Journal of Convex Analysis
Volume 33 (2026), No. 1&2, 441–454

Inequalities for Generalized Convex Functions with the Use of the Hesselager Theorem

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Received: September 16, 2024
Accepted: August 6, 2025

We observe that the theorem of Hesselager may be used to obtain new proofs of Hermite-Hadamard inequalities for functions that are convex with respect to a Chebyshev system. Such inequalities were proved by Bessenyei and Páles for integrals with density function, in our approach, the first function from the system involved must be constant but the main theorem is valid for the integral with respect to any measure. Additionally, we present examples of inequalities that follow from Hesselager's theorem but are not of the Hermite-Hadamard type.

Keywords: Chebyshev systems, higher-order convexity, generalized convexity, stochastic orderings.

2020 Mathematics Subject Classification: 60E15, 26A51,26D10, 39B62.

1. Introduction

In this paper we connect several topics, therefore, the introduction will be divided into subsections.

1.1. Convex orderings

The method that we use in this paper is based on two steps. First, we write given expressions as integrals with respect to suitably constructed measures, and then we study the properties of cumulative distribution functions (shortly CDFs) generated by these measures.

In recent years it turned out that many new results concerning convex functions may be obtained using the following Ohlin lemma.

Lemma 1.1. (Ohlin [19]) *Let X_1, X_2 be two random variables such that $\mathbb{E}X_1 = \mathbb{E}X_2$ and let F_1, F_2 be their cumulative distribution functions. If F_1, F_2 satisfy for some x_0 the following inequalities*

$$F_1(x) \le F_2(x) \text{ if } x < x_0 \text{ and } F_1(x) \ge F_2(x) \text{ if } x > x_0$$

then, for all convex functions $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}f(X_1) \le \mathbb{E}f(X_2). \tag{1}$$

It was observed by Rajba in [23] that this lemma may be used to prove inequalities satisfied by convex functions and this approach was used in many recent papers, see for example [1], [15], [17], [18].

ISSN 0944-6532 / $ 2.50

A natural extension of convexity is the following convexity of higher orders. Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a function, we define recursively

$$f[x_1] = f(x_1)$$

and
$$f[x_1, \dots, x_{n+1}] = \frac{f[x_2, \dots, x_{n+1}] - f[x_1, \dots, x_n]}{x_{n+1} - x_1}.$$

We say that f is *convex of order* n if for all $x_1, \dots, x_{n+2} \in I$ we have

$$f[x_1, \dots, x_{n+2}] \geq 0.$$

Observe that, using this definition for $n = 1$ we obtain the notion of increasing functions and for $n = 2$ we get the usual convexity.

To get a better understanding of this notion it is useful to note that a (sufficiently smooth) function is convex of order n if and only if $f^{(n+1)} \geq 0$.

For some recent results concerning higher-order convex functions see for example [7], [8], [11], [16].

Similarly, as the Ohlin lemma may be used to obtain inequalities for convex functions, a result from [10] may be used to get inequalities for functions that are convex of orders higher than one. Such an approach was used among others in papers [26], [25] and [27]. We will not go into details concerning that result since in the current paper we will be dealing with a more general notion of convexity with respect to Chebyshev systems and the main tool used here will be more general than the mentioned result from [10]. Thus in the next section of the introduction, we say a few words concerning Chebyshev systems.

1.2. Chebyshev systems

The notion of convexity of higher orders yields a particular case of convexity with respect to Chebyshev system which we briefly describe in this section.

We will obtain inequalities satisfied by functions that are convex with respect to Chebyshev systems. Thus we need to start with the definitions connected with the notion of the Chebyshev systems.

Let a, b, $a < b$ be given numbers and let $u_0 \dots, u_n : [a, b] \to \mathbb{R}$ be continuous functions. We say that $(u_0, \dots, u_n)$ is a *Chebyshev system* if

$$\begin{vmatrix} u_0(t_0) & u_0(t_1) & \dots & u_0(t_n) \\ u_1(t_0) & u_0(t_1) & \dots & u_0(t_n) \\ \vdots & & \ddots & \\ u_n(t_0) & u_n(t_1) & \dots & u_n(t_n) \end{vmatrix} > 0, \quad \text{for all } a < t_0 < \dots < t_n < b. \tag{2}$$

In the next definition, we do not assume that all points $t_i, i = 0, \dots, n$ are pairwise different. Instead, we assume that functions $u_i, i = 0, \dots, n$ are of the class C^n. Thus let $u_0, ..., u_n \in C^n[a, b]$ and let $a \leq t_0 \leq ... \leq t_n \leq b$. If for some p we have $t_{p-1} < t_p = t_{p+1} = ... = t_{p+q} < t_{p+q+1}$, then we replace in (2) $u_i(t_{p+q})$ by the derivatives $u_i^{(r)}(t_p)$, $r = 0, 1, \dots, q$.

