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A Note on the Carathéodory Number of the Joint Numerical Range

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We show that the Carathéodory number of the joint numerical range of d many bounded self-adjoint operators is at most $d-1$, and even at most $d-2$ if the underlying Hilbert space has dimension at least 3. This extension of the classical convexity results for numerical ranges shows that also joint numerical ranges are significantly less non-convex than general sets.

Keywords: Joint numerical range, convexity, Carathéodory number.

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1. Introduction and Preliminaries

Let $T_1, \ldots, T_d \in \mathcal{B}(H)_{\text{sa}}$ be self-adjoint bounded linear operators on the Hilbert space H. Their *joint numerical range* is defined as

$$\mathcal{W}(T_1, \ldots, T_d) := \{(\langle T_1 h, h\rangle, \ldots, \langle T_d h, h\rangle) \mid h \in H, \|h\| = 1\} \subseteq \mathbb{R}^d.$$

For $d \leqslant 2$ the joint numerical range is always convex, a famous result by Toeplitz and Hausdorff [5, 8]. This is still true if $d = 3$ and $\dim(H) \geqslant 3$ [1], but fails for $d \geqslant 4$ in general.

An interesting measure for the non-convexity of a set $S \subseteq \mathbb{R}^d$ is its *Carathéodory number* $\mathfrak{c}(S)$. It is defined as the smallest positive integer n, such that every element in the convex hull of S is a convex combination of at most n elements from S. A set is clearly convex if and only if its Caratheodory number is 1. Caratheodory's Theorem [3, 7] states that $\mathfrak{c}(S) \leqslant d+1$ holds for any set $S \subseteq \mathbb{R}^d$, and any set of $d+1$ affinely independent points shows that this general bound is optimal. However, if $S \subseteq \mathbb{R}^d$ is connected, then $\mathfrak{c}(S) \leqslant d$ [2, 4].

As the image of the unit sphere in H under a continuous map, $\mathcal{W}(T_1, \ldots, T_d)$ is clearly path-connected, and thus $\mathfrak{c}(\mathcal{W}(T_1, \ldots, T_d)) \leqslant d$. In this note we observe that this can be improved by one, and even by two if $\dim(H) \geqslant 3$ (Theorem 2.2). This shows that joint numerical ranges are significantly less non-convex than general connected sets. Our proof is a combination of some of the standard methods to prove convexity of the classical numerical range, but suitably applied to joint numerical ranges.

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2. Main result

The proof of the following proposition contains the initial idea from [5], combined with [6], and in the multidimensional setup. It shows that joint numerical ranges are path-connected in a strong sense.

Proposition 2.1. *For any $T_1, \ldots, T_d \in \mathcal{B}(H)_{\mathrm{sa}}$ and for any affine hyperplane $P \subseteq \mathbb{R}^d$,*

$$\mathcal{W}(T_1, \ldots, T_d) \cap P$$

is path-connected. If $\dim(H) \geqslant 3$, *this also holds when* $\operatorname{codim}(P) = 2$.

Proof. For the affine hyperplane P we can assume without loss of generality that $P = \{x_d = 0\}$. For $p_0, p_1 \in \mathcal{W}(T_1, \ldots, T_d) \cap P$ there are unit vectors $h_0, h_1 \in H$ with

$$p_i = (\langle T_1 h_i, h_i \rangle, \ldots, \underbrace{\langle T_d h_i, h_i \rangle}_{=0}).$$

Now as already shown and used in [5], the set

$$L := \{h \in H \mid \|h\| = 1, \langle T_d h, h \rangle = 0\}$$

is path-connected. So there is a continuous path h_t from h_0 to h_1 in L, and

$$p_t := (\langle T_1 h_t, h_t \rangle, \ldots, \underbrace{\langle T_d h_t, h_t \rangle}_{=0})$$

is thus a continuous path from p_0 to p_1 in $\mathcal{W}(T_1, \ldots, T_d) \cap P$.

Now in the case $\operatorname{codim}(P) = 2$ we can assume $P = \{x_{d-1} = 0 \wedge x_d = 0\}$. We repeat the same argument, this times using that

$$\begin{aligned} L &= \{h \in H \mid \|h\| = 1, \langle T_{d-1} h, h \rangle = \langle T_d h, h \rangle = 0\} \\ &= \{h \in H \mid \|h\| = 1, \langle (T_{d-1} + iT_d) h, h \rangle = 0\} \end{aligned}$$

is path connected. This was proven in [6] for the case that $\dim(H) \geqslant 3$. □

We can now prove our main result:

Theorem 2.2. *For $d \geqslant 2$ and any $T_1, \ldots, T_d \in \mathcal{B}(H)_{\mathrm{sa}}$ we have*

$$\mathfrak{c}\,(\mathcal{W}(T_1, \ldots, T_d)) \leqslant d - 1.$$

In case $d \geqslant 3$ and $\dim(H) \geqslant 3$ *we even have* $\mathfrak{c}\,(\mathcal{W}(T_1, \ldots, T_d)) \leqslant d - 2$.

Proof. Let $p \in \operatorname{conv}(\mathcal{W}(T_1, \ldots, T_d))$. From [4, 2] we already know that

$$\mathfrak{c}\,(\mathcal{W}(T_1, \ldots, T_d)) \leqslant d,$$

thus p is a convex combination of d points from $\mathcal{W}(T_1, \ldots, T_d)$. Let P be an affine hyperplane in $\mathbb{R}^d$ containing all these points. Then p is already in the convex hull of $\mathcal{W}(T_1, \ldots, T_d) \cap P$. Since $\mathcal{W}(T_1, \ldots, T_d) \cap P$ is connected (Proposition 2.1) and lives in the $(d-1)$-dimensional space P, we obtain $\mathfrak{c}\left(\mathcal{W}(T_1,\ldots,T_d)\cap P\right) \leqslant d-1$, again from [4, 2]. Thus p is a convex combination of at most $d-1$ points from $\mathcal{W}(T_1,\ldots,T_d)$, which proves the first claim.

In case $\dim(H) \geqslant 3$ we can iterate this argument once. Write p as a convex combination of $d-1$ many points from $\mathcal{W}(T_1,\ldots T_d)$. These points live in an affine subspace P of dimension $d-2$, and since $\mathcal{W}(T_1,\ldots,T_d)\cap P$ is still connected by Proposition 2.1, [4, 2] again implies that p is a convex combination of at most $d-2$ elements from $\mathcal{W}(T_1,\ldots,T_d)$. □

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