mathbb{E}(\delta.\varphi|\mathcal{C}).$$

Corollary 3.8. Assume that A is a compact metric space and $\mathcal{C}$ is μ -coarser than $\mathcal{A}$, then $D(\Omega, A)$ is dense in $\mathcal{R}(\Omega, A)$ for the topology $\mathcal{T}$.

The proof is omitted, since the result follows straightforwardly from Corollary 3.6.

An interesting particular case is that of $\mathcal{C} = \{\emptyset, \Omega\}$, where $\mathbb{E}(f|\mathcal{C}) = \int_\Omega f d\mu$, the topology $\mathcal{T}$ is not else but the Stable topology. Therefore, if μ is nonatomic, which also means that $\mathcal{C} = \{\emptyset, \Omega\}$ is μ -coarser than $\mathcal{A}$, we recover by Corollary 3.8 the well-known denseness result of $D(\Omega, A)$ in $\mathcal{R}(\Omega, A)$ with respect to the Stable topology [36]. Refer to [4] (Theorem 2.2.3, p. 40) for a more general version.

4. Appendix

We begin by recalling some results used in the proofs. In this appendix, X will denote a Banach space.

Theorem 4.1. (Knowles [29] or see Theorem 4, p. 263 in [6]) Let $G : \mathcal{A} \rightarrow X$ be a countably additive vector measure and μ a real positive finite measure on $\mathcal{A}$ such that for all $E \in \mathcal{A}$,

$$\mu(E) = 0 \Leftrightarrow G(E \cap F) = 0, \forall F \in \mathcal{A}. \quad (5)$$

Then, the following assertions are equivalent:

⁶ Theorem III.38 in Castaing and Valadier [5] establishes $\hat{\mathcal{A}}$ measurable selections, where $\hat{\mathcal{A}}$ is the universal completion of $\mathcal{A}$. But, since $\mathcal{A}$ is assumed to be μ -complete, these selections are necessarily $\mathcal{A}$ -measurable.

- (1) For all $E \in \mathcal{A}^+$, the operator $f \mapsto \int_E f dG$ from $L^\infty(E, \mu)$ into X is not a one-to-one.
- (2) For all $A \in \mathcal{A}$, $G(\mathcal{A}|_A) = \{G(F \cap A) : F \in \mathcal{A}\}$ is convex and weakly compact.

The previous theorem works by assuming instead of (5), that $G \ll \mu$, see for instance [2], Remark 1.

Theorem 4.2. (Corollary 7, p.265 in [6]) *A countably additive vector measure $G : \mathcal{A} \rightarrow X$ is Lyapunov if and only if for every $E \in \mathcal{A}$, there is $F \in \mathcal{A}$ such that $G(E \cap F) = G(E)/2$.*

Lemma 4.3. (Lemma 2 in [2]) *Let $G : \mathcal{A} \rightarrow X$ be a Lyapunov vector measure and $\alpha_i : \Omega \rightarrow [0, 1]$, $i = 1, \dots, n$, be measurable functions, such that $\sum_{i=1}^n \alpha_i(\omega) = 1$, G -a.e. on Ω .⁷ Then, there exists a measurable partition $E_i, i = 1, \dots, n$, of Ω such that, for every $i = 1, \dots, n$, $\int_\Omega \alpha_i(\omega) dG = G(E_i)$.*

Theorem 4.4. (Theorem IV.11, p. 101, in [5]) *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, where $\mathcal{A}$ is μ -complete and $T : \Omega \rightrightarrows \mathbb{R}^n$ a measurable set-valued map with nonempty convex compact values. Define the set-valued map $\text{ext}(T) : \Omega \rightrightarrows \mathbb{R}^n$ by*

$$\text{ext}(T)(\omega) = \{a \in \mathbb{R}^n : a \text{ is an extreme point of } T(\omega)\}.$$

Let $s : \Omega \rightarrow X$ be a measurable selection of T . Then, there exist $n + 1$ measurable selections $h_i : \Omega \rightarrow \mathbb{R}^n$, $i = 1, \dots, n + 1$, of $\text{ext}(T)$ and $n + 1$ (weight) mesurables functions from Ω to $[0, 1]$, such that,

$$\sum_{i=1}^{n+1} \alpha_i(\omega) = 1, \text{ and } s(\omega) = \sum_{i=1}^{n+1} \alpha_i(\omega) h_i(\omega), \text{ for every } \omega \in \Omega.$$

4.1. Proofs of Theorems 2.4 and 2.5

We produce thereafter a further elementary proof of Theorem 2.4 and deduce from it Theorem 2.5. Let us start with some needed lemmas.

