Journal of Convex Analysis
Volume 33 (2026), No. 1&2, 497–516
https://doi.org/10.68381/jca33031

Subdifferential Determination of a Primal Lower Regular Function on a Banach Space

Milen Ivanov
Radiant Life Technologies Ltd., Nicosia, Cyprus
milen@radiant-life-technologies.com

Matey Konstantinov, Nadia Zlateva
Faculty of Mathematics and Informatics, Sofia University, Bulgaria
matey@fmi.uni-sofia.bg, zlateva@fmi.uni-sofia.bg

Received: February 19, 2025
Accepted: October 2, 2025

We generalize to Banach spaces the Thibault-Zagrodny Theorem that if f and g are primal lower regular functions and $\partial f = \partial g$ locally, then f and g locally differ by a constant. Our method consists in a reduction to considerations of slopes.

Keywords: Primal lower regular function, slope, determination of a function.
2020 Mathematics Subject Classification: 49J52, 47J22.

1. Introduction

The primal lower regular functions are introduced by R. A. Poliquin in [18] under the name primal lower nice functions, for a modern account see [21, Chapter 11]. They are a useful abstraction of the framework for non-convex optimization proposed by R. T. Rockafellar, see e.g. [20], that is, the compositions of convex lower semicontinuous and twice smooth functions. The latter framework is really apt for the purpose, however doing calculus directly with those so called *convexly composite* functions, e.g. [21, 11.1.2] is cumbersome, because of having to account for two maps. In this respect primal lower regular functions come in handy.

In [18] R. A. Poliquin shows that primal lower regular functions share many properties with the convex functions, including a local version of Moreau-Rockafellar integrability theorem. Namely, if the subdifferentials of two primal lower regular functions coincide in a neighbourhood of some point, then in a neighbourhood of that point the functions differ by a constant. In [18] this property was proved in finite dimensions and later in [22] it was extended to Hilbert space setting, see also [21, p. 1113]. These proofs, however, rely on the special properties of Moreau-Yosida regularization which allow transferring the subdifferential equality to the regularized functions, when the norm is Hilbertian. This does not work in a general Banach space.

As the title suggests, this work is devoted to extending this result to a general Banach space setting.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

The problem of extending the subdifferential determination of primal lower regular functions to Banach spaces is surprisingly tough. One of the reasons being that the domain of a primal lower regular function could potentially be quite irregular: only approximate convexity can be perhaps claimed [15] and even that is not fully established.

The new method we employ here is based on *slopes*, see e.g. [14] and the references therein, and it works, because it does not require linear (or convex) structure, so the irregularity of the domain is not such an issue.

Let us recall the definition of a primal lower regular function.

Definition 1.1. (cf. [21], Definition 11.1) Let $(E, \|\cdot\|)$ be a Banach space. Let the function $f : E \to \mathbb{R} \cup \{+\infty\}$ be proper and lower semicontinuous. We say that f is *primal lower regular at* $\bar{x} \in E$ with coefficient $c > 0$ and radius $\delta > 0$, and write

$$\bar{x} \in \mathrm{plr}_{c,\delta} f,$$

if
$$f(y) \geq f(x) + p(y - x) - c(1 + \|p\|)\|y - x\|^2, \tag{1}$$

for all $x, y \in B^\circ(\bar{x}; \delta)$, and all $p \in \partial f(x)$.

The function f is called *primal lower regular* if

$$\bigcup_{c,\delta>0} \mathrm{plr}_{c,\delta} f = E.$$

As usual, $B^\circ(\bar{x}; \delta)$ denotes the open ball centered at $\bar{x}$ with radius δ.

We will now formulate our main result.

Theorem 1.2. *Let $(E, \|\cdot\|)$ be a Banach space and assume that the functions $f, g \colon E \to \mathbb{R} \cup \{+\infty\}$ are proper and lower semicontinuous. Let*

$$\bar{x} \in \mathrm{plr}_{c,\delta} f \cap \mathrm{plr}_{c,\delta} g \cap \mathrm{dom}\, f,$$

for some $c, \delta > 0$. Let

$$\partial f(x) = \partial g(x), \quad \forall x \in B^\circ(\bar{x}, \delta). \tag{2}$$

Then there is a constant $a \in \mathbb{R}$ such that

$$f(x) = g(x) + a, \quad \forall x \in B^\circ(\bar{x}, \hat{\delta}), \tag{3}$$

where
$$\hat{\delta} := \min\left\{\frac{\delta}{2}, \frac{1}{18c}\right\}. \tag{4}$$

Here $\mathrm{dom}\, f := \{x \in X : f(x) \neq +\infty\}$ is the domain of f, and ∂ denotes the Clarke-Rockafellar subdifferential, see Section 3 for a brief review of the latter.

This work is organized as follows. In Section 2 we develop the slope techniques we need. In Section 3 we recall what we use from the theory of Clarke-Rockafellar subdifferential and retrieve how subdifferential inclusions imply slopes inequalities. Finally, in Section 4 we prove the main result.

2. Slope regularity

Let (X, d) be a complete metric space. Functions like $g : X \to \mathbb{R} \cup \{+\infty\}$ are assumed lower semicontinuous, and proper, i.e. such that $\operatorname{dom} g$ is a non-empty set.

For a function $g : X \to \mathbb{R}$ and a convex set $U \subset X$ define

$$\operatorname{Lip} g(U) := \sup \left\{ \frac{|g(y) - g(x)|}{d(y, x)} : \ x \neq y \in U \right\},$$

and for $x \in X$ $$\operatorname{Lip} g(x) := \inf_{\delta > 0} \operatorname{Lip} g(B(x; \delta)),$$

where $B(x; \delta)$ stands for the closed ball centered at x with radius δ.

Estimating the Lipschitz constant of the kind of functions will be using, is straightforward.

Lemma 2.1. *Let $x_0 \in X$, and $g(x) := \varphi(d(x, x_0))$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a convex, increasing smooth function such that $\varphi(0)$=0. Then*

$$\operatorname{Lip} g(x) \leq \varphi'(d(x, x_0)), \quad \forall x \in X.$$

Proof. Fix $x \in X$ and $\delta > d(x, x_0)$.

We have that φ' is positive and increasing. Therefore, $\operatorname{Lip} \varphi([0, \delta]) = \varphi'(\delta)$.

So, if $x, y \in B(x_0; \delta)$, that is, $d(x, x_0), d(y, x_0) \in [0, \delta]$, then

$$|g(y) - g(x)| \leq \varphi'(\delta)|d(y, x_0) - d(x, x_0)| \leq \varphi'(\delta) d(y, x),$$

from the triangle inequality. So, $\operatorname{Lip} g(B(x_0; \delta)) \leq \varphi'(\delta)$.

But we have $\operatorname{Lip} g(x) \leq \operatorname{Lip} g(B(x_0; \delta))$ and $\inf_{\delta > d(x, x_0)} \varphi'(\delta) = \varphi'(d(x, x_0))$, because φ is smooth. □

The *local slope* of a function $f : X \to \mathbb{R} \cup \{+\infty\}$ at $x \in \operatorname{dom} f$ is

$$|\nabla f|(x) = \limsup_{\substack{y \to x \\ y \neq x}} \frac{[f(x) - f(y)]^+}{d(x, y)},$$

where $[t]^+ := \max\{0, t\} = (t + |t|)/2$. Therefore, $|\nabla f|(x) = 0$ if x is a local minimum to f, and if not

$$|\nabla f|(x) = \limsup_{\substack{y \to x \\ y \neq x}} \frac{f(x) - f(y)}{d(x, y)}.$$

We explicit that $|\nabla f|(x) = 0$ at the isolated points of $\operatorname{dom} f$, where the limit above is undefined. This notion was introduced by De Giorgi, Marino and Tosques [7] and used thereafter for study of concepts as descent curves (e.g. [8]), error bounds (e.g. [2]), subdifferential calculus and metric regularity (e.g. the monograph [9] and references therein). For determination of functions, local slopes recently appear as a useful tool, see e.g. [4, 5, 6, 14], as well as, in the convex case subgradients of minimal norm, see [16], and global slopes, see [23].

