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Quantitative Stability of Brunn-Minkowski Inequalities for the p-Torsional Rigidity

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We establish the quantitative Brunn-Minkowski inequality and the quantitative Urysohn inequality for p-torsional rigidity. Here the p-torsional rigidity can be formulated by the weak solution of the boundary value problem of p-Laplace equation. In order to prove the main results, we establish the relation between the Borell-Brascamp-Lieb deficit and the Brunn-Minkowski deficit, and then use the relation to prove two different stability of the Borell-Brascamp-Lieb inequality for p-concave functions with $p > 0$. The constants in our stability results are independent on the parameter p and the integrals of given functions, and thus also improve the results given by Ghilli and Salani about the case $p = 2$.

Keywords: Stability, Brunn-Minkowski inequality, Borell-Brascamp-Lieb inequality, p-torsional rigidity, p-Laplace operator.

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1. Introduction

In this paper, we study the stability of Brunn-Minkowski inequalities involving p-Laplacian, which belongs to the theory of Brunn-Minkowski type inequalities for functionals of the Calculus of Variation, see for example [14, 18, 32, 46].

We start by recalling the classical Brunn-Minkowski inequality. Let K_0 and K_1 be two convex bodies (compact convex sets with non-empty interior) in $\mathbb{R}^n$. The classical Brunn-Minkowski inequality states that

$$|K_\lambda|^{\frac{1}{n}} \geq (1-\lambda)\,|K_0|^{\frac{1}{n}} + \lambda\,|K_1|^{\frac{1}{n}}\,, \tag{1.1}$$

where $|\cdot|$ denotes the Lebesgue measure (i.e., the n-dimensional volume), and

$$K_\lambda = (1-\lambda)K_0 + \lambda K_1 = \{(1-\lambda)x + \lambda y : x \in K_0, y \in K_1\},$$

denotes the Minkowski convex combination of K_0 and K_1, which is still a convex body. In addition, equality in (1.1) holds if and only if K_0 and K_1 are homothetics, i.e., $K_0 = \lambda K_1 + x$, for some $\lambda > 0$ and $x \in \mathbb{R}^n$. Its importance, both to convexity and to other areas of mathematics, can hardly be overstated, see [30].

This paper makes attention to the stability version of Brunn-Minkowski inequality for p-torsional rigidity for $p > 1$. Let K be a convex body with C^2 boundary, Ω be its interior, and $p > 1$.

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The p-torsional rigidity $\tau_p(K)$ of Ω is defined by

$$\frac{1}{\tau_p(K)} = \inf\left\{ \frac{\int_\Omega |\nabla w(x)|^p dx}{\left[\int_\Omega |w(x)|dx\right]^p} : w \in W_0^{1,p}(\Omega), \int_\Omega |w(x)|dx > 0 \right\}.$$

The minimizer in $W_0^{1,p}(\Omega)$ is a weak solution to the boundary value problem

$$\begin{cases} \Delta_p u = -1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Δ_p is the p-Laplace operator.

Let $p > 1$. Colesanti, Cuoghi and Salani [16] proved the Brunn-Minkowski inequality for the p-torsional rigidity as

$$\tau_p(K_\lambda)^{\frac{1}{p+n(p-1)}} \geq (1-\lambda)\tau_p(K_0)^{\frac{1}{p+n(p-1)}} + \lambda\tau_p(K_1)^{\frac{1}{p+n(p-1)}}, \tag{1.3}$$

with equality if and only if K_1 is homothetic to K_0. In the case $p = 2$, we note that the inequality was proved by Borell [5] and the the characterization of the equality conditions was given by Colesanti [14].

Since (1.3) is obtained through an application of the Borell-Brascamp-Lieb inequality, a suitable stability result for the latter can be used to get the desired stability of the former.

Throughout the paper we use u_0 and u_1 to denote real nonnegative bounded functions belonging to $L^1(\mathbb{R}^n)$ $(n \geq 1)$ with compact supports Ω_0 and Ω_1 respectively. To avoid the trivial case of $u_i \equiv 0, i = 0, 1$, let us assume that

$$I_i = \int_{\mathbb{R}^n} u_i dx > 0, \quad \text{for } \ i = 0, 1.$$

The classical Borell-Brascamp-Lieb inequality (BBL inequality below) claims that

Theorem 1.1. (BBL inequality) *Let $0 < \lambda < 1, -\frac{1}{n} \leq p \leq +\infty$. Let u_0 and u_1 be real nonnegative bounded functions in $L^1(\mathbb{R}^n), n \geq 1$, with compact supports Ω_0 and Ω_1. Let $h \in L^1(\mathbb{R}^n)$ satisfies that, for all $x \in \Omega_0, y \in \Omega_1$,*

$$h((1-\lambda)x + \lambda y) \geq \mathcal{M}_p(u_0(x), u_1(y), \lambda). \tag{1.4}$$

Then

$$\int_{\Omega_\lambda} h(x)dx \geq \mathcal{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda). \tag{1.5}$$

Note that the number $p/(pn+1)$ has to be suitably interpreted in the extremal cases (namely, it is equal to $-\infty$ when $p = -1/n$, and to $1/n$ when $p = +\infty$). The quantity $\mathcal{M}_q^\lambda(x, y)$ denotes the (λ-weighted) q-mean of x, y. That is, for $x, y \geq 0$ and $xy = 0$, we set $\mathcal{M}_q^\lambda(x, y) = 0$ for every $q \in \mathbb{R} \cup \{\pm\infty\}$. For all $x, y > 0$ and $\lambda \in (0, 1)$, in the case $q \neq 0$ we set

$$\mathcal{M}_q^\lambda(x, y) = ((1-\lambda)x^q + \lambda y^q)^{1/q},$$

and in the case $q = 0$, we set $\mathcal{M}_0^\lambda(x, y) = x^{(1-\lambda)} y^\lambda$.

The BBL inequality was first proved for $p > 0$ (in a slightly different form) by Henstock and Macbeath [37] and by Dinghas [19], and was generalized by Brascamp and Lieb [8] and by Borell [4]. When $p = 0$, the inequality is known as the Prékopa-Leindler inequality, which was originally established by Prékopa [42] and Leindler [39], and later rediscovered by Brascamp and Lieb in [9]. See [11, 57] for the extension to the other spaces.

In order to properly formulate the quantitative question, we recall some notion. A nonnegative function u is called p-concave for some $p \in [-\infty, +\infty]$ if

$$u((1-\lambda)x + \lambda y) \geq \mathcal{M}_p(u(x), u(y), \lambda) \quad \text{for every } x, y \in \mathbb{R}^n \text{ and every } \lambda \in (0,1).$$

Let u_0, u_1 be real nonnegative bounded integral functions with compact supports Ω_0, Ω_1 in $\mathbb{R}^n$. The (p, λ)-convolution of u_0 and u_1 is defined by

$$u_{p,\lambda}(x) = \sup\{\mathcal{M}_p\left(u_0(x_0), u_1(x_1), \lambda\right) : x = (1-\lambda)x_0 + \lambda x_1, x_i \in \bar{\Omega}_i, i = 0, 1\}.$$

The Brunn-Minkowski deficit (BM deficit below) of Ω_0 and Ω_1 is defined by

$$\delta_\lambda(\Omega_0, \Omega_1) := \frac{|\Omega_\lambda|}{\mathcal{M}_{\frac{1}{n}}\left(|\Omega_0|, |\Omega_1|, \lambda\right)} - 1.$$

The Borell-Brascamp-Lieb deficit (BBL deficit below) of u_0, u_1 is defined by

$$\delta_\lambda(u_0, u_1, p) := \frac{\int_{\Omega_\lambda} u_{p,\lambda}(x)dx}{\mathcal{M}_{\frac{p}{np+1}}\left(I_0, I_1, \lambda\right)} - 1.$$

Note that (1.1) and Theorem 1.1 implies that $\delta_\lambda(\Omega_0, \Omega_1) \geq 0$ and $\delta_\lambda(u_0, u_1, p) \geq 0$.

When u_0, u_1 are non-negative power concave functions and $p > 0$, the stability question we want to ask is: if u_0, u_1 almost reach equality in (1.5) (where h has required only to satisfy (1.4)), does it follow that imply u_0 and u_1 are close to each other, if possible in some quantitative way? In this paper we consider the "quantitative" stability for supports sets of functions, we mean to estimate the distances of support sets Ω_0, Ω_1 of u_0, u_1 in terms of the BBL deficit.

