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HLPK Type Theorem for Triangle Functionals Induced by Convex Functions and Further Results on Refining Jensen's Inequality

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We study the triangle functionals induced by convex functions. To this end the vectorial majorization intended for comparing two tuples of vectors is used. A Hardy-Littlewood-Pólya-Karamata (HLPK) type theorem is proved for the triangle functionals. As applications, some refinements of the celebrated Jensen's inequality are established. In doing so, the specification of HLPK Theorem for two triples of vectors is utilized. Special attention is paid to convex functions that are simultaneously concave (affine) on some subset of their domains. In particular, functions that are both convex and piecewise affine are considered. Some applications for convex sequences are also provided.

Keywords: Majorization, HLPK inequality, convex function, Jensen's inequality, perspective function, triangle functional, piecewise affine function, convex sequence.

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1. Introduction, preliminaries and summary

In the last few decades convex functions have been attracted interest from many researchers [2, 4, 6, 5, 7, 8, 10, 13, 16, 17, 21, 24, 26]. Applications of convex functions can be found in optimization theory, programming, mathematical analysis, nonlinear analysis, probability, statistics, mathematical economics, engineering, information theory, etc. In this context perspective functions proved to be a convenient tool for the study of convexity (see e.g., [3, 9, 27]). In the present paper we focus on applications of perspective functions and their triangle functionals to refining Jensen's inequality.

Before giving our results, we now introduce some needed notation, terminology and facts.

Throughout V is a real linear space with null vector $\mathbf{0}$, and $\mathbb{R} \times V$ is the cartesian product of $\mathbb{R}$ and V. The set $\mathbb{R} \times V$ is a real linear space with the null vector $(0, \mathbf{0})$ (briefly denoted also by $\mathbf{0}$) and with the standard algebraic operations: for $(p, \mathbf{x}), (q, \mathbf{y}) \in \mathbb{R} \times V$ and $t \in \mathbb{R}$,

$$(p, \mathbf{x}) + (q, \mathbf{y}) = (p + q, \mathbf{x} + \mathbf{y}),$$
$$t(p, \mathbf{x}) = (tp, t\mathbf{x}).$$

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Given a (nonempty) convex set $I \subset V$ and a convex function $f : I \to \mathbb{R}$ on I, we define the *perspective function* of f by

$$g(p, \mathbf{x}) = pf\left(\frac{\mathbf{x}}{p}\right) \quad \text{for all pairs } (p, \mathbf{x}) \in \mathbb{R} \times V \text{ with } p > 0 \text{ and } \frac{\mathbf{x}}{p} \in I.$$

We also use the convention that $g(0, \mathbf{0}) = 0f\left(\frac{\mathbf{0}}{0}\right) = 0$.

For convenience, we denote the domain of g by

$$D_g = \{(p, \mathbf{x}) \in \mathbb{R} \times V : p > 0\ ,\ \frac{\mathbf{x}}{p} \in I\} \cup \{(0, \mathbf{0})\}.$$

The set D_g is *additive*, i.e.,

$$(p, \mathbf{x}), (q, \mathbf{y}) \in D_g \quad \Rightarrow \quad (p, \mathbf{x}) + (q, \mathbf{y}) = (p + q, \mathbf{x} + \mathbf{y}) \in D_g.$$

In fact, if $(p, \mathbf{x}) = (0, \mathbf{0})$ or $(q, \mathbf{y}) = (0, \mathbf{0})$ then this implication is valid. If we have $(p, \mathbf{x}), (q, \mathbf{y}) \in D_g \setminus \{(0, \mathbf{0})\}$ then $p, q > 0$ and $\frac{\mathbf{x}}{p}, \frac{\mathbf{y}}{q} \in I$. So we have $p + q > 0$ and

$$\frac{\mathbf{x} + \mathbf{y}}{p + q} = \frac{p}{p + q}\frac{\mathbf{x}}{p} + \frac{q}{p + q}\frac{\mathbf{y}}{q} \in I, \tag{1}$$

since I is convex. From this $(p + q, \mathbf{x} + \mathbf{y}) \in D_g$.

The set D_g is *positively homogeneous*, i.e.,

$$t \geq 0 \quad \text{and} \quad (p, \mathbf{x}) \in D_g \quad \Rightarrow \quad t(p, \mathbf{x}) = (tp, t\mathbf{x}) \in D_g$$

because this implication holds trivially for $t = 0$ and for $(p, \mathbf{x}) = (0, \mathbf{0})$, and in the case when $t > 0$ and $(p, \mathbf{x}) \in D_g \setminus \{(0, \mathbf{0})\}$ we have $tp > 0$ and $\frac{t\mathbf{x}}{tp} = \frac{\mathbf{x}}{p} \in I$.

In summary, D_g is a convex set in the linear space $\mathbb{R} \times V$.

The convexity of the function $f : I \to \mathbb{R}$ via (1) guarantees that the perspective map $(p, \mathbf{x}) \mapsto g(p, \mathbf{x})$ is *subadditive* on D_g, i.e.,

$$g((p, \mathbf{x}) + (q, \mathbf{y})) \leq g(p, \mathbf{x}) + g(q, \mathbf{y}) \quad \text{for } (p, \mathbf{x}), (q, \mathbf{y}) \in D_g, \tag{2}$$

or equivalently,

$$(p + q)f\left(\frac{\mathbf{x} + \mathbf{y}}{p + q}\right) \leq pf\left(\frac{\mathbf{x}}{p}\right) + qf\left(\frac{\mathbf{y}}{q}\right) \quad \text{for } (p, \mathbf{x}), (q, \mathbf{y}) \in D_g. \tag{3}$$

The map $(p, \mathbf{x}) \mapsto g(p, \mathbf{x})$ is *positively homogeneous* in $(p, \mathbf{x}) \in D_g$, i.e.,

$$g(t(p, \mathbf{x})) = tg(p, \mathbf{x}) \quad \text{for } t \in [0, \infty) \text{ and } (p, \mathbf{x}) \in D_g, \tag{4}$$

because
$$(tp)f\left(\frac{t\mathbf{x}}{tp}\right) = tpf\left(\frac{\mathbf{x}}{p}\right).$$

In the sequel, for $(p, \mathbf{x}), (q, \mathbf{y}), (r, \mathbf{z}) \in D_g$ we often use the abbreviations:

$$\mathbf{a} = (p, \mathbf{x})\ ,\quad \mathbf{b} = (q, \mathbf{y}) \quad \text{and} \quad \mathbf{c} = (r, \mathbf{z}).$$

For a convex function $f : I \to \mathbb{R}$ and its perspective function $g : D_g \to \mathbb{R}$, we introduce the *triangle functional* of g by

$$G(\mathbf{a}, \mathbf{b}) = g(\mathbf{a}) + g(\mathbf{b}) - g(\mathbf{a} + \mathbf{b}) \quad \text{for } \mathbf{a}, \mathbf{b} \in D_g. \tag{5}$$

Inequality (2) ensures that

$$G(\mathbf{a},\mathbf{b}) \geq 0 \quad \text{for } \mathbf{a},\mathbf{b} \in D_g. \tag{6}$$

That is, the triangle functional is nonnegative on $D_g \times D_g$.

