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Equability and Strong Equability in Banach Spaces

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D. Yost [*Intersecting balls and proximinal subspaces in Banach spaces*, Ph. D. Thesis, University of Edinburgh (1979)] introduced a notion called equability for subspaces of a Banach space, as a sufficient condition for the proximinality and lower semi-continuity of the metric projection. This notion was later modified by S. Lalithambigai [*Ball proximinality of equable spaces*, Collect. Math. 60/1 (2009) 79–88] to study ball proximinality. In this article, we study both the equability notions. We establish some geometric characterizations and strengthen the existing best approximation theoretic consequences of equability notions. Specifically, we prove that these equability notions are related to quasi uniform rotundity and uniform rotundity. We observe that equability notions are transitive through M-summands and further explore their stability under ℓ_p-direct sum for $1 \leq p \leq \infty$. Additionally, we show that equability can be lifted to the space of all vector valued bounded (and continuous) functions defined on topological spaces.

Keywords: Equability, strong equability, quasi uniform rotundity, uniform rotundity, uniform strong proximinality, stability.

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1. Introduction

Let X be a real Banach space and X^* be its dual. The closed unit ball and the unit sphere of X are denoted by B_X and S_X respectively. Let Y be a closed subspace of X. For any $x \in Y$ and $r > 0$, we define $B_Y[x,r] = \{y \in Y : \|x-y\| \leq r\}$ and we denote $B_X[x,r]$ as $B[x,r]$. For any nonempty subset A of X and $x \in X$, the set of all best approximants of x from A is defined as $P_A(x) = \{a \in A : \|x-a\| = d(x,A)\}$, where $d(x,A) = \inf\{\|x-a\| : a \in A\}$. We say the subspace Y is proximinal on X if $P_Y(x)$ is nonempty for every $x \in X$. The map $x \mapsto P_Y(x)$ is called the metric projection of X onto Y. Throughout the article, we assume all subspaces to be closed.

Several researchers in best approximation theory have extensively investigated the problem of the existence of best approximants from subspaces and the continuity of the corresponding metric projection. It is widely known that well-posedness of the

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minimum norm(distance) problem leads to continuity of the metric projection, while generalized well-posedness leads to upper semi-continuity. Since this problem has numerous applications, many authors proposed various sufficient criteria to provide an affirmative answer to the mentioned problem.

Some of these sufficient conditions include quasi uniform rotundity by Calder et al. [5], U-proximinality by Lau [11], property P_2 by Mach [12], property-(A) by Prolla and Chiacchio [14] and uniform strong proximinality by Dutta et al. [7]. In this direction, in 1979, Yost [19] introduced a notion called equability for a subspace and established that if Y is equable in X, then Y is proximinal on X and the metric projection P_Y is Hausdorff continuous on X. The definition is as follows.

Definition 1.1. [19] Let Y be a subspace of X. We say that Y is *equable* in X if for every $\epsilon > 0$, there exist a $\delta > 0$ and a map $\psi_\epsilon : Y \to Y$ such that for every $y \in Y$, we have $\|y - \psi_\epsilon(y)\| \leq \epsilon$ and $B_X \cap B[y, 1 + \delta] \subseteq B[\psi_\epsilon(y), 1]$.

Later, Bandyopadhyay et al. [3] introduced a notion called ball proximinality of subspaces. The subspace Y is ball proximinal on X if $P_{B_Y}(x)$ is nonempty for all $x \in X$. In [3], it is observed that ball proximinal subspaces are proximinal, but the converse is not true. Naturally, one could ask whether there exists a more robust version of Yost's equability that would serve as a sufficient condition for ball proximinality. To address this question, Lalithambigai [10] presented a modified version of Definition 1.1 as follows. Since Lalithambigai referred this modified form also as equable, in this article, it is appropriate to rename it as "strongly equable" to avoid ambiguity. In [10], it is proved that if Y is strongly equable in X, then it is ball proximinal on X.

Definition 1.2. [10] Let Y be a subspace of X. We say that Y is *strongly equable* in X if for every $\epsilon > 0$, there exist a $\delta > 0$ and a map $\phi_\epsilon : Y \to [0, 1]$ such that for every $y \in Y$, we have $\|y - \phi_\epsilon(y)y\| \leq \epsilon$ and $B_X \cap B[y, 1 + \delta] \subseteq B[\phi_\epsilon(y)y, 1]$.

We say that the space X is equable (respectively, strongly equable) if X is equable (respectively, strongly equable) in X.

The main objective of this article is to examine the concepts of equability and strong equability. In [10, 19], these equability concepts are explored using the best approximation theoretic properties which are known to be closely associated to the geometric aspects of the space. Also, Yost observed that in an uniformly rotund space every subspace is equable in the whole space. Motivated from this, in this article, we attempt to connect the concepts of equability with certain geometric properties. Further, we focus on the stability of these equability notions. The structure of the article is as follows.

In Section 2, we present some fundamental results on equability notions and observe that strong equability is stronger than equability. Some equivalent forms of equability notions are presented. We observe that subspaces do not necessarily have to inherit equability. Nevertheless, it is proved that a norm one projection on an equable space has an equable range.

In Section 3, we prove that equability and strong equability coincide for one-dimensional subspaces. We provide a sufficient condition for a finite dimensional subspace to be equable in the whole space. We improve some of the existing relations of these equability notions with best approximation theoretic properties, which further enhance the continuity properties of the metric projection. In terms of certain geometric properties, we establish various characterizations of the equability notions. Indeed, we show that equability and strong equability are equivalent to quasi uniform rotundity and uniform rotundity, respectively. The latter equivalence leads to a characterization of spaces which are uniformly rotund in every direction in terms of strong equability. We provide an example demonstrating that certain equivalences in [1, Proposition 2.2] are only partially valid. Additionally, our geometric relations reveal that the stability results on strong equability in [10, Theorems 3.1 and 3.2] do not hold. We prove that the equability notions are transitive through M-summands. Further, we extend our study to examine the stability of the equability notions. In particular, we show that under finite ℓ_∞-direct sum the notion equability is stable but not strong equability. In addition, we establish that equability can be lifted to the space of bounded (and continuous) functions. As a consequence, the stability of equability under ℓ_∞ and c_0 product of a space is obtained.

2. Preliminaries

We begin this section with some essential observations on equability and strong equability.

Remark 2.1. Let Z and Y be two subspaces of X such that $Z \subseteq Y \subseteq X$. Then the following statements hold.

(1) If Y is strongly equable in X, then Y is equable in X. However, the converse is not true in general (see, Example 2.2).

(2) If Y is strongly equable in X, then Z is strongly equable in X. But the similar statement need not be true for equability (see, Example 2.2). Also, the converse need not be true for both equability and strong equability (see, Example 3.21).

(3) If Z is equable (respectively, strongly equable) in X, then Z is equable (respectively, strongly equable) in Y. But, the converse need not be true in general (see, Example 3.15).

(4) If Y is equable (respectively, strongly equable) in X, then Y is equable (respectively, strongly equable). The converse is not true for equability and strong equability (see, Example 3.15).

