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The Derivative of a Nonlinear Positively Homogeneous Variational Inequality is a Complementarity Problem

Samir Adly, Loïc Bourdin
Institut de Recherche XLIM, UMR CNRS 7252, Université de Limoges, France
samir.adly@unilim.fr, loic.bourdin@unilim.fr

Dedicated to Professor R. T. Rockafellar on the occasion of his 90th birthday, in recognition of his profound contributions to variational analysis and optimization.

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Using advanced concepts and techniques from convex and variational analysis (such as the notion of twice epi-differentiability), we establish, under a set of appropriate conditions (including a polyhedricity assumption), that the solution to a parameterized nonlinear variational inequality associated with a positively homogeneous function is differentiable, and furthermore that its derivative is the solution to a corresponding complementarity problem. We illustrate our main result through a series of examples and counterexamples. In particular we introduce an example of a projection operator onto a nonempty closed convex cone that is not directionally differentiable, which is, to our best knowledge, new in the literature.

Keywords: Variational inequalities, complementarity problems, semi-differentiability, twice epi-differentiability, polyhedricity.

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1. Introduction

In this paper the symbol $\mathcal{H}$ denotes a general real Hilbert space, with $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ representing the associated inner product and $\| \cdot \|_{\mathcal{H}}$ the corresponding norm. This introductory section uses standard notations and terminologies from convex and variational analysis (see Section 2.1 for reminders). In particular we denote by $\mathcal{A}(\mathcal{H})$ the set of all Lipschitz continuous and strongly monotone operators $A : \mathcal{H} \to \mathcal{H}$, and by $\Gamma_0(\mathcal{H})$ the set of all proper closed convex extended real-valued functions $\Phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$. Additionally we recall that a function $\Phi \in \Gamma_0(\mathcal{H})$ is said to be *positively homogeneous* if $\Phi(\lambda v) = \lambda \Phi(v)$ for all $v \in \mathcal{H}$ and all $\lambda > 0$. Finally we use $\mathbb{N}^*$ to denote the set of all positive integers and the symbols $\mathbb{R}_+$, $\mathbb{R}_+^*$, $\mathbb{R}_-$, $\mathbb{R}_-^*$ represent the sets of nonnegative, positive, nonpositive, negative real numbers, respectively.

1.1. Variational inequalities and complementarity problems

Concepts and brief overview. In the literature a *variational inequality* is defined as a mathematical problem whose goal is to find an element $u \in \mathcal{H}$ which satisfies

$$\text{(VI)} \qquad \langle A(u) - f, v - u \rangle_{\mathcal{H}} + \Phi(v) - \Phi(u) \geq 0, \quad \forall v \in \mathcal{H},$$

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or equivalently the inclusion $f \in A(u) + \partial\Phi(u)$, where $(A, \Phi) \in \mathcal{A}(\mathcal{H}) \times \Gamma_0(\mathcal{H})$ and $f \in \mathcal{H}$ is a specified element. When A is linear (resp. nonlinear), the variational inequality (VI) is said to be *linear* (resp. *nonlinear*). Also, when $\Phi = \iota_C$ is the indicator function to a nonempty closed convex subset $C \subset \mathcal{H}$, the variational inequality (VI) reduces to find an element $u \in \mathcal{H}$ such that

$$u \in C \quad \text{and} \quad \langle A(u) - f, v - u\rangle_{\mathcal{H}} \geq 0, \quad \forall v \in C.$$

In that case, the variational inequality (VI) is said to be *of the first kind*. Otherwise, it is said to be *of the second kind*.

The theory of variational inequalities was originally introduced by Fichera [12] and Stampacchia [26] in the fields of potential theory and mechanics. Since then, it has been extensively developed and is now an important area within applied mathematics. We refer to the works by Brézis [4], Browder [6], Lions [16], Mosco [20], Stampacchia [27] and references therein. This theory serves as an effective tool for analyzing a broad spectrum of linear and nonlinear mathematical problems within a unified framework. The literature is enriched with many significant theoretical advancements and numerical treatments, and notable contributions underscore the practical applications of variational inequalities, demonstrating the theory's capacity to deal with complex challenges in various scientific domains such as contact mechanics, optimal control theory, traffic network equilibrium, financial mathematics, image processing, etc.

An important special case of (VI) is given when $\Phi = \iota_K$ is the indicator function to a nonempty closed convex cone $K \subset \mathcal{H}$. In that case, the variational inequality (VI) is of the first kind and moreover reduces to what is known in the literature as a *complementarity problem* which consists in finding an element $u \in \mathcal{H}$ such that

$$u \in K, \quad \langle u, A(u) - f\rangle_{\mathcal{H}} = 0 \quad \text{and} \quad A(u) - f \in K^\star,$$

which can be succinctly reformulated in a compact form as

$$\text{(CP)} \qquad K \ni u \perp A(u) - f \in K^\star,$$

where $K^\star$ stands for the *dual cone* (also called positive polar cone) of K defined by

$$K^\star := \{w \in \mathcal{H} \mid \langle w, v\rangle_{\mathcal{H}} \geq 0, \ \forall v \in K\}.$$

For an exhaustive exploration of complementarity problems and their significant implications in the domains of engineering and economics, the reader is directed to the seminal work [11] by Ferris and Pang.

Examples in optimization. Variational inequalities play a central role in the mathematical field of optimization. For example, the general variational inequality (VI) encompasses the classical characterization of the projection $u = \mathrm{proj}_C(f)$ of f onto a nonempty closed convex subset $C \subset \mathcal{H}$, that is, of the unique element $u \in C$ that minimizes the $\|\cdot\|_{\mathcal{H}}$-distance to f among all elements of C. This is obtained when (VI) is applied with the linear identity operator $A = \mathrm{Id}_{\mathcal{H}}$ and the indicator function $\Phi = \iota_C$.

Another significant example is given when A is derived from a potential, that is, when $A = \nabla\Psi$ is the (possibly nonlinear) gradient of a Fréchet differentiable convex function $\Psi : \mathcal{H} \to \mathbb{R}$ and $f = 0_{\mathcal{H}}$. In that context, the variational inequality (VI) characterizes the fact for an element $u \in \mathcal{H}$ to be a solution to the so-called composite optimization problem given by

$$\underset{v\in\mathcal{H}}{\text{minimize}}\ \ \Psi(v) + \Phi(v).$$

This kind of composite optimization problem is useful in many areas, like image processing and machine learning. Also, in statistical analysis, a well-known convex form of the LASSO problem, which is important for sparse linear regression, is given by

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \ \frac{1}{2}\|Rx - c\|^2_{\mathbb{R}^m} + \|x\|_1,$$

where n, $m \in \mathbb{N}^*$, $R \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^m$ and $\|\cdot\|_1$ denotes the classical 1-norm on $\mathbb{R}^n$.

Finally a complementarity problem arises, for example, when considering a quadratic optimization problem subject to linear inequality constraints of the form

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \ \frac{1}{2}\langle Qx, x\rangle_{\mathbb{R}^n} + \langle a, x\rangle_{\mathbb{R}^n}\ \ \text{subject to}\ \left\{\begin{array}{l} Px - b \in \mathbb{R}^m_+, \\ x \in \mathbb{R}^n_+, \end{array}\right.$$

where n, $m \in \mathbb{N}^*$, $Q \in \mathbb{R}^{n\times n}$ is symmetric, $P \in \mathbb{R}^{m\times n}$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. In that setting, the Karush-Kuhn-Tucker (KKT) conditions for a solution x write as the existence of a Lagrange multiplier $p \in \mathbb{R}^m$ such that the pair $u = (x, p)$ is a solution to (CP) with $\mathcal{H} = \mathbb{R}^{n+m}$, $K = \mathbb{R}^{n+m}_+$ and

$$A = \begin{pmatrix} Q & -P^\top \\ P & 0_{\mathbb{R}^{m\times m}} \end{pmatrix}\ \ \text{and}\ \ f = \begin{pmatrix} -a \\ b \end{pmatrix}.$$

1.2. Sensitivity analysis of variational inequalities in the literature

From a general point of view, the *sensitivity analysis* of a mathematical problem is concerned with the regularity (continuity, differentiability, etc.) of its solution with respect to some parameters. For optimization issues, a *first-order sensitivity analysis* (concerned with first-order differentiability results) is usually required in order to establish first-order necessary conditions for optimal parameters, or for numerical purposes (for example, for the implementation of gradient descent methods in order to numerically find optimal parameters).