Consequently, we say that $(u_0, \ldots, u_n)$ is an *extended Chebyshev system* if

$$\begin{vmatrix} u_0(t_0) & \ldots & u_0(t_p) & u_0'(t_p) & \ldots & u_0^{(q)}(t_p) & \ldots & u_0(t_n) \\ \vdots & & & & & & & \vdots \\ u_n(t_0) & \ldots & u_n(t_p) & u_n'(t_p) & \ldots & u_n^{(q)}(t_p) & \ldots & u_n(t_n) \end{vmatrix} > 0.$$

We say that $(u_0, ..., u_n)$ is a *complete extended Chebyshev system* if $(u_0, ..., u_k)$ is an extended Chebyshev system for all $k \leq n$. We will refer to complete extended Chebyshev systems using the abbreviation ECT-system (the letter T appears here because of the formerly used spelling of the name of Chebyshev).

A function f is said to be *convex with respect to the ECT-system* $(u_0, \ldots, u_n)$ if $(u_0, \ldots, u_n, f)$ is an ECT-system.

Remark 1.2. It is well known that a function f is convex of order n if and only if

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ t_0 & t_1 & \ldots & t_{n+1} \\ t_0^2 & t_1^2 & \ldots & t_n^2 \\ \vdots & & \ddots & \\ t_0^n & t_1^n & \ldots & t_{n+1}^n \\ f(t_0) & f(t_1) & \ldots & f(t_{n+1}) \end{vmatrix} > 0,$$

thus higher-order convexity is just the Chebyshev convexity with respect to the monomial system $(1, \ldots, x^n)$. □

For an extensive study of Chebyshev systems see [14] and for some recent results concerning functions that are convex with respect to Chebyshev systems see for example [8], [20], [21] and [29].

1.3. Inequalities for functions that are convex with respect to Chebyshev systems

Let f be a given function that is convex with respect to an ECT-system $(u_0, \ldots, u_n)$. We are interested in inequalities of the form

$$\int f(t)\, dF(t) > \int f(t)\, dG(t), \tag{3}$$

where F and G are CDFs of some measures μ, ν and we have

$$\int u_i(t)\, dF(t) = \int u_i(t)\, dG(t), \quad i = 0, ..., n. \tag{4}$$

In this paper, we will use the main theorem from the paper by Hesselager [13]. An important role in that paper is played by the number of crossing points of two functions which is nothing else but the number of sign changes of their difference. Therefore, now we introduce the definition of the number of sign changes of a given function.

Definition 1.3. Let $F : [a,b] \to \mathbb{R}$ be a given function. We say that F has n *sign changes* if there exist $a = t_0 < \cdots < t_{n+1} = b$ such that the sign of F on each consecutive intervals (t_i, t_{i+1}), (t_{i+1}, t_{i+2}), $i = 0, \ldots, n-1$ is different and F is not identically zero at each of these intervals. The number of sign changes of the function F will be denoted by $S(F)$. Note that the points of sign changes are not necessarily uniquely determined (but their number is). $\square$

Now, using this notion, we present the main result from [13].

Theorem 1.4. (O. Hesselager [13]) *Assume that the function f is convex with respect to the ECT-system $(1, u_1, \ldots, u_n)$ and let F and G satisfy (4). If $S(F-G) = n$ and the last sign changes is from $+$ to $-$ then (3) holds.*

The following lemma was also proved in [13].

Lemma 1.5. *Let $(1, u_1, ..., u_n)$ be an ECT-system. If the cumulative distribution functions F, G (of some measures) satisfy* (4) *with respect to this system, then $S(F-G) \geq n$.*

1.4. Hermite-Hadamard type inequalities

The result we will obtain may be viewed as a generalization of the classical Hermite-Hadamard inequality. This inequality in its simplest form states that for every convex function $f : [a,b] \to \mathbb{R}$ we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)\, dt \leq \frac{f(a)+f(b)}{2}.$$

The Hermite-Hadamard inequality was generalized by Bessenyei and Páles in [4] where higher-order version was proved. Then the same authors in [5] obtained inequalities of the Hermite-Hadamard type for functions that are convex with respect to the Chebyshev system consisting of two functions and in [2] one can find results for the system of three functions.