The following examples illustrate that strong equability is stronger than equability and the equability need not be inherited by subspaces.

Example 2.2. Let $X = \ell_\infty$ and $Y = c_0$.

(1) We show that Y is equable in X, but Y is not strongly equable. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$. For any $x = (x_i)_{i\in\mathbb{N}} \in Y$, define $\psi_\epsilon(x) = (\psi_\epsilon(x)_i)_{i\in\mathbb{N}}$, where

$$\psi_\epsilon(x)_i = \begin{cases} x_i + \epsilon, & \text{if } x_i < -\epsilon; \\ 0, & \text{if } |x_i| \leq \epsilon; \\ x_i - \epsilon, & \text{if } \epsilon < x_i, \end{cases}$$

for every $i \in \mathbb{N}$. It is easy to verify that

$$\psi_\epsilon(x) \in Y,\ \|x - \psi_\epsilon(x)\| \leq \epsilon \text{ and } B_X \cap B[x, 1+\delta] \subseteq B[\psi_\epsilon(x), 1] \text{ for every } x \in Y.$$

Thus, Y is equable in X. Suppose Y is strongly equable. Then, for $\epsilon \in (0,1)$ there exist a $\delta = \delta(\epsilon) > 0$ and a map $\phi_\epsilon : Y \to [0,1]$ such that $\|x - \phi_\epsilon(x)x\| \leq \epsilon$ and $B_Y \cap B_Y[x, 1+\delta] \subseteq B_Y[\phi_\epsilon(x)x, 1]$ for every $x \in Y$. Let $x = \delta e_1 - (2+\delta)e_2 \in Y$ and $w = -e_1 - e_2$. Since $w \in B_Y \cap B_Y[x, 1+\delta]$, we have $\|w - \phi_\epsilon(x)x\| \leq 1$, i.e., $1 + \phi_\epsilon(x)\delta \leq 1$ which implies $\phi_\epsilon(x) = 0$. Thus, $2 + \delta = \|x\| \leq \epsilon$, which is a contradiction. Therefore, Y is not strongly equable. Hence, by Remark 2.1, Y is not strongly equable in X.

(2) Let $f = \left(1, 1, \frac{1}{2}, \frac{1}{2^2}, \dots\right) \in \ell_1$. Consider $Z = ker(f) \subseteq Y$. Since $supp(f)$ is not finite and $2\|f\|_\infty < \|f\|_1$, by [17, Theorem 2], Z does not admit Chebyshev centers. Therefore, by [19, Proposition 1.7.1], Z is not equable. Hence, Z is not equable in X. However, by (1), we have Y is equable in X. □

Now, we provide some equivalent forms of equability, which will be used in the next section.

Proposition 2.3. *Let Y be a subspace of X. Then the following statements are equivalent.*

(1) *Y is equable in X.*

(2) *For every $\epsilon > 0$ there exist a $\delta \in (0,1)$ and a map $\gamma_\epsilon : Y \to Y$ such that for every $y \in Y$, we have $\|y - \gamma_\epsilon(y)\| \leq (1-\delta)\epsilon$ and*
$$B[0, 1-\delta] \cap B[y, 1] \subseteq B[\gamma_\epsilon(y), 1-\delta].$$

(3) *For every $\epsilon > 0$ there exists a $\delta > 0$ such that for each $y \in Y$ and $\beta > 0$, there exists a $z_{(y,\beta)} \in B[y, \epsilon] \cap Y$ satisfying $B_X \cap B[y, 1+\delta] \subseteq B[z_{(y,\beta)}, 1+\beta]$.*

(4) *For every summable sequence (ϵ_n) of positive real numbers, i.e., $\sum_{n=1}^\infty \epsilon_n < \infty$, there exists a sequence (δ_n) of positive real numbers with $\delta_n \to 0$ such that for any $y \in Y$ and $n \in \mathbb{N}$, there exists a $z \in B[y, \epsilon_n] \cap Y$ satisfying*
$$B_X \cap B[y, 1+\delta_n] \subseteq B[z, 1+\delta_{n+1}].$$

Proof. (1) ⇒ (2): Let $\epsilon > 0$. Consider δ and ψ_ϵ from Definition 1.1. Now, choose $\delta_1 = \frac{\delta}{1+\delta} > 0$ and consider the map $\gamma_\epsilon : Y \to Y$ defined as $\gamma_\epsilon(y) = \frac{1}{1+\delta}\psi_\epsilon((1+\delta)y)$ for every $y \in Y$. Let $y \in Y$. Observe that,

$$\|y - \gamma_\epsilon(y)\| = \frac{1}{1+\delta}\|(1+\delta)y - \psi_\epsilon((1+\delta)y)\| \leq (1-\delta_1)\epsilon.$$

Take $z \in B[0, 1-\delta_1] \cap B[y, 1]$. Since

$$(1+\delta)z \in B_X \cap B[(1+\delta)y, 1+\delta] \subseteq B[\psi_\epsilon((1+\delta)y), 1],$$

we have
$$\|z - \gamma_\epsilon(y)\| = \frac{1}{1+\delta}\|(1+\delta)z - \psi_\epsilon((1+\delta)y)\| \leq \frac{1}{1+\delta} = 1 - \delta_1.$$

Thus, $z \in B[\gamma_\epsilon(y), 1-\delta_1]$.

(2) ⇒ (3): Let $\epsilon > 0$. Take δ and γ_ϵ as in the statement (2). Choose $\delta_0 = \frac{\delta}{1-\delta} > 0$.

Let $y \in Y$ and $\beta > 0$. Consider $z_{(y,\beta)} = \frac{1}{1-\delta}\gamma_\epsilon((1-\delta)y)$. Observe that

$$\|y - z_{(y,\beta)}\| = \frac{1}{1-\delta}\|(1-\delta)y - \gamma_\epsilon((1-\delta)y)\| \le \frac{1}{1-\delta}(1-\delta)\epsilon = \epsilon.$$

Thus, $z_{(y,\beta)} \in B[y,\epsilon] \cap Y$. Now, let $w \in B_X \cap B[y, 1+\delta_0]$. Notice that

$$(1-\delta)w \in B[0, 1-\delta] \cap B[(1-\delta)y, 1].$$

By (2), it follows that $(1-\delta)w \in B[\gamma_\epsilon((1-\delta)y), 1-\delta]$. Therefore,

$$\|w - z_{(y,\beta)}\| \le 1 < 1 + \beta.$$

(3) $\Rightarrow$ (4): Let (ϵ_n) be a sequence of positive real numbers such that $\sum_{n=1}^\infty \epsilon_n < \infty$ and choose $0 < \delta_n < \epsilon_n$ from the assumption. Clearly, $\delta_n \to 0$. Now, let $y \in Y$ and $n \in \mathbb{N}$. For $\beta = \delta_{n+1}$, by (3), there exists a $z_{(y,n)} \in B[y, \epsilon_n] \cap Y$ satisfying $B_X \cap B[y, 1+\delta_n] \subseteq B[z_{(y,n)}, 1+\delta_{n+1}]$.