Concerning variational inequalities, it can be proved (see, e.g., [3, Proposition 31]) that the variational inequality (VI) admits a unique solution given by $u = \mathrm{prox}_{A,\Phi}(f)$, where $\mathrm{prox}_{A,\Phi} := (A + \partial\Phi)^{-1} : \mathcal{H} \to \mathcal{H}$ is the well-known Lipschitz continuous *extended proximal operator* (corresponding to an extension of the classical *proximal operator* $\mathrm{prox}_\Phi := (\mathrm{Id}_{\mathcal{H}} + \partial\Phi)^{-1} : \mathcal{H} \to \mathcal{H}$ introduced by Moreau [19]). In particular we notice that the unique solution u to (VI) depends Lipschitz continuously on the element f.

Now, in order to develop the sensitivity analysis of (VI) to the first-order, we will focus in the rest of this paper on the differentiability at $t = 0$ of the map $t \mapsto u_t$, where $u_t \in \mathcal{H}$ stands for the unique solution to the *parameterized* variational inequality

$$(\mathrm{VI}_t) \qquad \langle A(u_t) - f_t, v - u_t \rangle_{\mathcal{H}} + \Phi(v) - \Phi(u_t) \geq 0, \quad \forall v \in \mathcal{H},$$

where the (fixed) element f has been replaced by a *parameterized* element f_t with $t \geq 0$. Note that, when assuming that the map $t \mapsto f_t$ is differentiable at $t = 0$, this investigation corresponds to the study of the *Hadamard directional differentiability* (see, e.g., [2, Definition 2.45]) of the extended proximal operator $\mathrm{prox}_{A,\Phi}$.

The case of linear variational inequalities of the first kind. The pioneer works concerned with first-order sensitivity analysis of linear variational inequalities of the first kind are due to Haraux [13] and Mignot [17]. Actually these works are dedicated to the *directional differentiability* (see, e.g., [2, Definition 2.44]), also called *conical differentiability*, of the projection operator proj_C onto a nonempty closed convex subset $C \subset \mathcal{H}$. Precisely, assuming that C is *polyhedric* at $\mathrm{proj}_C(f)$ for $f - \mathrm{proj}_C(f)$, the authors establish the expansion formula

$$\mathrm{proj}_C(f + tf') = \mathrm{proj}_C(f) + t \ \mathrm{proj}_{\mathrm{T}_C(\mathrm{proj}_C(f)) \cap (f - \mathrm{proj}_C(f))^\perp}(f') + o(t),$$

for all $t \geq 0$ and all $f' \in \mathcal{H}$, where $\mathrm{T}_C(\mathrm{proj}_C(f))$ stands for the tangent cone to C at $\mathrm{proj}_C(f)$ and o stands for the usual Landau notation.

In fact, the above expansion formula has been obtained in [17, Theorem 2.2] in the generalized context of a a-projection operator proj^a_C where $a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is a (possibly nonsymmetric) bilinear continuous form. In that generalized setting, the expansion formula is obtained under a-polyhedricity of C at $\mathrm{proj}^a_C(f)$ for $f - \mathrm{proj}^a_C(f)$, and by replacing proj_C by proj^a_C everywhere and the classical orthogonal $(f - \mathrm{proj}_C(f))^\perp$ by the a^*-orthogonal $(f - \mathrm{proj}^a_C(f))^\perp_{a^*}$ where $a^* : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ stands for the conjugate form to a.

Finally, since Lipschitz continuity and directional differentiability imply together Hadamard directional differentiability (see, e.g., [2, Proposition 2.49]), if the map $t \mapsto f_t$ is differentiable at $t = 0$ and C is a-polyhedric at $u_0 := \mathrm{proj}^a_C(f_0)$ for $f_0 - u_0$, then the map $t \mapsto u_t := \mathrm{proj}^a_C(f_t)$ is differentiable at $t = 0$ with

$$u'_0 = \mathrm{proj}^a_{\mathrm{T}_C(u_0) \cap (f_0 - u_0)^\perp_{a^*}}(f'_0).$$

The case of linear variational inequalities of the second kind with positive homogeneity. As we saw earlier, a significant part of variational inequalities in the literature are associated with positively homogeneous functions. We can cite, for example, complementarity problems (when $\Phi = \iota_K$ is the indicator function to a nonempty closed convex cone $K \subset \mathcal{H}$) or sparse linear regression problems in statistics (when $\Phi = \| \cdot \|_1$ is the classical 1-norm on $\mathcal{H} = \mathbb{R}^n$).

Christof and Wachsmuth [8, Proposition 2.1] have recently observed that, when $A \in \mathcal{A}(H)$ is linear and $\Phi \in \Gamma_0(\mathcal{H})$ is positively homogeneous, the unique solution u_t to (VI_t) satisfies

$$u_t = A^{-1}\left(f_t - \mathrm{proj}^a_{\partial\Phi(0_{\mathcal{H}})}(f_t)\right), \tag{1}$$

for all $t \geq 0$, involving the subgradient $\partial\Phi(0_{\mathcal{H}})$ and the form $a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ defined by $a(v_1, v_2) := \langle A^{-1}v_1, v_2 \rangle_{\mathcal{H}}$ for all v_1, $v_2 \in \mathcal{H}$. Using the results of the previous paragraph, the authors deduce that, if the map $t \mapsto f_t$ is differentiable at $t = 0$ and

$\partial\Phi(0_{\mathcal{H}})$ is a-polyhedric at $\mathrm{proj}^a_{\partial\Phi(0_{\mathcal{H}})}(f_0)$ for $f_0 - \mathrm{proj}^a_{\partial\Phi(0_{\mathcal{H}})}(f_0)$ (which is equivalent to classical polyhedricity of $\partial\Phi(0_{\mathcal{H}})$ at $f_0 - Au_0$ for u_0), then the map $t \mapsto u_t$ is differentiable at $t = 0$ with

$$u'_0 = A^{-1}\left(f'_0 - \mathrm{proj}^a_{\mathrm{T}_{\partial\Phi(0_{\mathcal{H}})}(f_0 - Au_0)\cap u_0^{\perp}}(f'_0)\right). \tag{2}$$

This equality leads, together with some basic computations and reformulations, to [8, Theorem 2.3] characterizing u'_0 as the unique solution to the variational inequality of the first kind given by

$$u'_0 \in \left(\mathrm{T}_{\partial\Phi(0_{\mathcal{H}})}(f_0 - Au_0) \cap u_0^{\perp}\right)^{\circ}$$

$$\text{and} \qquad \langle Au'_0 - f'_0, v - u'_0\rangle_{\mathcal{H}} \geq 0, \quad \forall v \in \left(\mathrm{T}_{\partial\Phi(0_{\mathcal{H}})}(f_0 - Au_0) \cap u_0^{\perp}\right)^{\circ}. \tag{3}$$

We point out here that [8, Theorem 2.3] corresponds to a generalization of [14, Theorem 4.2] that was previously established in the particular context of a linear variational inequality associated with a specific positively homogeneous function.

The case of nonlinear variational inequalities. The previous paragraphs are both concerned with first-order sensitivity analysis of *linear* variational inequalities and are both based on the Haraux and Mignot's expansion formula for projection operators. Concerning *nonlinear* variational inequalities (that is, when $A \in \mathcal{A}(\mathcal{H})$ is nonlinear), even if $\Phi \in \Gamma_0(\mathcal{H})$ is an indicator function or positively homogeneous, such an approach cannot be achieved since a simple expression of u_t involving a projection operator (such as (1)) is not available in the nonlinear context.

