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Approximation of Spherical Convex Bodies of Constant Width $\pi/2$

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Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$ and let $C \subset \mathbb{S}^2$ be a spherical convex body of constant width τ. It is known that (i) if $\tau < \pi/2$ then for any $\varepsilon > 0$ there exists a spherical convex body C_ε of constant width τ whose boundary consists only of arcs of circles of radius τ such that the Hausdorff distance between C and C_ε is at most ε; (ii) if $\tau > \pi/2$ then for any $\varepsilon > 0$ there exists a spherical convex body C_ε of constant width τ whose boundary consists only of arcs of circles of radius $\tau - \frac{\pi}{2}$ and great circle arcs such that the Hausdorff distance between C and C_ε is at most ε. In this paper, we present an approximation of the remaining case $\tau = \pi/2$, that is, if $\tau = \pi/2$ then for any $\varepsilon > 0$ there exists a spherical polygon $\mathcal{P}_\varepsilon$ of constant width $\pi/2$ such that the Hausdorff distance between C and $\mathcal{P}_\varepsilon$ is at most ε.

Keywords: Constant width, approximation, spherical polytgon, Hausdorff distance.

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1. Introduction

Blaschke presented an approximation theorem saying that for every convex body of constant width $\tau > 0$ in the Euclidean plane $\mathbb{R}^2$ and every $\varepsilon > 0$ there exists a convex body of constant width τ whose boundary consists only of pieces of circles of radius τ such that the Hausdorff distance between the two bodies is at most ε (see [2, 3]). A generalization of this fact for normed planes is given by Lassak ([7]). In [12], Lassak presented an approximation of spherical reduced body of thickness below $\pi/2$. As a corollary of this fact, also a spherical version of the theorem of Blaschke is presented as follows:

Theorem 1.1. [12] *For any spherical convex body $C \subset \mathbb{S}^2$ of constant width $\tau < \pi/2$, and for any $\varepsilon > 0$ there exists a body C_ε of constant width τ whose boundary consists only of arcs of circles of radius τ such that*

$$h(C, C_\varepsilon) \leq \varepsilon,$$

where $h(C_1, C_2)$ means the Hausdorff distance between C_1 and C_2 .

A counterpart of Theorem 1.1 is given as follows:

Theorem 1.2. [5] *For any spherical convex body $C \subset \mathbb{S}^2$ of constant width $\tau > \pi/2$, and for any $\varepsilon > 0$ there exists a body C_ε of constant width τ whose boundary consists only of arcs of circles of radius $\tau - \frac{\pi}{2}$ and great circle arcs, such that*

$$h(C, C_\varepsilon) \leq \varepsilon,$$

where h is the Hausdorff distance.

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In the Euclidean plane, a *PC curve*, or *piecewise circular curve*, is defined as a finite sequence of circular arcs or line segments, with the endpoint of one arc coinciding with the beginning point of the next ([1, 4]). Then the boundary of C_ε from Theorem 1.2 is a spherical analog of PC curve.

In this paper, we present an approximation of the remaining case $\tau = \pi/2$.

Theorem 1.3. *For any spherical convex body $C \subset \mathbb{S}^2$ of constant width $\pi/2$, and for any $\varepsilon > 0$ there exists a spherical polygon $\mathcal{P}_\varepsilon$ of constant width $\pi/2$, such that*

$$h(C, \mathcal{P}_\varepsilon) \leq \varepsilon,$$

where h denotes the Hausdorff distance.

In [5], a conjecture was posed as "any spherical convex body of constant width $\pi/2$ in $\mathbb{S}^n$ can be approximated by a sequence of spherical convex polygons of constant width $\pi/2$". Thus Theorem 1.3 asserts that the conjecture is true if $n = 2$.

This paper is organized as follows. In Section 2, preliminaries are given. The proof of Theorem 1.3 is given in Section 3.