It is interesting to remark that Balogh and Kristály [2] provided an estimate for the BBL deficit in terms of the gap-function, which also supplied a new approach to Dubuc's characterization, see [21]. However it does not supply a very direct way to estimate the distance of functions u_0, u_1 in terms of the BBL deficit, since the gap-function defined in [2, Lemma 2.1] is a bit deep. It is natural to ask a direct way to estimate the distances of functions, especially the distances of their support sets, by using the BBL deficit.

The main step to solve the problem is to estimate the BM deficit of Ω_0, Ω_1 in terms of BBL deficit (and thus the final result follows by combining this estimation with the stability of the classical Brunn-Minkowski inequality). Recently, Ghilli and Salani [32] studied a stability result of the BBL inequality. It is shown in [32, Theorem 4.1] that in the same assumption of Theorem 1.1 and if u_i is p-concave function, $p > 0$, and for some (small enough) $\varepsilon > 0$ and

$$\int_{\Omega_\lambda} h(x)dx \leq \mathcal{M}_{\frac{p}{np+1}}\left(I_0, I_1, \lambda\right) + \varepsilon,$$

then it holds that $$|\Omega_\lambda| \le \mathcal{M}_{\frac{1}{n}}\left(|\Omega_0|, |\Omega_1|, \lambda\right)\left[1 + \eta\varepsilon^{\frac{p}{p+1}}\right],$$

where the constant η dependents on n, λ, p, I_0 and I_1, more precisely,

$$\eta \le 2(n + \mathcal{M}_{\frac{p}{np+1}}\left(I_0, I_1, \lambda\right)^{-1}).$$

The first aim of this paper is to contribute some further valid estimate toward the above mentioned main step to the quantitative question. Using the notion of deficits, the first main result of this paper explicitly provides an estimation of the constant depending only on n, but not depending on p and the integrals I_0, I_1.

Theorem 1.2. *Assume that* $0 < \lambda < 1$*,* $p > 0$*, and* $u_0,\ u_1$ *are nonnegative bounded and* p*-concave functions in* $\mathbb{R}^n$ *with convex compact supports* $\Omega_0,\ \Omega_1$ *respectively. If* $\delta_\lambda(u_0, u_1, p) > 0$ *is small enough, then*

$$\delta_\lambda(\Omega_0, \Omega_1) \le C\delta_\lambda(u_0, u_1, p)^{\frac{p}{p+1}}, \tag{1.6}$$

where $C \le 2(1 + n)$*.*

Note that the result will be proved as an application of Theorem 3.1, see Section 3. The constant C in Theorem 1.2 is only dependent on the dimension n. We remark that Theorem 3.1 is a carefully modification of the result in [32, Theorem 4.1]. We will also remark after Theorem 3.1 that our result in Theorem 3.1 is stronger than [32, Theorem 4.1] in the case when $\mathcal{M} := \mathcal{M}_{\frac{p}{np+1}}\left(I_0, I_1, \lambda\right) > 1$. By the definition of $\mathcal{M}$, an example for this case is the nonnegative functions u_0, u_1 supported on Ω_0, Ω_1 and satisfying

$$I_i = \int_{\mathbb{R}^n} u_i dx > 1, \quad \text{for} \quad i = 0, 1.$$

To state the stability of the BBL inequality, we need to define what "close" means. In this paper, we use the concepts of the normalized Hausdorff distance and the relative asymmetry of two bodies.

The Hausdorff distance $H(K, L)$ between two sets $K, L \subseteq \mathbb{R}^n$ is defined by

$$\begin{aligned} H(K, L) &:= \max\left\{|h_K(\theta) - h_L(\theta)| : \theta \in S^{n-1}\right\} \\ &= \inf\left\{r \ge 0 : K \subseteq L + r\overline{B}_2^n, K \subseteq L + r\overline{B}_2^n\right\}, \end{aligned}$$

where $h_M(x)$ denotes the support function of M (see Section 2.2), and B_2^n is the Euclidean unit ball in $\mathbb{R}^n$. And the normalized Hausdorff distance $\bar{H}(K, L)$ between two sets $K, L \subseteq \mathbb{R}^n$ is defined by

$$\bar{H}(K, L) := H(\phi_0 K, \phi_1 L), \tag{1.7}$$

where ϕ_0, ϕ_1 are two homotheties (i.e. a composition of a translation with a dilation) such that $|\phi_0 K| = |\phi_1 L| = 1$ and the centroids of $\phi_0 K$ and $\phi_1 L$ coincide.

In Section 4, we use Theorem 1.2 to establish the following result.

Theorem 1.3. *Assume that* $0 < \lambda < 1$*,* $p > 0$*, and* $u_0,\ u_1$ *are nonnegative bounded and* p*-concave functions in* $\mathbb{R}^n$ *with convex compact supports* $\Omega_0,\ \Omega_1$ *respectively. If* $\delta_\lambda(u_0, u_1, p) > 0$ *is small enough, then*

$$\bar{H}\left(\Omega_0, \Omega_1\right) \le \alpha\delta_\lambda(u_0, u_1, p)^{\frac{p}{(n+1)(p+1)}}, \tag{1.8}$$

where α *is a constant depending only on* n, λ*, and diameters and measures of* Ω_0 *and* Ω_1*.*

We next define the relative asymmetry by using the symmetric difference of two bodies with the same volumes. Let K and L be two convex bodies in $\mathbb{R}^n$. The relative asymmetry $\mathcal{A}(K,L)$ of K and L is defined as

$$\mathcal{A}(K,L) := \inf_{x\in\mathbb{R}^n} \left\{ \frac{V\left(K\Delta(\lambda L + x)\right)}{V(K)} : \lambda = \left(\frac{V(K)}{V(L)}\right)^{1/n} \right\}, \tag{1.9}$$

where $K\Delta L = (K \setminus L) \cup (L \setminus K)$ is the symmetric difference.

Using the concept of relative asymmetry, we have the following stability result of BBL inequality.

Theorem 1.4. *Assume that $0 < \lambda < 1$, $p > 0$, and u_0, u_1 are nonnegative bounded and p-concave functions in $\mathbb{R}^n$ with convex compact supports Ω_0, Ω_1 respectively. If $\delta_\lambda(u_0, u_1, p) > 0$ is small enough, then*

$$\mathcal{A}(\Omega_0,\Omega_1)^2 \leq \beta\delta_\lambda(u_0,u_1,p)^{\frac{p}{p+1}}, \tag{1.10}$$

where β is a constant depending only on n, λ and the measure of Ω_0 and Ω_1.

Remark 1.5. Note that the constant α in Theorem 1.3 and β in Theorem 1.4 are not dependent on p and the integrals I_0, I_1, which improves the relevant results in [32]. We also can provide the explicit estimates for these constants. In fact we have (1.8) holds with

$$\alpha = (2(1+n))^{\frac{1}{n+1}} \left[\gamma_n \left(\frac{M}{m}\frac{1}{\sqrt{\lambda(1-\lambda)}} + 2\right)\tilde{d}\right], \tag{1.11}$$

and (1.10) holds with

$$\beta = 2(1+n)\frac{\left(181n^7\right)^2}{\left(2-2^{\frac{n-1}{n}}\right)^3}\frac{\Lambda M}{m}, \tag{1.12}$$

which follows from the estimate (4.1), and can be improved to

$$\beta = cn^{11+o(1)}\frac{\Lambda M}{m}, \tag{1.13}$$

by using the refinement estimate (4.2). Here, c is a absolute constant, and

$$\Lambda = \max\{\lambda/(1-\lambda), (1-\lambda)/\lambda\},$$

$$d_i = d\left(\Omega_i\right) = \text{ the diameter of } \Omega_i, \quad \nu_i = \left|\Omega_i\right|^{1/n}, \text{ for } i = 0, 1,$$

$$\tilde{d} = \max\left\{\frac{d_0}{\nu_0}, \frac{d_1}{\nu_1}\right\}, \quad M = \max\left\{\nu_0, \nu_1\right\}, \quad m = \min\left\{\nu_0, \nu_1\right\},$$

and

$$\gamma_n = \left(1 + \frac{1}{3\cdot 2^{13}}\right) 3^{\frac{n-1}{n}} 2^{\frac{n+2}{n+1}} n < 6.00025n. \tag{1.14}$$

Remark 1.6. Throughout the paper we consider only the compactly supported functions, which is the only meaningful case of the BBL inequality for $p > 0$. Indeed when at least one of u_0 and u_1 has infinite support, the BBL inequality holds trivially, since if $|\Omega_0| = +\infty$ and there exists x_1 such that $u_1(x_1) = \varepsilon > 0$ (u_1 not identically vanishing), then $h(x) \geq \lambda^{\frac{1}{p}}\varepsilon$ for $x \in (1-\lambda)\Omega_0 + \lambda x_1$.