Nonetheless, using mathematical tools introduced and thoroughly studied by Rockafellar, such as the notion of *twice epi-differentiability* [23], it can be proved under appropriate conditions (see Lemma 2.4 for details) that the map $t \mapsto u_t$, where $u_t \in \mathcal{H}$ stands for the unique solution to (VI_t), is differentiable at $t = 0$ and that its derivative u'_0 is the unique solution to the variational inequality

$$(\mathrm{VI}'_0) \qquad \langle D_sA(u_0)(u'_0) - f'_0, v - u'_0\rangle_{\mathcal{H}} + \frac{1}{2}\Phi''(v) - \frac{1}{2}\Phi''(u'_0) \geq 0, \quad \forall v \in \mathcal{H},$$

where $D_sA(u_0) \in \mathcal{A}(\mathcal{H})$ stands for the semi-differential of A at u_0,

$$\Phi'' := d_e^2\Phi(u_0|f_0 - A(u_0)) \in \Gamma_0(\mathcal{H})$$

stands for the second-order epi-derivative of Φ at u_0 for $f_0 - A(u_0)$ and f'_0 stands for the derivative of $t \mapsto f_t$ at $t = 0$. Roughly speaking we say that the variational inequality (VI'_0) is the "derivative" at $t = 0$ of the parameterized nonlinear variational inequality (VI_t).

For further details on the sensitivity analysis of variational inequalities, we refer the reader to [7, 9].

1.3. Main result, other contributions and structure of the present paper

The main objective of the present paper is to address the first-order sensitivity analysis of *nonlinear* variational inequalities associated with *positively homogeneous functions*. Roughly speaking, our main result (Theorem 1.1) asserts that, under appropriate conditions (including a polyhedricity assumption), the "derivative" of a parameterized nonlinear variational inequality associated with a positively homogeneous function is a complementarity problem. It is stated as follows and will be proved later (see Section 2.2).

Theorem 1.1. *Let* $(A,\Phi) \in \mathcal{A}(\mathcal{H}) \times \Gamma_0(\mathcal{H})$ *and* $u_t \in \mathcal{H}$ *be the unique solution to the parameterized nonlinear variational inequality*

$$(\mathrm{VI}_t) \qquad \langle A(u_t) - f_t, v - u_t \rangle_{\mathcal{H}} + \Phi(v) - \Phi(u_t) \geq 0, \quad \forall v \in \mathcal{H},$$

where $f_t \in \mathcal{H}$*, for all* $t \geq 0$*. If* A *is semi-differentiable at* u_0*,* Φ *is positively homogeneous with* $\partial\Phi(0_{\mathcal{H}})$ *polyhedric at* $f_0 - A(u_0)$ *for* u_0 *and* $t \mapsto f_t$ *is differentiable at* $t = 0$*, then* $t \mapsto u_t$ *is differentiable at* $t = 0$ *and its derivative* u_0' *is the unique solution to the complementarity problem*

$$(\mathrm{CP}_0') \quad \mathrm{T}_{\mathrm{N}_{\partial\Phi(0_{\mathcal{H}})}(f_0 - A(u_0))}(u_0) \ni u_0' \perp D_s A(u_0)(u_0') - f_0' \in \mathrm{T}_{\mathrm{N}_{\partial\Phi(0_{\mathcal{H}})}(f_0 - A(u_0))}(u_0)^\star,$$

where $D_s A(u_0) \in \mathcal{A}(\mathcal{H})$ *stands for the semi-differential of* A *at* u_0 *and* f_0' *stands for the derivative of* $t \mapsto f_t$ *at* $t = 0$*. In that context we say, roughly speaking, that the complementarity problem* (CP_0') *is the "derivative" at* $t = 0$ *of the parameterized nonlinear variational inequality* (VI_t)*.*

Contributions with respect to the existing literature. Hereafter we provide a list of comments intended to highlight the contributions of the present paper in comparison with preceding works in the literature:

(i) First of all, for any nonempty closed convex subset $C \subset \mathcal{H}$, it can be proved by bipolar theorem that $(\mathrm{T}_C(v) \cap w^\perp)^\circ = \mathrm{T}_{\mathrm{N}_C(v)}(w)$ for any $v \in C$ and any $w \in \mathrm{N}_C(v)$ (see [17, Proposition 2.2]). As a consequence, by comparing (3) and (CP_0'), one can easily see that our main result (Theorem 1.1) corresponds to a *strict* extension of [8, Theorem 2.3] by Christof and Wachsmuth to the nonlinear case (that is, when $A \in \mathcal{A}(\mathcal{H})$ is nonlinear).

Furthermore, and as mentioned earlier, we insist on the fact that this extension to the nonlinear case is not a simple replica. Its proof (developed in Section 2.2) requires a novel approach based on different tools (such as the Rockafellar's concept of twice epi-differentiability) from that of the linear case based essentially, thanks to linearity, on the expression (1) and on the Haraux and Mignot's expansion formula for projection operators.

(ii) Secondly, even if the variational inequality (3) is of the first kind and associated with a cone (precisely the polar cone $(\mathrm{T}_{\partial\Phi(0_{\mathcal{H}})}(f_0 - Au_0) \cap u_0^\perp)^\circ$), it is not explicitly mentioned in [8] that (3) corresponds to a complementarity problem. On the contrary, Theorem 1.1 identifies clearly the complementarity structure of (CP_0') which can serve as a foundation for further developments (for instance, for the design of numerical algorithms that exploit the complementarity structure). Additionally, the different presentation of the associated cone $\mathrm{T}_{\mathrm{N}_{\partial\Phi(0_{\mathcal{H}})}(f_0 - A(u_0))}(u_0)$ in Theorem 1.1 makes the geometric content more transparent and directly linked to classical objects (precisely, a tangent cone to a normal cone), which may not be as readily visible in [8, Theorem 2.3] (involving a polar cone to an intersection between a tangent cone and an hyperplane). In our opinion, the structural clarity of our paper is one of its key contributions.

(iii) Thirdly, our proof of Theorem 1.1 (developed in Section 2.2) is constructed so that it easily and directly follows from the application of three lemmas known in the literature (with two of them being taken from the literature on twice epi-differentiability). Furthermore, this proof naturally makes the complementarity problem (CP_0') emerge, while the proof of [8, Theorem 2.3] requires a few computations and reformulations to establish (1), (2) and (3).

(iv) Fourthly, Theorem 1.1 is based, like [8, Theorem 2.3], on a polyhedricity assumption made on the subgradient $\partial\Phi(0_{\mathcal{H}})$. A contribution of the present paper is to provide several examples and counterexamples illustrating the requirement of this assumption (see Section 2.3).

(v) Finally we emphasize that one of our counterexamples, precisely Counterexample 2.9 in Section 2.3, may be of independent interests for mathematicians studying generalized differentiability properties of the projection operator. Indeed, two seminal works [15, 25] have provided examples of a nonempty closed convex subset $C \subset \mathcal{H}$ for which the projection operator proj_C fails to exhibit directional differentiability. In both papers, the set C is not conic. To our best knowledge, no example of a nonempty closed convex cone $K \subset \mathcal{H}$ for which the projection operator proj_K is not directionally differentiable has been provided in the literature yet. Counterexample 2.9 in Section 2.3 fills this gap.

Here we underline that Counterexample 2.9 is strongly inspired from the counterexample developed by Shapiro in [25]. Precisely, in Counterexample 2.9, the nonempty closed convex cone $K := \mathbb{R}_+(C \times \{1\}) \subset \mathbb{R}^3$ corresponds to a three-dimensional conical expansion of the two-dimensional nonempty closed convex subset $C \subset \mathbb{R}^2$ constructed by Shapiro. Therefore, since proj_C is not directionally differentiable on $\mathbb{R}^2$, the non-directional differentiability of proj_K on $\mathbb{R}^3$ can be anticipated. Nevertheless we insist on the fact that the proof of the non-directional differentiability of proj_K is not trivial and is not a simple adaptation of Shapiro's proof. It requires several technical adjustments that are discussed in Remark 2.11.

Structure of the paper. The rest of this paper is structured as follows. Section 2.1 is dedicated to required preliminaries from convex and variational analysis, and specific reminders on the notion of polyhedricity are provided in Remark 2.1. Section 2.2 contains the proof of Theorem 1.1 and some perspectives for extensions are discussed in Remark 2.5. Section 2.3 proposes several examples and counterexamples. It also includes Remark 2.11 that is dedicated to the technical adjustments required in Counterexample 2.9 compared to the counterexample developed by Shapiro [25]. Finally the paper closes with a conclusion in Section 3 and Appendix A contains a short proof of a technical lemma.