We say as usual that an m-tuple $\mathbf{x} = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$ *majorizes* an m-tuple $\mathbf{y} = (y_1, y_2, \dots, y_m) \in \mathbb{R}^m$ (written as $\mathbf{y} \prec \mathbf{x}$), if the sum of j largest entries of $\mathbf{y}$ does not exceed the sum of j largest entries of $\mathbf{x}$ for all $j = 1, 2, \dots, m$ with equality for $j = m$. That is, $\mathbf{y} \prec \mathbf{x}$ iff

$$\sum_{i=1}^{j} y_{[i]} \leq \sum_{i=1}^{j} x_{[i]} \quad \text{for } j = 1, 2, \dots, m, \quad \text{and} \quad \sum_{i=1}^{m} y_i = \sum_{i=1}^{m} x_i,$$

where $x_{[i]}$ and $y_{[i]}$ denote the ith largest entry of $\mathbf{x}$ and $\mathbf{y}$, respectively (see [13, p. 8]).

An $m \times m$ real matrix $\mathbf{S} = (s_{ij})$ is said to be *doubly stochastic* if all its entries are nonnegative and all its row- and column-sums are equal to 1, that is, if $s_{ij} \geq 0$ for $i, j = 1, 2, ..., m$, and $\sum\limits_{j=1}^{m} s_{ij} = 1$ for $i = 1, 2, ..., m$, and $\sum\limits_{i=1}^{m} s_{ij} = 1$ for $j = 1, 2, ..., m$ (see [13, Chapter 2]).

It is well known by Birkhoff and Rado's theorems (see [13, Theorem A.2, p. 30, and Proposition C.1, p. 162]) that

$$\mathbf{y} \prec \mathbf{x} \quad \text{iff} \quad \mathbf{y} = \mathbf{x}\mathbf{S}$$

for some $m \times m$ doubly stochastic matrix $\mathbf{S}$ (see [13, Theorem B.6, p. 35]).

A square real matrix $\mathbf{P} = (p_{ij})$ is called a *permutation matrix*, if its each entry is equal to 0 or 1, and, in addition, its each row and column contains exactly one entry equal to 1.

Evidently, each permutation matrix is doubly stochastic.

Theorem 1.1. (Schur [28], Hardy, Littlewood, Pólya [13, p. 92] and Karamata [11].) *Let $f : I \to \mathbb{R}$ be a convex function defined on an interval $I \subset \mathbb{R}$ and let $\mathbf{x} = (x_1, x_2, \dots, x_m) \in I^m$ and $\mathbf{y} = (y_1, y_2, \dots, y_m) \in I^m$. Then*

$$\mathbf{y} \prec \mathbf{x} \quad \text{implies} \quad \sum_{i=1}^{m} f(y_i) \leq \sum_{i=1}^{m} f(x_i). \tag{7}$$

In the present paper our goal is to provide a technique for refining Jensen's inequality for convex functions. We use a framework based on the vectorial majorization intended for comparing two $n+1$-tuples of vectors. To do so, we employ the triangle functional generated by the perspective function of a given convex one, and prove Hardy-Littlewood-Pólya-Karamata (HLPK) type theorem for the functional (see Theorem 2.1). As applications, by making use of the specification of HLPK Theorem for two triples of vectors (the case $n = 2$), we formulate a condition leading to some refinements of Jensen's inequality (see Theorem 3.1 and Corollary 3.2). We show that such condition is satisfied for the class of convex functions that are concave (affine) on some subset of their domains (see Corollary 3.3). For instance, it concerns convex functions that are piecewise affine (see Corollary 3.4). The usage of such functions is useful in order to establish some estimations for convex sequences (see Section 4). The methods used in this work develop the ones applied in [19, 20, 22].

2. HLPK type theorem for the triangle functional

In the sequel, we employ majorization theory for vectorial tuples rather than for real ones (see [18, 21, 15]). Therefore some further notation and terminology are given below.

Let W be a real linear space, and $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in W$ and $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m \in W$. We say that $(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m) \in W^m$ is *majorized* by $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m) \in W^m$, in notation $(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m) \prec (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m)$, if there exists a doubly stochastic matrix $\mathbf{S} = (s_{ij})$ of size $m \times m$ such that

$$(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m) = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m)\mathbf{S}, \tag{8}$$

in the sense that (see [15, 18, 21])

$$\mathbf{b}_j = s_{1j}\mathbf{a}_1 + s_{2j}\mathbf{a}_2 + \ldots + s_{mj}\mathbf{a}_m \quad \text{for } j = 1, 2, \ldots, m. \tag{9}$$

In the context of (8)–(9) we observe that each $\mathbf{b}_j$ lies in the convex hull spanned by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$, that is,

$$\mathbf{b}_j \in \operatorname{conv}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\} \quad \text{for } j = 1, 2, \ldots, m,$$

because $\mathbf{S}$ is doubly stochastic with $s_{1j}, s_{2j}, \ldots, s_{mj} \geq 0$ and $s_{1j} + s_{2j} + \ldots + s_{mj} = 1$.

We are now in a position to state and prove our first result.

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be a convex function on a convex set $I \subset V$, $g : D_g \to \mathbb{R}$ be the perspective function of f, and $G : D_g \times D_g \to \mathbb{R}$ be the triangle functional of g defined by* (5).