(4) $\Rightarrow$ (1): Let $\epsilon > 0$. Consider a sequence (ϵ_n) of positive real numbers such that $\sum_{n=1}^\infty \epsilon_n < \epsilon$ and (δ_n) is a sequence of positive real numbers satisfying statement (4). Let $y \in Y$. By assumption, there exists a $z_1 \in B[y, \epsilon_1] \cap Y$ such that

$$B_X \cap B[y, 1+\delta_1] \subseteq B[z_1, 1+\delta_2].$$

Now, corresponding to z_1, there exists a $z_2 \in B[z_1, \epsilon_2] \cap Y$ such that

$$B_X \cap B[z_1, 1+\delta_2] \subseteq B[z_2, 1+\delta_3].$$

Thus, $B_X \cap B[y, 1+\delta_1] \subseteq B[z_2, 1+\delta_3]$ and $\|z_2 - z_1\| \le \epsilon_2$. By induction, we get a sequence (z_n) in Y such that

$$B_X \cap B[y, 1+\delta_1] \subseteq B[z_n, 1+\delta_{n+1}] \text{ for every } n \in \mathbb{N}$$

and $\|z_n - z_{n-1}\| \le \epsilon_n$ for every $n \ge 2$. It is easy to verify that (z_n) is a Cauchy sequence in Y and hence converges to some point $z_y \in Y$. Now consider the map $\psi_\epsilon : Y \to Y$ defined as $\psi_\epsilon(y) = z_y$ for every $y \in Y$ and choose $\delta = \delta_1$. Observe that

$$\|\psi_\epsilon(y) - y\| = \lim_{n\to\infty} \|z_n - y\| \le \lim_{n\to\infty} \left(\sum_{k=2}^n \|z_k - z_{k-1}\| + \|z_1 - y\| \right) \le \lim_{n\to\infty} \sum_{k=1}^n \epsilon_k < \epsilon$$

and $B_X \cap B[y, 1+\delta_1] \subseteq \cap_{n\in\mathbb{N}} B[z_n, 1+\delta_{n+1}] \subseteq B[\psi_\epsilon(y), 1]$ for every $y \in Y$. Therefore, Y is equable in X. $\square$

Proposition 2.4. *Let Y be a subspace of X. Consider the following statements.*

(1) *Y is equable in X.*

(2) *For every $\epsilon > 0$, there exist a $\delta > 0$ and a map $\psi_\epsilon : Y \to Y$ such that for every $y \in Y$, we have $\|y - \psi_\epsilon(y)\| \le \epsilon$ and $B_Z \cap B_Z[y, 1+\delta] \subseteq B_Z[\psi_\epsilon(y), 1]$, whenever $Z = span\{Y, x\}$ for every $x \in X$.*

(3) *Y is equable in $span\{Y, x\}$ for every $x \in X$.*

Then, (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2): This implication follows from Definition 1.1.

(2) $\Rightarrow$ (1): Let $\epsilon > 0$. Choose $\delta > 0$ and $\psi_\epsilon : Y \to Y$ from the assumption. Let $y \in Y$. Observe that $\|y - \psi_\epsilon(y)\| \leq \epsilon$. Now, let $z \in B_X \cap B[y, 1 + \delta]$ and consider $Z = span\{Y, z\}$. Then, by assumption, we have

$$z \in B_Z \cap B_Z[y, 1 + \delta] \subseteq B_Z[\psi_\epsilon(y), 1] \subseteq B[\psi_\epsilon(y), 1].$$

Therefore, $B_X \cap B[y, 1 + \delta] \subseteq B[\psi_\epsilon(y), 1]$. Hence, Y is equable in X.

(2) $\Rightarrow$ (3): Obvious. □

The statement (2) of the preceding proposition can be considered as Y is equable in $span\{Y, x\}$ uniformly for every $x \in X$. In general, the implication (3) $\Rightarrow$ (1) of the preceding proposition need not be true (see, Example 3.15). It is easy to verify that Propositions 2.3 and 2.4 hold analogously for strong equability.

The assumption Y is equable in X leads to the following inclusions which are required to simplify some of the proofs in the next section.

Lemma 2.5. *Let Y be equable in X and $\epsilon > 0$. Assume $\delta = \delta(\epsilon)$ and $\psi_\epsilon : Y \to Y$ as in Definition 1.1. Then*

(1) *for any $w \in X$, $y \in Y$ and $r > 0$, we have*
$$B[w, r] \cap B[ry + w, r(1 + \delta)] \subseteq B[r\psi_\epsilon(y) + w, r];$$

(2) *for any $u, v \in Y$ and $r_1 > r_2 > 0$ with $\frac{r_1}{r_2} - 1 \leq \delta$, there exists a $z \in B[u, r_2\epsilon] \cap Y$ satisfying $B[u, r_1] \cap B[v, r_2] \subseteq B[z, r_2]$.*

Proof. (1): This inclusion is easy to verify.

(2): Let $u, v \in Y$ and consider r_1, r_2 as in the hypothesis. By letting $r = r_2, y = \frac{u-v}{r_2}$ and $w = v$ in (1), we have

$$B[v, r_2] \cap B[u, r_1] \subseteq B[v, r_2] \cap B[u, r_2(1 + \delta)] \subseteq B\Big[r_2\psi_\epsilon\Big(\frac{u - v}{r_2}\Big) + v, r_2\Big].$$

Choose $z = r_2\psi_\epsilon\big(\frac{u-v}{r_2}\big) + v$. Observe that $z \in Y$ and $\|u - z\| = r_2\Big\|\frac{u-v}{r_2} - \psi_\epsilon\left(\frac{u-v}{r_2}\right)\Big\| \leq r_2\epsilon$. Hence the proof. □

As mentioned in Remark 2.1, equability need not be inherited by subspaces. However, we have the following result.

Theorem 2.6. *Let Y be a subspace of X and X be equable. If Y is the range of a norm one projection on X, then Y is equable.*

Proof. Let $\epsilon > 0$. Since X is equable, there exist a $\delta > 0$ and a map $\psi_\epsilon : X \to X$ such that for every $x \in X$, we have $\|x - \psi_\epsilon(x)\| \leq \epsilon$ and $B_X \cap B[x, 1+\delta] \subseteq B[\psi_\epsilon(x), 1]$. Consider $P : X \to X$ such that $P^2 = P$, $\|P\| = 1$ and $P(X) = Y$. Define $\chi_\epsilon : Y \to Y$ as $\chi_\epsilon(y) = P(\psi_\epsilon(y))$ for every $y \in Y$. Let $y \in Y$. Notice that

$$\|y - \chi_\epsilon(y)\| = \|Py - P(\psi_\epsilon(y))\| \leq \|y - \psi_\epsilon(y)\| \leq \epsilon.$$

Now, consider $z \in B_Y \cap B_Y[y, 1 + \delta]$. By assumption, we have $\|z - \psi_\epsilon(y)\| \leq 1$ which implies $\|z - \chi_\epsilon(y)\| = \|Pz - P(\psi_\epsilon(y))\| \leq 1$. Therefore, $z \in B_Y[\chi_\epsilon(y), 1]$. Hence, Y is equable. □

The following examples illustrate that the assumption of norm one of the projection in the preceding theorem cannot be relaxed and the hypothesis need not imply Y is equable in X.