2. Proof of Theorem 1.1 and illustrative examples

2.1. Reminders from convex and variational analysis

In addition to the notations and concepts recalled in Introduction, this section is devoted to reminders from convex and variational analysis that are required all along the paper. We refer to standard references such as [5, 22, 24]. Recall that the *domain* of a function $\Phi \in \Gamma_0(\mathcal{H})$ is defined by $\mathrm{dom}(\Phi) := \{v \in \mathcal{H} \mid \Phi(v) < +\infty\}$, the associated (set-valued) *subdifferential map* $\partial\Phi : \mathcal{H} \rightrightarrows \mathcal{H}$ is defined by

$$\partial\Phi(v) := \begin{cases} \{w \in \mathcal{H} \mid \forall v' \in \mathcal{H},\ \langle w, v'-v\rangle_{\mathcal{H}} \leq \Phi(v') - \Phi(v)\} & \text{if } v \in \operatorname{dom}(\Phi), \\ \emptyset & \text{otherwise,} \end{cases}$$

for all $v \in \mathcal{H}$, and the associated *Legendre-Fenchel conjugate* $\Phi^\star \in \Gamma_0(\mathcal{H})$ by

$$\forall w \in \mathcal{H}, \quad \Phi^\star(w) := \sup_{v\in\mathcal{H}} \langle w, v\rangle_{\mathcal{H}} - \Phi(v).$$

In this paper we denote by $\overline{S}$ the closure of any nonempty subset $S \subset \mathcal{H}$. The *polar cone*, the *dual cone* and the *orthogonal* to S are respectively defined by

$$S^\circ := \{w \in \mathcal{H} \mid \forall v \in S,\ \langle w, v\rangle_{\mathcal{H}} \leq 0\}, \ \ S^\star := -S^\circ \ \text{ and } \ S^\perp := S^\circ \cap S^\star.$$

They are all nonempty closed convex cones.

When C is a nonempty closed convex subset of $\mathcal{H}$, we denote by $\iota_C \in \Gamma_0(\mathcal{H})$ the *indicator function* to C, defined by $\iota_C(v) := 0$ if $v \in C$ and $\iota_C(v) := +\infty$ otherwise, and by $\sigma_C := \iota_C^\star \in \Gamma_0(\mathcal{H})$ the *support function* to C, given by $\sigma_C(w) = \sup_{v\in C}\langle w, v\rangle_{\mathcal{H}}$ for all $w \in \mathcal{H}$. Furthermore, the *radial cone*, the *normal cone* and the *tangent cone* to C at some $v \in C$ are respectively defined by

$$\begin{aligned} &\mathrm{R}_C(v) := \mathbb{R}_+^*(C - v), \\ &\mathrm{N}_C(v) := \partial\iota_C(v) = (C-v)^\circ = \mathrm{R}_C(v)^\circ \\ \text{and} \qquad &\mathrm{T}_C(v) := \mathrm{N}_C(v)^\circ = \mathrm{R}_C(v)^{\circ\circ} = \overline{\mathrm{R}_C(v)}. \end{aligned}$$

Finally recall that C is said to be *polyhedric* at some $v \in C$ for some $w \in \mathrm{N}_C(v)$ if

$$\mathrm{T}_C(v) \cap w^\perp = \overline{\mathrm{R}_C(v) \cap w^\perp}.$$

Then C is said to be *globally polyhedric* if it is polyhedric at any $v \in C$ and for any $w \in \mathrm{N}_C(v)$.

Remark 2.1. For an overview on polyhedricity, we refer to [28] by Wachsmuth, and references therein. Hereafter we just recall a few properties:

(i) A sufficient condition for a nonempty closed convex subset $C \subset \mathcal{H}$ to be polyhedric at some $v \in C$ for any $w \in \mathrm{N}_C(v)$ is the closedness of $\mathrm{R}_C(v)$.

(ii) Denote by $\overline{\mathrm{B}}_1$ (resp. $\overline{\mathrm{B}}_{\mathbb{R}^n}$, $\overline{\mathrm{B}}_\infty$) the closed unit ball of $\mathbb{R}^n$, for some $n \in \mathbb{N}^*$, with respect to the 1-norm $\|\cdot\|_1$ (resp. the Euclidean norm $\|\cdot\|_{\mathbb{R}^n}$, the ∞-norm $\|\cdot\|_\infty$). Then $\overline{\mathrm{B}}_1$ and $\overline{\mathrm{B}}_\infty$ are both globally polyhedric. For $n = 1$, $\overline{\mathrm{B}}_{\mathbb{R}^n} = \overline{\mathrm{B}}_1 = \overline{\mathrm{B}}_\infty = [-1, 1]$ is globally polyhedric. On the contrary, for $n \geq 2$, $\overline{\mathrm{B}}_{\mathbb{R}^n}$ is not globally polyhedric (more precisely, it is not polyhedric at any $v \in \overline{\mathrm{B}}_{\mathbb{R}^n}$ such that $\|v\|_{\mathbb{R}^n} = 1$ and for any $w \in \mathrm{N}_{\overline{\mathrm{B}}_{\mathbb{R}^n}}(v)\backslash\{0_{\mathbb{R}^n}\}$).

(iii) If $S \subset \mathcal{H}$ is a *polyhedral set* (that is, a nonempty set that can be expressed as the intersection of a finite number of closed halfspaces), then S is globally polyhedric. The converse is not true in general (see [28, Example 4.24]).

(iv) If K is a nonempty closed convex cone of $\mathbb{R}^2$ with $K \neq \mathbb{R}^2$, then K is polyhedral (and thus is globally polyhedric). This result is not true in general in higher dimensions. $\square$

2.2. Proof of Theorem 1.1

Our proof follows directly from the application of three lemmas (Lemmas 2.2, 2.3, 2.4) extracted from the literature. Precisely Lemma 2.2 is a classical fact in convex analysis (related to results such as [18, Exercise 4.6] and [24, Example 11.4]), whose proof is recalled in Appendix A for the reader's convenience. On the contrary, Lemmas 2.3 and 2.4, that can be found for example in [10, Example 2.10] and [1, Theorem 4.15] respectively, relate on the advanced notion of *twice epi-differentiability.* Despite the fundamental role played by this concept, the statement and the proof of Theorem 1.1 are constructed in the present paper to stand independent of this technical notion. Hence, for the sake of brevity and to ensure accessibility, no reminder on twice epi-differentiability is provided here. Interested readers are directed to references such as [1, 10, 23].

Lemma 2.2. *If $\Phi \in \Gamma_0(\mathcal{H})$ is positively homogeneous, then $\Phi = \sigma_{\partial\Phi(0_{\mathcal{H}})}$.*

Lemma 2.3. *Let C be a nonempty closed convex subset of $\mathcal{H}$. If C is polyhedric at some $v \in C$ for some $w \in \mathrm{N}_C(v)$, then σ_C is twice epi-differentiable at w for v with*

$$d_e^2\sigma_C(w|v) = \iota_{\mathrm{T}_{\mathrm{N}_{C(v)}}(w)},$$

where $d_e^2\sigma_C(w|v) \in \Gamma_0(\mathcal{H})$ stands for the second-order epi-derivative of σ_C at w for v.

Lemma 2.4. *Let $(A, \Phi) \in \mathcal{A}(\mathcal{H}) \times \Gamma_0(\mathcal{H})$. Define $u_t := \mathrm{prox}_{A,\Phi}(f_t)$, where $f_t \in \mathcal{H}$, for all $t \geq 0$. If A is semi-differentiable at u_0, Φ is twice epi-differentiable at u_0 for $f_0 - A(u_0)$ and $t \mapsto f_t$ is differentiable at $t = 0$, then $t \mapsto u_t$ is differentiable at $t = 0$ and its derivative u_0' satisfies*

$$u_0' = \mathrm{prox}_{D_sA(u_0), \frac{1}{2}d_e^2\Phi(u_0|f_0 - A(u_0))}(f_0'),$$

where $D_sA(u_0) \in \mathcal{A}(\mathcal{H})$ stands for the semi-differential of A at the point u_0, $d_e^2\Phi(u_0|f_0 - A(u_0)) \in \Gamma_0(\mathcal{H})$ stands for the second-order epi-derivative of Φ at u_0 for $f_0 - A(u_0)$ and f_0' stands for the derivative of $t \mapsto f_t$ at $t = 0$.