Lemma 4.5. *Let $\mathcal{C}$ be a sub- σ -algebra of $\mathcal{A}$ and $E \in \mathcal{A}^+$. Then,*

- (1) *The σ -algebra $\sigma(\mathcal{C} \cup \{E\})$, generated by $\mathcal{C} \cup \{E\}$, is given by*

$$\mathcal{C}_E \oplus \mathcal{C}_{E^c} := \{A \subset \Omega : A \cap E \in \mathcal{C}_E \text{ and } A \cap E^c \in \mathcal{C}_{E^c}\}.$$

Hence, $B \in \sigma(\mathcal{C} \cup \{E\})$ iff there is $C_1, C_2 \in \mathcal{C}$ such that

$$B = (C_1 \cap E) \cup (C_2 \cap E^c).$$

- (2) *$\mathcal{C}$ is μ -coarser than $\mathcal{A}$ iff $\sigma(\mathcal{C} \cup \{E\})$ is μ -coarser than $\mathcal{A}$. Consequently, $\mathcal{C}$ is μ -coarser than $\mathcal{A}$ iff $\sigma(\mathcal{C} \cup \{E_i : i \text{ in a finite set } I\})$ is μ -coarser than $\mathcal{A}$.*

Proof. (1) If $A \subset \Omega$ is such that $A \cap E \in \mathcal{C}_E$ and $A \cap E^c \in \mathcal{C}_{E^c}$, that is, $A \cap E = C \cap E$ and $A \cap E^c = C' \cap E^c$ for $C, C' \in \mathcal{C}$, then $A = (C \cap E) \cup (C' \cap E^c) \in \sigma(\mathcal{C} \cup \{E\})$, which establishes the inclusion $\mathcal{C}_E \oplus \mathcal{C}_{E^c} \subset \sigma(\mathcal{C} \cup \{E\})$. As obviously $\mathcal{C}_E \oplus \mathcal{C}_{E^c}$ is a σ -algebra, it remains to prove $\mathcal{C} \cup \{E\} \subset \mathcal{C}_E \oplus \mathcal{C}_{E^c}$. To do so, let $A \in \mathcal{C} \cup \{E\}$.

⁷ Except on a G -null set. Note that $A \in \mathcal{A}$ is said to be G -null iff $G(A \cap F) = 0$ for every $F \in \mathcal{A}$.

If $A = E$, $A \cap E = E \in \mathcal{C}_E$ and $A \cap E^c = \emptyset \in \mathcal{C}_{E^c}$.

If $A \in \mathcal{C}$, $A = (A \cap E) \cup (A \cap E^c) \in \mathcal{C}_E \oplus \mathcal{C}_{E^c}$.

(2) Suppose that $\mathcal{C}$ is μ -coarser than $\mathcal{A}$. Let $D \in \mathcal{A}^+$. Suppose that $\mu(D \cap E) > 0$. Then, there exists an $\mathcal{A}$ -measurable subset D_0 of $D \cap E$ such that

$$\mu(D_0 \Delta (D \cap E \cap C)) > 0, \quad \forall C \in \mathcal{C}. \quad (6)$$

Let $B \in \sigma(\mathcal{C} \cup \{E\})$, i.e. $B = (C_1 \cap E) \cup (C_2 \cap E^c)$ for $C_1, C_2 \in \mathcal{C}$. Thus, $D_0 \Delta (D \cap B) = D_0 \Delta [(D \cap E \cap C_1) \cup (D \cap E^c \cap C_2)]$ and since $D_0 \subset D \cap E$, $D_0 \Delta (D \cap E \cap C_1) \subset [D_0 \Delta (D \cap B)]$.

Then, from (6),

$$\mu(D_0 \Delta (D \cap B)) > 0, \quad \forall B \in \sigma(\mathcal{C} \cup \{E\}).$$

The remaining case $\mu(E \cap D) = 0$ can be handled in the same manner as previously by reasoning on $E^c \cap D$ which is non-negligible. This means that $\sigma(\mathcal{C} \cup \{E\})$ is μ -coarser than $\mathcal{A}$. The converse is trivial, since $\mathcal{C} \subset \sigma(\mathcal{C} \cup \{E\})$. $\square$

Lemma 4.6. *Let $\mathcal{C} \subset \mathcal{A}$ be a sub- σ -algebra that is μ -coarser than $\mathcal{A}$. For every $f \in L^1(\Omega, \mathcal{A}, \mu)$, the vector measure defined from $\mathcal{A}$ to $L^1(\Omega, \mathcal{C}, \mu)$ by*

$$fG_{\mathcal{C}}(E) = \mathbb{E}(f\chi_E|\mathcal{C})$$

is well defined, countably additive and Lyapunov.