We may consider $|\nabla f|$ as a function from $\operatorname{dom} f$ to $[0,\infty]$. Moreover,

$$|\nabla(rf)|(x) = r|\nabla f|(x), \quad \forall r \geq 0,\ \forall x \in \operatorname{dom} f.$$

In some sense, not a strict one, the local slope $|\nabla f|(x)$ can be regarded as an one-sided, or *unilateral*, analogy of $\operatorname{Lip} f(x)$. Formally, $|\nabla f|(x)$ is an unilaterisation of $\limsup_{y\to x} |f(x)-f(y)|/d(x,y)$. What we will be using is the following relationship.

Lemma 2.2. *Let $f : X \to \mathbb{R}$ be proper and lower semicontinuous and let $g : X \to \mathbb{R}$ be locally Lipschitz. If $f+g$ has a local minimum at x_0 then*

$$|\nabla f|(x_0) \leq \operatorname{Lip} g(x_0). \tag{5}$$

Proof. We can assume without loss of generality that $f(x_0) = g(x_0) = 0$. So, there is $\varepsilon > 0$ such that

$$f(x) + g(x) \geq 0, \quad \forall x \in B(x_0;\varepsilon).$$

Then for each $\delta \in (0,\varepsilon)$

$$f(x) \geq -g(x) \geq -\operatorname{Lip} g(B(x_0;\delta))d(x,x_0), \quad \forall x \in B(x_0;\delta).$$

By definition $\qquad |\nabla f|(x_0) \leq \operatorname{Lip} g(B(x_0;\delta)),$

and taking infimum over $\delta \in (0,\varepsilon)$, we get (5). □

Later on we will be using the following simple observation.

Lemma 2.3. *Let f be proper and lower semicontinuous and let $x_0 \in \operatorname{dom} f$. Consider the sublevel set*

$$Y := \{x \in X :\ f(x) \leq f(x_0)\},$$

and the restriction of f to it: $\overline{f} := f \restriction Y$. Then

$$|\nabla f|(x) = |\nabla \overline{f}|(x), \quad \forall x \in Y. \tag{6}$$

Proof. The definition of $|\nabla f|(x)$ can be rewritten as

$$|\nabla f|(x) = \inf_{\varepsilon>0} \sup \left\{ \frac{f(x)-f(y)}{d(x,y)} :\ y \in B(x;\varepsilon),\ f(y) \leq f(x) \right\}.$$

Obviously, if $x \in Y$, then $f(y) \leq f(x)$ implies $y \in Y$ as well, and this gives (6). □

Recall the famous Ekeland Variational Principle, e.g. [17, p. 45], or [21, p. 198].

Theorem 2.4. (Ekeland) *If $f : X \to \mathbb{R} \cup \{+\infty\}$ is proper, lower semicontinuous, and bounded below, and $x_0 \in \operatorname{dom} f$, then for each $\lambda > 0$ there exists x_λ such that*

$$\lambda d(x_\lambda, x_0) \leq f(x_0) - f(x_\lambda),$$
$$f(x) + \lambda d(x_\lambda, x) > f(x_\lambda), \quad \forall x \neq x_\lambda.$$

Lemma 2.5. *Let $f : X \to \mathbb{R}$ be a proper and lower semicontinuous function, and let $g : X \to \mathbb{R}$ be locally Lipschitz. If $f + g$ is bounded below, then for each $\varepsilon > 0$ there exists x_ε such that*

$$(f + g)(x_\varepsilon) \le \inf(f + g) + \varepsilon, \tag{7}$$

and

$$|\nabla f|(x_\varepsilon) \le \operatorname{Lip} g(x_\varepsilon) + \varepsilon. \tag{8}$$

Proof. Fix $\varepsilon > 0$ and pick some x_0 such that

$$(f + g)(x_0) \le \inf(f + g) + \varepsilon.$$

By Ekeland's Variational Principle – Theorem 2.4 – applied for $(f + g)$ (instead of f), and for ε (instead of λ), there is x_ε which satisfies

$$\varepsilon d(x_\varepsilon, x_0) \le (f + g)(x_0) - (f + g)(x_\varepsilon), \tag{9}$$

$$(f + g)(x) + \varepsilon d(x_\varepsilon, x) > (f + g)(x_\varepsilon), \quad \forall x \ne x_\varepsilon. \tag{10}$$

Now, (9) implies $(f + g)(x_\varepsilon) \le (f + g)(x_0)$, and we get (7).

On the other hand, (10) means that the function

$$x \to f(x) + (g(x) + \varepsilon d(x_\varepsilon, x))$$

attains its minimum at x_ε. Obviously, $\operatorname{Lip}(g(\cdot) + \varepsilon d(\cdot, x_\varepsilon))(x_\varepsilon) \le \operatorname{Lip} g(x_\varepsilon) + \varepsilon$, so Lemma 2.2 gives (8). □

Definition 2.6. The function $f : X \to \mathbb{R} \cup \{+\infty\}$ is *regularly sloped of order* 2 *with coefficient* $c > 0$ whenever for each $x \in \operatorname{dom} |\nabla f|$,

$$f(y) \ge f(x) - |\nabla f|(x) d(y, x) - c(1 + |\nabla f|(x)) d^2(y, x), \quad \forall y \in X. \tag{11}$$

Theorem 2.7. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be regularly sloped of order* 2 *with coefficient $c > 0$. Then for each $\bar{x}$ with*

$$0 < |\nabla f|(\bar{x}) < \infty$$

there is a sequence $x_n \to \bar{x}$ such that

$$|\nabla f|(\bar{x}) = \lim_{n \to \infty} \frac{f(\bar{x}) - f(x_n)}{d(\bar{x}, x_n)}, \tag{12}$$

and

$$|\nabla f|(x_n) - |\nabla f|(\bar{x}) < 2c(|\nabla f|(\bar{x}) + 2) d(\bar{x}, x_n), \quad \forall n > 1/|\nabla f|(\bar{x}). \tag{13}$$

Proof. Fix $\bar{x} \in \operatorname{dom} |\nabla f|$ such that $r := |\nabla f|(\bar{x}) > 0$. To shorten the annotations, we can assume without loss of generality that $f(\bar{x}) = 0$. So, (11) gives

$$f(x) \ge -r d(x, \bar{x}) - c(r + 1) d^2(x, \bar{x}), \quad \forall x \in X. \tag{14}$$

Fix an arbitrary $n \in \mathbb{N}$ such that $n > 1/r$, thus

$$r_n := r - 1/n > 0. \tag{15}$$

Consider the function $g_n(x) := r_n d(x, \bar{x}) + c(r + 2) d^2(x, \bar{x})$.

Lemma 2.1 gives that

$$\operatorname{Lip} g_n(x) \le r_n + 2c(r + 2) d(x, \bar{x}), \quad \forall x \in X. \tag{16}$$

Note that the function $(f + g_n)$ is bounded below. Indeed, from (14) it follows that

$$f(x) + g_n(x) \geq -\frac{d(x,\bar{x})}{n} + cd^2(x,\bar{x}) \geq \min_{t\geq 0}\big(ct^2 - \frac{t}{n}\big).$$

It is not possible for the function $(f+g_n)$ to have a local minimum at $\bar{x}$, for otherwise Lemma 2.2 and (16) would give $r = |\nabla f|(\bar{x}) \leq r_n < r$, contradiction. So we have $(f + g_n)(\bar{x}) = 0 > \inf(f + g_n)$.