We are now in a position to develop the (short) proof of Theorem 1.1. For this purpose, let us consider the framework of Theorem 1.1. Since $\Phi \in \Gamma_0(\mathcal{H})$ is positively homogeneous, Lemma 2.2 ensures that $\Phi = \sigma_{\partial\Phi(0_{\mathcal{H}})}$ and, since $\partial\Phi(0_{\mathcal{H}})$ is polyhedric at $f_0 - A(u_0)$ for u_0, Lemma 2.3 ensures that Φ is twice epi-differentiable at u_0 for $f_0 - A(u_0)$ with

$$d_e^2\Phi(u_0|f_0 - A(u_0)) = \iota_{\mathrm{T}_{\mathrm{N}_{\partial\Phi(0_{\mathcal{H}})}(f_0 - A(u_0))}(u_0)}. \tag{4}$$

Since u_t is the unique solution to (VI_t), it can be expressed as $u_t = \mathrm{prox}_{A,\Phi}(f_t)$ for all $t \geq 0$. Then, Lemma 2.4 ensures that $t \mapsto u_t$ is differentiable at $t = 0$ and its derivative u_0' satisfies

$$u_0' = \mathrm{prox}_{D_sA(u_0), \frac{1}{2}d_e^2\Phi(u_0|f_0 - A(u_0))}(f_0'). \tag{5}$$

We get from (4) and (5) that u_0' is the unique solution to the complementarity problem (CP_0'). The proof of Theorem 1.1 is complete.

Remark 2.5. From the above proof of Theorem 1.1, there are a number of possible perspectives:

(i) From Lemma 2.4, one can easily see that the assumptions made on Φ in Theorem 1.1 can be relaxed. Indeed it is sufficient to assume that Φ is twice epi-differentiable at u_0 for $f_0 - A(u_0)$ with its second-order epi-derivative $d_e^2\Phi(u_0|f_0 - A(u_0)) = \iota_K$ being the indicator function to a nonempty closed convex cone $K \subset \mathcal{H}$. In that context we obtain that $t \mapsto u_t$ is differentiable at $t = 0$ and its derivative u_0' is the unique solution to the complementarity problem

$$K \ni u_0' \perp D_s A(u_0)(u_0') - f_0' \in K^\star.$$

However a characterization of the functions $\Phi \in \Gamma_0(\mathcal{H})$ satisfying the above assumptions, furthermore with an explicit expression of K, seems untractable. One of the main contributions of the present work has been to gather known results from the literature to reveal that the positively homogeneous functions $\Phi \in \Gamma_0(\mathcal{H})$ with $\partial\Phi(0_\mathcal{H})$ polyhedric at $f_0 - A(u_0)$ for u_0 satisfy the above assumptions, furthermore with $K = \mathrm{T}_{\mathrm{N}_{\partial\Phi(0_\mathcal{H})}(f_0 - A(u_0))}(u_0)$.

(ii) In the same spirit than the previous item, a first step towards the relaxation of the assumptions made on Φ in Theorem 1.1 could concern the polyhedricity assumption, by characterizing the nonempty closed convex subsets $C \subset \mathcal{H}$ such that σ_C is twice epi-differentiable and, furthermore, with a second-order epi-derivative being the indicator function to a nonempty closed convex cone. This open question is beyond the scope of the present paper, but may constitute an interesting perspective for further researches.

(iii) Finally we bring the reader's attention to the reference [1, Theorem 4.15] in which Lemma 2.4 is actually stated in the more general context of parameterized nonlinear variational inequalities of the form

$$\langle A_t(u_t) - f_t, v - u_t\rangle_\mathcal{H} + \Phi_t(v) - \Phi_t(u_t) \geq 0, \quad \forall v \in \mathcal{H},$$

in which all elements are parameterized, including the pair (A_t, Φ_t). Therefore a natural perspective for future works could be the extension of Theorem 1.1 to this generalized context.

2.3. Examples and counterexamples

In this section, for illustration, we apply Theorem 1.1 on a parameterized LASSO problem (Example 2.6) and on a parameterized complementarity problem (Example 2.7), for which the polyhedricity assumption is satisfied in both situations. Above all, in a second stage, we prove with two counterexamples that the conclusions of Theorem 1.1 may fail if the polyhedricity assumption is not satisfied. Precisely, Counterexample 2.8 provides a situation where the map $t \mapsto u_t$ is differentiable at $t = 0$, but its derivative u_0' is not a solution to the complementarity problem (CP_0'). Then, Counterexample 2.9 provides a situation where the map $t \mapsto u_t$ is not even differentiable at $t = 0$.

Example 2.6. Consider the parameterized LASSO problem given by

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \ \frac{1}{2}\|Rx - c_t\|^2_{\mathbb{R}^m} + \|x\|_1,$$

where n, $m \in \mathbb{N}^*$, $R \in \mathbb{R}^{m\times n}$ is such that $R^\top R \in \mathbb{R}^{n\times n}$ is invertible, $c_t \in \mathbb{R}^m$ for all $t \geq 0$, and where $\|\cdot\|_1$ stands for the classical 1-norm on $\mathbb{R}^n$. It corresponds to the parameterized variational inequality (VI$_t$) when considering $\mathcal{H} = \mathbb{R}^n$, the linear operator $A \in \mathcal{A}(\mathbb{R}^n)$ given by $A(x) := R^\top Rx$ for all $x \in \mathbb{R}^n$, the positively homogeneous function $\Phi := \|\cdot\|_1 \in \Gamma_0(\mathbb{R}^n)$ and $f_t := R^\top c_t$ for all $t \geq 0$. In the sequel we denote by x_t its unique solution for all $t \geq 0$.

It is known that $\partial\|\cdot\|_1(0_{\mathbb{R}^n}) = \overline{\mathrm{B}}_\infty$ which is globally polyhedric (see Remark 2.1). Hence we deduce from Theorem 1.1 that, if $t \mapsto c_t$ is differentiable at $t = 0$, then $t \mapsto x_t$ is differentiable at $t = 0$ and its derivative x'_0 is the unique solution to the complementarity problem

$$\mathrm{T}_{\mathrm{N}_{\overline{\mathrm{B}}_\infty}(R^\top(c_0-Rx_0))}(x_0) \ni x'_0 \perp R^\top(Rx'_0 - c'_0) \in \mathrm{T}_{\mathrm{N}_{\overline{\mathrm{B}}_\infty}(R^\top(c_0-Rx_0))}(x_0)^\star.$$

Let us explore here a simple illustration of this result by considering $n = m = 2$, $R = \mathrm{Id}_{\mathbb{R}^2}$ and $c_t = 2(\cos(t), \sin(t))$ for all $t \geq 0$. In that context, from the closed-form expression of $\mathrm{prox}_{\|\cdot\|_1}$ (see, e.g., [21]), it can easily be established that we have $x_t = (2\cos(t) - 1, 0)$ for all $t \geq 0$. Therefore $c_0 = (2, 0)$, $x_0 = (1, 0)$ and one can easily obtain that

$$\mathrm{T}_{\mathrm{N}_{\overline{\mathrm{B}}_\infty}(R^\top(c_0-Rx_0))}(x_0) = \mathbb{R} \times \{0\} \quad \text{and} \quad \mathrm{T}_{\mathrm{N}_{\overline{\mathrm{B}}_\infty}(R^\top(c_0-Rx_0))}(x_0)^\star = \{0\} \times \mathbb{R}.$$

With $c'_0 = (0, 2)$, we indeed see that $x'_0 = (0, 0)$ is the unique solution to the complementarity problem

$$\mathbb{R} \times \{0\} \ni x'_0 \perp x'_0 - c'_0 \in \{0\} \times \mathbb{R}. \qquad \square$$