Let $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in D_g$ and $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \in D_g$ be such that

$$(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n) \prec (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n). \tag{10}$$

Let $\widetilde{\mathbf{a}}_0, \widetilde{\mathbf{a}}_1, \widetilde{\mathbf{a}}_2, \ldots, \widetilde{\mathbf{a}}_n, \widetilde{\mathbf{a}}_{n+1} \in D_g$ and $\widetilde{\mathbf{b}}_0, \widetilde{\mathbf{b}}_1, \widetilde{\mathbf{b}}_2, \ldots, \widetilde{\mathbf{b}}_n, \widetilde{\mathbf{b}}_{n+1} \in D_g$ be such that

$$\widetilde{\mathbf{a}}_0 = \mathbf{0}\ , \quad \widetilde{\mathbf{a}}_i = \mathbf{a}_{i-1} + \widetilde{\mathbf{a}}_{i-1} \quad \textit{for } i = 1, 2, \ldots n, n+1, \tag{11}$$

$$\widetilde{\mathbf{b}}_0 = \mathbf{0}\ , \quad \widetilde{\mathbf{b}}_i = \mathbf{b}_{i-1} + \widetilde{\mathbf{b}}_{i-1} \quad \textit{for } i = 1, 2, \ldots n, n+1. \tag{12}$$

Then
$$\begin{aligned} &G(\mathbf{b}_0, \widetilde{\mathbf{b}}_0) + G(\mathbf{b}_1, \widetilde{\mathbf{b}}_1) + G(\mathbf{b}_2, \widetilde{\mathbf{b}}_2) + \ldots + G(\mathbf{b}_n, \widetilde{\mathbf{b}}_n) \\ &\leq G(\mathbf{a}_0, \widetilde{\mathbf{a}}_0) + G(\mathbf{a}_1, \widetilde{\mathbf{a}}_1) + G(\mathbf{a}_2, \widetilde{\mathbf{a}}_2) + \ldots + G(\mathbf{a}_n, \widetilde{\mathbf{a}}_n) \end{aligned} \tag{13}$$

with $G(\mathbf{a}_0, \widetilde{\mathbf{a}}_0) = 0$ and $G(\mathbf{b}_0, \widetilde{\mathbf{b}}_0) = 0$.

In particular, inequality (13) *holds for any permutation $(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ of the $n+1$-tuple $(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$.*

Proof. It is clear that

$$G(\mathbf{a}_0, \widetilde{\mathbf{a}}_0) = G(\mathbf{a}_0, \mathbf{0}) = g(\mathbf{a}_0) + g(\mathbf{0}) - g(\mathbf{a}_0 + \mathbf{0}) = g(\mathbf{0}) = 0f\left(\frac{\mathbf{0}}{0}\right) = 0.$$

Likewise, we have $G(\mathbf{b}_0, \widetilde{\mathbf{b}}_0) = 0$.

By (11) we get $\widetilde{\mathbf{a}}_0 = \mathbf{0}$, $g(\widetilde{\mathbf{a}}_0) = 0$ and further

$$\sum_{i=0}^{n} G(\mathbf{a}_i, \widetilde{\mathbf{a}}_i) = \sum_{i=0}^{n} (g(\mathbf{a}_i) + g(\widetilde{\mathbf{a}}_i) - g(\mathbf{a}_i + \widetilde{\mathbf{a}}_i))$$
$$= \sum_{i=0}^{n} (g(\mathbf{a}_i) + g(\widetilde{\mathbf{a}}_i) - g(\widetilde{\mathbf{a}}_{i+1})) = \sum_{i=0}^{n} g(\mathbf{a}_i) + \sum_{i=0}^{n} (g(\widetilde{\mathbf{a}}_i) - g(\widetilde{\mathbf{a}}_{i+1})$$
$$= \sum_{i=0}^{n} (g(\mathbf{a}_i) + g(\widetilde{\mathbf{a}}_i) - g(\widetilde{\mathbf{a}}_{i+1})) = \sum_{i=0}^{n} g(\mathbf{a}_i) + g(\widetilde{\mathbf{a}}_0) - g(\widetilde{\mathbf{a}}_{n+1}) = \sum_{i=0}^{n} g(\mathbf{a}_i) - g(\widetilde{\mathbf{a}}_{n+1}).$$

Furthermore, by (11) used $n+1$ times, we get

$$\begin{aligned} g(\widetilde{\mathbf{a}}_{n+1}) &= g(\mathbf{a}_n + \widetilde{\mathbf{a}}_n) = g(\mathbf{a}_n + \mathbf{a}_{n-1} + \widetilde{\mathbf{a}}_{n-1}) = g(\mathbf{a}_n + \mathbf{a}_{n-1} + \mathbf{a}_{n-2} + \widetilde{\mathbf{a}}_{n-2}) \\ &= \ldots = g(\mathbf{a}_n + \mathbf{a}_{n-1} + \mathbf{a}_{n-2} + \ldots + \mathbf{a}_2 + \widetilde{\mathbf{a}}_2) \\ &= g(\mathbf{a}_n + \mathbf{a}_{n-1} + \mathbf{a}_{n-2} + \ldots + \mathbf{a}_2 + \mathbf{a}_1 + \widetilde{\mathbf{a}}_1) \\ &= g(\mathbf{a}_n + \mathbf{a}_{n-1} + \mathbf{a}_{n-2} + \ldots + \mathbf{a}_2 + \mathbf{a}_1 + \mathbf{a}_0 + \widetilde{\mathbf{a}}_0) \\ &= g(\mathbf{a}_n + \mathbf{a}_{n-1} + \mathbf{a}_{n-2} + \ldots + \mathbf{a}_2 + \mathbf{a}_1 + \mathbf{a}_0) = g\Big(\sum_{i=0}^{n} \mathbf{a}_i\Big). \end{aligned}$$

In summary, we have

$$\begin{aligned} G(\mathbf{a}_0, \widetilde{\mathbf{a}}_0) + G(\mathbf{a}_1, \widetilde{\mathbf{a}}_1) + G(\mathbf{a}_2, \widetilde{\mathbf{a}}_2) + \ldots + G(\mathbf{a}_{n-1}, \widetilde{\mathbf{a}}_{n-1}) + G(\mathbf{a}_n, \widetilde{\mathbf{a}}_n) \\ = \sum_{i=0}^{n} g(\mathbf{a}_i) - g\Big(\sum_{i=0}^{n} \mathbf{a}_i\Big). \end{aligned} \tag{14}$$