Proof. As noticed in section 2, $fG_{\mathcal{C}}$ is bounded, countably additive, and for every $g \in L^\infty(\Omega, \mathcal{A}, \mu)$, $\int_E gdfG_{\mathcal{C}} = \mathbb{E}(fg\chi_E|\mathcal{C})$.

According to Theorem 4.1, we have to prove that given an $E \in \mathcal{A}^+$, there exists a function $\bar{g} \in L^\infty(\Omega, \mathcal{A})$ vanishing outside of E , with $\|\bar{g}\|_\infty > 0$ and such that $\mathbb{E}(\bar{g}f\chi_E|\mathcal{C}) = 0$. Let $E \in \mathcal{A}^+$. If $f = 0, \mu$ -a.e. on E , the result is obvious. Otherwise, there is $E' \subset E$, $E' \in \mathcal{A}^+$ and $\varepsilon > 0$, such that $|f| > \varepsilon$ on E' . Without loss of generality, we assume that $E' = E$ below, since we can perform the analysis below on E' instead of E .

The sub- σ -algebra $\mathcal{C}$ being μ -coarser than $\mathcal{A}$, there exists $E_0 \in \mathcal{A}$, $E_0 \subset E$ such that $\mu(E_0 \Delta F) > 0$ for all $F \in \mathcal{C}_E$.

Consider the function g defined by $g = \chi_{E_0} - \mathbb{E}(\chi_{E_0}|\sigma(\mathcal{C} \cup \{E\}))$. Observe that $\mathbb{E}(g\chi_E|\mathcal{C}) = \mathbb{E}[\mathbb{E}(g\chi_E|\sigma(\mathcal{C} \cup \{E\}))|\mathcal{C}] = 0$. From $0 \leq \chi_{E_0} \leq \chi_E$, it follows that $0 \leq \mathbb{E}(\chi_{E_0}|\sigma(\mathcal{C} \cup \{E\})) \leq \mathbb{E}(\chi_E|\sigma(\mathcal{C} \cup \{E\})) = \chi_E$ for μ -almost every $\omega \in \Omega$, which implies that g vanishes outside of E . The set F defined by

$$F = E \cap \{\omega \in \Omega : \mathbb{E}(\chi_{E_0}|\sigma(\mathcal{C} \cup \{E\}))(\omega) = 1\}$$

is a subset of E and is $\sigma(\mathcal{C} \cup \{E\})$ -measurable. Hence, by Lemma 4.5, $F \in \mathcal{C}_E$. Consequently, we have $\mu(E_0 \Delta F) > 0$. Since $g \neq 0$ on $E_0 \Delta F$, it follows that $\|g\|_\infty > 0$.

Set $\bar{g} = \frac{g}{f}$. Then, $\bar{g} \in L^\infty(\Omega, \mathcal{A}, \mu)$, $\|\bar{g}\|_\infty > 0$, $\bar{g}$ vanishes outside of E and $\mathbb{E}(f\bar{g}\chi_E|\mathcal{C}) = \mathbb{E}(g\chi_E|\mathcal{C}) = 0$. This ends the proof. $\square$

Proof of Theorem 2.4. To shorten notation, we write G (respectively G_E , fG , fG_E) instead of $G_{\mathcal{C}}$ (respectively $G_{\sigma(\mathcal{C} \cup \{E\})}$, $fG_{\mathcal{C}}$, $fG_{\sigma(\mathcal{C} \cup \{E\})}$).