Below, we present the (more general) results that may be found in [3] or [6]. We start with the result for the Chebyshev system consisting of an even number of functions.

Theorem 1.6. *Let $(u_0, u_1, \ldots, u_{2m-1})$ be a T-system on the interval $[a,b]$ and $\rho : [a,b] \to [0,\infty)$ be an integrable function, then there exist uniquely determined points $\xi_1, ..., \xi_m$ and $\eta_1, ..., \eta_{m-1}$ in the interval (a,b) and uniquely determined positive constants $\alpha_1, ..., \alpha_m$ and $\beta_0, ..., \beta_m$ such that*

$$\sum_{k=1}^{m} \alpha_k\, u_i(\xi_k) = \int_a^b u_i(t)\rho(t)\, dt = \beta_0\, u_i(a) + \sum_{k=1}^{m-1} \beta_k\, u_i(\eta_k) + \beta_m\, u_i(b). \tag{5}$$

For every function $f : [a,b] \to \mathbb{R}$ convex with respect to the T-system $(u_0, u_1, ..., u_{2m-1})$ the following Hermite-Hadamard inequalities hold:

$$\sum_{k=1}^{m} \alpha_k\, f(\xi_k) \leq \int_a^b f(t)\rho(t)\, dt \leq \beta_0\, f(a) + \sum_{k=1}^{m-1} \beta_k\, f(\eta_k) + \beta_m\, f(b). \tag{6}$$

Note that in this theorem on the left-hand side of (6) we have a Gauss-type quadrature, whereas, on the right-hand side, there is a quadrature of the Lobatto type. In the next result, where the number of functions in the Chebyshev system is odd, the quadratures of Gauss and Lobatto are replaced by the left and right Radau quadratures.

Theorem 1.7. *Let $(u_0, u_1, \dots, u_{2m})$ be a T-system on the interval $[a, b]$ and $\rho : [a, b] \to [0, \infty)$ be an integrable function, then there exist uniquely determined points $\xi_1, \dots, \xi_m$ and $\eta_1, \dots, \eta_m$ in (a, b) and uniquely determined positive constants $\alpha_0, \dots, \alpha_m$ and $\beta_0, \dots, \beta_m$ such that*

$$\alpha_0\, u_i(a) + \sum_{k=1}^{m} \alpha_k\, u_i(\xi_k) = \int_a^b u_i(t)\rho(t)\, dt = \sum_{k=1}^{m} \beta_k\, u_i(\eta_k) + \beta_{m+1}\, u_i(b). \tag{7}$$

For every function $f : [a, b] \to \mathbb{R}$ convex with respect to T-system $(u_0, u_1, \dots, u_{2m})$ the following Hermite-Hadamard inequalities hold:

$$\alpha_0\, f(a) + \sum_{k=1}^{m} \alpha_k\, f(\xi_k) \le \int_a^b f(t)\rho(t)\, dt \le \sum_{k=1}^{m} \beta_k\, f(\eta_k) + \beta_{m+1}\, f(b). \tag{8}$$

2. The main result

In this section, we will use methods similar to those of papers [25] and [26] to provide a generalization of Theorems 1.6 and 1.7. Namely, we show that the inequalities (6) and (8) may be proved with integrals

$$\int_a^b f(t)\rho(t)dt$$

replaced by the integral of f with respect to a given measure. This will allow (among others) to obtain inequalities between two quadratures.

In the first theorem, we consider the case of the Chebyshev system which consists of an even number of functions.

Theorem 2.1. *Let $a, b; a < b$ be real numbers and let μ be a Borel probability measure concentrated on the interval $[a, b]$. Further, let $(u_0, u_1, \dots, u_{2m-1})$ be an ECT-system with $u_0 = 1$ and let $\xi_1, ..., \xi_m$, $\eta_1, ..., \eta_{m-1}$, $\alpha_1, ..., \alpha_m$ and $\beta_0, ..., \beta_m$ be such that the equalities*

$$\sum_{k=1}^{m} \alpha_k\, u_i(\xi_k) = \int_a^b u_i\, d\mu = \beta_0\, u_i(a) + \sum_{k=1}^{m-1} \beta_k\, u_i(\eta_k) + \beta_m\, u_i(b), \tag{9}$$

for $i = 0, \dots, 2m - 1$ are satisfied. Then for every function f which is convex with respect to the ECT-system $(1, u_1 \dots, u_{2m-1})$ we have

$$\sum_{k=1}^{m} \alpha_k\, f(\xi_k) < \int_a^b f\, d\mu < \beta_0\, f(a) + \sum_{k=1}^{m-1} \beta_k\, f(\eta_k) + \beta_m\, f(b). \tag{10}$$