In a similar manner we obtain

$$\begin{aligned} G(\mathbf{b}_0, \widetilde{\mathbf{b}}_0) + G(\mathbf{b}_1, \widetilde{\mathbf{b}}_1) + G(\mathbf{b}_2, \widetilde{\mathbf{b}}_2) + \ldots + G(\mathbf{b}_{n-1}, \widetilde{\mathbf{b}}_{n-1}) + G(\mathbf{b}_n, \widetilde{\mathbf{b}}_n) \\ = \sum_{j=0}^{n} g(\mathbf{b}_j) - g\Big(\sum_{j=0}^{n} \mathbf{b}_j\Big). \end{aligned} \tag{15}$$

By (10) we infer that there exists a doubly stochastic matrix $\mathbf{S} = (s_{ij})_{i,j=0,1,\ldots,n}$ of size $(n+1) \times (n+1)$ such that

$$(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n) = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)\mathbf{S},$$

that is,
$$\mathbf{b}_j = \sum_{i=0}^{n} s_{ij}\mathbf{a}_i \quad \text{for } j = 0, 1, \ldots, n. \tag{16}$$

For this reason we have

$$\sum_{j=0}^{n} \mathbf{b}_j = \sum_{j=0}^{n} \sum_{i=0}^{n} s_{ij}\mathbf{a}_i = \sum_{i=0}^{n} \sum_{j=0}^{n} s_{ij}\mathbf{a}_i = \sum_{i=0}^{n} \Big(\sum_{j=0}^{n} s_{ij}\Big)\mathbf{a}_i = \sum_{i=0}^{n} \mathbf{a}_i,$$

the last equality being a consequence of the fact that the row-sums of $\mathbf{S}$ are equal to 1.

Thus we find that
$$g\Big(\sum_{i=0}^{n}\mathbf{b}_i\Big)=g\Big(\sum_{i=0}^{n}\mathbf{a}_i\Big). \tag{17}$$

Since f is convex, and g is subadditive and positively homogeneous on D_g (see (2) and (4)), g is also convex on D_g. Hence, via (16), we deduce for $j=0,1,\ldots,n$ that

$$g(\mathbf{b}_j)=g\Big(\sum_{i=0}^{n}s_{ij}\mathbf{a}_i\Big)\le\sum_{i=0}^{n}s_{ij}g\left(\mathbf{a}_i\right).$$

Therefore we derive the inequality

$$\begin{aligned}\sum_{j=0}^{n}g\big(\mathbf{b}_j\big)&\le\sum_{j=0}^{n}\sum_{i=0}^{n}s_{ij}g\big(\mathbf{a}_i\big)=\sum_{i=0}^{n}\sum_{j=0}^{n}s_{ij}g\big(\mathbf{a}_i\big)\\&=\sum_{i=0}^{n}\Big(\sum_{j=0}^{n}s_{ij}\Big)g\left(\mathbf{a}_i\right)=\sum_{i=0}^{n}g\left(\mathbf{a}_i\right),\end{aligned} \tag{18}$$

the last equality being a consequence of the double stochasticity of $\mathbf{S}$.

In consequence, by (14), (15), (17) and (18) we get the inequality

$$\begin{aligned}&G(\mathbf{b}_0,\widetilde{\mathbf{b}}_0)+G(\mathbf{b}_1,\widetilde{\mathbf{b}}_1)+G(\mathbf{b}_2,\widetilde{\mathbf{b}}_2)+\ldots+G(\mathbf{b}_n,\widetilde{\mathbf{b}}_n)\\&\le G(\mathbf{a}_0,\widetilde{\mathbf{a}}_0)+G(\mathbf{a}_1,\widetilde{\mathbf{a}}_1)+G(\mathbf{a}_2,\widetilde{\mathbf{a}}_2)+\ldots+G(\mathbf{a}_n,\widetilde{\mathbf{a}}_n)\end{aligned} \tag{19}$$

with $G(\mathbf{a}_0,\widetilde{\mathbf{a}}_0)=0$ and $G(\mathbf{b}_0,\widetilde{\mathbf{b}}_0)=0$, as claimed.

To show the last statement of Theorem 2.1 we notice that if

$$(\mathbf{b}_0,\mathbf{b}_1,\mathbf{b}_2,\ldots,\mathbf{b}_n)=(\mathbf{a}_0,\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_n)\mathbf{P}$$

for some permutation matrix $\mathbf{P}$ of size $(n+1)\times(n+1)$, then (10) is met, because $\mathbf{P}$ is doubly stochastic. Next, it is enough to apply the proved part of Theorem 2.1. This completes the proof. □

3. Refining Jensen's inequality

In this section we are going to refine Jensen's inequality (3). We begin our discussion with a specialization of Theorem 2.1 for $n=2$.

Theorem 3.1. *Let $f: I\to\mathbb{R}$ be a convex function on a convex set $I\subset V$, $g: D_g\to\mathbb{R}$ be the perspective function of f, and $G: D_g\times D_g\to\mathbb{R}$ be the triangle functional of g defined by* (5).

Let $\mathbf{a}_0,\mathbf{a}_1,\mathbf{a}_2\in D_g$ and $\mathbf{b}_0,\mathbf{b}_1,\mathbf{b}_2\in D_g$ be such that

$$(\mathbf{b}_0,\mathbf{b}_1,\mathbf{b}_2)\prec(\mathbf{a}_0,\mathbf{a}_1,\mathbf{a}_2). \tag{20}$$

Denote
$$\mathbf{a}_3=\mathbf{a}_0+\mathbf{a}_1\quad\text{and}\quad\mathbf{b}_3=\mathbf{b}_0+\mathbf{b}_1. \tag{21}$$

Then

$$0\le\max\{G(\mathbf{b}_0,\mathbf{b}_1),G(\mathbf{b}_2,\mathbf{b}_3)\}\le G(\mathbf{b}_0,\mathbf{b}_1)+G(\mathbf{b}_2,\mathbf{b}_3)\le G(\mathbf{a}_0,\mathbf{a}_1)+G(\mathbf{a}_2,\mathbf{a}_3). \tag{22}$$