In our results, the p-concavity assumption is essential and necessary, since we will estimate the relevant deficit in terms of the distance of support sets. Without the assumption, the support sets Ω_0 and Ω_1 can be wildly modified without affecting the integrals I_0 and I_1, thus (1.6), (1.8) and (1.10) would be false. □

Using the results we obtained in Theorems 1.2, 1.3 and 1.4, we derive quantitative versions of the Brunn-Minkowski and Urysohn inequality for p-torsional rigidity.

Let Ω_0 and Ω_1 be open bounded convex sets in $\mathbb{R}^n$, $\lambda \in (0,1)$, $p > 1$, and let $\Omega_\lambda = (1-\lambda)\Omega_0 + \lambda\Omega_1$. The deficit of p-torsional rigidity of Ω_0 and Ω_1 is defined by

$$\delta_\lambda(\Omega_0, \Omega_1, \tau_p) := \frac{\tau_p(\Omega_\lambda)^{\frac{1}{p-1}}}{\left(\mathcal{M}_{\frac{1}{p+n(p-1)}}\left(\tau_p\left(\Omega_0\right), \tau_p\left(\Omega_1\right), \lambda\right)\right)^{\frac{1}{p-1}}} - 1.$$

We establish the quantitative version of the inequality (1.3) as follows.

Theorem 1.7. *Let Ω_0 and Ω_1 be open bounded convex sets in $\mathbb{R}^n$, $\lambda \in (0,1)$, $p > 1$, and $\Omega_\lambda = (1-\lambda)\Omega_0 + \lambda\Omega_1$. Then we get*

$$\bar{H}\left(\Omega_0, \Omega_1\right) \leq \alpha\delta_\lambda(\Omega_0, \Omega_1, \tau_p)^{\frac{p-1}{(n+1)(2p-1)}},$$

and

$$\mathcal{A}(\Omega_0, \Omega_1)^2 \leq \beta\delta_\lambda(\Omega_0, \Omega_1, \tau_p)^{\frac{p-1}{2p-1}}.$$

An interesting consequence of (1.3) is the Urysohn's isoperimetric inequality for p-torsional rigidity, which states that in the class of convex sets with given mean width, the p-torsional rigidity attains the maximum value on balls. Namely,

$$\tau_p(\Omega) \leq \tau_p(\Omega^\sharp), \tag{1.15}$$

where $\Omega^\sharp$ represents the ball with the same mean-width of Ω.

For an open bounded convex set Ω, and $p > 1$, we define the Urysohn deficit for τ_p by

$$\delta_w(\Omega, \tau_p) = \left(\frac{\tau_p\left(\Omega^\sharp\right)}{\tau_p(\Omega)}\right)^{\frac{1}{p-1}} - 1. \tag{1.16}$$

The following results present two quantitative versions of equation (1.15) in terms of the Hausdorff distance and the relative asymmetry. These results can be interpreted as a consequence of Theorems 1.3, 1.4, and 1.7.

Theorem 1.8. *Let Ω be an open bounded convex set in $\mathbb{R}^n$ and $p > 1$. If $\delta_w(\Omega, \tau_p)$ is small enough, then it holds*

$$H \leq \alpha'\delta_w(\Omega, \tau_p)^{\frac{p-1}{(n+1)(2p-1)}},$$

and

$$\mathcal{A}^2 \leq \beta'\delta_w(\Omega, \tau_p)^{\frac{p-1}{2p-1}},$$

where $H = H(\Omega, \Omega^\sharp)$ and $\mathcal{A} = \max\{\mathcal{A}(\Omega, \Omega_\rho) : \rho$ any rotation in $\mathbb{R}^n\}$.

Here α' is a constant depending only on n and the diameter $d(\Omega)$ of Ω, and β' is a constant depending only on n.

Note that the constants α' and β' are independent on τ_p. More precisely, we have

$$\alpha' = (2(1+n))^{\frac{1}{n+1}} 4\gamma_n d(\Omega) \quad \text{and} \quad \beta' = \frac{2(1+n)\left(181n^7\right)^2}{\left(2-2^{\frac{n-1}{n}}\right)^3}.$$

In the case of $p = 2$, we have that τ_p is the classical torsional rigidity τ, and the related constants α, β in Theorem 1.7 and α', β' in Theorem 1.8 are independent on $\tau(\Omega_i)$ and $\tau(\Omega)$, respectively (and thus strengthen the previous results about torsional rigidity).

The paper is organized as follows. In Section 2, we introduce some necessary notation and recall some classical and useful results. In Section 3, we prove Theorem 1.2 which will be the key part in the proof of Theorems 1.3 and 1.4, proved in Section 4. In Section 5, we give applications with p-Laplace operator. Indeed we give the proof of Theorems 1.7 and 1.8.

2. Notation and preliminaries

2.1. Means of nonnegative numbers

The definition of p-mean of two nonnegative numbers has been given in the Introduction. Here we just recall some useful facts, see [35] for more details. For all p and λ, it is clear that $\mathcal{M}_p(a, b, \lambda)$ is non-decreasing with respect to a and b. And using Jensen's inequality, $\mathcal{M}_p(a, b, \lambda)$ is non-decreasing with respect to p, that is,

$$\mathcal{M}_p(a, b, \lambda) \leq \mathcal{M}_q(a, b, \lambda), \quad \text{if } p \leq q.$$

We recall the following technical lemma.

Lemma 2.1. *Let $0 < \lambda < 1$ and a, b, c, d be nonnegative numbers. If $p + q > 0$, then*

$$\mathcal{M}_p(a, b, \lambda)\mathcal{M}_q(c, d, \lambda) \geq \mathcal{M}_s(ac, bd, \lambda),$$

where $s = \frac{pq}{p+q}$. Moreover, the result holds trivially with $s = 0$ if $p = q = 0$.

2.2. Convex bodies

In this paper, we use K (possibly with subscripts) to denote a convex body in $\mathbb{R}^n$, and Ω (possibly with subscripts) the interior of K. Denoted by $\mathcal{K}_0^n$ the class of convex bodies in $\mathbb{R}^n$.

For a convex set $K \subset \mathbb{R}^n$, its support function $h(K, \cdot)$ is defined as

$$h(K, x) = \sup\{\langle x, y\rangle : y \in K\}, \quad x \in \mathbb{R}^n.$$

If $K \in \mathcal{K}_0^n$, the latter supremum attains in fact a maximum and we write

$$h(K, x) = \max\{\langle x, y\rangle : y \in K\}, \quad x \in \mathbb{R}^n.$$

For convenience, we often use $h_K(\cdot) := h(K, \cdot)$ for the support function of K. Note that the support function $h_K(\cdot)$ is positively homogeneous and subadditive, that is, for $x, y \in \mathbb{R}^n$,

$$h_K(\alpha x) = \alpha h_K(x), \quad \alpha \geq 0,$$

and

$$h_K(x + y) \leq h_K(x) + h_K(y).$$

Moreover, for $K_1, K_2 \in \mathcal{K}_0^n$, we have for $\alpha, \beta \geq 0$,

$$h_{\alpha K_1 + \beta K_2}(\cdot) = \alpha h_{K_1}(\cdot) + \beta h_{K_2}(\cdot).$$

The width of $K \in \mathcal{K}_0^n$ for the direction θ is the distance between the two support hyperplanes of K orthogonal to θ, that is,

$$w(K, \theta) = h_K(\theta) + h_K(-\theta), \quad \theta \in S^{n-1},$$

where S^{n-1} is the sphere of the n-dimensional unit ball B_2^n. The diameter of K is the maximum of the width function, which is

$$d(K) = \max\{w(K, \theta) : \theta \in S^{n-1}\}.$$

We use $w(K)$ to denote the mean width of K, which is given by

$$w(K) = \frac{1}{n\omega_n} \int_{S^{n-1}} w(K, \theta) d\theta = \frac{2}{n\omega_n} \int_{S^{n-1}} h(K, \theta) d\theta.$$

Clearly, $w(B_2^n) = 2$. Urysohn's inequality gives an isoperimetric type inequality for $w(K)$, which states

$$\frac{|K|}{w(K)^n} \leq \frac{|B_2^n|}{2^n}. \tag{2.1}$$

The classical Brunn-Minkowski inequality (1.1) can be written in the following equivalent multiplicative form

$$|K_\lambda| \geq |K_0|^{1-\lambda} |K_1|^\lambda.$$

It is well known that the Brunn-Minkowski inequality is closely related to other important inequalities, including the isoperimetric inequality, the Sobolev inequality, and the Prékopa-Leindler inequality [6, 7, 29, 30, 47]. It has been applied to various other problems in convexity, such as convex functions, convex geometry, probability theory on convex sets, and computational complexity, see [1, 41, 47, 53, 54, 55, 56]. The Brunn-Minkowski inequality and its functional version have been generalized to many different spaces, see [10, 15, 17, 18, 45, 58, 59, 60] for example.