Proof. Let H be the cumulative distribution function of the measure μ, then

$$\int_a^b f d\mu = \int_a^b f dH.$$

We will use Theorem 1.4. We need to prove two inequalities, we start with

$$\sum_{k=1}^{m} \alpha_k \, f(\xi_k) < \int_a^b f(t) \, dH(t). \tag{11}$$

To this end define
$$F_1(t) = \begin{cases} 0 & t < \xi_1, \\ \alpha_1 & t \in [\xi_1, \xi_2), \\ \alpha_1 + \alpha_2 & t \in [\xi_2, \xi_3), \\ \vdots & \\ 1 & t \geq \xi_m . \end{cases}$$

Observe that the function H is non-decreasing and F_1 is constant on each interval $(\xi_i, \xi_{i+1}), i = 1, \ldots, m-1$. Therefore, there can be at most one point of sign change of $F_1 - H$ in each interval $(\xi_i, \xi_{i+1}), i = 1, \ldots, m-1$. Moreover, there can be a sign change of $F_1 - H$ in each node $\xi_i, i = 1, \ldots, m$. Thus we have

$$S(F_1 - H) \leq m - 1 + m = 2m - 1. \tag{12}$$

On the other hand, from Lemma 1.5 we have $S(F_1 - H) \geq 2m - 1$ which, together with (12), yields $S(F_1 - H) = 2m - 1$. Thus we may use Theorem 1.4 and we obtain

$$\int_a^b f(t) dF_1(t) < \int_a^b f(t) dH(t).$$

Since
$$\int_a^b f dF_1 = \sum_{k=1}^{m} \alpha_k \, f(\xi_k),$$

the proof of (11) is finished. Now, we prove

$$\int_a^b f(t) \, dH(t) < \beta_0 \, f(a) + \sum_{k=1}^{m-1} \beta_k \, f(\eta_k) + \beta_m \, f(b). \tag{13}$$

We define
$$F_2(t) = \begin{cases} 0 & t < a, \\ \beta_0 & t \in [a, \eta_1), \\ \beta_0 + \beta_1 & t \in [\eta_1, \eta_2), \\ \vdots & \\ 1 & t \geq b. \end{cases}$$

Put $\eta_0 = a$, $\eta_m = b$. Similarly as before, there can be at most one point of sign change of $H - F_2$ in each interval $(\eta_{i-1}, \eta_i), i = 1, \ldots, m$. Moreover there can be a sign change of $H - F_2$ in each node $\eta_i, i = 1, \ldots, m-1$. As we can see

$$S(H - F_2) \leq m + m - 1 = 2m - 1$$

Again, by Lemma 1.5, we obtain $S(H–F_2) \geq 2m–1$, which yields $S(H–F_2) = 2m-1$.

Thus we may use Theorem 1.4 to get

$$\int_a^b f dH < \int_a^b f dF_2.$$

This, in view of $\displaystyle\int_a^b f dF_2 = \beta_0\ f(a) + \sum_{k=1}^{m-1} \beta_k\ f(\eta_k) + \beta_m\ f(b)$ finishes the proof of the ineqality (13). □

In the next theorem, we prove analog results concerning the Chebyshev system consisting of an odd number of functions.

Theorem 2.2. *Let $a, b; a < b$ be real numbers and let μ be a Borel probability measure concentrated on the interval $[a, b]$. Let $(u_0, u_1, \ldots, u_{2m})$ be an ECT-system with $u_0 = 1$, let $\xi_1, ..., \xi_m$, $\eta_1, ..., \eta_m$ and $\alpha_0, ..., \alpha_m$, $\beta_1, ..., \beta_{m+1}$ be such that*

$$\alpha_0\ u_i(a) + \sum_{k=1}^{m} \alpha_k\ u_i(\xi_k) = \int_a^b u_i\ d\mu = \sum_{k=1}^{m} \beta_k\ u_i(\eta_k) + \beta_{m+1}\ u_i(b), \tag{14}$$

for $i = 0, ..., 2m$ is satisfied. Then for every function f which is convex with respect to ECT-system $(1, u_1, ..., u_{2m})$ we have

$$\alpha_0\ f(a) + \sum_{k=1}^{m} \alpha_k\ f(\xi_k) < \int_a^b f\ d\mu < \sum_{k=1}^{m} \beta_k\ f(\eta_k) + \beta_{m+1}\ f(b). \tag{15}$$