If in addition $G(\mathbf{a}_0, \mathbf{a}_1) = 0$ *then*

$$0 \le \max\{G(\mathbf{b}_0, \mathbf{b}_1), G(\mathbf{b}_2, \mathbf{b}_3)\} \le G(\mathbf{b}_0, \mathbf{b}_1) + G(\mathbf{b}_2, \mathbf{b}_3) \le G(\mathbf{a}_2, \mathbf{a}_3). \tag{23}$$

If $G(\mathbf{a}_2, \mathbf{a}_3) = 0$ *then*

$$0 \le \max\{G(\mathbf{b}_0, \mathbf{b}_1), G(\mathbf{b}_2, \mathbf{b}_3)\} \le G(\mathbf{b}_0, \mathbf{b}_1) + G(\mathbf{b}_2, \mathbf{b}_3) \le G(\mathbf{a}_0, \mathbf{a}_1). \tag{24}$$

In particular, inequalities (22), (23) *and* (24) *are satisfied for any permutation* $(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2)$ *of the triple* $(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$.

Proof. Thanks to (20) we have $(\mathbf{b}_1, \mathbf{b}_0, \mathbf{b}_2) \prec (\mathbf{a}_1, \mathbf{a}_0, \mathbf{a}_2)$.

By applying Theorem 2.1 for $n = 2$ we obtain the inequality:

$$G(\mathbf{b}_1, \mathbf{0}) + G(\mathbf{b}_0, \mathbf{b}_1) + G(\mathbf{b}_2, \mathbf{b}_3) \le G(\mathbf{a}_1, \mathbf{0}) + G(\mathbf{a}_0, \mathbf{a}_1) + G(\mathbf{a}_2, \mathbf{a}_3) \tag{25}$$

with $G(\mathbf{a}_1, \mathbf{0}) = 0$ and $G(\mathbf{b}_1, \mathbf{0}) = 0$. We now conclude from (6) and (25) that

$$\max\{G(\mathbf{b}_0, \mathbf{b}_1), G(\mathbf{b}_2, \mathbf{b}_3)\} \le G(\mathbf{b}_0, \mathbf{b}_1) + G(\mathbf{b}_2, \mathbf{b}_3) \le G(\mathbf{a}_0, \mathbf{a}_1) + G(\mathbf{a}_2, \mathbf{a}_3).$$

Thus the proof of inequalities (22) is complete. Inequalities (23) and (24) follow easily from (22). Finally, the last part of Theorem 3.1 is a consequence of the last part of Theorem 2.1. □

Corollary 3.2. *Let* $f : I \to \mathbb{R}$ *be a convex function on a convex set* $I \subset V$, g *be the perspective function of* f, *and* G *be the triangle functional of* g. *Let* $\mathbf{c}, \mathbf{d} \in D_g$ *be given. If* $\mathbf{a}, \mathbf{b} \in D_g$ *are such that*

$$\mathbf{d} = \mathbf{a} + \mathbf{b} \tag{26}$$

and

$$G(\mathbf{a}, \mathbf{b}) = 0, \tag{27}$$

then

$$0 \le \max\{G(\mathbf{a}, \mathbf{c}), G(\mathbf{b}, \mathbf{a} + \mathbf{c})\} \le G(\mathbf{c}, \mathbf{d}). \tag{28}$$

Proof. Assuming that (26) and (27) are fulfilled, we choose the permutation $(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2) = (\mathbf{c}, \mathbf{a}, \mathbf{b})$ of $(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2) = (\mathbf{a}, \mathbf{b}, \mathbf{c})$. We also put

$$\mathbf{a}_3 = \mathbf{a}_0 + \mathbf{a}_1 = \mathbf{a} + \mathbf{b} = \mathbf{d} \text{ and } \mathbf{b}_3 = \mathbf{b}_0 + \mathbf{b}_1 = \mathbf{c} + \mathbf{a}.$$

Since $G(\mathbf{a}, \mathbf{b}) = G(\mathbf{a}_0, \mathbf{a}_1) = 0$, it follows from Theorem 3.1 that

$$0 \le \max\{G(\mathbf{c}, \mathbf{a}), G(\mathbf{b}, \mathbf{c} + \mathbf{a})\} = \max\{G(\mathbf{b}_0, \mathbf{b}_1), G(\mathbf{b}_2, \mathbf{b}_3)\}$$

$$\le G(\mathbf{a}_2, \mathbf{a}_3) = G(\mathbf{c}, \mathbf{d}). \tag{29}$$

Now, we get (28) from (29), as wanted. □

As seen in Corollary 3.2, a statement of type (27) is crucial to get a required refinement. To avoid verification of such a condition and provide any certainty of its validity, we can limit our interest to some special classes of convex functions.

In the next corollary we present a refinement of the classical subadditivity inequality (3) for the perspective function of f in the case when f is convex on I and concave (affine) on $J \subset I$. This automatically guarantees the validity of the requirement $G(\mathbf{c}, \mathbf{d}) = 0$.

Corollary 3.3. *Let $f : I \to \mathbb{R}$ be a convex function on a convex set $I \subset V$ such that f restricted to some convex subset $J \subset I$ is concave (affine) on J. Let $\mathbf{x}, \mathbf{y} \in V$ and $p, q > 0$ be given such that $\frac{\mathbf{x}}{p}, \frac{\mathbf{y}}{q} \in I$ and*

$$\frac{\mathbf{x}+\mathbf{y}}{p+q} \in J. \tag{30}$$

Assume that there exist $\mathbf{z} \in V$ and $r > 0$ satisfying

$$\frac{\mathbf{z}}{r} \in J. \tag{31}$$

Then

$$\begin{aligned}
&(p+q) f\left(\frac{\mathbf{x}+\mathbf{y}}{p+q}\right) \\
&\le \max\left\{(p+q) f\left(\frac{\mathbf{x}+\mathbf{y}}{p+q}\right) + p f\left(\frac{\mathbf{x}}{p}\right) + r f\left(\frac{\mathbf{z}}{r}\right) - (p+r) f\left(\frac{\mathbf{x}+\mathbf{z}}{p+r}\right),\right. \\
&\qquad \left. q f\left(\frac{\mathbf{y}}{q}\right) + (p+r) f\left(\frac{\mathbf{x}+\mathbf{z}}{p+r}\right) - r f\left(\frac{\mathbf{z}}{r}\right)\right\} \\
&\le p f\left(\frac{\mathbf{x}}{p}\right) + q f\left(\frac{\mathbf{y}}{q}\right).
\end{aligned} \tag{32}$$