Choose an $\varepsilon_n > 0$ such that $\quad \inf(f + g_n) + \varepsilon_n < 0, \tag{17}$

and $$\varepsilon_n < \frac{1}{n}. \tag{18}$$

From Lemma 2.5 there is x_n such that

$$(f + g_n)(x_n) \leq \inf(f + g_n) + \varepsilon_n, \tag{19}$$

and $$|\nabla f|(x_n) \leq \operatorname{Lip} g_n(x_n) + \varepsilon_n. \tag{20}$$

From (19) and (17) it follows that $(f + g_n)(x_n) < 0$, so we have

$$f(x_n) < -g_n(x_n) \leq -r_n d(x_n,\bar{x}).$$

Thus, $$\frac{f(\bar{x}) - f(x_n)}{d(\bar{x},x_n)} = -\frac{f(x_n)}{d(\bar{x},x_n)} > r_n. \tag{21}$$

Using again $(f + g_n)(x_n) < 0$ together with (14), we estimate

$$0 > f(x_n) + g_n(x_n) \geq -d(x_n,\bar{x})/n + cd^2(x_n,\bar{x}),$$

so $$0 > d(x_n,\bar{x})(cd(x_n,\bar{x}) - 1/n).$$

Clearly, this implies $$d(x_n,\bar{x}) < \frac{1}{cn}. \tag{22}$$

From (16) and (20) we have that

$$|\nabla f|(x_n) - |\nabla f|(\bar{x}) \leq (r_n + 2c(r+2)d(x_n,\bar{x}) + \varepsilon_n) - r = (r_n - r + \varepsilon_n) + 2c(r+2)d(x_n,\bar{x}).$$

From (15) and (18) it follows that $(r_n - r + \varepsilon_n) = \varepsilon_n - 1/n < 0$. Thus,

$$|\nabla f|(x_n) - |\nabla f|(\bar{x}) < 2c(r + 2)d(x_n,\bar{x}). \tag{23}$$

Having constructed the sequence $(x_n)_{n>1/r}$, we claim that it satisfies the conclusion of the theorem. Indeed, (22) means that $x_n \to \bar{x}$, as $n \to \infty$, and (21) plus $r_n \to r$, as $n \to \infty$, gives (12). Finally, (23) immediately implies (13). □

Lemma 2.8. *If $f : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and regularly sloped of order 2 with coefficient $c > 0$, then the function*

$$x \to |\nabla f|(x)$$

is lower semicontinuous.

Proof. Fix $x \in X$ and let $x_n \to x$ as $n \to \infty$. We have to show that

$$|\nabla f|(x) \le \liminf_{n\to\infty} |\nabla f|(x_n).$$

If $|\nabla f|(x_n) \to \infty$ this holds, so let $r \in \mathbb{R}$ be such that

$$r > \liminf_{n\to\infty} |\nabla f|(x_n).$$

There is a subsequence $x_{n_k} \to x$ as $k \to \infty$, such that $|\nabla f|(x_{n_k}) < r$ for all $k \in \mathbb{N}$. From (11) it follows that

$$f(y) \ge f(x_{n_k}) - rd(y, x_{n_k}) - c(1+r)d^2(y, x_{n_k})$$

for any fixed $y \in X$ and all $k \in \mathbb{N}$. Passing to limit as $k \to \infty$ and taking into account that f is lower semicontinuous, we get

$$f(y) \ge f(x) - rd(y, x) - c(1+r)d^2(y, x),$$

so, by definition $r \ge |\nabla f|(x)$. □

Lemma 2.9. *If $f : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and regularly sloped of order 2 with coefficient $c > 0$, and the sequence $x_n \to x$ is such that the sequence $(|\nabla f|(x_n))_{n\in\mathbb{N}}$ is bounded, then*

$$\lim_{n\to\infty} f(x_n) = f(x). \tag{24}$$

Proof. Let $r > 0$ be such that $|\nabla f|(x_n) < r$ for all $n \in \mathbb{N}$. From (11) it follows that

$$f(x) \ge f(x_n) - rd(x, x_n) - c(r+1)d^2(x, x_n), \quad \forall n \in \mathbb{N}.$$

Since $d(x, x_n) \to 0$, as $n \to \infty$, we have

$$\limsup_{n\to\infty} f(x_n) \le f(x),$$

and (24) follows from the lower semicontinuity of f. □

The following estimate is crucial for our approach.

Lemma 2.10. *Let $a_n \ge 0$ and $b_n > 0$ be such that*

$$\sum_{n=0}^{\infty} a_n < \infty, \tag{25}$$

and for some $c > 0$: $\quad b_{n+1} - b_n < 2c(2 + b_n)a_n, \quad \forall n \ge 0.$ (26)

Then the sequence $(b_n)_0^\infty$ is bounded.

Proof. Set $s := \sum_{n=0}^{\infty} a_n$. Let $A := \{0\} \cup \{n \in \mathbb{N} : \ b_n \le 1\}$. Fix an arbitrary $n \in \mathbb{N} \setminus A$. Let $k := \max\{i \in A : \ i < n\}$. This means that $b_k \le b := \max\{b_0, 1\}$ and $b_i > 1$, for all i such that $k + 1 \le i \le n$.

So, we derive from (26) that

$$\frac{b_{i+1}}{b_i} < 1 + 2c\big(1 + \frac{2}{b_i}\big)a_i < 1 + 6ca_i, \quad \text{for all } i \text{ such that } k+1 \le i \le n-1.$$

Since $\ln(1+t) \le t$ for $t \ge 0$, we have

$$\ln b_{i+1} - \ln b_i < 6ca_i, \quad \text{for all } i \text{ such that } k+1 \le i \le n-1.$$

Summing these, we get

$$\ln b_n - \ln b_{k+1} < 6c \sum_{i=k+1}^{n-1} a_i < 6cs.$$

Therefore, $$b_n < b_{k+1} e^{6cs}.$$

But because $b_k \le b$, (26) gives $b_{k+1} \le b + 2c(2+b)a_k < b + 2c(2+b)s$, so

$$b_n < (b + 2c(2+b)s)e^{6cs}, \quad \forall n \in N \text{ such that } \ b_n > 1. \qquad \square$$

We will prove here a variant of the Long Orbit Empty Value (LOEV) method, see [10, 13]. It concerns multivalued maps $S : X \rightrightarrows X$, where (X, d) is a complete metric space. As usual, $\operatorname{dom} S := \{x \in X : \ S(x) \neq \varnothing\}$.

An *S-orbit* is called any finite or infinite sequence $x_0, x_1, x_2 \ldots$ such that $x_{i+1} \in S(x_i)$ for all i's with a successor. We call x_0 *the starting point* of the orbit. We differentiate between *finite* and *infinite* orbits. It stands to reason that for a finite orbit $x_0, \ldots, x_n$, with $x_{i+1} \in S(x_i)$, for all i such that $0 \le i \le n-1$, the last point x_n would be called *the end point* of the orbit. Obviously, each finite orbit $(x_i)_0^n$ is of *finite length*

$$|(x_i)_0^n| := \sum_{i=0}^{n-1} d(x_{i+1}, x_i),$$

if $n \ge 1$. The length of an orbit consisting of one point is, of course, zero.

Considering infinite orbits, $(x_i)_0^\infty$ with $x_{n+1} \in S(x_n)$ for all $n \ge 0$ is called orbit of *finite length* if

$$|(x_i)_0^\infty| := \sum_{n=0}^{\infty} d(x_{n+1}, x_n) < \infty,$$

and, respectively, of *infinite length* if the above series is divergent.

Since X is complete, each infinite orbit of finite length is convergent as a sequence, that is, there is $x_\infty \in X$ such that

$$\lim_{n\to\infty} d(x_n, x_\infty) = 0.$$

We call x_∞ the *end point* of the orbit $(x_i)_0^\infty$.

Definition 2.11. Let (X, d) be a complete metric space and let $S : X \rightrightarrows X$ be a multivalued map. We say that S satisfies property $(*)$ if

$$x \notin S(x), \quad \forall x \in X, \tag{27}$$

and for any infinite S-orbit $(x_n)_0^\infty$ of finite length with end point x_∞, and any $y \in S(x_\infty)$, there exists $N \in \mathbb{N}$ such that

$$y \in S(x_n), \quad \forall n > N.$$

Theorem 2.12. *Let (X, d) be a complete metric space. Let $S : X \rightrightarrows X$ satisfy property $(*)$ and let $x_0 \in X$ be fixed.*

Then at least one of the two following properties holds:

(a) *There is an S-orbit of finite length starting at x_0 and ending outside of* $\operatorname{dom} S$.