Proof. Let H be the cumulative distribution function generated by the measure μ i.e.

$$\int_a^b f d\mu = \int_a^b f dH.$$

We will use Theorem 1.4. We need to prove two inequalities, we start with

$$\alpha_0\ f(a) + \sum_{k=1}^{m} \alpha_k\ f(\xi_k) < \int_a^b f(t)\ dH(t). \tag{16}$$

To this end, define

$$R_1(t) = \begin{cases} 0 & t < a, \\ \alpha_0 & t \in [a, \xi_1), \\ \alpha_0 + \alpha_1 & t \in [\xi_1, \xi_2), \\ \vdots & \\ 1 & t \geq \xi_m. \end{cases}$$

Put $\xi_0 = a$. Observe that the function H is non-decreasing, therefore (similarly as in the proof of Theorem 2.1), there can be at most one point of sign change of $R_1 - H$ in each interval (ξ_i, ξ_{i+1}), $i = 0, \ldots, m-1$. Moreover, there can be a sign change of $R_1 - H$ in each node ξ_i, $i = 1, \ldots, m$. Thus we have

$$S(R_1 - H) \leq m + m = 2m. \tag{17}$$

On the other hand, in view of (14), from Lemma 1.5 we have

$$S(R_1 - H) \geq 2m$$

which, together with (17), yields $S(R_1 - H) = 2m$. Thus we may use Theorem 1.4 and we obtain

$$\int_a^b f(t)dR_1(t) < \int_a^b f(t)dH(t).$$

Since, $\displaystyle\int_a^b f dR_1 = \alpha_0\, f(a) + \sum_{k=1}^{m} \alpha_k\, f(\xi_k)$, the proof of (16) is finished.

Now, we prove
$$\int_a^b f(t)\, dH(t) < \sum_{k=1}^{m} \beta_k\, f(\eta_k) + \beta_{m+1}\, f(b). \tag{18}$$

We define
$$R_2(t) = \begin{cases} 0 & t < \eta_1, \\ \beta_1 & t \in [\eta_1, \eta_2), \\ \beta_1 + \beta_2 & t \in [\eta_2, \eta_3), \\ \vdots & \\ 1 & t > b. \end{cases}$$

Put $\eta_{m+1} = b$. Similarly as before, there can be at most one point of sign change of $H - R_2$ in each interval $(\eta_i, \eta_{i+1}), i = 1, \ldots, m$. Further, there can be a sign change of $H - R_2$ in each node $\eta_i, i = 1, \ldots, m$. As we can see

$$S(H - R_2) \leq m + m = 2m$$

Again, from Lemma 1.5 we have

$$S(H - R_2) \geq 2m$$

which yields $S(H - R_2) = 2m$ thus we may use Theorem 1.4 to get

$$\int_a^b f dH < \int_a^b f dR_2.$$

This, given

$$\int_a^b f R_2 = \sum_{k=1}^{m} \beta_k\, f(\eta_k) + \beta_{m+1}\, f(b)$$

finishes the proof of (13). $\square$

Note that in our results we have to assume that the function u_0 is constant, equal to one. This assumption was not needed in the results obtained in paper [3]. But on the other hand, we consider the integral with respect to any measure. Consequently, we will be able to obtain inequalities between quadrature operators. In the next corollary, we note that the quadratures of Gauss and Lobatto are respectively, the smallest and the biggest possible among all quadratures with the same accuracy. This is expressed in the following corollary which follows directly from Theorem 2.1.

Corollary 2.3. *Let a, b, $a < b$, be real numbers and let μ be a Borel probability measure concentrated on the interval $[a, b]$. Let the quadrature Q be given by*

$$Q(f) = \sum_{k=1}^{l} \gamma_k \; f(\theta_k),$$

where $\gamma_i > 0$, $\theta_i \in [a, b]$. Further, let $(u_0, u_1, \ldots, u_{2m-1})$ be an ECT-system with $u_0 = 1$ and suppose that we have

$$Q(u_i) = \int_a^b u_i \; d\mu$$

for $i = 0, \ldots, 2m-1$. If $\xi_1, ..., \xi_m$, $\eta_1, ..., \eta_{m-1}$, $\alpha_1, ..., \alpha_m$ and $\beta_0, ..., \beta_m$ are such that (9) holds, then

$$\sum_{k=1}^{m} \alpha_k \; f(\xi_k) < Q(f) < \beta_0 \; f(a) + \sum_{k=1}^{m-1} \beta_k \; f(\eta_k) + \beta_m \; f(b),$$

for every function f that is convex with respect to the system $(1, u_1, \ldots, u_{2m-1})$.