The Steiner point $s(K)$ of a convex body $K \in \mathcal{K}_0^n$ is defined by

$$s(K) = \frac{1}{\omega_n} \int_{S^{n-1}} \theta h(K, \theta) d\mathcal{H}^{n-1}(\theta), \tag{2.2}$$

where ω_n is the volume of the n-dimensional unit ball, $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure (for more [49]). The following helpful properties of the mapping $s : \mathcal{K}_0^n \to \mathbb{R}^n$ will be used later.

(i) If $s(K) = 0$ (the origin), then $s(\alpha K) = 0$ for all α.

(ii) $s(K_1 + K_2) = s(K_1) + s(K_2)$ for $K_1, K_2 \in \mathcal{K}_0^n$.

(iii) $s(AK) = As(K)$ for $K \in \mathcal{K}_0^n$ and each similarity $A : \mathbb{R}^n \to \mathbb{R}^n$.

(iv) $s(K)$ is a continuous function of K.

Here, the similarity A is a composition of a dilation with a rigid motion, where a rigid motion can be understood as a combination of translation, rotation, and reflection operations, refer to [31] for more details.

2.3. Power concave function and (p, λ)-convolution of nonnegative functions

Let Ω be a convex set in $\mathbb{R}^n$ and $p \in [-\infty, +\infty]$. A nonnegative function u defined in Ω is called *p-concave* if

$$u((1-\lambda)x + \lambda y) \geq \mathcal{M}_p(u(x), u(y), \lambda),$$

for every $x, y \in \Omega$ and $\lambda \in (0,1)$. Equivalently, u is p-concave if it has convex support Ω and satisfying that u^p is concave (or convex) in Ω for $p > 0$ (or $p < 0$); $\log u$ is concave in Ω for $p = 0$; u is a nonnegative constant in Ω, for $p = +\infty$; or u is quasi-concave for $p = -\infty$ i.e. all its superlevel sets are convex.

For every $q \leq p$, we have u is q-concave if u is p-concave. And the concavity properties of a function u can be expressed in terms of its level sets as follows.

Proposition 2.2. *A nonnegative function u is p-concave in a convex domain Ω for some $p \in [-\infty, +\infty]$ if and only if*

$$\{x \in \Omega : u(x) \geq \mathcal{M}_p(t_0, t_1, \lambda)\} \supseteq (1-\lambda)\{x \in \Omega : u(x) \geq t_0\} + \lambda \{x \in \Omega : u(x) \geq t_1\},$$

for every $t_0, t_1 \geq 0$ and every $\lambda \in (0,1)$.

For a nonnegative measurable function u on Ω, its distribution function μ is given by

$$\mu(t) = |\{x \in \Omega : u \geq t\}|.$$

The Brunn-Minkowski inequality together with Proposition 2.2 implies that

Proposition 2.3. *If u is p-concave for some $p \neq 0$, then*

$$\mu(t^{1/p})^{1/n} \quad \textit{is concave in } t.$$

Let $p \in (-\infty, +\infty)$ and u_0, u_1 be nonnegative functions with compact convex support Ω_0 and Ω_0. The (p, λ)-convolution of u_0 and u_1 is the function defined as

$$u_{p,\lambda}(x) = \sup\{\mathcal{M}_p(u_0(x_0), u_1(x_1), \lambda) : x = (1-\lambda)x_0 + \lambda x_1, x_i \in \bar{\Omega}_i, i = 0, 1\}.$$

From above the definition of $u_{p,\lambda}$ and the monotonicity of p-mean with respect to p, we obtain

$$u_{p,\lambda} \leq u_{q,\lambda}, \quad \text{for } -\infty \leq p \leq q \leq +\infty.$$

Note that the support of $u_{p,\lambda}$ is $\Omega_\lambda = (1-\lambda)\Omega_0 + \lambda\Omega_1$ and the continuity of u_0 and u_1 yields the continuity of $u_{p,\lambda}$. See [43, 50] for the details.

3. Proof of Theorem 1.2

The goal of this section is to prove Theorem 3.1, which is a modification of the result given by Ghilli and Salani in [32, Theorem 4.1].

Theorem 3.1. *In the same assumptions and notation of Theorem* 1.2*, if for some (small enough)* $\varepsilon > 0$*,*

$$\int_{\Omega_\lambda} h(x)\ dx \leq (1+\varepsilon)\mathcal{M}_{\frac{p}{np+1}}\left(\int_{\Omega_0} u_0(x)\ dx, \int_{\Omega_1} u_1(x)\ dx, \lambda\right), \tag{3.1}$$

then

$$|\Omega_\lambda| \leq \left[1 + 2(1+n)\varepsilon^{\frac{p}{p+1}}\right] \mathcal{M}_{\frac{1}{n}}\left(|\Omega_0|, |\Omega_1|, \lambda\right). \tag{3.2}$$

Remark that in the case that $\int_{\Omega_0} u_0(x)\ dx = \int_{\Omega_1} u_1(x)\ dx = 1$, Theorem 3.1 coincides with the result given in [32, Theorem 4.1] by Ghilli and Salani. And in the case of

$$\mathcal{M} := \mathcal{M}_{\frac{p}{np+1}}\left(\int_{\Omega_0} u_0(x)\ dx, \int_{\Omega_1} u_1(x)\ dx, \lambda\right) > 1,$$

Theorem 3.1 is stronger than [32, Theorem 4.1]. In fact, replacing ε by $\varepsilon\mathcal{M}$, it follows from [32, Theorem 4.1] that the same assumptions hold with Theorem 3.1, and the result is

$$|\Omega_\lambda| \leq \left[1 + 2(\mathcal{M}^{-1}+n)\mathcal{M}^{\frac{p}{p+1}}\varepsilon^{\frac{p}{p+1}}\right] \mathcal{M}_{\frac{1}{n}}\left(|\Omega_0|, |\Omega_1|, \lambda\right),$$

which is slightly weaker than (3.2), since

$$\left[1 + 2(\mathcal{M}^{-1}+n)\mathcal{M}^{\frac{p}{p+1}}\varepsilon^{\frac{p}{p+1}}\right] \geq \left[1 + 2(\mathcal{M}^{-1}+n)\mathcal{M}\varepsilon^{\frac{p}{p+1}}\right] \geq \left[1 + 2(1+n)\varepsilon^{\frac{p}{p+1}}\right],$$

by applying $\mathcal{M} > 1$.

Proof. For simplicity, we recall some notation and concepts. For the given bounded functions $u_i, i = 0, 1$, their distribution functions are given by

$$\mu_i(s) = \left|\{u_i \geq s\}\right|,\ s \in [0, L_i],$$

where $L_i = \max\limits_{\Omega_i} u_i$. And for $\lambda \in (0,1)$ and $p > 0$, we set $L_\lambda = \max\limits_{\Omega_\lambda} u_{p,\lambda}$, and

$$\mu_\lambda(s) = \left|\{u_{p,\lambda} \geq s\}\right|,$$

for $s \in [0, L_\lambda]$. From the p-concavity of u_0, u_1 and $u_{p,\lambda}$, we have the distribution functions μ_0, μ_1 and μ_λ are continuous.

The Brunn-Minkowski inequality yields $|\Omega_\lambda| \geq \mathcal{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda)$.

In the case of equality, the final result holds trivially. Then we may assume

$$|\Omega_\lambda| = (1+\tau)\mathcal{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda), \tag{3.3}$$

for some $\tau > 0$. Now we want to find a function to estimate τ depending on ε, that is, to search for a function f such that $\lim\limits_{\varepsilon\to 0} f(\varepsilon) = 0$ and $\tau < f(\varepsilon)$.