2. Preliminaries

Let $\mathbb{S}^n$ be the unit sphere centered at the origin of $n+1$ dimensional Euclidean space $\mathbb{R}^{n+1}$. For any point P of $\mathbb{S}^n$, the hemisphere centered at P is denoted by $H(P)$, that is

$$H(P) = \{Q \in \mathbb{S}^n \mid P \cdot Q \geq 0\},$$

where the dot stands for the standard dot product of two vectors $P, Q \in \mathbb{R}^{n+1}$. For any points P, Q of $\mathbb{S}^n$ such that $P \neq -Q$, denote by PQ the spherical great circle arc with end points P, Q such that $-P, -Q$ are not contained in it. We say that a non empty subset W of $\mathbb{S}^n$ is *hemispherical* if there exists a point $P \in \mathbb{S}^n$ such that $W \cap H(P) = \emptyset$; a hemispherical subset C is *spherical convex* if PQ is contained in C for any $P, Q \in C$; and a convex set C is a *spherical convex body* if it contains an interior point. Denote by ∂C the boundary of C. Let P be a point of ∂C. If

$$C \subset H(Q) \text{ and } P \in C \cap \partial H(Q),$$

then we say the hemisphere $H(Q)$ *supports* C at P, and $H(Q)$ is a *supporting hemisphere* of C at P. The intersection $H(P) \cap H(Q)$ is called a *lune* of $\mathbb{S}^n$ provided $P \neq Q$. The *thickness of the lune* $H(P) \cap H(Q)$ is given by $\pi - \arccos(P \cdot Q)$, and denoted by $\Delta(H(P) \cap H(Q))$. If $H(P)$ is a supporting hemisphere of a spherical convex body C, *the width of* C with respect to $H(P)$, denoted by $\mathrm{width}_{H(P)}(C)$, is defined by

$$\mathrm{width}_{H(P)}(C) = \min\{\Delta(H(P) \cap H(Q)) | H(Q) \text{ supports } C\}.$$

We say that a spherical convex body C is *of constant width*, if all widths of C are equal. Following [8], *the thickness* of a convex body $C \subset \mathbb{S}^n$, denoted by $\Delta(C)$, is the minimum of $\mathrm{width}_{H(P)}(C)$ over all supporting hemispheres $H(P)$ of C. Namely,

$$\Delta(C) = \min\{\mathrm{width}_K(C) | K \text{ is a supporting hemisphere of } C\}.$$

We say that a spherical convex body C is of constant diameter $\tau > 0$, if the diameter of C is τ, and for every point P on the boundary of C there exists a point Q of C such that $\arccos(P \cdot Q) = \tau$.

For any non-empty subset $W \subset \mathbb{S}^n$, the *spherical polar set of* W, denoted by W°, is defined as follows:

$$W^\circ = \bigcap_{P \in W} H(P).$$

We say a spherical convex body C is *self-dual* if $C = C^\circ$.

Proposition 2.1. [6, 9] *Let $C \subset \mathbb{S}^n$ be a spherical convex body. Then, $C = C^\circ$ if and only if C is of constant width $\pi/2$.*

More details on spherical convex body of constant width, see for instance [11].

Lemma 2.2. [5] *Let C be a spherical convex body in $\mathbb{S}^n$. The following two assertions are equivalent:*

(1) *the hemisphere $H(P)$ supports C° at Q;*

(2) *the hemisphere $H(Q)$ supports C at P.*

3. Proof of Theorem 1.3

Proposition 3.1. [10, 13] *Let C be a spherical convex body in $\mathbb{S}^n$. The following two statements are equivalent:*

1. *C is of constant diameter $\pi/2$.*
2. *C is of constant width $\pi/2$.*

Lemma 3.2. *Let P_1, P_2 be two points of $\mathbb{S}^2$ such that* $\arccos(P_1 \cdot P_2) = \pi/2$. *Let P_3 be a point of $H(P_1) \cap H(P_2)$. Then*

$$\arccos(Q_1 \cdot Q_2) < \pi/2,$$

where Q_1, Q_2 are two points of the spherical triangle P_1, P_2, P_3 such that $Q_i \neq P_j$, $i = 1, 2$, $j = 1, 2, 3$.