Proof. To prove the corollary it is enough to use Theorem 2.1 with the measure ν given by $\nu = \gamma_1 \delta_{\theta_1} + \cdots + \gamma_l \delta_{\theta_l}$ instead of μ. □

A version of this corollary valid for systems consisting of an odd number of functions may also be presented. We will provide examples of concrete inequalities of this form in the next section.

Remark 2.4. As convexity with respect to Chebyshev systems generalizes the notion of convexity of higher orders, Corollary 2.3 generalizes the result obtained by Brass and Schmeisser in [9].

3. Examples

In many papers, we can find results where quadrature operators are compared (see for example [28],[9]). Now, we provide examples of such inequalities involving quadratures connected with Chebyshev systems

Example 3.1. The 1-point Gauss quadrature operator with respect to ECT-system $(1, x^2)$ on the interval $[0, 1]$ is given by

$$\mathcal{G}_1[0, 1] = f\Big(\sqrt{\tfrac{1}{3}}\Big).$$

The 2-points Radau quadrature operator on the interval $[0, 1]$ containing the point a as a node with respect to the ECT-system $(1, x^2, x^4)$ is given by

$$\mathcal{R}_2^l[0, 1](f) = \tfrac{4}{9} f(0) + \tfrac{5}{9} f\Big(\sqrt{\tfrac{3}{5}}\Big).$$

Then from Corollary 2.3 it follows that the inequality

$$f\left(\sqrt{\frac{1}{3}}\right) < \frac{4}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right) \tag{19}$$

is satisfied by all functions f which are strictly convex with respect to the Chebyshev system $(1, x^2)$. □

Before we proceed to the next example we interpret the above inequality from the point of view of a result from [3]. Namely, it was proved in [3] that a function f is convex with respect to the system (u_0, u_1) if and only if the function

$$g = \frac{f}{u_0} \circ \left(\frac{u_1}{u_0}\right)^{-1}$$

is convex. Thus in our situation, we obtain that (19) is satisfied by all functions f such that $g(x) = f(\sqrt{x})$ is convex. Using this form of g, we may present (19) in the form

$$g\left(\frac{1}{3}\right) < \frac{4}{9}\, g\,(0) + \frac{5}{9}\, g\left(\frac{3}{5}\right)$$

which is trivially satisfied by every convex function, since

$$\frac{1}{3} = \frac{4}{9} \cdot 0 + \frac{5}{9} \cdot \frac{3}{5}.$$

We presented here only a particular case concerning 1-point Gaus quadrature a 2-point Radau quadrature, and a Chebyshev system consisting of two functions. In the next example, we go a little bit further.

Example 3.2. The 2-point Radau quadrature with respect to the ECT-system $(1, e^x, e^{2x})$ on the interval $\left[0, \frac{1}{2}\right]$ is given by

$$\mathcal{R}_2^l\left[0, \frac{1}{2}\right] = \left(\frac{-5 + 8\sqrt{e} - 3e}{(\sqrt{e} - 2)^2}\right) f\,(0) + \frac{(3 - 2\sqrt{e})^2}{(\sqrt{e} - 2)^2} f\left(2\log(\sqrt{e} - 1) - \log(2\sqrt{e} - 3)\right)$$

The 2-point Lobatto quadrature with respect to the ECT-system $(1, e^x)$ on the interval $\left[0, \frac{1}{2}\right]$ is given by

$$\mathcal{L}_2\left[0, \frac{1}{2}\right] = \frac{2 - \sqrt{e}}{\sqrt{e} - 1} f(0) + \frac{2\sqrt{e} - 3}{\sqrt{e} - 1} f\left(\frac{1}{2}\right)$$

It follows from Theorem 2.2 that

$$\mathcal{R}_2^l\left[0, \frac{1}{2}\right] < \mathcal{L}_2\left[0, \frac{1}{2}\right]$$

for every function which is convex with respect to the ECT-system $(1, e^x)$.

Example 3.3. Consider the 2-point Gauss quadrature with respect to ECT-system $(1, x^2, x^4, x^6)$ on the interval $[0, 1]$ i.e.

$$\mathcal{G}_2[0, 1] = \frac{3 - \sqrt{\frac{5}{6}}}{6} f\left(\frac{3 + 2\sqrt{\frac{6}{5}}}{7}\right) + \frac{3 + \sqrt{\frac{5}{6}}}{6} f\left(\frac{3 - 2\sqrt{\frac{6}{5}}}{7}\right)$$

and the 2-point Radau quadrature with respect to the ECT-system $(1, x^2, x^4)$ on the same interval

$$\mathcal{R}_2^l[0,1] = \frac{4}{5}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right).$$

Using Theorem 2.1, we get $\mathcal{R}_2^l[0,1] < \mathcal{G}_2[0,1]$ for all functions f which are strictly convex with respect to the system $(1, x^2, x^4)$.