(b) *There is an S-orbit of infinite length starting at x_0.*

Proof. We construct the desired S-orbit by induction, starting at x_0.

If x_n is already chosen, then:

If $S(x_n) = \varnothing$ then (a) is fulfilled and we are done.

If, on the other hand, $S(x_n) \neq \varnothing$, let

$$\nu_n := \sup\{d(y, x_n) : \ y \in S(x_n)\} > 0. \tag{28}$$

Take $x_{n+1} \in S(x_n)$ such that $\quad d(x_{n+1}, x_n) > \min\{1, \nu_n/2\}$. (29)

Assume that we obtain in this way an infinite orbit $(x_n)_0^\infty$ with $x_{n+1} \in S(x_n)$, $\forall n \geq 0$.

If $(x_n)_0^\infty$ is of infinite length then (b) is fulfilled and we are done.

If $(x_n)_0^\infty$ is of finite length, then let x_∞ be the end point of $(x_n)_0^\infty$, i.e.

$$\lim_{n\to\infty} x_n = x_\infty.$$

Assume that there is some $y \in S(x_\infty)$. We know from $(*)$ property, see (27), that $y \neq x_\infty$, and $y \in S(x_n)$ for all but finitely many n's. Then (28) gives us $\nu_n \geq d(y, x_n) \to d(y, x_\infty) > 0$ and (29) implies $\liminf_{n\to\infty} d(x_{n+1}, x_n) > 0$, so $|(x_n)_0^\infty| = \infty$, a contradiction. Therefore, $S(x_\infty) = \varnothing$, and (a) holds. □

Lemma 2.13. (see [14]) *Let (X, d) be a complete metric space. Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous. Let f also be proper and bounded below. Then*

$$\inf_{\operatorname{dom} f}(f - g) = \inf_{\operatorname{dom} |\nabla f|}(f - g). \tag{30}$$

In [14, Lemma 3.8.] the equality (30) is proved for the global slope, which dominates the local slope, so the formula there is stronger, but we do not need it in full.

Definition 2.14. Let (X, d) be a complete metric space. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous and bounded below.

We say that $\bar{x}$ is the *sharp minimum* of f, if it is the minimum:

$$f(\bar{x}) = \min f,$$

and, moreover, $$0 < \inf_{x\neq\bar{x}} |\nabla f|(x). \tag{31}$$

If f has a sharp minimum, then f cannot have any other local minima, because $|\nabla f|(x) = 0$ at each local minimum x. It is easy to check that the sharp minimum is *strong*, that is, each minimising sequence converges to it, but a strong minimum is not necessarily sharp.

Proposition 2.15. *Let (X, d) be a complete metric space and assume that the $f, g : X \to \mathbb{R} \cup \{+\infty\}$ are proper and lower semicontinuous. Let f be regularly sloped of order 2 with coefficient $c > 0$ and have sharp minimum at $\bar{x}$. If*

$$|\nabla g|(x) < |\nabla f|(x), \quad \forall x \in \operatorname{dom} |\nabla f| \setminus \{\bar{x}\}, \tag{32}$$

and

$$g(y) \geq g(x) - |\nabla f|(x) d(y, x) - c(|\nabla f|(x) + 1) d^2(y, x), \tag{33}$$

for all $x \in \operatorname{dom} |\nabla f|$ and all $y \in X$, then

$$\min_{\operatorname{dom} f} (f - g) = f(\bar{x}) - g(\bar{x}). \tag{34}$$

Proof. If $\bar{x} \notin \operatorname{dom} g$ then the right hand side of (34) is $-\infty$, this is the minimum, and (34) is trivial.

So, let $g(\bar{x}) < \infty$ and let us assume for simplicity – and without loss of generality – that $f(\bar{x}) = g(\bar{x}) = 0$. To verify (34) we need to prove that $f \geq g$. But from Lemma 2.13 we know that it is enough to show that

$$f(x) \geq g(x), \quad \forall x \in \operatorname{dom} |\nabla f|.$$

Therefore, fix an arbitrary $x_0 \in X$ such that $f(x_0) < \infty$. In what follows we will show that $f(x_0) \geq g(x_0)$.

Since $\bar{x}$ is a sharp minimum, from (31) there is $\varepsilon > 0$ such that

$$|\nabla f|(x) > \varepsilon, \quad \forall x \in \operatorname{dom} |\nabla f|(x) \setminus \{\bar{x}\}. \tag{35}$$

For any $x \in \operatorname{dom} |\nabla f|$ such that $x \neq \bar{x}$, define the following subsets of $\operatorname{dom} |\nabla f|$:

$$S_1(x) := \{y \in \operatorname{dom} |\nabla f| : \ f(y) - g(y) < f(x) - g(x)\} \tag{36}$$

(note that because of (32), $x \in \operatorname{dom} |\nabla g| \subset \operatorname{dom} g$, so $f(x) - g(x)$ is well defined);

$$S_2(x) := \{y \in \operatorname{dom} |\nabla f| : \ f(y) < f(x) - \varepsilon d(y, x)\}; \tag{37}$$

$$S_3(x) := \{y : \ |\nabla f|(y) < |\nabla f|(x) + 2c(2 + |\nabla f|(x)) d(y, x)\}, \tag{38}$$

and

$$S(x) := S_1(x) \cap S_2(x) \cap S_3(x).$$

In this way we define a multivalued map $S : X \rightrightarrows X$. We will study its properties.

First, the way it is defined ensures that $\operatorname{dom} S \subset \operatorname{dom} |\nabla f| \setminus \{\bar{x}\}$. We claim that in fact

$$\operatorname{dom} S = \operatorname{dom} |\nabla f| \setminus \{\bar{x}\}. \tag{39}$$

Indeed, fix $x \in \operatorname{dom} |\nabla f|$ such that $x \neq \bar{x}$, so

$$r := |\nabla f|(x) \in (\varepsilon, \infty),$$

see (35). From (32) we have $|\nabla g|(x) < r$, so there is $\delta > 0$ such that $r - \delta > 0$ and

$$g(y) \geq g(x) - (r - \delta) d(y, x), \quad \forall y \in B(x; \delta).$$

From Theorem 2.7 there is a sequence $x_n \to x$ satisfying (12) and (13). Now, (13) simply means that $x_n \in S_3(x)$ for all large n's. From (12) we get on the one hand

that, $f(x) - f(x_n) > \varepsilon d(x, x_n)$, because $|\nabla f|(x) > \varepsilon$. Therefore, $x_n \in S_2(x)$ for all large n's. On the other hand, $x_n \in B(x; \delta)$ for all large n's, so using the above inequality for g and again (12) we get

$$\frac{f(x) - f(x_n)}{d(x, x_n)} > r - \delta \geq \frac{g(x) - g(x_n)}{d(x, x_n)} \Rightarrow f(x_n) - g(x_n) < f(x) - g(x),$$

for all n large enough. This means that $x_n \in S_1(x)$ for all large n's. So, $x_n \in S(x)$ for all but finitely many n's and, therefore, $S(x) \neq \varnothing$ and (39) is verified.

Next, each S-orbit starting at x_0 is of finite length. Indeed, if $(x_n)_0^\infty$ is a S-orbit, it is also a S_2-orbit, so iterating on (37) we get

$$\sum_{i=0}^{n} d(x_{i+1}, x_i) < \varepsilon^{-1} \sum_{i=0}^{n} (f(x_i) - f(x_{i+1})) = \varepsilon^{-1}(f(x_0) - f(x_{n+1})),$$

and, since $f \geq 0$, we get $$|(x_n)_0^\infty| \leq f(x_0)/\varepsilon. \tag{40}$$

The same estimate is valid, of course, for the lengths of the finite S-orbits starting at x_0 (but we do not need it).