From the definition of $u_{p,\lambda}$, we obtain

$$\{u_{p,\lambda} \geq \mathcal{M}_p (s_0, s_1, \lambda)\} \supseteq (1-\lambda) \{u_0 \geq s_0\} + \lambda \{u_1 \geq s_1\}$$

for $s_0 \in [0, L_0], s_1 \in [0, L_1]$. Thus, the Brunn-Minkowski inequality implies

$$\mu_\lambda(\mathcal{M}_p (s_0, s_1, \lambda)) \geq \mathcal{M}_{\frac{1}{n}} (\mu_0(s_0), \mu_1(s_1), \lambda) . \tag{3.4}$$

For $i = 0, 1$, we set functions $s_i : [0,1] \to [0, L_i]$ such that

$$\int_0^{s_i(t)} \mu_i(s)ds = tI_i, \quad \text{for } t \in [0,1], \tag{3.5}$$

where $I_i = \int_{\mathbb{R}^n} u_i(x)dx$. Note that s_i is strictly increasing, then it is differentiable almost everywhere. Then by differentiating (3.5), we have, for a.e. $t \in [0,1]$,

$$\mu_i(s_i(t))s_i'(t) = I_i. \tag{3.6}$$

Since $s_i(t)$ is continuous, the above formula claims the derivative s_i' coincides with a continuous function almost everywhere in $[0,1]$, therefore s_i' is continuous in the whole interval $[0,1]$. We now define the map $s_\lambda : [0,1] \to [0, L_\lambda]$

$$s_\lambda(t) = \mathcal{M}_p(s_0(t), s_1(t), \lambda), \quad t \in [0,1]. \tag{3.7}$$

Thanks to (3.4), we get

$$\mu_\lambda(s_\lambda(t)) \geq \mathcal{M}_{\frac{1}{n}}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda), \quad t \in [0,1]. \tag{3.8}$$

By (3.7) and $s_i \in C^1([0,1])$, $i = 0, 1$, we have $s_\lambda \in C^1([0,1])$ and

$$s_\lambda'(t) = \left((1-\lambda)s_0'(t)s_0(t)^{p-1} + \lambda s_1'(t)s_1(t)^{p-1}\right) s_\lambda(t)^{1-p}, \quad t \in [0,1]. \tag{3.9}$$

Now applying Lemma 2.1 with $p = \frac{1}{n}$ and $q = 1$, we obtain for $s_0(t) \in [0, L_0]$, $s_1(t) \in [0, L_1]$,

$$\begin{aligned} &\mathcal{M}_{\frac{1}{n}} (\mu_0 (s_0(t)), \mu_1 (s_1(t)), \lambda) \mathcal{M}_1 \left(s_0'(t)s_0(t)^{p-1}, s_1'(t)s_1(t)^{p-1}, \lambda\right) \\ &\geq \mathcal{M}_{\frac{1}{n+1}} \left(\mu_0 (s_0(t)) s_0(t)^{p-1}s_0'(t), \mu_1 (s_1(t)) s_1(t)^{p-1}s_1'(t), \lambda\right). \end{aligned}$$

Then the identity

$$s_\lambda(t)^{1-p} = \mathcal{M}_p (s_0(t), s_1(t), \lambda)^{1-p} = \mathcal{M}_{\frac{p}{1-p}} \left(s_0(t)^{1-p}, s_1(t)^{1-p}, \lambda\right),$$

together with Lemma 2.1 and (3.6), imply

$$\begin{aligned} &\mathcal{M}_{\frac{1}{n}} (\mu_0 (s_0(t)), \mu_1 (s_1(t)), \lambda) \mathcal{M}_1 \left(s_0'(t)s_0(t)^{p-1}, s_1'(t)s_1(t)^{p-1}, \lambda\right) s_\lambda(t)^{1-p} \\ &\geq \mathcal{M}_{\frac{1}{n+1}} \left(\mu_0(s_0(t))s_0(t)^{p-1}s_0'(t), \mu_1(s_1(t))s_1(t)^{p-1}s_1'(t), \lambda\right) \mathcal{M}_{\frac{p}{1-p}} \left(s_0(t)^{1-p}, s_1(t)^{1-p}, \lambda\right) \\ &\geq \mathcal{M}_{\frac{p}{np+1}} (\mu_0 (s_0(t)) s_0'(t), \mu_1 (s_1(t)) s_1'(t), \lambda) \\ &= \mathcal{M}_{\frac{p}{np+1}} (I_0, I_1, \lambda) . \end{aligned} \tag{3.10}$$

Next, for given any $\delta > 0$, set

$$F_\delta = \left\{ t \in [0,1] : \mu_\lambda(s_\lambda(t)) > (1+\delta)\mathcal{M}_{\frac{1}{n}}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda) \right\}, \tag{3.11}$$

and
$$\Gamma_\delta = \{s_\lambda(t) : t \in F_\delta\}. \tag{3.12}$$

Note that the set Γ_δ is measurable, by Proposition 2.3 and the monotonicity and regularity of the s_i. Then we have

$$\begin{aligned}
I_\lambda &= \int_0^{L_\lambda} \mu_\lambda(s)ds = \int_0^1 \mu_\lambda\left(s_\lambda(t)\right) s_\lambda'(t)dt \\
&= \int_{F_\delta} \mu_\lambda\left(s_\lambda(t)\right) s_\lambda'(t)dt + \int_{[0,1]\setminus F_\delta} \mu_\lambda\left(s_\lambda(t)\right) s_\lambda'(t)dt \\
&\geq \int_{F_\delta} (1+\delta)\mathcal{M}_{\frac{1}{n}}\left(\mu_0\left(s_0(t)\right), \mu_1\left(s_1(t)\right), \lambda\right) s_\lambda'(t)dt + \int_{[0,1]\setminus F_\delta} \mu_\lambda\left(s_\lambda(t)\right) s_\lambda'(t)dt \\
&= \int_0^1 \mathcal{M}_{\frac{1}{n}}\left(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda\right) s_\lambda'(t)dt + \delta\int_{F_\delta} \mathcal{M}_{\frac{1}{n}}\left(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda\right) s_\lambda'(t)dt.
\end{aligned}$$

From (3.9) and (3.10), it follows that

$$\int_0^1 \mathcal{M}_{\frac{1}{n}}\left(\mu_0\left(s_0(t)\right), \mu_1\left(s_1(t)\right), \lambda\right) s_\lambda'(t)dt \geq \mathcal{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda),$$

and
$$\int_{F_\delta} \mathcal{M}_{\frac{1}{n}}\left(\mu_0\left(s_0\right), \mu_1\left(s_1\right), \lambda\right) s_\lambda'dt \geq \mathcal{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda)|F_\delta|.$$

Therefore,
$$I_\lambda \geq (1+\delta|F_\delta|)\mathcal{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda).$$

By comparison with (3.1), we have

$$(1+\varepsilon)\mathcal{M}_{\frac{p}{np+1}}\left(I_0, I_1, \lambda\right) \geq I_\lambda \geq (1+\delta|F_\delta|)\mathcal{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda),$$

which implies that $|F_\delta| \leq \frac{\varepsilon}{\delta}$. We now take $\delta = \varepsilon^\alpha$ for some $0 < \alpha < 1$. Thus

$$|F_{\varepsilon^\alpha}| \leq \varepsilon^{1-\alpha}. \tag{3.13}$$

Letting ε small enough, (3.13) implies that the set $[0,1] \setminus F_{\varepsilon^\alpha}$ is big enough. Then there exists $\bar{t} \notin F_{\varepsilon^\alpha}$ such that $s_\lambda(\bar{t}) \leq \eta$, for any small η. Therefore, there exists $\bar{t} \in [0,1]$ such that

$$s_\lambda(\bar{t}) \leq \varepsilon^{1-\alpha}L_\lambda, \tag{3.14}$$

and
$$\mu_\lambda(s_\lambda(\bar{t})) \leq (1+\varepsilon^\alpha)\mathcal{M}_{\frac{1}{n}}(\mu_0(s_0(\bar{t})), \mu_1(s_1(\bar{t})), \lambda). \tag{3.15}$$

Let
$$\xi = \left(\frac{s_\lambda(\bar{t})}{L_\lambda}\right)^p. \tag{3.16}$$

From (3.14) we have
$$\xi \leq \varepsilon^{(1-\alpha)p}. \tag{3.17}$$

By the assumption that u_0 and u_1 are p-concave, we have $u_{p,\lambda}$ is p-concave. Hence,

$$\{z : u_{p,\lambda}(z) \geq \mathcal{M}_p(0, L_\lambda, \xi)\} \supseteq (1-\xi)\{x : u_{p,\lambda}(x) \geq 0\} + \xi\{y : u_{p,\lambda}(y) \geq L_\lambda\}.$$

Clearly, $s_\lambda(\bar{t}) = \mathcal{M}_p(0, L_\lambda, \xi)$. Then the above inequality can be written as

$$\{u_{p,\lambda} \geq s_\lambda(\bar{t})\} \supseteq (1-\xi)\Omega_\lambda + \xi\{u_{p,\lambda} \geq L_\lambda\}.$$