Example 2.7. (1) Let $X = c_0$ and consider $Z = ker(f)$ where $f = \left(1, 1, \frac{1}{2}, \frac{1}{2^2}, ...\right) \in \ell_1$. Then, from [13, Theorem 3.2.18], we have Z is the range of a projection on X. However, in Example 2.2, we observed that X is equable, but Z is not equable.

(2) Let $X = c_0$ and consider $Y = ker(g)$, where $g = \left(\frac{1}{2}, \frac{1}{2^2}, \ldots\right) \in \ell_1$. Then, by [4, Theorem 1], we have Y is the range of a norm one projection on X. Since $supp(g)$ is not finite, by [17, Fact 1], we have Y is not proximinal on X. Therefore, by [19, Theorem 1.7.2], Y is not equable in X.

3. Main results

It is easy to verify that every one-dimensional space is equable. Yost [19, Theorem 1.7.2] proved that if a subspace is equable in the whole space, then the corresponding metric projection is Hausdorff continuous on the entire space. It is known that, [6, pg. 193], there exists a three-dimensional space with a one-dimensional subspace having discontinuous metric projection. Thus, in general, a one-dimensional subspace need not be equable in the whole space.

In the following result, we observe that the notions equability and strong equability coincide for one-dimensional subspaces.

Proposition 3.1. *Let Y be a one-dimensional subspace of X. Then Y is equable in X if and only if Y is strongly equable in X.*

Proof. It is enough to prove the necessary part. Let Y be a one-dimensional subspace of X which is equable in X and $\epsilon > 0$. Then, there exist a $\delta > 0$ with $\delta < \epsilon$ and a map $\psi_\epsilon : Y \to Y$ such that for every $y \in Y$, we have $\|y - \psi_\epsilon(y)\| \leq \epsilon$ and $B_X \cap B[y, 1 + \delta] \subseteq B[\psi_\epsilon(y), 1]$. We will show that for every $y \in Y$, there exists a $\lambda \in [0, 1]$ such that $\|y - \lambda y\| \leq \epsilon$ and $B_X \cap B[y, 1 + \delta] \subseteq B[\lambda y, 1]$. Let $y \in Y$.

Case (i): If $\|y\| \leq \epsilon$, then choose $\lambda = 0$ and if $\|y\| > 2 + \delta$, then choose $\lambda = 1$. Hence, the conditions satisfy trivially.

Case (ii): Assume $\epsilon < \|y\| \leq 2 + \delta$. Since Y is a one-dimensional subspace, by assumption, we have $\psi_\epsilon(y) = \lambda y$ for some $\lambda \in \mathbb{R}$. Now, it is enough to show that $\lambda \in [0, 1]$. Since $\|y - \lambda y\| \leq \epsilon$ and $\|y\| > \epsilon$, it follows that $\lambda \in (0, 2)$. It is easy to verify that $(1 - \frac{1+\delta}{\|y\|})y \in B_X \cap B[y, 1+\delta]$. Thus, by assumption, $\|(1 - \frac{1+\delta}{\|y\|})y - \lambda y\| \leq 1$, which further implies $|\|y\| - 1 - \delta - \lambda\|y\|| \leq 1$. Therefore, $\delta \leq (1 - \lambda)\|y\|$, which leads to $\lambda < 1$. Hence the proof. $\square$

The preceding result need not be true for subspaces of dimension two (see, Example 3.25). Now, we observe that if every one-dimensional subspace is equable in the whole space, then the space is equable whenever it has finite dimension.

Theorem 3.2. *Let Y be a subspace of X with $dim(Y) = m$ and $k \in \mathbb{N}$ with $k < m$. If every k-dimensional subspace of Y is equable in X, then Y is equable in X.*

Proof. Suppose Y is not equable in X. Then, there exists an $\epsilon > 0$ such that for every $n \in \mathbb{N}$, there exists a $y_n \in Y$ such that whenever $z \in B[y_n, \epsilon] \cap Y$, we have $B_X \cap B\left[y_n, 1 + \frac{1}{n}\right] \nsubseteq B[z, 1]$. Notice that $\|y_n\| \leq 3$ for all $n \in \mathbb{N}$. Since (y_n) is a bounded sequence in Y, by assumption, (y_n) has a convergent subsequence (y_{n_j}), i.e., $y_{n_j} \to y_0$ for some $y_0 \in Y$. Consider a k-dimensional subspace Z of X such that $y_0 \in Z \subseteq Y$. Since Z is equable in X, there exist $0 < \delta < \frac{\epsilon}{2}$ and a map $\psi_{\frac{\epsilon}{2}} : Z \to Z$ such that for every $u \in Z$, we have

$$\left\|u - \psi_{\frac{\epsilon}{2}}(u)\right\| \leq \frac{\epsilon}{2} \quad \text{and} \quad B_X \cap B[u, 1+\delta] \subseteq B\left[\psi_{\frac{\epsilon}{2}}(u), 1\right].$$

Therefore, we have $\left\|y_0 - \psi_{\frac{\epsilon}{2}}(y_0)\right\| \leq \frac{\epsilon}{2}$ and $B_X \cap B[y_0, 1+\delta] \subseteq B\left[\psi_{\frac{\epsilon}{2}}(y_0), 1\right]$. Now, choose $k_1 \in \mathbb{N}$ such that $\|y_{n_j} - y_0\| \leq \frac{\delta}{2}$ and $\frac{1}{n_j} \leq \frac{\delta}{2}$ for every $j \geq k_1$. Clearly, $B_X \cap B[y_{n_j}, 1 + \frac{1}{n_j}] \subseteq B_X \cap B[y_0, 1+\delta] \subseteq B\left[\psi_{\frac{\epsilon}{2}}(y_0), 1\right]$ for every $j \geq k_1$. This contradicts the assumption, since

$$\left\|y_{n_j} - \psi_{\frac{\epsilon}{2}}(y_0)\right\| \leq \left\|y_{n_j} - y_0\right\| + \left\|y_0 - \psi_{\frac{\epsilon}{2}}(y_0)\right\| \leq \frac{\delta}{2} + \frac{\epsilon}{2} < \epsilon$$

for every $j \geq k_1$. Hence, Y is equable in X. □

In the preceding theorem, the assumption, Y is of finite dimension cannot be omitted (see, Example 3.21). The following result is a direct consequence of the preceding theorem.

Corollary 3.3. *If every one-dimensional subspace of X is equable in X, then every finite dimensional subspace of X is equable in X.*

Later, in Corollary 3.20, we will observe that a statement analogous to Corollary 3.3 in terms of strong equability characterizes a well-known geometric property. However, in Example 3.25, we see that Theorem 3.2 and Corollary 3.3 do not hold for strong equability.

Now, we relate equability and strong equability with best approximation theoretic notions. In [19, Theorem 1.7.2] and [10, Theorem 2.4], it is proved that if Y is equable (respectively, strongly equable) in X, then Y is proximinal (respectively, ball proximinal) on X. Here, we present strengthened versions of these statements. To see this, we need the following definitions.