Let $(x_n)_0^\infty$ be an arbitrary infinite S-orbit starting at x_0. We already know that it is of finite length, so it has an end point, say x_∞. We claim that

$$\lim_{n \to \infty} (f(x_n), g(x_n)) = (f(x_\infty), g(x_\infty)). \tag{41}$$

Indeed, set $a_n := d(x_{n+1}, x_n)$ and $b_n := |\nabla f|(x_n)$. We know that a_n's satisfy (25) and because $(x_n)_0^\infty$ is a S_3-orbit, (38) translates to (26). From Lemma 2.10 we get that the sequence $(|\nabla f|(x_n))_0^\infty$ is bounded and then Lemma 2.9 ensures that $f(x_n) \to f(x_\infty)$. We repeat the proof of Lemma 2.9 for g using (33). Thus (41) is verified.

Now we can show that S has property $(*)$. Because of the strict inequality, S_1 satisfies (27), thus so does S.

Let $(x_n)_0^\infty$ be an infinite S-orbit of finite length ending at x_∞ and let $y \in S(x_\infty)$. Because $y \in S_1(x_\infty)$, we have by (36) that $f(y) - g(y) < f(x_\infty) - g(x_\infty)$, then (41) gives that $f(y) - g(y) < f(x_n) - g(x_n)$ for all large n's. That is, eventually $y \in S_1(x_n)$.

In a similar fashion, since $y \in S_2(x_\infty)$, we have $f(y) < f(x_\infty) - \varepsilon d(y, x_\infty)$, see (37). Since the function $x \to f(x) - f(y) - \varepsilon d(y, x)$ is lower semicontinuous and with positive value at x_∞, we have $f(x_n) - f(y) - \varepsilon d(y, x_n) > 0$ for all but finitely many n's, ergo $y \in S_2(x_n)$ eventually.

Since $y \in S_3(x_\infty)$, we have by (38) that

$$|\nabla f|(y) < |\nabla f|(x_\infty) + 2c(2 + |\nabla f|(x_\infty))d(y, x_\infty).$$

From Lemma 2.8 we have that the function $x \to |\nabla f|(x)$ is lower semicontinuous, thus the function $x \to |\nabla f|(x) + (2c + |\nabla f|(x))d(y, x)$ is lower semicontinuous, so if we replace x_∞ by x_n in the above inequality, it will still be valid for large n's, meaning that $y \in S_3(x)$ eventually.

Property $(*)$ of S being established, we can now apply Theorem 2.12. From (40) it follows that for S only (a) is possible.

If a finite S-orbit starting at x_0, say $x_0, x_1, \ldots, x_n$, ends outside of $\operatorname{dom} S$, then because S maps $\operatorname{dom}|\nabla f| \setminus \{\bar{x}\}$ into $\operatorname{dom}|\nabla f|$, necessarily $x_n \in \operatorname{dom}|\nabla f|$. Since $S(x_n) = \varnothing$, from (39) it follows that $x_n = \bar{x}$. Iterating (36), we get $f(x_0) - g(x_0) > 0$. Finally, let $(x_n)_0^\infty$ be an infinite S-orbit of finite length starting at x_0 and ending at x_∞ with $S(x_\infty) = \varnothing$. In the same way as in the proof of (41) we see that $(|\nabla f|(x_n))_0^\infty$ is a bounded sequence, so that Lemma 2.8 implies $x_\infty \in \operatorname{dom}|\nabla f|$. Since $S(x_\infty) = \emptyset$, we have $x_\infty = \bar{x}$. Since $(x_n)_0^\infty$ is a S_1-orbit, we have $f(x_n) - g(x_n) < f(x_0) - g(x_0)$ for all $n \geq 0$ and (41) yields $f(x_0) \geq g(x_0)$. (We can actually obtain strict inequality, but this is not important.) □

Theorem 2.16. *Let (X, d) be a complete metric space and let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper and lower semicontinuous functions. Let f be regularly sloped of order 2 with coefficient $c > 0$ and have sharp minimum at $\bar{x}$. If*

$$|\nabla g|(x) \leq |\nabla f|(x), \quad \forall x \in X, \tag{42}$$

and (33) is satisfied, then

$$f(x) \geq (f(\bar{x}) - g(\bar{x})) + g(x), \quad \forall x \in X. \tag{43}$$

Proof. Since $\bar{x}$ is the minimum of f, we have $|\nabla f|(\bar{x}) = 0$. From (42) it follows that $|\nabla g|(\bar{x}) = 0$ and in particular $g(\bar{x}) < \infty$. This means that the right hand side of (43) is well defined, and also we can assume without loss of generality that

$$f(\bar{x}) = g(\bar{x}) = 0.$$

We have to prove that $$f(x) \geq g(x), \quad \forall x \in X. \tag{44}$$

Since $f \geq 0$, $$f = \inf_{\delta > 0} (1 + \delta) f.$$

For $\delta > 0$ set $f_\delta := (1 + \delta) f$. Since (44) is trivial if $f(x) = \infty$, it is sufficient to prove that

$$f_\delta(x) \geq g(x), \quad \forall x \in \operatorname{dom} f, \ \forall \delta > 0.$$

Fix an arbitrary $\delta > 0$. By definition,

$$|\nabla f_\delta|(x) = (1 + \delta)|\nabla f|(x), \quad \forall x \in X.$$

From this and (42) it follows that

$$|\nabla f_\delta|(x) > |\nabla g|(x), \quad \forall x \in \operatorname{dom}|\nabla f_\delta| \setminus \{\bar{x}\},$$

that is – because $\operatorname{dom}|\nabla f_\delta| = \operatorname{dom}|\nabla f|$ – the functions f_δ and g satisfy (32). Clearly, $\bar{x}$ is a sharp minimum to f_δ, so Proposition 2.15 gives (see (34))

$$f_\delta(x) - g(x) \geq 0, \quad \forall x \in \operatorname{dom} f.$$ □

3. Clarke-Rockafellar subdifferential

To the end of the article we work in a Banach space denoted by $(E, \|\cdot\|)$ with closed unit ball B_E and dual E^*.

Provided $f : E \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous at $x \in \operatorname{dom} f$, the Clarke-Rockafellar directional derivative, e.g. [3, p.97], at the point x at direction $h \in E$ is:

$$f^\circ(x;h) := \sup_{\varepsilon>0} f^\circ_\varepsilon(x;h), \text{ where} \tag{45}$$

$$f^\circ_\varepsilon(x;h) := \limsup_{y \to_f x, t \searrow 0} \frac{\inf f(y + t(h + \varepsilon B_X)) - f(y)}{t}.$$

The Clarke-Rockafellar subdifferential of f at $x \in \operatorname{dom} f$ is defined as:

$$\partial f(x) := \{p \in E^* : \ p(h) \le f^\circ(x;h), \ \forall h \in E\}. \tag{46}$$

Of course, $\partial f(x) = \varnothing$ if $f(x) = +\infty$. It can be shown that

$$\begin{aligned} \operatorname{dom} \partial f &= \{x \in \operatorname{dom} f : \ f^\circ(x;h) > -\infty, \ \forall h \in E\} \\ &= \{x \in \operatorname{dom} f : \ \exists c > 0 : \ f^\circ(x;h) \ge -c\|h\|, \ \forall h \in E\}, \end{aligned}$$

and that on the latter set the function $h \to f^\circ(x;h)$ is sublinear, thus convex, and lower semicontinuous function from E to $(-\infty, \infty]$. It is clear that $\partial f(x)$ is a convex and w^*-closed subset of E^*.

Note that if f is λ-Lipschitz around x, then $\inf f(y + t(h + \varepsilon B_E)) \ge f(y + th) - \lambda t\varepsilon$, so for a Lipschitz function (45) reduces to the much simpler Clarke's formula:

$$f^\circ(x;h) = \limsup_{\substack{y \to x \\ t \searrow 0}} \frac{f(y + th) - f(y)}{t}.$$

We consider also the Fréchet subdifferential of f at $x \in \operatorname{dom} f$:

$$p \in \partial_F f(x) \iff \liminf_{\|h\| \to 0} \frac{f(x + h) - f(x) - p(h)}{\|h\|} \ge 0,$$

otherwise $\partial_F(x) = \varnothing$. It is easy to check that $\partial_F \subset \partial$, see e.g. [21, Proposition 4.8(a)].