By the Brunn-Minkowski inequality we get

$$|\{u_{p,\lambda} \geq s_\lambda(\bar{t})\}| \geq \left((1-\xi)\,|\Omega_\lambda|^{\frac{1}{n}} + \xi\,|\{u_\lambda \geq L_\lambda\}|^{\frac{1}{n}}\right)^n.$$

Since the set $\{u_{p,\lambda} \geq L_\lambda\} = (1-\lambda)\{u_0 \geq L_0\} + \lambda\{u_1 \geq L_1\}$ is a single point set under the assumption that the involved functions are strictly p-concave, we then have $\{u_{p,\lambda} \geq L_\lambda\}$ has zero measure. Then using (3.3), we get

$$|\{u_{p,\lambda} \geq s_\lambda(\bar{t})\}| \geq (1-\xi)^n\,|\Omega_\lambda| \geq (1-\xi)^n(1+\tau)\mathcal{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda).$$

Then, by (3.15) we have

$$(1+\varepsilon^\alpha)\mathcal{M}_{\frac{1}{n}}(\mu_0(s_0(\bar{t})), \mu_1(s_1(\bar{t})), \lambda) \geq (1-\xi)^n(1+\tau)\mathcal{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda).$$

Since $\mu_0(s_0(\bar{t})) \leq |\Omega_0|$ and $\mu_1(s_1(\bar{t})) \leq |\Omega_1|$, the monotonicity of the mean $\mathcal{M}_{\frac{1}{n}}$ implies

$$(1+\varepsilon^\alpha) \geq (1-\xi)^n(1+\tau).$$

Hence,

$$\tau \leq \frac{1+\varepsilon^\alpha}{(1-\xi)^n} - 1.$$

Since $(1-\xi)^n \geq 1 - n\xi \geq \frac{1}{2}$ for $0 \leq \xi \leq \frac{1}{2n}$, we get

$$\tau \leq \frac{1+\varepsilon^\alpha - (1-\xi)^n}{(1-\xi)^n} \leq 2(\varepsilon^\alpha + n\xi).$$

Taking $\alpha = \frac{p}{p+1}$ and ε small enough (more precisely, $\varepsilon \leq (\frac{1}{2n})^{\frac{p+1}{p}}$), the above inequality together with (3.17) implies that

$$\tau \leq 2(\varepsilon^\alpha + n\xi) \leq 2\varepsilon^{\frac{p}{p+1}}(1+n). \tag{3.18}$$

Therefore, combining it with (3.3), we have

$$|\Omega_\lambda| \leq \left(1 + 2\,(1+n)\,\varepsilon^{\frac{p+1}{p}}\right)\mathcal{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda).$$

The proof is complete. □

Proof of Theorem 1.2. We choose

$$\varepsilon = \delta_\lambda(u_0, u_1, p) = \frac{\int_{\Omega_\lambda} u_{p,\lambda}(x)dx}{\mathcal{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda)} - 1 > 0$$

small enough. Then the condition (3.1) holds in Theorem 3.1. Thus we get

$$|\Omega_\lambda| \leq \left[1 + 2(1+n)\varepsilon^{\frac{p}{p+1}}\right]\mathcal{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda).$$

Then the final result follows from the definition of $\delta_\lambda(\Omega_0, \Omega_1)$ and the choice of ε. □

4. Proof of Theorem 1.3 and Theorem 1.4

For convex sets K_0 and K_1, the first stability forms of the Brunn-Minkowski inequality were due to Minkowski himself (see Groemer [34]). In terms of the Hausdorff distance of convex sets K_0 and K_1, Diskant [20] and Groemer [33] provided close to optimal stability versions (see Groemer [34]). In a cornerstone work, Figalli and Jerison [23] obtained the first quantitative stability for the Brunn-Minkowski inequality (see also [22, 24]). The recent sharp stability result for the Brunn-Minkowski inequality is due to Figalli, van Hintum, and Tiba [27, 28].

We now recall the following stability result in terms of the Hausdorff distance.

Theorem 4.1. [33] *Let $K_0, K_1 \in \mathcal{K}_0^n$, $n \geq 2$, $\lambda \in (0,1)$ and let $K_\lambda = (1-\lambda)K_0 + \lambda K_1$. Then*

$$|K_\lambda| \geq \mathcal{M}_{\frac{1}{n}}\left(|K_0|, |K_1|, \lambda\right)\left(1 + \omega \bar{H}\left(K_0, K_1\right)^{(n+1)}\right),$$

where $\bar{H}$ is the normalized Hausdorff distance defined in (1.7) *and:*

$$\omega = \left(\gamma_n \left(\frac{M}{m}\frac{1}{\sqrt{\lambda(1-\lambda)}} + 2\right)\tilde{d}\right)^{-(n+1)},$$

$\tilde{d} = \max\left\{\frac{d(K_0)}{\nu_0}, \frac{d(K_1)}{\nu_1}\right\}$, $d(\cdot)$ is the diameter, $\nu_i = |K_i|^{\frac{1}{n}}$, $M = \max\{\nu_0, \nu_1\}$, $m = \min\{\nu_0, \nu_1\}$, and γ_n given in (1.14).

Proof of Theorem 1.3. Since Theorem 1.2 holds in the same assumptions of Theorem 1.3, we have

$$\delta_\lambda(\Omega_0, \Omega_1) \leq 2(1+n)\delta_\lambda(u_0, u_1, p)^{\frac{p}{p+1}}.$$

Using the definition of $\delta_\lambda(\Omega_0, \Omega_1)$, it follows from Theorem 4.1 that

$$\omega \bar{H}\left(\Omega_0, \Omega_1\right)^{(n+1)} \leq \delta_\lambda(\Omega_0, \Omega_1).$$

Thus we get

$$\bar{H}\left(\Omega_0, \Omega_1\right)^{(n+1)} \leq 2(1+n)\omega^{-1}\delta_\lambda(u_0, u_1, p)^{\frac{p}{p+1}}.$$

The proof is complete. □

In terms of the relative asymmetry, and the optimal Brunn-Minkowski inequality is due to Figalli, Maggi, Pratelli [25, 26].

Theorem 4.2. [25, 26] *Let $K_0, K_1 \in \mathcal{K}_0^n, \lambda \in (0,1)$ and let $K_\lambda = (1-\lambda)K_0 + \lambda K_1$. Then*

$$|K_\lambda| \geq \mathcal{M}_{\frac{1}{n}}\left(|K_0|, |K_1|, \lambda\right)\left(1 + \frac{m}{\Lambda M}\left(\frac{\mathcal{A}(K_0, K_1)}{C_n}\right)^2\right),$$

where $\mathcal{A}\left(K_0, K_1\right)$ is defined in (1.9)*, m and M are defined as before, C_n is a constant depending only on n, and $\Lambda = \max\{\lambda/(1-\lambda), (1-\lambda)/\lambda\}$.*

Note that the exponent 2 of $\mathcal{A}(K_0, K_1)^2$ is optimal, see Figalli, Maggi, Pratelli [26]. In the same paper [26], the constant C_n has been estimated as

$$C_n \leq \frac{181 n^7}{\left(2 - 2^{\frac{n-1}{n}}\right)^{\frac{3}{2}}}. \tag{4.1}$$

This bound was improved to cn^7 by Segal [48], and to $cn^{5.5}$ by Kolesnikov, Milman [38, Theorem 12.12]. The best bound for C_n up to now is

$$cn^{5+o(1)}, \tag{4.2}$$

which follows by combining the general estimate of Kolesnikov, Milman [38, Theorem 12.2] with the bound $n^{o(1)}$ on the Cheeger constant of a convex body in isotropic position that follows from Chen's work [13]. We also note that Harutyunyan [36] conjectured that $C(n) = cn^2$ is the optimal order of the constant, and showed that it cannot be of smaller order, and that Segal [48] proved that Dar's conjecture would imply that we may choose $C(n) = cn^2$ for some absolute constant $c > 0$. Therefore C_n has polynomial growth in n as $n \to +\infty$.

Now we can prove Theorem 1.4.