Example 2.7. Let $A \in \mathcal{A}(\mathcal{H})$, $K \subset \mathcal{H}$ be a nonempty closed convex cone and $u_t \in \mathcal{H}$ be the unique solution to the parameterized complementarity problem

$$K \ni u_t \perp A(u_t) - f_t \in K^\star, \tag{CP$_t$}$$

where $f_t \in \mathcal{H}$, for all $t \geq 0$. Applying Theorem 1.1 to $\Phi = \iota_K$ for which $\partial\Phi(0_{\mathcal{H}}) = K^\circ$, and recalling that K° is polyhedric at $f_0 - A(u_0)$ for u_0 if and only if K is polyhedric at u_0 for $f_0 - A(u_0)$ (see [28, Lemma 3.2]), we obtain the following result: if A is semi-differentiable at u_0, K is polyhedric at u_0 for $f_0 - A(u_0)$ and $t \mapsto f_t$ is differentiable at $t = 0$, then $t \mapsto u_t$ is differentiable at $t = 0$ and its derivative u'_0 is the unique solution to the complementarity problem

$$\mathrm{T}_{\mathrm{N}_{K^\circ}(f_0-A(u_0))}(u_0) \ni u'_0 \perp D_sA(u_0)(u'_0) - f'_0 \in \mathrm{T}_{\mathrm{N}_{K^\circ}(f_0-A(u_0))}(u_0)^\star.$$

Let us give a simple illustration of this result by considering $\mathcal{H} = \mathbb{R}^2$, $A = \mathrm{Id}_{\mathbb{R}^2}$, $K = \mathbb{R}^2_+$ and $f_t = (t, t-1)$ for all $t \geq 0$. In that context it is clear that $K^\circ = \mathbb{R}^2_-$ and $u_t = (t, 0)$ for all $t \geq 0$. Therefore $f_0 = (0, -1)$, $u_0 = (0, 0)$ and one can easily obtain that

$$\mathrm{T}_{\mathrm{N}_{K^\circ}(f_0-A(u_0))}(u_0) = \mathbb{R}_+ \times \{0\} \quad \text{and} \quad \mathrm{T}_{\mathrm{N}_{K^\circ}(f_0-A(u_0))}(u_0)^\star = \mathbb{R}_+ \times \mathbb{R}.$$

With $f'_0 = (1, 1)$, we indeed see that $u'_0 = (1, 0)$ is the unique solution to the complementarity problem

$$\mathbb{R}_+ \times \{0\} \ni u'_0 \perp u'_0 - f'_0 \in \mathbb{R}_+ \times \mathbb{R}. \qquad \square$$

Counterexample 2.8. In the literature, when considering a LASSO problem but replacing the 1-norm $\|\cdot\|_1$ on $\mathbb{R}^n$ by the Euclidean norm $\|\cdot\|_{\mathbb{R}^n}$, one obtains what is commonly called a ridge regression problem. Here we consider the parameterized ridge regression problem given by

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \frac{1}{2}\|Rx-c_t\|^2_{\mathbb{R}^m}+\|x\|_{\mathbb{R}^n},$$

where n, $m\in\mathbb{N}^*$, $R\in\mathbb{R}^{m\times n}$ is such that $R^\top R\in\mathbb{R}^{n\times n}$ is invertible and $c_t\in\mathbb{R}^m$ for all $t\geq 0$. It corresponds to the parameterized variational inequality (VI_t) when considering $\mathcal{H}=\mathbb{R}^n$, the linear operator $A\in\mathcal{A}(\mathbb{R}^n)$ given by $A(x):=R^\top Rx$ for all $x\in\mathbb{R}^n$, the positively homogeneous function $\Phi:=\|\cdot\|_{\mathbb{R}^n}\in\Gamma_0(\mathbb{R}^n)$ and $f_t:=R^\top c_t$ for all $t\geq 0$. In the sequel we denote by x_t its unique solution for all $t\geq 0$.

Here the polyhedricity assumption of Theorem 1.1 is not satisfied if $n\geq 2$. Indeed, in that context, it is known that $\partial\|\cdot\|_{\mathbb{R}^n}(0_{\mathbb{R}^n})=\overline{\mathrm{B}}_{\mathbb{R}^n}$ which is not globally polyhedric (see Remark 2.1) and thus the conclusions of Theorem 1.1 are not guaranteed, as shown below.

Consider $n=m=2$, $R=\mathrm{Id}_{\mathbb{R}^2}$ and $c_t=2(\cos(t),\sin(t))$ for all $t\geq 0$. In that context, from the closed-form expression of $\mathrm{prox}_{\|\cdot\|_{\mathbb{R}^n}}$ (see, e.g., [21]), it can easily be established that $x_t=(\cos(t),\sin(t))$ for all $t\geq 0$. Therefore $c_0=(2,0)$, $x_0=(1,0)$ and one can easily obtain that

$$\mathrm{T}_{\mathrm{N}_{\overline{\mathrm{B}}_{\mathbb{R}^n}}(R^\top(c_0-Rx_0))}(x_0)=\mathbb{R}\times\{0\}\quad\text{and}\quad\mathrm{T}_{\mathrm{N}_{\overline{\mathrm{B}}_{\mathbb{R}^n}}(R^\top(c_0-Rx_0))}(x_0)^\star=\{0\}\times\mathbb{R}.$$

With $c_0'=(0,2)$, we observe that, even if $t\mapsto x_t$ is differentiable at $t=0$, its derivative $x_0'=(0,1)$ is not a solution to the complementarity problem

$$\mathbb{R}\times\{0\}\ni x_0'\perp x_0'-c_0'\in\{0\}\times\mathbb{R}.\qquad\square$$

Counterexample 2.9. Consider the parameterized complementarity problem (CP_t) of Example 2.7 with $A=\mathrm{Id}_{\mathbb{R}^n}$. In that context the unique solution to (CP_t) is given by $u_t=\mathrm{proj}_K(f_t)$ for all $t\geq 0$. Hereafter our aim is to construct a counterexample showing that, even if $t\mapsto f_t$ is differentiable at $t=0$, if K is not polyhedric at u_0 for f_0-u_0, then $t\mapsto u_t$ is not differentiable at $t=0$ in general. Furthermore, in our counterexample below, f_t is *t-affine* (in the sense that $f_t=f_0+tf_0'$ for all $t\geq 0$, for some f_0, $f_0'\in\mathcal{H}$) which implies that proj_K is not directionally differentiable at f_0 in the direction f_0'. To our best knowledge, this is the first time in the literature that an example of non-directionally differentiable projection operator onto a nonempty closed convex *cone* is provided. We refer to Item (v) at the end of Introduction for additional comments and related references.

Inspired by the work [25] by Shapiro, we consider now the nonempty closed convex subset $C\subset\mathbb{R}^2$ defined as the convex envelope of $(0,0)$, $(1,0)$ and the points $(\cos(\gamma_k),\sin(\gamma_k))_{k\in\mathbb{N}^*}$ where $\gamma_k:=\frac{\pi}{2^k}$ for all $k\in\mathbb{N}^*$ (see Figure 2). Note that C is not polyhedric at $(1,0)$ for $(1,0)$. In [25], it is proved that proj_C is not directionally differentiable at $(2,0)$ (whose projection onto C is $(1,0)$) in the direction $(0,1)$. However, with respect to our objective, C is not *conic*. Furthermore, we know that any nonempty closed convex cone in $\mathbb{R}^2$ is globally polyhedric (see Remark 2.1).

Therefore, to construct our counterexample, our idea is to consider the nonempty closed convex cone $K := \mathbb{R}_+(C \times \{1\})$ of $\mathbb{R}^3$ (see Figure 1) and to prove that proj_K is not directionally differentiable at $f_0 := (2,0,0)$ (whose projection onto K is $u_0 := (1,0,1)$) in the direction $f_0' := (0,1,0)$. However, to our best knowledge, an exploitable expression of proj_K in function of proj_C is not tractable. As a consequence, to fill our objective, we need to develop a similar strategy than the one proposed in [25]. However we emphasize that this extension is not a simple replica and requires several technical adjustments (see Remark 2.11 for details).