The function $f : E \to \mathbb{R} \cup \{+\infty\}$ is called *F-regular* at $x \in \operatorname{dom} f$ if

$$\partial_F f(x) = \partial f(x),$$

and it is called *F-regular* if it is so at any point $x \in \operatorname{dom} f$.

The basic example of F-regular functions are the convex continuous functions.

We will use the following Sum Theorem.

Theorem 3.1. (Sum Theorem) *Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous and F-regular function. Let $h : E \to \mathbb{R}$ be locally Lipschitz and F-regular function. Then $f + h$ is F-regular, and*

$$\partial(f + h)(x) = \partial f(x) + \partial h(x), \quad \forall x \in E. \tag{47}$$

Proof. First, let $x \in \operatorname{dom} f$. From [21, Theorem 2.98] we have that

$$\partial(f + h)(x) \subset \partial f(x) + \partial h(x).$$

To get the opposite inclusion, we use that

$$\partial f(x) + \partial h(x) = \partial_F f(x) + \partial_F h(x)$$

by F-regularity of both functions, and that

$$\partial_F f(x) + \partial_F h(x) \subset \partial_F (f+h)(x),$$

see e.g. [21, Proposition 4.11]. Hence,

$$\partial f(x) + \partial h(x) \subset \partial_F (f+h)(x) \subset \partial (f+h)(x), \quad \forall x \in \operatorname{dom} f.$$

If $x \notin \operatorname{dom} f$, then $x \notin \operatorname{dom}(f+h)$ either, hence $\partial(f+h)(x) = \partial f(x) = \varnothing$. Then $\partial f(x) + \partial h(x) = \varnothing$ by definition, and (47) is established. □

F-regularity provides a concise condition allowing the transfer of subdifferential inclusions to slope inequalities.

Proposition 3.2. *Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous and F-regular. Then*

$$\operatorname{dom} |\nabla f| = \operatorname{dom} \partial f, \tag{48}$$

and

$$|\nabla f|(x) = \min\{\|p\| : \ p \in \partial f(x)\}, \quad \forall x \in \operatorname{dom} |\nabla f|. \tag{49}$$

Proof. Note that, because the intersection of $\partial f(x)$ with a dual ball is w^*-compact, and the dual norm is w^* upper semicontinuous, the minimum in the right hand side of (49) is really attained.

Let $x_0 \in \operatorname{dom} |\nabla f|$ and let $r > |\nabla f|(x_0)$ be arbitrary. By definition, the function

$$u(x) := f(x) + r\|x - x_0\|$$

attains a local minimum at x_0, so $0 \in \partial u(x_0)$. By the Sum Theorem – Theorem 3.1 – there are $\bar{p} \in \partial f(x_0)$ and $q \in \partial r\|\cdot - x_0\|(x_0)$ such that $\|q\| \le r$, and $\bar{p} + q = 0$. Clearly, $\|\bar{p}\| \le r$. Thus we get

$$\operatorname{dom} |\nabla f| \subset \operatorname{dom} \partial f, \quad \text{and} \quad r \ge \|\bar{p}\| \ge \min\{\|p\| : p \in \partial f(x_0)\}. \tag{50}$$

The latter being true for any $r > |\nabla f|(x_0)$, it results

$$|\nabla f|(x_0) \ge \min\{\|p\| : \ p \in \partial f(x_0)\}. \tag{51}$$

Note that for these we did not use the F-regularity of f.

Let now $x_0 \in \operatorname{dom} \partial f$ and $p_0 \in \partial f(x_0)$ be such that $\|p_0\| = \min\{\|p\| : \ p \in \partial f(x_0)\}$. Since f is F-regular, $p_0 \in \partial_F f(x_0)$, that is,

$$f(x) - f(x_0) \ge p_0(x - x_0) + \alpha(x)\|x - x_0\|,$$

where $\alpha(x) \to 0$, as $x \to x_0$. So, $f(x) - f(x_0) \ge -(\|p_0\| - \alpha(x))\|x - x_0\|$, and

$$\liminf_{x \to x_0} \frac{f(x) - f(x_0)}{\|x - x_0\|} \ge -\|p_0\|,$$

meaning that

$$|\nabla f|(x_0) \le \|p_0\|. \tag{52}$$

So, $x_0 \in \operatorname{dom}|\nabla f|$ and, since $x_0 \in \operatorname{dom}\partial f$ was arbitrary, we get $\operatorname{dom}\partial f \subset \operatorname{dom}|\nabla f|$, which together with (50) gives (48). On the other hand, (51) and (52) give (49). □

4. Primal lower regular functions

We start with some preliminary results concerning primal lower regular functions defined on a Banach space $(E, \|\cdot\|)$.

First, observe that if $\bar{x} \in \operatorname{plr}_{c,\delta} f$, then f is F-regular on $B^\circ(\bar{x};\delta)$, see [21, Theorem 11.16(a)].

Lemma 4.1. *If $\bar{x} \in \operatorname{plr}_{c,\delta} f$ for some $c, \delta > 0$, and $\alpha \in (0,1)$, then*

$$\bar{x} \in \operatorname{plr}_{c,\delta}(\alpha f).$$

Proof. Let $x, y \in B^\circ(\bar{x};\delta)$ and $p \in \partial f(x)$ be arbitrary. Multiplying (1) by $\alpha > 0$, and using that $p_1 := \alpha p \in \partial(\alpha f)(x)$, we get

$$\begin{aligned} \alpha f(y) &\geq \alpha f(x) + p_1(y-x) - c(\|p_1\| + \alpha)\|y-x\|^2 \\ &\geq \alpha f(x) + p_1(y-x) - c(\|p_1\| + 1)\|y-x\|^2. \end{aligned}$$

Because $\partial(\alpha f) = \alpha \partial f$, we are done. □

Lemma 4.2. *Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper and lower semicontinuous function, and let $h : E \to \mathbb{R}$ be a convex and continuous function.*

If $$\bar{x} \in \operatorname{plr}_{c,\delta} f \quad \text{and} \quad \operatorname{Lip} h(B^\circ(\bar{x};\delta)) = L,$$

then $$\bar{x} \in \operatorname{plr}_{c(L+1),\delta}(f+h).$$

Proof. Denote $u := f + h$ and fix arbitrary $x \in B^\circ(\bar{x};\delta)$ and $\xi \in \partial u(x)$.

From the Sum Theorem – Theorem 3.1 – we have that $\xi = p + q$ where $p \in \partial f(x)$ and $q \in \partial h(x)$. The latter means

$$h(y) \geq h(x) + \langle q, y - x\rangle, \ \forall y \in X.$$

Adding this and (1) together yields

$$u(y) \geq u(x) + \langle \xi, y-x\rangle - c\,(\|p\| + 1)\,\|y-x\|^2, \quad \forall y \in B^\circ(\bar{x};\delta). \tag{53}$$

Note that, since $\|p\| = \|\xi - q\| \leq \|\xi\| + \|q\|$,

$$\|p\| + 1 \leq \|\xi\| + \|q\| + 1 \leq (\|q\| + 1)(\|\xi\| + 1),$$

and because $\|q\| \leq L$, the Lipshitz constant of h around $\bar{x}$, (53) gives

$$u(y) \geq u(x) + \langle \xi, y-x\rangle - c(L+1)(\|\xi\| + 1)\|y-x\|^2, \quad \forall y \in B^\circ(\bar{x};\delta). \qquad \square$$