Proof. Denote $\mathbf{a} = (p, \mathbf{x})$, $\mathbf{b} = (q, \mathbf{y})$, $\mathbf{c} = (r, \mathbf{z})$ and $\mathbf{d} = (s, \mathbf{v}) = \mathbf{a} + \mathbf{b}$, where $s = p + q > 0$ and $\mathbf{v} = \mathbf{x} + \mathbf{y}$. Then $\mathbf{a} \in D_g$, $\mathbf{b} \in D_g$, $\mathbf{c} \in D_g$ by (31). In addition, (30) gives

$$\frac{\mathbf{v}}{s} = \frac{\mathbf{x}+\mathbf{y}}{p+q} \in J, \tag{33}$$

whence $\mathbf{d} \in D_g$ by $p + q > 0$ and $J \subset I$.

Since f is affine on J and $\frac{\mathbf{z}}{r}, \frac{\mathbf{v}}{s} \in J$, we have

$$f\left(\frac{\mathbf{z}+\mathbf{v}}{r+s}\right) = f\left(\frac{r}{r+s}\frac{\mathbf{z}}{r} + \frac{s}{r+s}\frac{\mathbf{v}}{s}\right) = \frac{r}{r+s} f\left(\frac{\mathbf{z}}{r}\right) + \frac{s}{r+s} f\left(\frac{\mathbf{v}}{s}\right). \tag{34}$$

Denote by G the triangle functional of g. It now follows from (34) that

$$\begin{aligned}
G(\mathbf{c}, \mathbf{d}) &= g(\mathbf{c}) + g(\mathbf{d}) - g(\mathbf{c}+\mathbf{d}) \\
&= r f\left(\frac{\mathbf{z}}{r}\right) + s f\left(\frac{\mathbf{v}}{s}\right) - (r+s) f\left(\frac{\mathbf{z}+\mathbf{v}}{r+s}\right) = 0,
\end{aligned} \tag{35}$$

which gives $G(\mathbf{c}, \mathbf{d}) = 0$. By making use of inequality (24) in Theorem 3.1 for the permutation $(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2) = (\mathbf{c}, \mathbf{a}, \mathbf{b})$ of $(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2) = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ with

$$\mathbf{a}_3 = \mathbf{a}_0 + \mathbf{a}_1 = \mathbf{a} + \mathbf{b} = \mathbf{d} \quad \text{and} \quad \mathbf{b}_3 = \mathbf{b}_0 + \mathbf{b}_1 = \mathbf{c} + \mathbf{a},$$

we obtain from $G(\mathbf{a}_2, \mathbf{a}_3) = G(\mathbf{c}, \mathbf{d}) = 0$ that

$$\begin{aligned}
0 &\le \max\{G(\mathbf{c}, \mathbf{a}), G(\mathbf{b}, \mathbf{c}+\mathbf{a})\} = \max\{G(\mathbf{b}_0, \mathbf{b}_1), G(\mathbf{b}_2, \mathbf{b}_3)\} \\
&\le G(\mathbf{a}_0, \mathbf{a}_1) = G(\mathbf{a}, \mathbf{b}).
\end{aligned} \tag{36}$$

Next, by using the fact $G(\mathbf{c}, \mathbf{a}) = G(\mathbf{a}, \mathbf{c})$, we get

$$0 \le \max\{G(\mathbf{a}, \mathbf{c}), G(\mathbf{b}, \mathbf{a}+\mathbf{c})\} \le G(\mathbf{a}, \mathbf{b}). \tag{37}$$

That is,

$$\begin{aligned}
0 &\le \max\{g(\mathbf{a}) + g(\mathbf{c}) - g(\mathbf{a}+\mathbf{c}), g(\mathbf{b}) + g(\mathbf{a}+\mathbf{c}) - g(\mathbf{a}+\mathbf{b}+\mathbf{c})\} \\
&\le g(\mathbf{a}) + g(\mathbf{b}) - g(\mathbf{a}+\mathbf{b}).
\end{aligned} \tag{38}$$

Identity (35) ensures that

$$g(\mathbf{a}+\mathbf{b}+\mathbf{c}) = g(\mathbf{c}+\mathbf{d}) = g(\mathbf{c}) + g(\mathbf{d}) = g(\mathbf{c}) + g(\mathbf{a}+\mathbf{b})$$

So, in light of (38) we see that

$$\begin{aligned} 0 &\le \max\{g(\mathbf{a}) + g(\mathbf{c}) - g(\mathbf{a}+\mathbf{c}), g(\mathbf{b}) + g(\mathbf{a}+\mathbf{c}) - g(\mathbf{c}) - g(\mathbf{a}+\mathbf{b})\} \\ &\le g(\mathbf{a}) + g(\mathbf{b}) - g(\mathbf{a}+\mathbf{b}). \end{aligned}$$

Hence,

$$g(\mathbf{a}+\mathbf{b}) \le \max\{g(\mathbf{a}+\mathbf{b})+g(\mathbf{a})+g(\mathbf{c})-g(\mathbf{a}+\mathbf{c}), g(\mathbf{b})+g(\mathbf{a}+\mathbf{c})-g(\mathbf{c})\} \le g(\mathbf{a})+g(\mathbf{b}).$$

or, equivalently,

$$\begin{aligned} &(p+q)f\left(\frac{\mathbf{x}+\mathbf{y}}{p+q}\right) \\ &\le \max\left\{(p+q)f\left(\frac{\mathbf{x}+\mathbf{y}}{p+q}\right) + pf\left(\frac{\mathbf{x}}{p}\right) + rf\left(\frac{\mathbf{z}}{r}\right) - (p+r)f\left(\frac{\mathbf{x}+\mathbf{z}}{p+r}\right),\right. \\ &\qquad \left. qf\left(\frac{\mathbf{y}}{q}\right) + (p+r)f\left(\frac{\mathbf{x}+\mathbf{z}}{p+r}\right) - rf\left(\frac{\mathbf{z}}{r}\right)\right\} \\ &\le pf\left(\frac{\mathbf{x}}{p}\right) + qf\left(\frac{\mathbf{y}}{q}\right), \end{aligned}$$

which proves (32), as desired. □

A function $f : I \to \mathbb{R}$ is called *piecewise affine* if there exist disjoint convex subsets $J_1, \ldots, J_l \subset I$ such that $J_1 \cup \ldots \cup J_l = I$ and f restricted to J_i is affine on J_i for each index $i \in \{1, \ldots, l\}$.