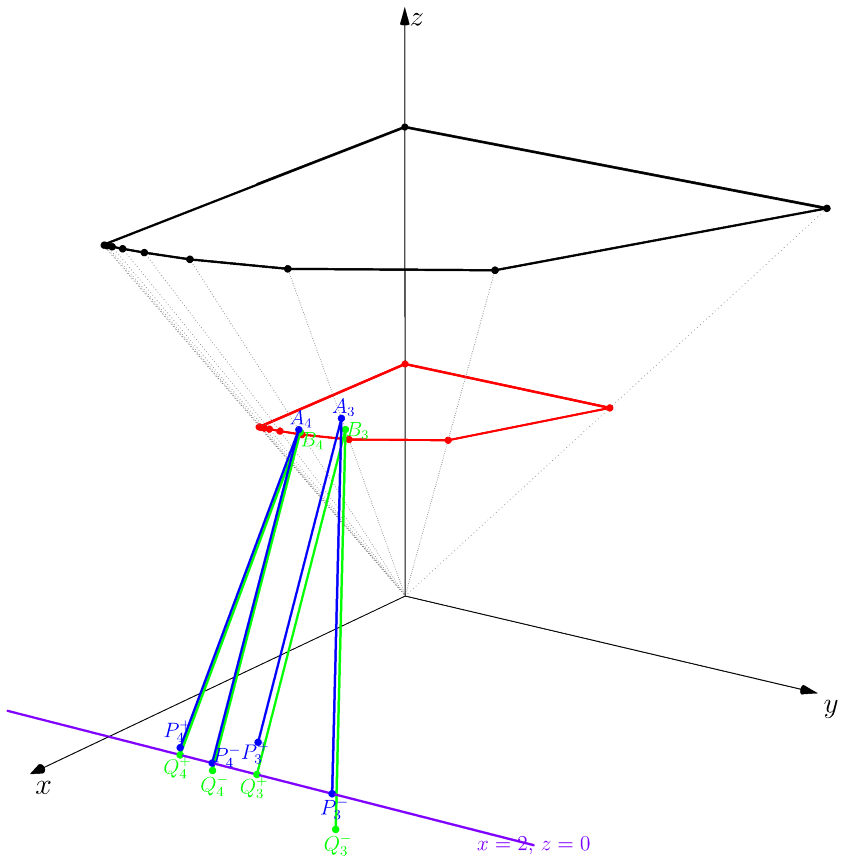

Figure 1: Illustration of Counterexample 2.9

Let us introduce the points

$$A_k := \alpha_k(\cos(\gamma_k), \sin(\gamma_k), 1) \in K \quad \text{and} \quad B_k := \beta_k(\cos(\gamma_k), \sin(\gamma_k), 1) \in K,$$

for some $\alpha_k > 0$ and $\beta_k > 0$ (to be determined later), for all $k \in \mathbb{N}^*$. Now fix $k \geq 2$. With basic computations, one can easily determine the equation of the plane containing $0_{\mathbb{R}^3}$, A_k and $A_{k\pm 1}$ and an associated orthogonal vector. One deduces that

$$P_k^\pm := A_k + \lambda_k^\pm \left(\frac{\sin(\gamma_{k\pm 1}) - \sin(\gamma_k)}{\sin(\gamma_{k\pm 1} - \gamma_k)}, \frac{\cos(\gamma_k) - \cos(\gamma_{k\pm 1})}{\sin(\gamma_{k\pm 1} - \gamma_k)}, -1 \right),$$

for any $\lambda^\pm \geq 0$ (to be determined later), is such that $\mathrm{proj}_K(P_k^\pm) = A_k$. Similarly

$$Q_k^\pm := B_k + \mu_k^\pm \left(\frac{\sin(\gamma_{k\pm 1}) - \sin(\gamma_k)}{\sin(\gamma_{k\pm 1} - \gamma_k)}, \frac{\cos(\gamma_k) - \cos(\gamma_{k\pm 1})}{\sin(\gamma_{k\pm 1} - \gamma_k)}, -1 \right),$$

for any $\mu^\pm \geq 0$ (to be determined later), is such that $\mathrm{proj}_K(Q_k^\pm) = B_k$.

Now we impose P_k^- and Q_k^+ to be on the line $f_0 + \mathbb{R}_+ f_0'$, that is with first component equal to 2 ($x = 2$) and last component equal to 0 ($z = 0$), by taking

$$\lambda_k^- = \alpha_k = 2\,\frac{\sin(\gamma_{k-1} - \gamma_k)}{\cos(\gamma_k)\sin(\gamma_{k-1} - \gamma_k) + \sin(\gamma_{k-1}) - \sin(\gamma_k)} > 0,$$

and
$$\mu_k^+ = \beta_k = 2\,\frac{\sin(\gamma_{k+1} - \gamma_k)}{\cos(\gamma_k)\sin(\gamma_{k+1} - \gamma_k) + \sin(\gamma_{k+1}) - \sin(\gamma_k)} > 0.$$

We refer to Figure 1 for illustration. Contrary to P_k^- and Q_k^+, note that P_k^+ and Q_k^- do not belong to the line $f_0 + \mathbb{R}_+ f_0'$.

It holds that $P_k^- = f_0 + r_k f_0'$ and $Q_k^+ = f_0 + s_k f_0'$ with

$$r_k := 2\frac{\sin(\gamma_{k-1} - \gamma_k)\sin(\gamma_k) + \cos(\gamma_k) - \cos(\gamma_{k-1})}{\cos(\gamma_k)\sin(\gamma_{k-1} - \gamma_k) + \sin(\gamma_{k-1}) - \sin(\gamma_k)} > 0,$$

and
$$s_k := 2\,\frac{\sin(\gamma_{k+1} - \gamma_k)\sin(\gamma_k) + \cos(\gamma_k) - \cos(\gamma_{k+1})}{\cos(\gamma_k)\sin(\gamma_{k+1} - \gamma_k) + \sin(\gamma_{k+1}) - \sin(\gamma_k)} > 0.$$

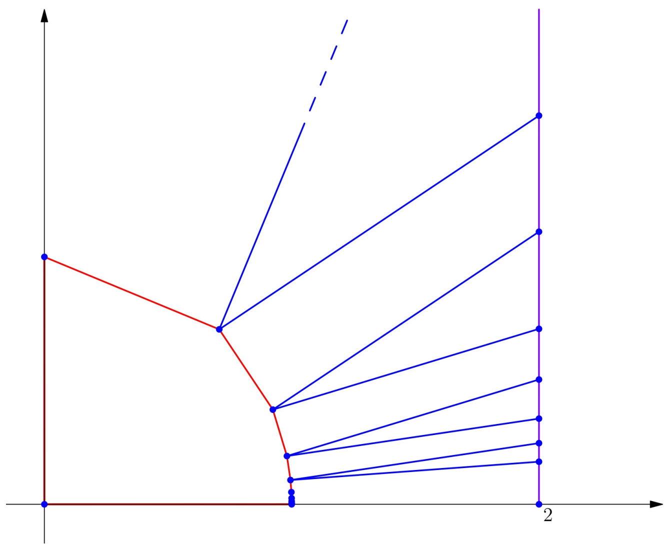

Figure 2: Illustration of the counterexample developed by Shapiro in [25].

The following technical lemma (that can be proved with basic Taylor expansion formulas) is now required.

Lemma 2.10. *Let* $\delta > 0$ *with* $\delta \neq 1$*, and let*

$$h_\delta(\rho) := 2\,\frac{\sin(\delta\rho - \rho)}{\cos(\rho)\sin(\delta\rho - \rho) + \sin(\delta\rho) - \sin(\rho)} > 0$$

and
$$H_\delta(\rho) := 2\,\frac{\sin(\delta\rho - \rho)\sin(\rho) + \cos(\rho) - \cos(\delta\rho)}{\cos(\rho)\sin(\delta\rho - \rho) + \sin(\delta\rho) - \sin(\rho)} > 0,$$

for any $\rho > 0$ *small enough. Then*

$$\lim_{\rho\to 0} h_\delta(\rho) = 1, \quad \lim_{\rho\to 0} H_\delta(\rho) = 0 \quad \text{and} \quad \lim_{\rho\to 0} \frac{H_\delta(\rho)}{\rho} = \frac{\delta+3}{2}.$$

In particular, since γ_k converges to 0 when $k \to +\infty$, we infer that $\alpha_k = h_2(\gamma_k)$ and $\beta_k = h_{1/2}(\gamma_k)$ both converge to 1 when $k \to +\infty$, and $r_k = H_2(\gamma_k)$ and $s_k = H_{1/2}(\gamma_k)$ both converge to 0 when $k \to +\infty$.