Lemma 4.3. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper and lower semicontinuous function. Let*

$$\bar{x} \in \operatorname{plr}_{c,\delta} f, \tag{54}$$

for some $c, \delta > 0$, and let $p \in \partial f(\bar{x})$ be such that

$$\|p\| < 1. \tag{55}$$

For the function $$f_1(x) := f(x) - p(x) + 4\|x - \bar{x}\| \tag{56}$$
it is fulfilled that $\bar{x} \in \mathrm{plr}_{c',\delta'} f_1$, *with*
$$c' = 6c, \quad \delta' = \min\left\{\delta, \frac{1}{9c}\right\}. \tag{57}$$
Moreover, $f_1(\bar{x}) \leq \inf f_1(B^\circ(\bar{x};\delta'))$, *and*
$$|\nabla f_1|(x) \geq 1, \quad \forall x \in B^\circ(\bar{x};\delta') \setminus \{\bar{x}\}. \tag{58}$$

Proof. Assume that, contrary to (58), there is $x \in B^\circ(\bar{x};\delta') \setminus \{\bar{x}\}$ such that $|\nabla f_1|(x) < 1$. From Proposition 3.2, see (49), there is $\xi \in \partial f_1(x)$ such that $\|\xi\| < 1$. By the Sum Theorem – Theorem 3.1 – we have that
$$\xi = q - p + r,$$
where $q \in \partial f(x)$ and $r \in \partial(4\|\cdot - \bar{x}\|)(x)$. It is immediate that, since $x \neq \bar{x}$,
$$\|r\| = 4, \text{ and } r(\bar{x} - x) = -4\|\bar{x} - x\|.$$
Since $\|\xi\| < 1$ and $\|p\| < 1$, we have
$$q(\bar{x} - x) = (\xi + p)(\bar{x} - x) - r(\bar{x} - x) \geq 2\|\bar{x} - x\|.$$
By definition and using that $\|q\| \leq 2 + \|r\| = 6$,
$$\begin{aligned} f(\bar{x}) &\geq f(x) + q(\bar{x} - x) - c(\|q\| + 1)\|\bar{x} - x\|^2 \\ &\geq f(x) + 2\|\bar{x} - x\| - 7c\|\bar{x} - x\|^2. \end{aligned}$$
But, again by definition, and using that $\|p\| < 1$,
$$\begin{aligned} f(x) &\geq f(\bar{x}) + p(x - \bar{x}) - c(\|p\| + 1)\|x - \bar{x}\|^2 \\ &> f(\bar{x}) - \|x - \bar{x}\| - 2c\|x - \bar{x}\|^2. \end{aligned}$$
Now, putting the above two together, we get
$$0 > \|x - \bar{x}\| - 9c\|x - \bar{x}\|^2,$$
meaning that $\|x - \bar{x}\| > 1/9c \geq \delta'$, see (57), contradiction.

Since the function $h(x) = -p(x) + 4\|x - \bar{x}\|$ is convex and globally Lipschitz with constant less or equal to $\|p\| + 4 < 5$, Lemma 4.2 immediately gives that $\bar{x} \in \mathrm{plr}_{c',\delta} f_1$ with c' given by (57). Since $\delta' \leq \delta$, we are done.

Moreover, for any $x \in B^\circ(\bar{x};\delta')$ it holds that
$$f(x) \geq f(\bar{x}) + p(x - \bar{x}) - c(\|p\| + 1)\|x - \bar{x}\|^2.$$
Hence, having in mind that $\|p\| < 1$,
$$f(x) - p(x) \geq f(\bar{x}) - p(\bar{x}) - 2c\|x - \bar{x}\|^2,$$
$$f(x) - p(x) + 4\|x - \bar{x}\| \geq f(\bar{x}) - p(\bar{x}) + 4\|x - \bar{x}\| - 2c\|x - \bar{x}\|^2,$$
$$f_1(x) \geq f_1(\bar{x}) + 2\|x - \bar{x}\|(2 - c\|x - \bar{x}\|).$$
Since $2c\|x - \bar{x}\| < 1$ for $x \in B^\circ(\bar{x};\delta')$, we have reached per force that
$$f_1(\bar{x}) \leq \inf f_1(B^\circ(\bar{x};\delta')).$$
□

Lemma 4.4. *Let the functions $f, g : E \to \mathbb{R} \cup \{+\infty\}$ be proper and lower semicontinuous, and let for some $c, \delta > 0$,*

$$\bar{x} \in \operatorname{plr}_{c,\delta} f \cap \operatorname{plr}_{c,\delta} g \cap \operatorname{dom} \partial f. \tag{59}$$

Let $$\partial f(x) \subset \partial g(x), \quad \forall x \in B^\circ(\bar{x}; \delta). \tag{60}$$

Then $$f(x) \geq g(x) + (f(\bar{x}) - g(\bar{x})), \quad \forall x \in B^\circ(\bar{x}; \delta'), \tag{61}$$

where δ' is given by (57).

Proof. First, let us note that from (59) and (60) it follows that $\bar{x} \in \operatorname{dom} g$, so the right hand side of (61) is well defined.

We can assume without loss of generality that $\bar{x} = 0$ and $f(\bar{x}) = g(\bar{x}) = 0$.

Fix an arbitrary $x_0 \in \delta' B_E^\circ$. We need to show that $f(x_0) \geq g(x_0)$, so we can assume without loss of generality that $f(x_0) < \infty$.

Let $\nu \in (0, 1)$ be such that $\nu + \|x_0\| < \delta'$.

From Proposition 3.2 it follows that $0 \in \operatorname{dom} |\nabla f|$. Pick $\alpha \in (0, 1)$ such that

$$|\nabla(\alpha f)|(0) < \min\{1, \nu/\delta'\}, \ \inf(\alpha f)(\delta' B_E) > -\nu, \ (\alpha f)(x_0) < \nu. \tag{62}$$

Note that by definition f is bounded below on $\delta' B_E$, so such choice is possible. From Lemma 4.1 and (59) we have that

$$0 \in \operatorname{plr}_{c,\delta}(\alpha f) \cap \operatorname{plr}_{c,\delta}(\alpha g). \tag{63}$$

From (62) and Proposition 3.2, see (49), it follows that there is a $p \in \partial(\alpha f)(0)$ such that

$$\|p\| < \min\{1, \nu/\delta'\}. \tag{64}$$

In particular, (55) holds. Therefore, Lemma 4.3 gives that

$$0 \in \operatorname{plr}_{c',\delta'} f_1,$$

where $f_1(x) = \alpha f(x) + h(x)$, $h(x) = -p(x) + 4\|x\|$, and $c' = 6c$, see (57). Moreover, $\inf f_1(\delta' B_E^\circ) \geq 0$ and

$$|\nabla f_1|(x) \geq 1, \quad \forall x \in \delta' B_E^\circ \setminus \{0\}. \tag{65}$$

Define $g_1(x) := \alpha g(x) + h(x)$.

Using (59), (60) and Lemma 4.3, we get

$$0 \in \operatorname{plr}_{c',\delta'} g_1. \tag{66}$$

Since $\partial(\alpha f) = \alpha \partial f$, and $\partial(\alpha g) = \alpha \partial g$ from (60) it holds that $\partial(\alpha f) \subset \partial(\alpha g)$. Now the Sum Theorem – Theorem 3.1 – gives that

$$\partial f_1(x) = \partial \alpha f(x) + \partial h(x) \subset \partial \alpha g(x) + \partial h(x) = \partial g_1(x), \quad \forall x \in \delta B_E^\circ.$$

From Proposition 3.2 it follows that

$$|\nabla f_1|(x) \geq |\nabla g_1|(x), \quad \forall x \in \delta B_E^\circ. \tag{67}$$

Set $$Y := \{x \in \delta' B_E^\circ : f_1(x) \le f_1(x_0)\}.$$

Since 0 is the minimum of f_1 on $\delta' B_E^\circ$, we have that

$$0 \in Y. \tag{68}$$

We claim that $Y \subset (\nu + \|x_0\|)B_E$.

Indeed, if x is such that $\nu + \|x_0\| < \|x\| < \delta'$, then, see (62) and (64),

$$f_1(x) = \alpha f(x) - p(x) + 4\|x\| > -\nu - \frac{\nu}{\delta'}\delta' + 4(\nu + \|x_0\|) = 2\nu + 4\|x_0\|,$$

while $f_1(x_0) < \nu + \frac{\nu}{\delta'}\delta' + 4\|x_0\| = 2\nu + 4\|x_0\|$, so $x \notin Y$.