Proof of Theorem 1.4. By Theorem 1.2, we have

$$\delta_\lambda(\Omega_0, \Omega_1) \leq 2(1+n)\delta_\lambda(u_0, u_1, p)^{\frac{p}{p+1}}.$$

Using the definition of $\delta_\lambda(\Omega_0, \Omega_1)$, it follows from Theorem 4.2 that

$$\frac{m}{\Lambda M}\left(\frac{\mathcal{A}(\Omega_0, \Omega_1)}{C_n}\right)^2 \leq \delta_\lambda(\Omega_0, \Omega_1).$$

Thus we get $$\frac{m}{\Lambda M}\left(\frac{\mathcal{A}(\Omega_0, \Omega_1)}{C_n}\right)^2 \leq 2(1+n)\delta_\lambda(u_0, u_1, p)^{\frac{p}{p+1}}.$$

The proof is complete. □

5. Proof of Theorems 1.7 and 1.8

In this section, we utilize Theorems 1.3 and 1.4 to prove the quantitative version of the Brunn-Minkowski and Urysohn inequalities for p-torsional rigidity.

Let us recall the definition of the p-torsional rigidity τ_p. Let Ω be a open convex set in $\mathbb{R}^n$ and $p > 1$. The p-torsional rigidity is defined by

$$\frac{1}{\tau_p(\Omega)} = \inf\left\{\frac{\int_\Omega |\nabla w(x)|^p dx}{\left[\int_\Omega |w(x)|dx\right]^p} : w \in W_0^{1,p}(\Omega), \int_\Omega |w(x)|dx > 0\right\}. \tag{5.1}$$

The p-torsional rigidity is positive homogeneous of degree $p + n(p-1)$, that is,

$$\tau_p(\lambda\Omega) = \lambda^{p+n(p-1)}\tau_p(\Omega), \text{ for every } \lambda > 0.$$

The above infimum admits a minimizer in $W_0^{1,p}(\Omega)$, which is a weak solution of the next boundary value problem

$$\begin{cases} \Delta_p u = -1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

where Δ_p is the p-Laplace operator. The p-Laplace equation is

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right),$$

where ∇u is the gradient of u and

$$|\nabla u|^{p-2} = \left(\left(\frac{\partial u}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial u}{\partial x_n}\right)^2\right)^{\frac{p-2}{2}}.$$

The solution u to (5.2) has some specific properties such as $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$ (refer to [51, 52]). See also [40] for Orlicz extension of torsional rigidity. Moreover, Sakaguchi proved the convexity of u in [44], which will be useful.

Theorem 5.1. [44] *If u is the solution to problem (5.2), then u is $\frac{p-1}{p}$-concave, i.e. the function* $v(x) = u(x)^{\frac{p-1}{p}}$ *is concave in Ω.*

Next we recall a comparison result for solutions of problem (5.2) in different domains (see [16]).

Theorem 5.2. [16] *Let Ω_0 and Ω_1 be convex bodies, $\lambda \in [0,1]$, $p > 1$ and $\Omega_\lambda = (1-\lambda)\Omega_0 + \lambda\Omega_1$. Let u_i be the solution of problem (5.2) in Ω_i, $i = 0, 1, \lambda$.*

Then $u_\lambda((1-\lambda)x + \lambda y)^{\frac{p-1}{p}} \geq (1-\lambda)u_0(x)^{\frac{p-1}{p}} + \lambda u_1(y)^{\frac{p-1}{p}}, \quad \forall x \in \Omega_0,\ y \in \Omega_1.$

Using the deficit of p-torsional rigidity, the inequality (1.3) is equivalent to

$$\delta_\lambda(\Omega_0, \Omega_1, \tau_p) \geq 0. \tag{5.3}$$

Note that Theorem 1.7 is the stability of (5.3), which will be proved by using Theorems 5.1 and 5.2. By the definition of the deficit of p-torsional rigidity, Theorem 1.7 can be rewritten as follows.

Theorem 5.3. *Let Ω_0 and Ω_1 be open bounded convex sets in $\mathbb{R}^n$, $\lambda \in (0,1)$, $p > 1$, and $\Omega_\lambda = (1-\lambda)\Omega_0 + \lambda\Omega_1$. Then we get*

$$\tau_p(\Omega_\lambda) \geq \mathcal{M}_{\frac{1}{p+n(p-1)}}\left(\tau_p(\Omega_0), \tau_p(\Omega_1), \lambda\right)\left(1 + \left(\alpha^{-1}\bar{H}(\Omega_0, \Omega_1)\right)^{\frac{(n+1)(2p-1)}{p-1}}\right)^{p-1},$$

and

$$\tau_p(\Omega_\lambda) \geq \mathcal{M}_{\frac{1}{p+n(p-1)}}\left(\tau_p(\Omega_0), \tau_p(\Omega_1), \lambda\right)\left(1 + \left(\beta^{-1}\mathcal{A}(\Omega_0, \Omega_1)^2\right)^{\frac{2p-1}{p-1}}\right)^{p-1},$$

where α and β are given in (1.11) and (1.12).

Proof. Let u be the solution to the elliptic boundary value problems (5.2). Applying the Gauss-Green formula, we obtain

$$\int_\Omega |\nabla u|^p dx = \int_\Omega u(x)dx.$$

This combining with the definition of p-torsional rigidity implies that

$$\tau_p(\Omega) = \left(\int_\Omega u(x)dx\right)^{p-1}. \tag{5.4}$$

The Brunn-Minkowski inequality regarding τ_p, given by inequality (5.3), can be expressed equivalently as

$$\tau_p(\Omega_\lambda) \geq \mathcal{M}_{\frac{1}{p+n(p-1)}}\left(\tau_p(\Omega_0), \tau_p(\Omega_1), \lambda\right).$$

Taking the power of $\frac{1}{p-1}$, we obtain the following expression which coincides with the BBL inequality in the case of $(p-1)/p$

$$\int_{\Omega_\lambda} u_\lambda(x)dx \geq \mathcal{M}_{\frac{p-1}{p+n(p-1)}}\left(\int_{\Omega_0} u_0(x)dx, \int_{\Omega_1} u_1(x)dx, \lambda\right).$$

By the definition of the deficit of p-torsional rigidity and the BBL deficit, we obtain

$$\begin{aligned}\delta_\lambda(\Omega_0, \Omega_1, \tau_p) &= \frac{\int_{\Omega_\lambda} u_\lambda(x)dx}{\mathcal{M}_{\frac{p-1}{p+n(p-1)}}\left(\int_{\Omega_0} u_0(x)dx, \int_{\Omega_1} u_1(x)dx, \lambda\right)} - 1 \\ &= \delta_\lambda\left(u_0, u_1, \frac{p-1}{p}\right).\end{aligned} \tag{5.5}$$

If u_i is the solution of problem (5.2) in Ω_i, $i = 0, 1, \lambda$, then Theorems 5.1 and 5.2 imply that u_i is $(p-1)/p$-concave for $i = 0, 1, \lambda$, and

$$u_\lambda((1-\lambda)x + \lambda y)^{\frac{p-1}{p}} \geq (1-\lambda)u_0(x)^{\frac{p-1}{p}} + \lambda u_1(y)^{\frac{p-1}{p}}, \forall x \in \Omega_0, y \in \Omega_1.$$

Thus the conditions of Theorems 1.3 and 1.4 are satisfied with $p' = (p-1)/p$. Therefore, it follows from Theorem 1.3 that

$$\bar{H}(\Omega_0, \Omega_1) \leq \alpha\delta_\lambda\left(u_0, u_1, \frac{p-1}{p}\right)^{\frac{p-1}{(n+1)(2p-1)}} = \alpha\delta_\lambda(\Omega_0, \Omega_1, \tau_p)^{\frac{p-1}{(n+1)(2p-1)}},$$

where α is a constant given in (1.11). From Theorem 1.4, it deduces that

$$\mathcal{A}(\Omega_0, \Omega_1)^2 \leq \beta\delta_\lambda\left(u_0, u_1, \frac{p-1}{p}\right)^{\frac{p-1}{2p-1}} = \beta\delta_\lambda(\Omega_0, \Omega_1, \tau_p)^{\frac{p-1}{2p-1}},$$

where β is a constant given in (1.12). We then have the final results by glancing at the definition of the deficits. $\square$

Now we are in the position to consider the Urysohn type inequality for p-torsional rigidity.

Theorem 5.4. *Let Ω be an open bounded convex set in $\mathbb{R}^n$, $p > 1$, and let $\Omega^\sharp$ be the ball with the same mean-width of Ω. Then it holds*

$$\tau_p(\Omega) \leq \tau_p(\Omega^\sharp), \tag{5.6}$$

with equality holds if and only if $\Omega = \Omega^\sharp$.

The above theorem can be derived from more general results, refer to [12, 3, 32]. For the sake of completeness, we provide a detailed and explicit proof in the following.

Proof. Taking into account the definition of τ_p, it becomes apparent that the quantity is invariant under translations. Consequently, we can apply a translation to Ω in such a manner that the origin coincides with the Steiner point s of Ω.