Proof. Without loss of generality, set $P_1 = (1, 0, 0), P_2 = (0, 1, 0)$ and $P_3 = (0, 0, 1)$. Then we can assume that $Q_1 = (x_1, y_1, z_1), Q_2 = (x_2, y_2, z_2)$ where $0 \leq x_i, y_i, z_i < 1$ such that $x_i^2 + y_i^2 + z_i^2 = 1, i = 1, 2$. Since $Q_1 \cdot Q_2 = x_1x_2 + y_1y_2 + z_1z_2 < 1$, it follows that $\arccos(Q_1 \cdot Q_2) < \pi/2$. □

If a spherical convex body C of constant width is a spherical polygon, it is nothing to prove. Hence, we assume that C is locally strictly convex, namely that the boundary of C always contains a spherical curve different from great circle arc.

Part I: Construct a spherical convex body C_ε^1.

We can parametrize ∂C by a mapping $c : [0, 2\pi] \to \mathbb{S}^2$ such that $c(0) = c(2\pi)$ and $c(t_1) \neq c(t_2)$ for any two different numbers $t_1, t_2 \in (0, 2\pi)$. Then, we could say that the curve $\widehat{R_1R_2}$ of ∂C is oriented from R_1 to R_2. Let $T'T''$ be a strictly convex part of ∂C with endpoints T', T''.

We insert a finite number of points $P_1, \dots, P_l$ into $T'T''$ such that P_iP_{i+1} is a pice of great circle arc of $\partial H(R_i)$ and

$$R_i \in B_s(C, \varepsilon) = \left\{ P \in \mathbb{S}^2 \mid \inf_{Q \in C} \{P \cdot Q < \varepsilon\} \right\},$$

for any $1 \le i \le l-1$. Here, $T' = P_1$ and $T'' = P_l$. Next, we gradually construct the convex bodies C_ε^i of constant width $\pi/2$ and remove these strictly convex parts $\widehat{P_iP}_{i+1}$. Since C is a body of constant width $\pi/2$, by Proposition 3.1, there exist two points $Q_1, Q_2 \in \partial C^\circ$ such that

$$\arccos(P_1 \cdot Q_1) = \arccos(P_2 \cdot Q_2) = \pi/2.$$

Since ε can be a sufficiently small number, we may assume that Q_1, Q_2 are contained in $H(R_1)$ (see Figure 1). Then we know that the great circle segment R_1Q_1 (resp. R_1Q_2) is a subset of the boundary of the supporting hemisphere $H(P_1)$ (resp. $H(P_2)$) of C. This means that R_1 is an intersection point of $\partial H(P_1) \cap \partial H(P_2)$. We denote by C_ε^1 the spherical convex hull of

$$\left(\partial C \backslash \widehat{P_1P_2}\right) \cup P_1P_2 \cup R_1.$$

Then by the discussion above, we know that ∂C_ε^1 is the union

$$\partial C \backslash \left(\widehat{P_1P_2} \cup \widehat{Q_1Q_2}\right) \cup P_1P_2 \cup R_1Q_1 \cup R_1Q_2,$$

where $\widehat{Q_1Q_2}$ (counterclockwise in the usual sense, see Figure 1) is the curve of ∂C with beginning point Q_1 and endpoint Q_2.

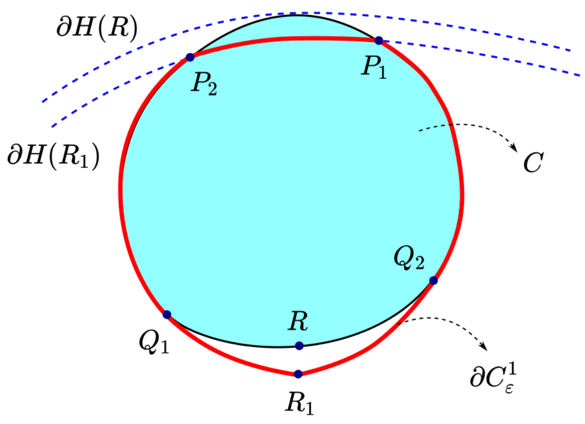

Figure 1: A spherical convex body C_ε^1.

Part II: The spherical convex body C_ε^1 is of constant width $\pi/2$.

By Proposition 3.1, it is sufficient to prove C_ε^1 is of constant diameter $\pi/2$.