This means that Y is closed, because f_1 is lower semicontinuous. So, with respect to the canonical metric induced by $\|\cdot\|$, Y is a complete metric space. In addition, Lemma 2.3 gives that for

$$\overline{f}_1 := f_1 \restriction Y,$$

we have $$|\nabla \overline{f}_1|(x) = |\nabla f_1|(x), \quad \forall x \in Y. \tag{69}$$

Set $$\overline{g}_1 := g_1 \restriction Y.$$

Obviously, $|\nabla \overline{g}_1|(x) \le |\nabla g_1|(x)$, for all $x \in Y$, so (67) gives

$$|\nabla \overline{f}_1|(x) \ge |\nabla \overline{g}_1|(x), \quad \forall x \in Y.$$

From this and (66) we have that for all $x \in \operatorname{dom} |\nabla \overline{f}_1| \subset \operatorname{dom} |\nabla \overline{g}_1|$, and all $y \in Y$ we have

$$\begin{aligned} \overline{g}_1(y) &\ge \overline{g}_1(x) - |\nabla \overline{g}_1|(x)\|y - x\| - c(|\nabla \overline{g}_1|(x) + 1)\|y - x\|^2 \\ &\ge \overline{g}_1(x) - |\nabla \overline{f}_1|(x)\|y - x\| - c(|\nabla \overline{f}_1|(x) + 1)\|y - x\|^2 \end{aligned}$$

Finally, since 0 is the minimum of $\overline{f}_1$ on Y, it is sharp by (65) and (69), and we can apply Theorem 2.16 to conclude that $\overline{f}_1 \ge \overline{g}_1$ on Y. Since $x_0 \in Y$, this means $\alpha f(x_0) + p(x_0) + 4\|x_0\| \ge \alpha g(x_0) + p(x_0) + 4\|x_0\|$, therefore $f(x_0) \ge g(x_0)$ and we are done. □

Proof of Theorem 1.2. We again assume that $\bar{x} = 0$.

From (4) and (57) it is clear that $\hat{\delta} = \delta'/2$.

For any fixed $x \in \hat{\delta} B_E^\circ$ such that $|\nabla f|(x) < \infty$ (and there are such x's because of the graphical density of $\operatorname{dom} |\nabla f|$ in $\operatorname{dom} f$, see e.g. [14, Lemma 3.7.]), Lemma 4.4 applied to f and g and then to g and f on $B^\circ(x; \hat{\delta})$ shows that:

$$f(y) = g(y) + (f(x) - g(x)), \quad \forall y \in B^\circ(x; \hat{\delta}).$$

But $0 \in B^\circ(x; \hat{\delta})$, so

$$f(0) = g(0) + (f(x) - g(x)) \iff f(x) - g(x) = f(0) - g(0).$$

That is, for $a := f(0) - g(0)$ we have

$$f(y) = g(y) + a, \quad \forall y \in B^\circ(x; \hat{\delta}),\ \forall x \in \hat{\delta}B_E^\circ \cap \operatorname{dom}|\nabla f|.$$

But because $\operatorname{dom}|\nabla f|$ is dense in $\operatorname{dom} f$ and $0 \in \operatorname{dom} f$

$$\hat{\delta}B_E^\circ \subset \left(\hat{\delta}B_E^\circ \cap \operatorname{dom}|\nabla f|\right) + \hat{\delta}B_E^\circ.$$

The proof is now complete. □

Acknowledgements. The authors dedicate this article to Prof. Lionel Thibault and express their sincere gratitude for his inspiration and constant support.

Funding. The work of M. Ivanov and N. Zlateva is supported by the European Union-NextGenerationEU, through the National Recovery and Resilience Plan of the Republic of Bulgaria, project SUMMIT BG-RRP-2.004-0008-C01 and by the Scientific Fund of Sofia University under grant 80-10-50/23.05.2025.

The work of M. Konstantinov is supported by the Bulgarian Ministry of Education and Science under the National Research Programme Young scientists and postdoctoral students approved by DCM #206/07.04.2022.

References

[1] D. Azé: *A unified theory for metric regularity of multifunctions*, J. Convex Analysis 13/2 (2006) 225–252.

[2] D. Azé, J.-N. Corvellec: *On some variational properties of metric spaces*, J. Fixed Point Theory 5 (2009) 185–200.

[3] F. H. Clarke: *Optimization and Nonsmooth Analysis*, Wiley, New York (1983).

[4] A. Daniilidis, T. M. Le, D. Salas: *Metric compatibility and determination in complete metric spaces*, Math. Zeitschrift 308/4 (2024), art.no. 62, 31p.

[5] A. Daniilidis, L. Miclo, D. Salas: *Descent modulus and applications*, J. Funct. Anal. 287/11 (2024), art. no. 110626, 54p.

[6] A. Daniilidis, D. Salas: *A determination theorem in terms of the metric slope*, Proc. Amer. Math. Soc. 150 (2022) 4325–4333.

[7] E. De Giorgi, A. Marino, M. Tosques: *Problems of evolution in metric spaces and maximal decreasing curve*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. 68/3 (1980) 180–187.

[8] D. Drusvyatskiy, A. D. Ioffe, A. S. Lewis: *Curves of descent*, SIAM J. Control Optim. 53/1 (2015) 114–138.

[9] A. Ioffe: *Variational Analysis of Regular Mappings. Theory and Applications*, Springer Monographs in Mathematics, Springer, Cham (2017).

[10] M. Ivanov, D. Kamburova, N. Zlateva: *On Long Orbit Empty Value (LOEV) principle*, arXiv: 2407.07189 (2024).

[11] M. Ivanov, N. Zlateva: *On primal lower nice property*, Compt. Rend. Acad. Bulg. Sci. 54/11 (2001) 5–10.

[12] M. Ivanov, N. Zlateva: *Subdifferential characterization of primal lower-nice functions on smooth Banach spaces*, Compt. Rend. Acad. Bulg. Sci. 57/5 (2004) 13–18.

[13] M. Ivanov, N. Zlateva: *Long Orbit or Empty Value principle, fixed point and surjectivity theorems*, Compt. Rend. Acad. Bulg. Sci. 69/5 (2016) 553–562.

[14] M. Ivanov, N. Zlateva: *Slopes and Moreau-Rockafellar theorem*, J. Convex Analysis 32/2 (2025) 359–374.

[15] M. Konstantinov, N. Zlateva: *Epigraphical characterization of uniformly lower regular functions in Hilbert spaces*, J. Convex Analysis 30/4 (2023) 1139–1153.

[16] P. Pérez-Aros, D. Salas, E. Vilches: *Determination of convex functions via subgradients of minimal norm*, Math. Programming A 190/1-2 (2021) 561–583.

[17] R. R. Phelps: *Convex Functions, Monotone Operators and Differentiability*, 2nd ed., Lecture Notes in Mathematics 1364, Springer, Berlin (1993).

[18] R. A. Poliquin: *Integration of subdifferentials of nonconvex functions*, Nonlinear Anal. Theor. Meth. Appl. 17 (1991) 385–398

[19] R. A. Poliquin, R. T. Rockafellar, L. Thibault: *Local differentiability of distance functions*, Trans. Amer. Math. Soc. 352/11 (2000) 5231–5249.

[20] R. T. Rockafellar: *First- and second-order epi-differentiability in nonlinear programming*, Trans. Amer. Math. Soc. 307/1 (1988) 75–108.

[21] L. Thibault: *Unilateral Variational Analysis in Banach Spaces. I: General Theory, II: Special Classes of Functions and Sets*, World Scientific, Singapore (2023).

[22] L. Thibault, D. Zagrodny: *Integration of subdifferentials of lower semicontinuous functions on Banach spaces*, J. Math. Analysis Appl. 189/1 (1995) 33–58.

[23] L. Thibault, D. Zagrodny: *Determining functions by slopes*, Comm. Contemp. Math. 25/7 (2023), art.no. 225014, 30p.