Using Hadwiger's Theorem (see [47]), there exists a sequence of rotations $\{\rho_k\}$ such that

$$\Omega_n = \frac{1}{n}\left(\rho_1\Omega + \ldots + \rho_n\Omega\right) \tag{5.7}$$

converges to a ball in the Hausdorff metric. Notice that, since the mean width is invariant under rigid motions and Minkowski additive (see [47, Section 1.7]), we get

$$w(\Omega_n) = w(\Omega) = b$$

for all n. Taking the limit $n \to \infty$, it follows that

$$w(\Omega^\sharp) = w(\Omega) = b.$$

Moreover, we recall that the Steiner point $s(\Omega)$ of a convex set Ω is defined in (2.2) as

$$s(\Omega) = \frac{1}{\omega_n}\int_{S^{n-1}} \theta h(\Omega,\theta) d\mathcal{H}^{n-1}(\theta).$$

Thanks to the property of Steiner point, we have $s(\Omega_n) = 0$ for all n. Therefore, $\Omega^\sharp$ is the ball with radius $r = \frac{b}{2}$ centered at 0.

Since τ_p is homogeneous of degree $p+n(p-1)$, it follows from (1.3) that for $\lambda_i \in [0,1]$, $i = 1, \cdots, n$ with $\sum\limits_{i=1}^{n} \lambda_i = 1$, and $K_i \in \mathcal{K}_o^n$,

$$\tau_p\left(\sum_{i=1}^{n} \lambda_i K_{\lambda_i}\right)^{\frac{1}{p+n(p-1)}} \geq \sum_{i=1}^{n} \lambda_i \tau_p\left(K_i\right)^{\frac{1}{p+n(p-1)}}, \tag{5.8}$$

with equality if and only if $K_i \in \mathcal{K}_o^n$ are homothetic to each other when $\lambda_i \in (0,1)$, $i = 1, \cdots, n$. Since $\tau_p(\rho\Omega) = \tau_p(\Omega)$ for any rotation ρ, it follows from (5.7) and (5.8) that

$$\tau_p(\Omega_n) \geq \tau_p(\Omega), \quad \text{for all } n > 0. \tag{5.9}$$

Since Ω_n converges to a ball $\Omega^\sharp$ in the Hausdorff metric as n tends to infinity, there exists m such that

$$\Omega_n \subseteq B(0, r + \frac{1}{n})$$

for all $n \geq m$. Since τ_p is increasing with respect to inclusion, it follows that

$$\tau_p\left(\Omega_n\right) \leq \tau_p\left(B(0, r + \frac{1}{n})\right). \tag{5.10}$$

Since $\tau_p\left(B(0, r + \frac{1}{n})\right)$ converges to $\tau_p(\Omega^\sharp)$ as n tends to infinity, it follows from (5.9) and (5.10) that for all $n > 0$,

$$\tau_p(\Omega) \leq \tau_p(\Omega_n) \leq \tau_p(\Omega^\sharp).$$

Let us now characterize the maximizers. If Ω is a ball we get the equality in (5.6). Conversely, if equality holds in (5.6), the above proof shows

$$\tau_p(\Omega) = \tau_p(\Omega_n) = \tau_p(\Omega^\sharp),$$

for all $n > 0$. And thanks to the equality case in (5.8), we can conclude that Ω is a ball. $\square$

Note that in the class of convex sets with fixed mean width and same Steiner point, there exists a unique ball whose p-torsional rigidity is maximal. In fact, assume that Ω belongs to this class satisfying $\tau_p(\Omega) = \tau_p(\Omega^\sharp)$ and Ω does not coincide with $\Omega^\sharp$. Then by (1.3) we have

$$\tau_p\left(\frac{\Omega+\Omega^\sharp}{2}\right)^{\frac{1}{p+n(p-1)}} > \frac{1}{2}\tau_p(\Omega)^{\frac{1}{p+n(p-1)}} + \frac{1}{2}\tau_p(\Omega^\sharp)^{\frac{1}{p+n(p-1)}} = \tau_p(\Omega^\sharp)^{\frac{1}{p+n(p-1)}},$$

and equivalently,
$$\tau_p\left(\frac{\Omega+\Omega^\sharp}{2}\right) > \tau_p(\Omega^\sharp). \tag{5.11}$$

The consequence (5.11) contradicts (5.6), since

$$w\left(\frac{\Omega+\Omega^\sharp}{2}\right) = w(\Omega^\sharp),$$

by the Minkowski additivity of the mean width.

Using the definition of the Urysohn deficit for τ_p, Theorem 1.8 can be rewritten as follows.

Theorem 5.5. *Let Ω be an open bounded convex set in $\mathbb{R}^n$, $p > 1$, and let $\Omega^\sharp$ be the ball with the same mean-width of Ω. If $\delta_w(\Omega, \tau_p)$ is small enough, then it holds*

$$\tau_p(\Omega^\sharp) \geq \tau_p(\Omega)\left(1 + \left(\alpha^{-1}|\Omega|^{-\frac{1}{n}}H\right)^{\frac{(n+1)(2p-1)}{p-1}}\right)^{p-1}, \tag{5.12}$$

and
$$\tau_p(\Omega^\sharp) \geq \tau_p(\Omega)\left(1 + \left(\beta^{-1}\mathcal{A}^2\right)^{\frac{2p-1}{p-1}}\right)^{p-1}, \tag{5.13}$$

where $H = H(\Omega, \Omega^\sharp)$ and $\mathcal{A} = \max\{\mathcal{A}(\Omega, \Omega_\rho) : \rho$ any rotation in $\mathbb{R}^n\}$.

Proof. We only prove (5.12), because (5.13) can be proved in the same way.

Assume Ω_ρ is a rotation of Ω with center at the centroid of Ω, and set

$$\hat{\Omega} = \frac{1}{2}\Omega + \frac{1}{2}\Omega_\rho.$$

Clearly, $w(\hat{\Omega}) = w(\Omega)$, and by (5.6) we have $\tau_p(\hat{\Omega}) \leq \tau_p(\Omega^\sharp)$.

Since $\tau_p(\Omega_\rho) = \tau_p(\Omega)$, Theorem 5.3 gives

$$\tau_p(\hat{\Omega}) \geq \tau_p(\Omega)\left(1 + \left(\alpha^{-1}\bar{H}\left(\Omega, \Omega_\rho\right)\right)^{\frac{(n+1)(2p-1)}{p-1}}\right)^{p-1}. \tag{5.14}$$

Since
$$\bar{H}(\Omega, \Omega_\rho) = \frac{H(\Omega, \Omega_\rho)}{|\Omega|^{\frac{1}{n}}},$$

(5.14) becomes

$$\tau_p(\hat{\Omega}) \geq \tau_p(\Omega) \left(1 + \left(\alpha^{-1}|\Omega|^{-\frac{1}{n}} H\left(\Omega, \Omega_\rho\right)\right)^{\frac{(n+1)(2p-1)}{p-1}}\right)^{p-1}.$$

We now choose a rotation Ω_{ρ_0} such that $H(\Omega, \Omega_{\rho_0}) \geq H(\Omega, \Omega^\sharp)$.

In fact, from the definition of $H(\Omega, \Omega^\sharp)$, we deduce

$$H\left(\Omega, \Omega^\sharp\right) = \max_{\theta \in S^{n-1}} |h_\Omega(\theta) - h_{\Omega^\sharp}(\theta)| = \max_{\theta \in S^{n-1}} |h_\Omega(\theta) - r|, \tag{5.15}$$

where r is the radius of $\Omega^\sharp$, that is,

$$r = \frac{w(\Omega)}{2} = \frac{1}{n\omega_n} \int_{S^{n-1}} h_\Omega(\theta) d\theta.$$

By the mean value theorem for integrals and the continuity of h_Ω, there exists θ_0 such that $h_\Omega(\theta_0) = r$.

Denote $\bar{\theta}$ such that the maximum in (5.15) is attained at $\bar{\theta}$ and let ρ_0 be a rotation with center in the centroid of Ω such that $h_{\Omega_{\rho_0}}(\bar{\theta}) = h_\Omega\left(\theta_0\right)$.

Then $h_{\Omega_{\rho_0}}(\bar{\theta}) = r$.

Thanks to (5.15) we get

$$H\left(\Omega, \Omega_{\rho_0}\right) \geq \left|h_\Omega(\bar{\theta}) - h_{\Omega_{\rho_0}}(\bar{\theta})\right| = \left|h_\Omega(\bar{\theta}) - r\right| = H\left(\Omega, \Omega^\sharp\right),$$

as desired. □

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