We are now in a position to conclude this counterexample. Define $f_t := f_0 + tf_0'$ and $u_t := \mathrm{proj}_K(f_t)$ for all $t \geq 0$. Our aim is to prove that the limit

$$\lim_{t\to 0} \frac{u_t - u_0}{t} = \lim_{t\to 0} \frac{\mathrm{proj}_K(f_0 + tf_0') - \mathrm{proj}_K(f_0)}{t},$$

does not exist. For this purpose, on the one hand, when considering the quotient

$$\frac{\mathrm{proj}_K(f_0 + r_k f_0') - \mathrm{proj}_K(f_0)}{r_k} = \frac{\mathrm{proj}_K(P_k^-) - \mathrm{proj}_K(f_0)}{r_k} = \frac{A_k - u_0}{r_k},$$

we observe that its second component is given by

$$\frac{\alpha_k \sin(\gamma_k)}{r_k} = \frac{h_2(\gamma_k)\sin(\gamma_k)}{H_2(\gamma_k)}$$

which converges to $2/5$ when $k \to +\infty$ from Lemma 2.10. On the other hand, when considering the quotient

$$\frac{\mathrm{proj}_K(f_0 + s_k f_0') - \mathrm{proj}_K(f_0)}{s_k} = \frac{\mathrm{proj}_K(Q_k^+) - \mathrm{proj}_K(f_0)}{s_k} = \frac{B_k - u_0}{s_k},$$

we observe that its second component is given by

$$\frac{\beta_k \sin(\gamma_k)}{s_k} = \frac{h_{1/2}(\gamma_k)\sin(\gamma_k)}{H_{1/2}(\gamma_k)}$$

which converges to $4/7$ when $k \to +\infty$ following Lemma 2.10. Since $2/5 \neq 4/7$, we conclude that $t \mapsto u_t$ is not differentiable at $t = 0$. At this occasion we also deduce that proj_K is not directionally differentiable at $f_0 = (2, 0, 0)$ in the direction $f_0' = (0, 1, 0)$, and that K is not polyhedric at $u_0 = (1, 0, 1)$ for $f_0 - u_0 = (1, 0, -1)$. □

Remark 2.11. This remark is dedicated to the primary differences and technical adjustments required in our Counterexample 2.9 compared to the one developed by Shapiro in [25]. Firstly, our counterexample involves a *conic* convex subset, unlike the one in [25]. For this purpose, we had to use a three-dimensional example, whereas [25] used a two-dimensional example. Secondly, it is important to note that the example in [25] is based on a single family of points on the boundary of C (see Figure 2), from which two orthogonal lines (shown in blue) intersect both the vertical line $[x = 2]$ (shown in violet). When dealing with the three-dimensional convex cone $K := \mathbb{R}_+(C \times \{1\})$, our initial approach was to follow a similar strategy by constructing a single family of points on the edges of K from which the two orthogonal lines intersect the same line. However, we mathematically reached that this approach is not feasible. To address this issue, we constructed two different families $(A_k)_{k\in\mathbb{N}^*}$ and $(B_k)_{k\in\mathbb{N}^*}$ of points on the edges of K. For each $k \in \mathbb{N}^*$, the points A_k and B_k are located on the same edge of K but at different heights (and not at height $z = 1$). From these points, one of the orthogonal lines (blue and green, respectively) intersects the violet line $[x = 2, z = 0]$, precisely at P_k^- and Q_k^+ respectively, while the other orthogonal line does not (see Figure 1). □

3. Conclusion

We recently became aware of the recent works [7, 8, 9] by Christof and Wachsmuth. After carefully examining their contributions, we observed some points of similarity, but also some important differences. In particular their analysis in [8, Theorem 2.3] is limited to *linear* operators and relies, thanks to this linearity, on some reformulations of the involved variational inequalities and on the classical first-order sensitivity analysis for projection operators by Haraux [13] and Mignot [17]. Unfortunately this method cannot be extended to the *nonlinear* setting. In contrast, our approach, based on the notion of twice epi-differentiability introduced by Rockafellar [23] and extended to infinite-dimensional spaces by Do [10], applies to nonlinear operators as well, leading to a more general and unified result (Theorem 1.1). Furthermore Theorem 1.1 provides a clear and explicit characterization of the structure of the derivative which turns out to be the unique solution to a complementarity problem. This result gives a transparent geometric interpretation and shows a strong link between sensitivity analysis of variational inequalities and complementarity theory.

The main contributions of the present work can be summarized as follows:

(i) A general first-order sensitivity analysis for nonlinear variational inequalities associated with positively homogeneous functions.

(ii) An explicit characterization of the Hadamard directional derivative of the solution map as the solution to a structured complementarity problem.

(iii) A simple and direct proof based on tools from epi-differential calculus.

(iv) Several examples and counterexamples showing the role and the necessity of the polyhedricity assumption, including what seems to be the first example in the literature of a projection onto a nonempty closed convex cone that is not directionally differentiable.

In addition to the above contributions, several questions remain open. First, as mentioned in Remark 2.5, it would be interesting to identify more general classes of functions $\Phi \in \Gamma_0(\mathcal{H})$, not necessarily positively homogeneous, for which the second-order epi-derivative is the indicator function to a nonempty closed convex cone. This would allow one to extend the present analysis beyond the positively homogeneous case. Second, the polyhedricity condition plays a key role in our result, but it could be restrictive in some situations. Finding alternative regularity conditions that would lead to the same kind of conclusion is an open problem that deserves further investigations. Third, it would be interesting to investigate whether our approach can be extended to more general settings, when replacing the *convex* subdifferential with limiting or proximal subdifferential. In particular, the class of prox-regular functions studied in [24] provides a natural framework where second-order epi-differentiability has already been studied. Adapting these tools to the analysis of Hadamard directional differentiability of the solutions to nonlinear *nonconvex* variational inequalities in this broader context is a promising direction for future works.

A. Proof of Lemma 2.2

Consider the framework of Lemma 2.2. Since Φ is positively homogeneous closed and proper, it is clear that $\Phi(0_{\mathcal{H}}) = 0$. Furthermore, a basic change of variable gives

$$\Phi^\star(w) = \sup_{v\in\mathcal{H}} \langle w, 2v\rangle_{\mathcal{H}} - \Phi(2v) = 2\Phi^\star(w)$$

for all $w \in \mathcal{H}$, which implies, since $\Phi^\star \in \Gamma_0(\mathcal{H})$ is proper, that $\Phi^\star$ is an indicator function and thus $\Phi^\star = \iota_{\mathrm{dom}(\Phi^\star)}$. Second, note that $w \in \mathrm{dom}(\Phi^\star)$ if and only if there exists $\mu \in \mathbb{R}$ such that $\langle w, v\rangle_{\mathcal{H}} - \Phi(v) \leq \mu$ for all $v \in \mathcal{H}$. Replacing v by λv with $\lambda > 0$ and dividing by $\lambda \to +\infty$, we obtain that $w \in \mathrm{dom}(\Phi^\star)$ if and only if $\langle w, v\rangle_{\mathcal{H}} - \Phi(v) \leq 0$ for all $v \in \mathcal{H}$, that is, since $\Phi(0_{\mathcal{H}}) = 0$, if and only if $w \in \partial\Phi(0_{\mathcal{H}})$. Hence we have obtained that $\Phi^\star = \iota_{\partial\Phi(0_{\mathcal{H}})}$. Finally, since $\Phi \in \Gamma_0(\mathcal{H})$, the Fenchel–Moreau theorem ensures that $\Phi = \Phi^{\star\star}$ and the proof of Lemma 2.2 is complete. □

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