We give a proof by induction on n . The statement $P(1)$ is nothing else but Lemma 4.6. Assume that $P(n)$ is true. Let $f = (f_1, f_2, \dots, f_{n+1}) \in L^1(\Omega, \mathcal{A}, \mu)^{n+1}$. Let $E \in \mathcal{A}^+$. According to Theorem 4.1, we have to construct a function $g \in L^\infty(\Omega, \mathcal{A}, \mu)$ vanishing outside of E , with $\|g\|_\infty > 0$, such that

$$\mathbb{E}(fg\chi_E|\sigma(\mathcal{C} \cup \{E\})) = \mathbb{E}(fg|\sigma(\mathcal{C} \cup \{E\})) = 0.$$

This will end the proof since

$$\mathbb{E}(fg|\mathcal{C}) = \mathbb{E}(\mathbb{E}(fg|\sigma(\mathcal{C} \cup \{E\}))|\mathcal{C}) = 0.$$

Without loss of generality, we can assume that $\sum_{i=1}^{n+1} |f_i| > 0$ for μ -almost every $\omega \in E$: otherwise, there is $E' \in \mathcal{A}^+$, $E' \subset E$, such that $f_i \equiv 0$ on E' , for every $i = 1, \dots, n+1$. Set in this case $g = \chi_{E'}$. We can assume as well that every $f_i, i = 1, \dots, n+1$, has a constant sign on E :

$$f_i = s_i |f_i| \text{ with } s_i \in \{-1, 0, 1\} \text{ for all } i = 1, \dots, n+1, \quad (7)$$

since we can always find such an $\mathcal{A}$ -measurable subset E' of E with a strictly positive measure and perform all the forthcoming analysis on E' instead of E . Let

$$h_i = \frac{|f_i|}{\sum_{j=1}^{n+1} |f_j|} \chi_E,$$

which clearly belongs to $L^1(\Omega, \mathcal{A}, \mu)$.

If $h_i = \mathbb{E}(h_i|\sigma(\mathcal{C} \cup \{E\})), \mu - a.e.$, that is, h_i is $\sigma(\mathcal{C} \cup \{E\})$ -measurable up-to a μ -null set, for all $i = 1, \dots, n+1$, then it results that

$$\mathbb{E}(h_i \bar{g}|\sigma(\mathcal{C} \cup \{E\})) = h_i \mathbb{E}(\bar{g}|\sigma(\mathcal{C} \cup \{E\})), \mu - a.e. \text{ for all } \bar{g} \in L^\infty(\Omega, \mathcal{A}, \mu).$$

As G_E is Lyapunov (Lemma 4.6, $f = 1$), there is some $\bar{g} \in L^\infty(\Omega, \mathcal{A}, \mu)$, $\|\bar{g}\|_\infty > 0$, vanishing outside of E and satisfying

$$\mathbb{E}(h_i \bar{g}|\sigma(\mathcal{C} \cup \{E\})) = 0, \mu - a.e., \quad \forall i = 1, \dots, n+1, \quad (8)$$

In the contrary case, there exists an $\mathcal{A}$ -measurable subset $A \subset E$, with $\mu(A) > 0$, where all the functions $h_i^0 := h_i - \mathbb{E}(h_i|\sigma(\mathcal{C} \cup \{E\}))$ keep a constant sign and where $|h_{i_0}^0| > 0$, for μ -almost every $\omega \in A$, for at least one i_0 . Note that as defined, the function $h_i^0, i = 1, \dots, n+1$, satisfy, μ -a.e.

$$\mathbb{E}(h_i^0|\sigma(\mathcal{C} \cup \{E\})) = \mathbb{E}(h_i|\sigma(\mathcal{C} \cup \{E\})) - \mathbb{E}(\mathbb{E}(h_i|\sigma(\mathcal{C} \cup \{E\}))|\sigma(\mathcal{C} \cup \{E\})) = 0, \quad (9)$$

Without loss of generality, assume that $i_0 \leq n$. Applying the induction hypothesis $P(n)$ with $H^0 = (h_1^0, \dots, h_n^0)$ and the sub- σ -algebra $\sigma(\mathcal{C} \cup \{E\})$, it follows that $H_0 G_E$ is Lyapunov. By Theorem 4.2, there is $A_0 \in \mathcal{A}$ subset of A such that

$$H^0 G_E(A_0) = \frac{1}{2} H^0 G_E(A). \quad (10)$$

Since A and A_0 are subsets of E , the function $\bar{g} = \varphi - \mathbb{E}(\varphi|\sigma(\mathcal{C} \cup \{E\}))$, with $\varphi = \chi_{A_0} - \chi_{A \setminus A_0}$, vanishes outside of E . Let us moreover show that $\|\bar{g}\|_\infty > 0$.