Remark 3.4. Note that inequalities presented in the above examples could not be obtained using Theorem 1.6, since the inequalities obtained in that theorem involve an integral with respect to a measure with density. □

In the above examples, we illustrated Theorems 2.1 and 2.2. However, Theorem 1.4 may be used in a more general situation, namely F and G may be CDFs connected with any measures – not necessarily with Gauss Lobatto or Radau quadratures. We give now examples of such inequalities.

Example 3.5. Consider the quadrature which is exact for functions $(1, x^2, x^4)$ with nodes $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, it is of the form

$$\mathcal{Q}_1[0,1](f) = -\frac{28}{45}f\left(\frac{1}{4}\right) + \frac{269}{225}f\left(\frac{1}{2}\right) + \frac{32}{75}f\left(\frac{3}{4}\right).$$

Similarly, the quadrature with nodes $0, \frac{1}{3}, \frac{2}{3}$ is given by

$$\mathcal{Q}_2[0,1](f) = \frac{13}{10}f(0) - \frac{7}{5}f\left(\frac{1}{3}\right) + \frac{11}{10}f\left(\frac{2}{3}\right).$$

We will show that $Q_1[0,1](f) > Q_2[0,1](f)$ for every function f that is convex with respect to the system $(1, x^2, x^4)$. However, observe that we cannot compare $\mathcal{Q}_2$ with $\mathcal{Q}_1$, using Theorem 1.4, since these quadratures have negative weights. It is however enough to move the terms with negative coefficients to the other side of the inequality getting

$$\frac{7}{5}f\left(\frac{1}{3}\right) + \frac{269}{225}f\left(\frac{1}{2}\right) + \frac{32}{75}f\left(\frac{3}{4}\right) > \frac{13}{10}f(0) + \frac{28}{45}f\left(\frac{1}{4}\right) + \frac{11}{10}f\left(\frac{2}{3}\right).$$

Dividing by $\frac{136}{45}$ we get the inequality

$$\frac{63}{136}f\left(\frac{1}{3}\right) + \frac{269}{680}f\left(\frac{1}{2}\right) + \frac{12}{85}f\left(\frac{3}{4}\right) > \frac{117}{272}f(0) + \frac{7}{34}f\left(\frac{1}{4}\right) + \frac{99}{272}f\left(\frac{2}{3}\right).$$

The cumulative distribution functions F_1, F_2 connected with the above expressions are presented in Figure 1A. It is easy to see that they have 2 crossing points, thus it follows from Theorem 1.4 that the inequality

$$\mathcal{Q}_2 < \mathcal{Q}_1$$

is satisfied for all functions f which are convex with respect to the system $(1, x^2, x^4)$. □

Note that the inequality obtained in the last example does not follow from any of the Theorems 1.6, 1.7, since all inequalities occurring in these theorems involve quadratures of Gauss, Radau or Lobatto.

(A) Graphs of F_1, F_2. (B) Graphs of the functions F and G.

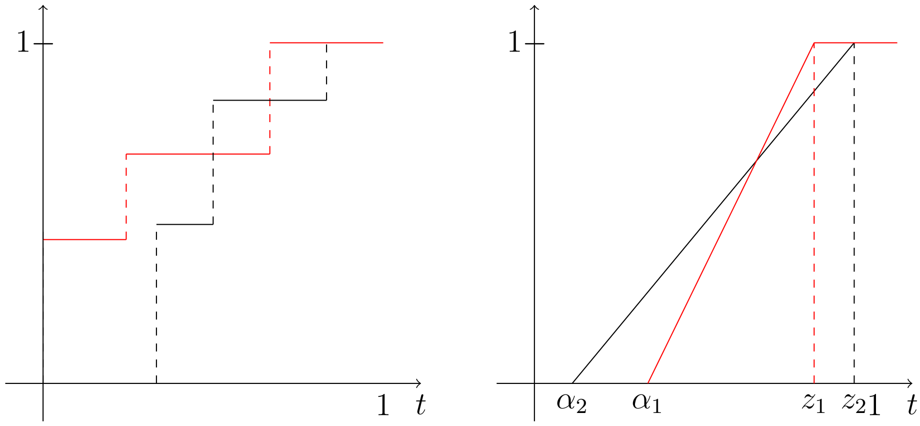

Figure 1: Graphs of CDFs needed in Examples 3.5 and 3.6.