Corollary 3.4. *Let $f : I \to \mathbb{R}$ be a convex and piecewise affine function on a convex set $I \subset V$, as above. Let $\mathbf{x}, \mathbf{y} \in V$ and $p, q > 0$ be such that $\frac{\mathbf{x}}{p}, \frac{\mathbf{y}}{q} \in I$.*

Assume that there exist $\mathbf{z} \in V$, $r > 0$ and an index $i \in \{1, \ldots, l\}$ satisfying

$$\frac{\mathbf{x}+\mathbf{y}}{p+q} \in J_i \quad \text{and} \quad \frac{\mathbf{z}}{r} \in J_i. \tag{39}$$

Then inequalities (32) *are satisfied.*

Proof. It is sufficient to apply Corollary 3.3 for $J = J_i$. □

4. Applications for convex sequences

A sequence $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ is said to be *convex* (see [29, p. 526]) if

$$u_i \le \frac{u_{i-1} + u_{i+1}}{2} \quad \text{for } i = 2, \ldots, n-1. \tag{40}$$

For applications of convex sequences, see e.g., [1, 12, 14, 25, 29].

It is a consequence of (40) that a sequence $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ is convex iff the following Jensen's type inequality holds:

$$(k-j)u_i \le (k-i)u_j + (i-j)u_k \quad \text{for } i, j, k \in \{1, \ldots, n\},\ j \le i \le k. \tag{41}$$

It is convenient to relate a convex sequence $\mathbf{u}$ to the following continuous convex function:

$$\varphi_{\mathbf{u}}(t) = \begin{cases} u_1 + (u_2 - u_1)(t-1), & t \in [1,2), \\ u_2 + (u_3 - u_2)(t-2), & t \in [2,3), \\ \dots & \dots \\ u_i + (u_{i+1} - u_i)(t-i), & t \in [i, i+1), \\ \dots & \dots \\ u_{n-1} + (u_n - u_{n-1})(t-n+1), & t \in [n-1, n] \end{cases} \tag{42}$$

(see [29]). Here $\varphi_{\mathbf{u}}(i) = u_i$ for $i = 1, 2 \dots, n$.

Corollary 4.1. *Let* $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$ *be a convex sequence. Let* $r \in \{1, \dots, n\}$.

Denote
$$i^* = \begin{cases} i+1 & \textit{if} \;\; i = 1, \\ i \pm 1 & \textit{if} \;\; i = 2, \dots, n-1, \\ i-1 & \textit{if} \;\; i = n. \end{cases}$$

Then the following refinement of Jensen's type inequality (41) *is satisfied:*

$$\begin{aligned} (k-j)u_i \le \max \Big\{ & (k-j)u_i + (k-i)u_j + r u_{i^*} - (k-i+r)\varphi_{\mathbf{u}}\left(\frac{j(k-i)+ri^*}{k-i+r}\right), \\ & (i-j)u_k + (k-i+r)\varphi_{\mathbf{u}}\left(\frac{j(k-i)+ri^*}{k-i+r}\right) - r u_{i^*} \Big\} \\ \le (k-i)u_j + (i-j)u_k \quad & \textit{for } i,j,k \in \{1, \dots, n\},\; j \le i \le k. \end{aligned} \tag{43}$$

Proof. Since $\mathbf{u}$ is convex, we use inequality (41). By invoking (42) we may rewrite (41) as

$$(k-j)\varphi_{\mathbf{u}}(i) \le (k-i)\varphi_{\mathbf{u}}(j) + (i-j)\varphi_{\mathbf{u}}(k) \quad \text{for } i,j,k \in \{1, \dots, n\},\; j \le i \le k,$$

or even

$$(p+q)\varphi_{\mathbf{u}}\left(\frac{x+y}{p+q}\right) \le p\varphi_{\mathbf{u}}\left(\frac{x}{p}\right) + q\varphi_{\mathbf{u}}\left(\frac{y}{q}\right) \quad \text{for } i,j,k \in \{1, \dots, n\},\; j \le i \le k,$$

with
$$p = k - i\,, \;\; q = i - j\,, \;\; p + q = k - j, \tag{44}$$

$$x = jp\,, \;\; y = kq\,, \;\; \frac{x+y}{p+q} = i. \tag{45}$$

Fix arbitrarily $i, j, k \in \{1, \dots, n\}$ with $j \le i \le k$.

The function $\varphi_{\mathbf{u}} : I = [1, n] \to \mathbb{R}$ is piecewise affine and convex as $\mathbf{u}$ is a convex sequence (see (42)). More precisely, $\varphi_{\mathbf{u}}$ is affine on each of the intervals

$$J_1 = [1,2),\; J_2 = [2,3), \dots, J_{n-2} = [n-2, n-1),\; J_{n-1} = [n-1, n].$$

Let $z \in \mathbb{R}$ be defined by $z = ri^*$.

It now follows from Corollary 3.4 for $V = \mathbb{R}$ that

$$
\begin{aligned}
&(p+q)\varphi_{\mathbf{u}}\left(\frac{x+y}{p+q}\right) \\
&\le \max\left\{(p+q)\varphi_{\mathbf{u}}\left(\frac{x+y}{p+q}\right) + p\varphi_{\mathbf{u}}\left(\frac{x}{p}\right) + r\varphi_{\mathbf{u}}\left(\frac{z}{r}\right) - (p+r)\varphi_{\mathbf{u}}\left(\frac{x+z}{p+r}\right),\right. \\
&\qquad\left. q\varphi_{\mathbf{u}}\left(\frac{y}{q}\right) + (p+r)\varphi_{\mathbf{u}}\left(\frac{x+z}{p+r}\right) - r\varphi_{\mathbf{u}}\left(\frac{z}{r}\right)\right\} \\
&\le p\varphi_{\mathbf{u}}\left(\frac{x}{p}\right) + q\varphi_{\mathbf{u}}\left(\frac{y}{q}\right). \qquad (46)
\end{aligned}
$$