For a nonempty subset A of X, $x \in X$ and $\delta \geq 0$, we define

$$P_A(x, \delta) = \{a \in A : \|x - a\| \leq d(x, A) + \delta\}.$$

Clearly, $P_A(x, 0) = P_A(x)$. Let B be a nonempty subset of X. We say A is proximinal on B if $P_A(x)$ is nonempty for every $x \in B$. We say A is uniformly strongly proximinal on B [7] if A is proximinal on B and for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $P_A(x, \delta) \subseteq P_A(x) + \epsilon B_X$ for every $x \in B$. For any $0 \leq \alpha \leq \beta$, we denote $D_\alpha^\beta(A) = \{x \in X : \alpha \leq d(x, A) \leq \beta\}$ and $D_\alpha^\alpha(A) = D_\alpha(A)$.

Theorem 3.4. *Let Y be a subspace of X and $\alpha > 0$. Then the following statements hold.*

(1) *If Y is equable in X, then Y is uniformly strongly proximinal on $D_0^\alpha(Y)$.*

(2) *If Y is strongly equable in X, then B_Y is uniformly strongly proximinal on $D_0^\alpha(B_Y)$.*

Proof. (1): Since Y is equable in X, by [19, Theorem 1.7.2], it follows that Y is proximinal on X. Let $0 < \epsilon < 3\alpha$. By assumption, there exist a $0 < \delta < 1$ and a map $\psi_{\frac{\epsilon}{\alpha}} : Y \to Y$ such that for every $y \in Y$, we have $\|y - \psi_{\frac{\epsilon}{\alpha}}(y)\| \leq \frac{\epsilon}{\alpha}$ and $B_X \cap B[y, 1 + \delta] \subseteq B[\psi_{\frac{\epsilon}{\alpha}}(y), 1]$. Choose $\delta_1 = \frac{\epsilon\delta}{3}$. We claim that $P_Y(x, \delta_1) \subseteq P_Y(x) + \epsilon B_X$ for every $x \in D_0^{\alpha}(Y)$. For this, consider $x \in D_0^{\alpha}(Y)$. Let $z_0 \in P_Y(x, \delta_1)$ and $y_0 \in P_Y(x)$. If $x \in D_0^{\frac{\epsilon}{3}}(Y)$, then

$$\|z_0 - y_0\| \leq \|z_0 - x\| + \|x - y_0\| \leq 2d(x, Y) + \delta_1 \leq \epsilon.$$

Thus, $P_Y(x, \delta_1) \subseteq P_Y(x) + \epsilon B_X$. Suppose $x \in D_{\frac{\epsilon}{3}}^{\alpha}(Y)$. Take $\lambda = d(x, Y)$. Then $\frac{x}{\lambda} \in B[\frac{y_0}{\lambda}, 1] \cap B[\frac{z_0}{\lambda}, 1 + \delta]$. By taking $w = \frac{y_0}{\lambda}, y = \frac{z_0 - y_0}{\lambda}$ and $r = 1$ in Lemma 2.5, we get

$$B\left[\frac{y_0}{\lambda}, 1\right] \cap B\left[\frac{z_0}{\lambda}, 1 + \delta\right] \subseteq B\left[\psi_{\frac{\epsilon}{\alpha}}\left(\frac{z_0 - y_0}{\lambda}\right) + \frac{y_0}{\lambda}, 1\right].$$

Thus, we have $\|\frac{x}{\lambda} - (\psi_{\frac{\epsilon}{\alpha}}(\frac{z_0 - y_0}{\lambda}) + \frac{y_0}{\lambda})\| \leq 1$ which implies $\lambda\psi_{\frac{\epsilon}{\alpha}}(\frac{z_0 - y_0}{\lambda}) + y_0 \in P_Y(x)$. Since $\|\frac{z_0 - y_0}{\lambda} - \psi_{\frac{\epsilon}{\alpha}}(\frac{z_0 - y_0}{\lambda})\| \leq \frac{\epsilon}{\alpha}$, we have $z_0 \in P_Y(x) + \epsilon B_X$. Hence the claim.

(2): Since Y is strongly equable in X, by [10, Theorem 2.4], we have B_Y is proximinal on X. Let $0 < \epsilon < 3\alpha$. By [10, Remark 2.3] and the assumption, there exist a $0 < \delta < 1$ and a map $\phi_{\frac{\epsilon}{\alpha}} : Y \to [0, 1]$ such that for every $s > 0$, $y, z \in B_Y$, we have $\left\|\frac{w}{s} - \frac{z}{s}\right\| \leq \frac{\epsilon}{\alpha}$ and

$$B\left[\frac{y}{s}, 1\right] \cap B\left[\frac{z}{s}, 1 + \delta\right] \subseteq B\left[\frac{w}{s}, 1\right],$$

where $w = s\left[\phi_{\frac{\epsilon}{\alpha}}\left(\frac{z-y}{s}\right)\left(\frac{z-y}{s}\right) + \frac{y}{s}\right] \in B_Y$. Choose $\delta_1 = \frac{\epsilon\delta}{3}$. We claim that $P_{B_Y}(x, \delta_1) \subseteq P_{B_Y}(x) + \epsilon B_X$ for every $x \in D_0^{\alpha}(B_Y)$. For this, consider $x \in D_0^{\alpha}(B_Y)$, $z_0 \in P_{B_Y}(x, \delta_1)$ and $y_0 \in P_{B_Y}(x)$. If $x \in D_0^{\frac{\epsilon}{3}}(B_Y)$, then

$$\|z_0 - y_0\| \leq \|z_0 - x\| + \|x - y_0\| \leq 2d(x, B_Y) + \delta_1 \leq \epsilon.$$

Thus, $P_{B_Y}(x, \delta_1) \subseteq P_{B_Y}(x) + \epsilon B_X$. Suppose $x \in D_{\frac{\epsilon}{3}}^{\alpha}(B_Y)$. Take $\lambda = d(x, B_Y)$. Then $\frac{x}{\lambda} \in B[\frac{y_0}{\lambda}, 1] \cap B[\frac{z_0}{\lambda}, 1 + \delta]$. Thus, from the previous argument, we have $\frac{x}{\lambda} \in B\left[\frac{w_0}{\lambda}, 1\right]$ and $\|\frac{w_0}{\lambda} - \frac{z_0}{\lambda}\| \leq \frac{\epsilon}{\alpha}$ where $w_0 = \lambda\left[\phi_{\frac{\epsilon}{\alpha}}\left(\frac{z_0 - y_0}{\lambda}\right)\left(\frac{z_0 - y_0}{\lambda}\right) + \frac{y_0}{\lambda}\right] \in B_Y$. Therefore, $w_0 \in P_{B_Y}(x)$ and $\|w_0 - z_0\| \leq \epsilon$. Thus, $z_0 \in P_{B_Y}(x) + \epsilon B_X$. Hence the claim. □

Recall that if Y is equable (respectively, strongly equable) in X, then the metric projection P_Y (respectively, P_{B_Y}) is Hausdorff continuous on X (see, [19, Theorem 1.7.2] and [10, Theorem 2.5]). However, as a consequence of Theorem 3.4, in the following result we show that the corresponding metric projection is uniformly Hausdorff continuous on every bounded subset of the space.