Now we give an example where two integrals will be compared. First, consider a convex function f defined on the interval $[0,1]$. It is well known that if $\alpha_1, \alpha_2 \in [0, \frac{1}{2})$ are such that $\alpha_1 < \alpha_2$ then

$$\frac{1}{1-2\alpha_1}\int_{\alpha_1}^{1-\alpha_1} f(t)dt \geq \frac{1}{1-2\alpha_2}\int_{\alpha_2}^{1-\alpha_2} f(t)dt.$$

In the next example, we give the analog result involving convexity with respect to the system $(1, x^2)$.

Example 3.6. Take any $\alpha_1, \alpha_2 \in [0,1]$,

$$z_1 = \frac{-\alpha_1 + \sqrt{4-3\alpha_1^2}}{2}, \quad a = \frac{1}{\frac{-\alpha_1+\sqrt{4-3\alpha_1^2}}{2} - \alpha_1}, \quad b = -\frac{\alpha_1}{\frac{-\alpha_1+\sqrt{4-3\alpha_1^2}}{2} - \alpha_1},$$
$$z_2 = \frac{-\alpha_2 + \sqrt{4-3\alpha_2^2}}{2}, \quad c = \frac{1}{\frac{-\alpha_2+\sqrt{4-3\alpha_2^2}}{2} - \alpha_2}, \quad d = -\frac{\alpha_2}{\frac{-\alpha_2+\sqrt{4-3\alpha_2^2}}{2} - \alpha_2}.$$

Let the function F be given by $F(x) = \begin{cases} 0 & x < \alpha_1, \\ ax+b & x \in (\alpha_1, z_1) \\ 1 & x > z_1 \end{cases}$.

Further let G be defined by $G(x) = \begin{cases} 0 & x < \alpha_2, \\ cx+d & x \in (\alpha_2, z_2) \\ 1 & x > z_2 \end{cases}$.

We have $F(\alpha_1) = a\alpha_1 + b = 0$, $F(z_1) = az_1 + b = 1$, $G(\alpha_2) = c\alpha_2 + d = 0$, $G(z_2) = cz_2 + d = 1$ (see Figure 1B). We have also

$$\int_0^1 x^2 dF(x) = \int_{\alpha_1}^{z_1} ax^2 dx = \int_{\alpha_1}^{z_1} \frac{1}{\frac{-\alpha_1+\sqrt{4-3\alpha_1^2}}{2} - \alpha_1} x^2 dx = \frac{1}{z_1 - \alpha_1}\int_{\alpha_1}^{z_1} x^2 dx$$
$$= \frac{1}{\frac{-\alpha_1+\sqrt{4-3\alpha_1^2}}{2} - \alpha_1}\frac{1}{3}\left(\left(\frac{-\alpha_1 + \sqrt{4-3\alpha_1^2}}{2}\right)^3 - \alpha_1^3\right) = \frac{1}{3},$$

and

$$\int_0^1 x^2 dG(x) = \int_{\alpha_2}^{z_2} cx^2 dx = \int_{\alpha_2}^{z_2} \frac{1}{\frac{-\alpha_2+\sqrt{4-3\alpha_2^2}}{2} - \alpha_2} x^2 dx = \frac{1}{z_2-\alpha_2}\int_{\alpha_2}^{z_2} x^2 dx$$

$$= \frac{1}{\frac{-\alpha_2+\sqrt{4-3\alpha_2^2}}{2} - \alpha_2} \frac{1}{3}\left(\left(\frac{-\alpha_2+\sqrt{4-3\alpha_2^2}}{2}\right)^3 - \alpha_2^3\right) = \frac{1}{3}.$$

Observe that functions F, G have one crossing point. Indeed, it is obvious that these functions cannot have more than one crossing point, and from Lemma 1.5 we know that they have at least one such point (see Figure 1 B). Therefore, using Theorem 1.4 we get

$$\frac{1}{z_1-\alpha_1}\int_{\alpha_1}^{z_1} f(t)dt < \frac{1}{z_2-\alpha_2}\int_{\alpha_2}^{z_2} f(t)dt.$$

for every f which is strictly convex with respect to $(1, x^2)$.

Remark 3.7. The calculations in all examples were done with the help of the WolframAlpha online tool.

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