In other words, by (44)–(45) we derive

$$
\begin{aligned}
&(k-j)\varphi_{\mathbf{u}}(i) \\
&\le \max\left\{(k-j)\varphi_{\mathbf{u}}(i) + (k-i)\varphi_{\mathbf{u}}(j) + r\varphi_{\mathbf{u}}(i^*) - (k-i+r)\varphi_{\mathbf{u}}\left(\frac{j(k-i)+ri^*}{k-i+r}\right),\right. \\
&\qquad\left. (i-j)\varphi_{\mathbf{u}}(k) + (k-i+r)\varphi_{\mathbf{u}}\left(\frac{j(k-i)+ri^*}{k-i+r}\right) - r\varphi_{\mathbf{u}}(i^*)\right\} \\
&\le (k-i)\varphi_{\mathbf{u}}(j) + (i-j)\varphi_{\mathbf{u}}(k). \qquad (47)
\end{aligned}
$$

It is now seen that (43) is an easy consequence of (47), because

$$\varphi_{\mathbf{u}}(i) = u_i,\ \ \varphi_{\mathbf{u}}(j) = u_j,\ \ \varphi_{\mathbf{u}}(i^*) = u_{i^*},\ \ \varphi_{\mathbf{u}}(k) = u_k. \qquad \square$$

References

[1] M. A. Abdellaoui, H. Mebarki, Z. Dahmani, M. Z. Sarikaya: *New results on symmetrized convex sequences*, Math. Inequal. Appl. 27/1 (2024) 173–184.

[2] W. W. Breckner, T. Trif: *Convex Functions and Related Functional Equations: Selected Topics*, Cluj University Press, Cluj (2008).

[3] B. Dacorogna, P. Maréchal: *The role of perspective functions in convexity, polyconvexity, rank-one convexity and separate convexity*, J. Convex Analysis 15/2 (2008) 271–284.

[4] S. S. Dragomir: *Some refinements of Jensen's inequality*, J. Math. Analysis Appl. 168 (1992) 518–522.

[5] S. S. Dragomir, *Some majorisation type discrete inequalities for convex functions*, Math. Inequal. Appl. 7/2 (2004) 207–216.

[6] S. S. Dragomir: *Bounds for the normalised Jensen functional*, Bull. Austral. Math. Soc. 74 (2006) 471–478.

[7] S. S. Dragomir: *A refinement of Jensen's inequality with applications for f-divergence measures*, Taiwanese J. Math. 14/1 (2010) 153–164.

[8] S. S. Dragomir, J. E. Pečarić, L. E. Persson: *Properties of some functionals related to Jensen's inequality*, Acta Math. Hung. 70/1-2 (1996) 129–143.

[9] E. G. Effros: *A matrix convexity approach to some celebrated quantum inequalities*, Proc. Natl. Acad. Sci. USA 106/4 (2009) 1006–1008.

[10] G. M. Hardy, J. E. Littlewood, G. Pólya: *Inequalities*, 2nd ed., Cambridge University Press, Cambridge (1952).

[11] J. Karamata: *Sur une inégalité rélative aux fonctions convexes*, Publ. Math. Univ. Belgrade 1 (1932) 145–148.

[12] Z. Latreuch, B. Belaïdi: *New inequalities for convex sequences with applications*, Int. J. Open Problems Comput. Math. 5/3 (2012) 15–27.

[13] A. W. Marshall, I. Olkin, B. C. Arnold: *Inequalities: Theory of Majorization and its Applications*, 2nd ed., Springer, New York (2011).

[14] G. M. Molnár, Z. Páles: *On convex and concave sequences and their applications*, Math. Inequal. Appl. 25/3 (2022) 727–750.

[15] C. T. Ng: *Functions generating Schur-convex sums*, Int. Ser. Num. Math. 80 (1985) 433–438.

[16] C. Niculescu, L.-E. Persson: *Convex Functions and Their Applications. A Contemporary Approach*, Springer, Berlin (2006).

[17] M. Niezgoda: *Inequalities for convex functions and doubly stochastic matrices*, Math. Inequal. Appl. 16/1 (2013) 221–232.

[18] M. Niezgoda: *Locally strong majorization and commutativity in C^*-algebras with applications*, Result. Math. 74/1 (2019), art.no. 46, 16p.

[19] M. Niezgoda: *On triangle inequality for Miranda-Thompson's majorization and gradients of increasing functions*, Adv. Oper. Theory 5/3 (2020) 647–656.

[20] M. Niezgoda: *Refinements of triangle like inequalities in Lie's framework*, Bull. Malaysian Math. Sci. Soc. 44/1 (2021) 243–250.

[21] M. Niezgoda: *G-majorization and Fejér and Hermite-Hadamard like inequalities for G-symmetrized convex functions*, J. Convex Analysis 29/1 (2022) 231–242.

[22] M. Niezgoda: *A complementary result to the superadditivity of operator means*, Commun. Algebra 50/8 (2022) 3338–3345.

[23] M. Niezgoda, J. Pečarić: *Hardy-Littlewood-Pólya-type theorems for invex functions*, Comput. Math. Appl. 64/4 (2012) 518–526.

[24] J. Pečarić, F. Proschan, Y. Y. Tong: *Convex Functions, Partial Ordering, and Statistical Applications*, Academic Press, New York (1992).

[25] R. Pereira, S. Plosker: *Dirichlet polynomials, majorization, and trumping*, J. Phys. A 46 (2015) 80–86.

[26] R. T. Rockafellar: *Convex Analysis*, Princeton University Press, Princeton (1970).

[27] T. Sano: *Matrix perspectives and matrix convexity*, Linear Algebra Appl. 666 (2023) 29–36.

[28] I. Schur: *Über eine Klasse von Mittelbildungen mit Anwendungen auf die Determinanten-Theorie*, Sitzungsber. Berlin Math. Gesellschaft 22 (1923) 9–20; reprinted in Issai Schur Collected Works, A. Brauer and H. Rohrbach (eds.), vol. II, Springer, Berlin (1973), 416–427.

[29] S. Wu, L. Debnath: *Inequalities for convex sequences and their applications*, Comput. Math. Appl. 54 (2007) 525–534.