Corollary 3.5. *The following statements hold.*

(1) *If Y is equable in X, then P_Y is uniformly Hausdorff continuous on every bounded subset of X.*

(2) *If Y is strongly equable in X, then P_{B_Y} is uniformly Hausdorff continuous on every bounded subset of X.*

Proof. (1): Let A be any bounded subset of X. Then $A \subseteq D_0^\alpha(Y)$ for some $\alpha > 0$. By assumption and Theorem 3.4, we have Y is uniformly strongly proximinal on $D_0^\alpha(Y)$. Therefore, by [7, Remark 2.3], we have P_Y is uniformly Hausdorff continuous on $D_0^\alpha(Y)$. Hence, P_Y is uniformly Hausdorff continuous on A.

(2): Let B be any bounded subset of X. Then $B \subseteq D_0^\beta(B_Y)$ for some $\beta > 0$. By assumption and Theorem 3.4, we have B_Y is uniformly strongly proximinal on $D_0^\beta(B_Y)$. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $P_{B_Y}(x, \delta) \subseteq P_{B_Y}(x) + \epsilon B_X$ for every $x \in D_0^\beta(B_Y)$. For any $x, y \in D_0^\beta(B_Y)$ satisfying $\|x - y\| < \frac{\delta}{4}$, we have $P_{B_Y}(x) \subseteq P_{B_Y}(y, \delta) \subseteq P_{B_Y}(y) + \epsilon B_X$. Thus, P_{B_Y} is uniformly continuous on $D_0^\beta(B_Y)$. Hence, P_{B_Y} is uniformly Hausdorff continuous on B. □

The following example illustrates that the converse of the statements in Theorem 3.4 are not true in general. To see this, we need the following definition.

Definition 3.6. [8] An *M-projection* on X is a projection $P : X \to X$ such that $\|x\| = \max\{\|Px\|, \|x - Px\|\}$ for every $x \in X$. A subspace Y of X is said to be an *M-summand* in X if it is the range of an M-projection on X.

Example 3.7. (1) Let Y, Z be two Banach spaces and Y be a non-equable space. Consider $X = Y \oplus_\infty Z$. Then, Y is an M-summand in X. Since every M-summand has $1\frac{1}{2}$- ball property, it follows from [7, Proposition 2.11] that Y is uniformly strongly proximinal on $D_\alpha(Y)$ in X for any $\alpha > 0$. However, by Remark 2.1, Y is not equable in X.

(2) Let $X = \ell_\infty$ and $Y = c_0$. As we observed in Example 2.2, Y is not strongly equable in X. It follows from [9, Theorem 3.1] that B_Y is strongly proximinal on X. However, a careful inspection of the proof shows that the δ can be chosen independent of $x \in X$. Thus, B_Y is uniformly strongly proximinal on X. □

Now, we proceed to relate equability and strong equability with some geometric properties. First, we establish a characterization for equability in terms of quasi uniform rotundity. The latter notion was introduced by Calder et al. [5, Definition 1.7]. However, we mainly use its equivalent form given by Veselý [18, Definition 2.1].

Definition 3.8. [18] Let Y be a subspace of X. We say X is quasi uniformly rotund (in short, QUR) with respect to Y, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $v \in Y$, there exists a $w \in Y$ satisfying $w \in B[0, \epsilon]$ and $B[0, 1+\delta] \cap B[v, 1] \subseteq B[w, 1]$. In particular, we say X is QUR if X is QUR with respect to X.

Theorem 3.9. *Let Y be a subspace of X. Then Y is equable in X if and only if X is QUR with respect to Y.*

Proof. To see the necessary part let $\epsilon > 0$. Then there exist a $\delta > 0$ and a map $\psi_\epsilon : Y \to Y$ such that for every $y \in Y$, we have $\|y - \psi_\epsilon(y)\| \leq \epsilon$ and $B_X \cap B[y, 1+\delta] \subseteq B[\psi_\epsilon(y), 1]$. Let $v \in Y$. Then, choose $w = v + \psi_\epsilon(-v) \in Y$. Observe that $\|w\| \leq \epsilon$ and $B_X \cap B[-v, 1 + \delta] \subseteq B[\psi_\epsilon(-v), 1]$. Therefore, by translation with v, we get $B[v, 1] \cap B[0, 1 + \delta] \subseteq B[w, 1]$. Hence, X is QUR with respect to Y.

To prove the converse let $\epsilon > 0$. By assumption, there exists a $\delta > 0$ such that for every $v \in Y$, there exists a $w \in Y$ satisfying $w \in B[0, \epsilon]$ and $B[0, 1 + \delta] \cap B[v, 1] \subseteq B[w, 1]$. Let $y \in Y$. Since $-y \in Y$, by assumption, there exists a $z_{-y} \in Y$ such that $z_{-y} \in B[0,\epsilon]$ and $B[0,1+\delta] \cap B[-y,1] \subseteq B[z_{-y},1]$. Define $\psi_\epsilon : Y \to Y$ as $\psi_\epsilon(y) = z_{-y} + y$ for every $y \in Y$. Since $\|y - \psi_\epsilon(y)\| \leq \epsilon$, by translation with y, we get $B_X \cap B[y,1+\delta] \subseteq B[\psi_\epsilon(y),1]$. Hence, Y is equable in X. □

Now, we present a few relations between strong equability and uniform rotundity.

Definition 3.10. [2] Let Y be a subspace of X. The modulus of rotundity of X with respect to Y, denoted by, $\delta_X^Y(\cdot)$, is defined as

$$\delta_X^Y(\epsilon) = \inf\left\{1 - \left\|\frac{u+v}{2}\right\| : u,v \in B_X,\ u-v \in Y \text{ with } \|u-v\| \geq \epsilon\right\}.$$

(1) We say that X is *rotund* with respect to Y if $\left\|\frac{u+v}{2}\right\| < 1$ whenever $u,v \in B_X$, $u \neq v$ and $u - v \in Y$.

(2) We say X is *uniformly rotund* (in short, UR) with respect to Y if $\delta_X^Y(\epsilon) > 0$ for every $\epsilon \in (0,2]$.

It is clear that the space X is rotund (respectively, UR) if X is rotund (respectively, UR) with respect to X. We say X is uniformly rotund in every direction (in short, URED) if X is UR with respect to every one-dimensional subspace of X. □

For any $x, y \in X$, the line segment joining x and y is denoted by $[x,y]$.

Theorem 3.11. *Let Y be a subspace of X. Consider the following statements.*

(1) *X is UR with respect to Y.*

(2) *Y is strongly equable in X.*

(3) *For every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that for every $v \in Y$, there exists a $w \in Y$ satisfying $w \in B[0,\epsilon]$ with $w \in [0,v]$ and $B_X \cap B[v,1-\delta] \subseteq B[w,1-\delta]$.*

Then (1) ⇒ (2) ⇔ (3).