Lemma 3.3. *The chords P_1Q_1, P_2Q_2 of C intersect each other.*

Proof. Assume the opposite that P_1Q_1, P_2Q_2 disjoint each other, then P_2, Q_2 are two points of a spherical triangle P_1Q_1R', where $R' \in \partial H(P_1) \cap \partial H(Q_1)$. By Lemma 3.2, $\arccos(P_2 \cdot Q_2) < \pi/2$. This constradicts the fact that $\arccos(P_2 \cdot Q_2) = \pi/2$. □

By the proof of Lemma 3.3, one obtains that any two chords of C whose length are equal to the diameter intersect each other. Let P be a relative interior point of $\widehat{P_2Q_1}$. Since $\arccos(P_1 \cdot Q_1) = \pi/2$ and C is a spherical convex body of constant diameter, by Lemma 3.3, there exists a point $P' \in \widehat{Q_1P_1}$ such that $\arccos(P \cdot P') = \pi/2$. On the other hand, since $\arccos(P_2 \cdot Q_2) = \pi/2$, applying Lemma 3.3 again, we know that PP' and P_2Q_2 intersect each other. Thus, we know that P' is a point of $\widehat{Q_2P_1}$. Similarly, for any point $Q \in \widehat{Q_2P_1}$ there exists a point $Q' \in \widehat{P_2Q_1}$ such that $\arccos(Q \cdot Q') = \pi/2$. From the above discussion we know that C_ε^1 is a body of constant diameter $\pi/2$, that is for any point $R \in \partial C_\varepsilon^1$ there exists a point $R' \in \partial C_\varepsilon^1$ such that $\arccos(R \cdot R') = \pi/2$.

Part III: $h(C, C_\varepsilon^1) < \varepsilon/2$.

By the construction in Part I, it is clear that C_ε^1 is a subset of $B_s(C, \varepsilon)$.

On the other hand, by the assumption $R_1 \in B_s(C, \varepsilon)$, there exists a point $R \in \partial C$ such that $\arccos(R \cdot R_1) < \varepsilon$. This implies obviously that the spherical convex hull of $P_1P_2 \cup \widehat{P_1P_2}$ is a subset of the lune $H(R) \cap H(R_1)$. Conclude that the thickness of the lune $H(R) \cap H(R_1)$ is smaller than ε, that is

$$\Delta(H(R) \cap H(R_1)) < \varepsilon.$$

Then we know that for any point P of P_1P_2 there exist a point $\widehat{P} \in \widehat{P_1P_2}$ such that $\arccos(P \cdot \widehat{P}) < \varepsilon$ (see Figure 2). This implies C_ε is a subset of $B_s(C_\varepsilon^1, \varepsilon)$. Thus we conclude $h(C, C_\varepsilon^1) < \varepsilon$.

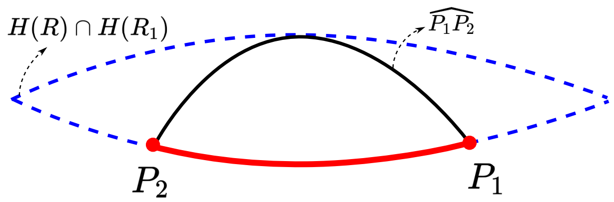

Figure 2: $\Delta(H(R) \cap H(R_1)) < \varepsilon$.

We repeat the steps in Part II. And then, after a finite number of steps we can remove the arc $T'T''$ and obtain a spherical polygon $C_{\varepsilon,1}$ of constant width $\pi/2$ such that

$$h(C, C_{\varepsilon,1}) < \varepsilon.$$

If $C_{\varepsilon,1}$ is a spherical polygon, then put $\mathcal{P}_\varepsilon = C_{\varepsilon,1}$. In the opposite case, we replace $C_{\varepsilon,1}$ (resp. ε) with C (resp. $\varepsilon/2$) and repeat the step in Part I. And then, after a finite number of m steps we get a spherical polygon $C_{\varepsilon,m}$ of constant width $\pi/2$, and

$$h(C, C_{\varepsilon,m}) < h(C, C_{\varepsilon,1}) + h(C, C_{\varepsilon,2}) + \cdots + h(C, C_{\varepsilon,m}) < \varepsilon + \frac{\varepsilon}{2} + \cdots + \frac{\varepsilon}{2^m} < 2\varepsilon.$$

Put $\mathcal{P}_\varepsilon = C_{\varepsilon,m}$. By the arbitrariness of ε, this completes the proof.

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