For the sake of contradiction, assume that $\bar{g} = 0$ μ -a.e.. Then, the $\sigma(\mathcal{C} \cup \{E\})$ -measurable set $B = \{\omega \in \Omega : \mathbb{E}(\varphi|\sigma(\mathcal{C} \cup \{E\}))(\omega) = 1\}$ is such that $\mu(B \triangle A_0) = 0$. Hence, by the non-nullity of $h_{i_0}^0$ over A_0 , $\int_B h_i^0 d\mu = \int_{A_0} h_i^0 d\mu \neq 0$, which contradicts (9). Moreover, we have for $i = 1, \dots, n$,

$$\begin{aligned} \mathbb{E}(h_i^0 \bar{g} | \sigma(\mathcal{C} \cup \{E\})) &= \mathbb{E}(h_i^0 \chi_{A_0} | \sigma(\mathcal{C} \cup \{E\})) - \mathbb{E}(h_i^0 \chi_{A \setminus A_0} | \sigma(\mathcal{C} \cup \{E\})) \\ &\quad - \mathbb{E}(h_i^0 \mathbb{E}(\varphi | \sigma(\mathcal{C} \cup \{E\})) | \sigma(\mathcal{C} \cup \{E\})) \\ &= h_i^0 G_E(A_0) - h_i^0 G_E(A \setminus A_0) - \mathbb{E}(\varphi | \sigma(\mathcal{C} \cup \{E\})) \mathbb{E}(h_i^0 | \sigma(\mathcal{C} \cup \{E\})) = 0. \end{aligned}$$

where the last equality follows from (10) and (9).

Furthermore, as $\mathbb{E}(\bar{g} | \sigma(\mathcal{C} \cup \{E\})) = \mathbb{E}(\varphi - \mathbb{E}(\varphi | \sigma(\mathcal{C} \cup \{E\})) | \sigma(\mathcal{C} \cup \{E\})) = 0$, we have for $i = 1, \dots, n$,

$$\begin{aligned} \mathbb{E}(h_i \bar{g} | \sigma(\mathcal{C} \cup \{E\})) &= \mathbb{E}(h_i^0 \bar{g} | \sigma(\mathcal{C} \cup \{E\})) + \mathbb{E}(\mathbb{E}(h_i | \sigma(\mathcal{C} \cup \{E\})) \bar{g} | \sigma(\mathcal{C} \cup \{E\})) \\ &= 0 + \mathbb{E}(h_i | \sigma(\mathcal{C} \cup \{E\})) \mathbb{E}(\bar{g} | \sigma(\mathcal{C} \cup \{E\})) = 0. \end{aligned}$$

This is also true for $i = n + 1$ by remarking that $h_{n+1} = \chi_E - \sum_{i=1}^n h_i$ and $\mathbb{E}(\chi_E \bar{g} | \sigma(\mathcal{C} \cup \{E\})) = \chi_E \mathbb{E}(\bar{g} | \sigma(\mathcal{C} \cup \{E\})) = 0$. Finally, by setting

$$g = \frac{\bar{g}}{\sum_{i=1}^{n+1} |f_i|}$$

it comes that $\mathbb{E}(|f_i|g | \sigma(\mathcal{C} \cup \{E\})) = 0$, for every $i = 1, \dots, n + 1$. Then, by (7), $\mathbb{E}(f_i g | \sigma(\mathcal{C} \cup \{E\})) = 0$, for all $i = 1, \dots, n + 1$. The function g has the desired properties. $\square$

Proof of Theorem 2.5. Set $\mu = \frac{1}{n} \sum_{i=1}^n \mu_i$.

Then, $\mu_i \ll \mu$ for every i and obviously $\mathcal{C}$ is μ -coarser than $\mathcal{A}$. Observe now that $F_{\mathcal{C}} \ll \mu$. Set, for every i , $h_i = \frac{d\mu_i}{d\mu}$ the Radon-Nikodym derivative of μ_i relatively to μ . Hence, $f_i h_i \in L^1(\Omega, \mathcal{A}, \mu)$ and from Theorem 2.4 and Theorem 4.1, for every fixed $E \in \mathcal{A}^+$, there exists $g \in L^\infty(\Omega, \mathcal{A}, \mu)$ vanishing outside of E with $\|g\|_\infty > 0$ and $\mathbb{E}(g f_i h_i \chi_E | \mathcal{C}) = 0$. But, for every $A \in \mathcal{C}$, $\int_A g f_i h_i \chi_E d\mu = \int_A g f_i \chi_E d\mu_i$. Therefore, $\mathbb{E}^{\mu_i}(g f_i \chi_E | \mathcal{C}) = 0$, for every i . It results from Theorem 4.1, that $F_{\mathcal{C}}$ is Lyapunov. $\square$

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