Proof. (1) ⇒ (2): Let $\epsilon \in (0,1)$. By assumption, choose

$$0 < \delta(\epsilon) < \min\left\{\delta_X^Y\left(\tfrac{\epsilon}{2}\right), \tfrac{2\epsilon}{3}\delta_X^Y(\epsilon)\right\}.$$

Note that $\delta \in (0,1)$. Consider the map $\phi_\epsilon : Y \to [0,1]$ defined as

$$\phi_\epsilon(y) = \begin{cases} 1 - \frac{\epsilon}{\|y\|}, & \text{if } \|y\| > \epsilon; \\ 0, & \text{if } \|y\| \leq \epsilon, \end{cases}$$

for every $y \in Y$. It is clear that $\|y - \phi_\epsilon(y)y\| \leq \epsilon$ for every $y \in Y$. Thus, it is enough to show that, $B_X \cap B[y,1+\delta] \subseteq B[\phi_\epsilon(y)y,1]$ holds for every $y \in Y$. To see this, let $y \in Y$.

Case (i): If either $\|y\| \leq \epsilon$ or $\|y\| > 2+\delta$, then the inclusion holds trivially.

Case (ii): Suppose $\epsilon < \|y\| \leq 2\epsilon$. Let $z \in B_X \cap B[y,1+\delta]$. Since $\frac{z}{1+\delta}, \frac{z-y}{1+\delta} \in B_X$, $\frac{z-y}{1+\delta} - \left(\frac{z}{1+\delta}\right) \in Y$ and

$$\left\|\frac{z-y}{1+\delta} - \frac{z}{1+\delta}\right\| > \frac{\epsilon}{1+\delta} \geq \frac{\epsilon}{2},$$

it follows that

$$\frac{1}{2}\left\|\frac{z-y}{1+\delta} + \frac{z}{1+\delta}\right\| \leq 1 - \delta_X^Y\left(\frac{\epsilon}{2}\right).$$

Therefore, $\frac{1}{2}\|z-y+z\| \le (1+\delta)\left(1-\delta_X^Y\left(\frac{\epsilon}{2}\right)\right) < 1-\delta^2 < 1$.

Since $0 \le \frac{2\epsilon}{\|y\|} - 1 < 1$, we have

$$\|z-\phi_\epsilon(y)y\| = \left\|\left(1-\left(\frac{2\epsilon}{\|y\|}-1\right)\right)\left(\frac{z-y+z}{2}\right) + \left(\frac{2\epsilon}{\|y\|}-1\right)z\right\| < 1.$$

Case (iii): Suppose $2\epsilon < \|y\| \le 2+\delta$. Let $z \in B_X \cap B[y, 1+\delta]$. Since $\frac{z}{1+\delta}, \frac{z-y}{1+\delta} \in B_X$, $\frac{z-y}{1+\delta} - \left(\frac{z}{1+\delta}\right) \in Y$ and

$$\left\|\frac{z-y}{1+\delta} - \frac{z}{1+\delta}\right\| > \frac{2\epsilon}{1+\delta} > \epsilon,$$

it follows that $\frac{1}{2}\left\|\frac{z-y}{1+\delta} + \frac{z}{1+\delta}\right\| \le 1 - \delta_X^Y(\epsilon)$. Thus, we get

$$\begin{aligned}\|z-\phi_\epsilon(y)y\| &= \left\|\left(1-\frac{2\epsilon}{\|y\|}\right)(z-y) + \frac{2\epsilon}{\|y\|}\left(\left(\frac{z-y}{2}\right)+\frac{z}{2}\right)\right\| \\ &\le \left(1-\frac{2\epsilon}{\|y\|}\right)(1+\delta) + \frac{2\epsilon}{\|y\|}(1+\delta)(1-\delta_X^Y(\epsilon)) \\ &\le (1+\delta)\left(1-\frac{2\epsilon}{3}\delta_X^Y(\epsilon)\right) < 1.\end{aligned}$$

Therefore, $B_X \cap B[y, 1+\delta] \subseteq B[\phi_\epsilon(y)y, 1]$ for every $y \in Y$. Hence, the implication holds.

(2) $\Rightarrow$ (3): Let $\epsilon > 0$. By (2), there exist a $\delta_1 > 0$ and a map $\phi_\epsilon : Y \to [0,1]$ such that for every $y \in Y$, we have $\|y - \phi_\epsilon(y)y\| \le \epsilon$ and $B_X \cap B[y, 1+\delta_1] \subseteq B[\phi_\epsilon(y)y, 1]$. We show that the statement (3) holds for $\delta = \frac{\delta_1}{1+\delta_1}$. To see this, let $v \in Y$ and consider $w = v - \phi_\epsilon\left(\frac{-v}{1-\delta}\right)v$. Observe that $w \in [0, v]$ and

$$\|w\| = (1-\delta)\left\|\frac{-v}{1-\delta} - \phi_\epsilon\left(\frac{-v}{1-\delta}\right)\left(\frac{-v}{1-\delta}\right)\right\| \le (1-\delta)\epsilon < \epsilon.$$

Now, it is enough to prove that $B_X \cap B[v, 1-\delta] \subseteq B[w, 1-\delta]$. Let $z \in B_X \cap B[v, 1-\delta]$. Since $\frac{z-v}{1-\delta} \in B_X \cap B\left[\frac{-v}{1-\delta}, 1+\delta_1\right]$, it follows that

$$\left\|\frac{z}{1-\delta} - \phi_\epsilon\left(\frac{-v}{1-\delta}\right)\left(\frac{-v}{1-\delta}\right) - \frac{v}{1-\delta}\right\| \le 1.$$

Thus, $\|z-w\| \le 1-\delta$. Hence, the implication holds.

(3) $\Rightarrow$ (2): Let $\epsilon \in (0,1)$. From (3), choose $\delta'\left(\frac{\epsilon}{2}\right) > 0$. Thus, by [1, Lemma 2.1], for $0 < \delta(\epsilon) < \min\left\{\frac{1}{2}, \delta'\left(\frac{\epsilon}{2}\right)\right\}$, we have for every $v \in Y$, there exists a $w \in Y$ satisfying $w \in B\left[0, \frac{\epsilon}{2}\right]$ with $w \in [0, v]$ and $B_X \cap B[v, 1-\delta] \subseteq B[w, 1-\delta]$. Choose $\eta = \frac{\delta}{1-\delta}$. Now, it is enough to prove that there exists a map $\phi_\epsilon : Y \to [0,1]$ such that for every $y \in Y$, we have $\|y - \phi_\epsilon(y)y\| \le \epsilon$ and $B_X \cap B[y, 1+\eta] \subseteq B[\phi_\epsilon(y)y, 1]$. For this, let $y \in Y$. From the preceding argument, for $-(1-\delta)y \in Y$ there exists a $\lambda_y \in [0,1]$ such that $\|-\lambda_y(1-\delta)y\| \le \frac{\epsilon}{2}$ and

$$B_X \cap B[-(1-\delta)y, 1-\delta] \subseteq B[-\lambda_y(1-\delta)y, 1-\delta].$$

Choose $\phi_\epsilon(y) = 1 - \lambda_y$ and observe that

$$\|y - \phi_\epsilon(y)y\| = \lambda_y\|y\| \le \frac{\epsilon}{2(1-\delta)} < \epsilon.$$

Further, let $z \in B_X \cap B[y, 1+\eta]$. Observe that $(1-\delta)(z-y) \in B_X \cap B[-(1-\delta)y, 1-\delta]$. Thus, it follows from the previous inclusion that $\|z - \phi_\epsilon(y)y\| \leq 1$. Therefore, $B_X \cap B[y, 1+\eta] \subseteq B[\phi_\epsilon(y)y, 1]$. Hence, Y is strongly equable in X. $\square$

Note that, in the preceding result, the proof of (1) $\Rightarrow$ (2) is similar to the proof of [19, Proposition 1.7.4]. Also, the statement (3) is equivalent to X is quasi uniformly rotund with respect to Y whilst w (in Definition 3.8) can be chosen from the line segment $[0, v]$. Now, we present an example to see that the implication (2) $\Rightarrow$ (1) (hence (3) $\Rightarrow$ (1)) of Theorem 3.11 need not be true in general.

Example 3.12. Let $X = (\mathbb{R}^3, \|.\|_c)$ where

$$\|(x_1, x_2, x_3)\|_c = \max\left\{\left(|x_1|^2 + |x_2|^2\right)^{\frac{1}{2}}, |x_3|\right\}.$$

Let $Y = \{(x_1, x_2, 0) \in X : x_1, x_2 \in \mathbb{R}\}$, $u = (-1, 0, 1)$ and $v = (1, 0, 1)$. Observe that $u, v \in B_X, u \neq v$, $u - v \in Y$ but $\frac{u+v}{2} \in S_X$. Therefore, X is not rotund with respect to Y. Since X has finite dimension, it follows that X is not UR with respect to Y. However, Y is strongly equable in X. For this, let $\epsilon > 0$. Since Y is UR, by Theorem 3.11, we have Y is strongly equable. Thus, there exist a $\delta > 0$ and a map $\phi_\epsilon : Y \to [0, 1]$ such that for every $y \in Y$, we have $\|y - \phi_\epsilon(y)y\|_c \leq \epsilon$ and $B_Y \cap B_Y[y, 1+\delta] \subseteq B_Y[\phi_\epsilon(y)y, 1]$. Now, it is easy to verify that $B_X \cap B[y, 1+\delta] \subseteq B[\phi_\epsilon(y)y, 1]$. $\square$

As a consequence of Theorem 3.11 and Example 3.12, we remark that the implication (c)$\Rightarrow$(a) in [1, Proposition 2.2] is not true.

In view of Examples 2.2 and 3.12 it is clear that the notion strong equability is stronger than equability and weaker to uniform rotundity. However, in the following result we prove that these three notions coincide under some assumptions.

Theorem 3.13. *Let Y be a subspace of X such that X is rotund with respect to Y. Then the following statements are equivalent.*

(1) *X is UR with respect to Y.*

(2) *Y is strongly equable in X.*

(3) *Y is equable in X.*

Proof. (1) $\Rightarrow$ (2): This implication follows from Theorem 3.11.

(2) $\Rightarrow$ (3): This implication follows from Remark 2.1.

(3) $\Rightarrow$ (1): Let $\epsilon > 0$. By assumption, there exist $\delta_1\left(\frac{\epsilon}{4}\right) > 0$ and a map $\psi_{\frac{\epsilon}{4}} : Y \to Y$ such that for every $z \in Y$, we have $\left\|z - \psi_{\frac{\epsilon}{4}}(z)\right\| \leq \frac{\epsilon}{4}$ and $B_X \cap B[z, 1+\delta_1] \subseteq B[\psi_{\frac{\epsilon}{4}}(z), 1]$. Now for $\delta(\epsilon) = \frac{\delta_1\left(\frac{\epsilon}{4}\right)}{1+\delta_1\left(\frac{\epsilon}{4}\right)}$, we claim that whenever $x, y \in B_X$ with $x - y \in Y$ and $\left\|\frac{x+y}{2}\right\| > 1 - \delta$, it follows that $\|x - y\| < \epsilon$. For this, let $d = 1 - \left\|\frac{x+y}{2}\right\|$, so $d \in [0, 1)$. For $\frac{x-y}{2(1-d)} \in Y$, by assumption, we have

$$\left\|\frac{x-y}{2(1-d)} - \psi_{\frac{\epsilon}{4}}\left(\frac{x-y}{2(1-d)}\right)\right\| \leq \frac{\epsilon}{4}$$

and
$$B_X \cap B\left[\frac{x-y}{2(1-d)}, 1+\delta_1\right] \subseteq B\left[\psi_{\frac{\epsilon}{4}}\left(\frac{x-y}{2(1-d)}\right), 1\right].$$

Let $u = \frac{x+y}{2(1-d)} - \psi_{\frac{\epsilon}{4}}\left(\frac{x-y}{2(1-d)}\right)$ and $v = \frac{x+y}{2(1-d)} + \psi_{\frac{\epsilon}{4}}\left(\frac{x-y}{2(1-d)}\right)$.

Since $\frac{x+y}{2(1-d)}, -\frac{x+y}{2(1-d)} \in B_X \cap B\left[\frac{x-y}{2(1-d)}, 1+\delta_1\right]$, we have $\|u\| \leq 1$ and $\|v\| \leq 1$.

Further, note that
$$\frac{1}{2}\|u+v\| = \left\|\frac{x+y}{2(1-d)}\right\| = 1.$$

Thus, it follows that $\|u\| = 1$ and $\|v\| = 1$. Since $u - v = -2\psi_{\frac{\epsilon}{4}}\left(\frac{x-y}{2(1-d)}\right) \in Y$, it follows from the hypothesis that $u = v$, i.e., $\psi_{\frac{\epsilon}{4}}\left(\frac{x-y}{2(1-d)}\right) = 0$.

Therefore, $\left\|\frac{x-y}{2(1-d)}\right\| \leq \frac{\epsilon}{4}$, which implies $\|x-y\| \leq \frac{\epsilon}{2}(1-d) < \epsilon$. Hence the claim, i.e., X is UR with respect to Y. □

Remark 3.14. (1) The implication (3) ⇒ (1) of Theorem 3.13 also follows from Theorem 3.9 and [1, Proposition 2.2]. However, the proof presented here is direct.

(2) As a consequence of Theorem 3.13, it is easy to see that whenever the space X is rotund, we have X is uniformly rotund if and only if it is strongly equable if and only if it is equable. However, in Theorem 3.19, we prove an improved equivalence among these notions. □

As mentioned in Remark 2.1, in the following example we observe that the equability (respectively, strong equability) of Z in Y need not imply the equability (respectively, strong equability) of Z in X.

Example 3.15. Let X be a rotund space but not URED [15, Example 1]. Therefore, there exists a one-dimensional subspace Z of X such that X is not UR with respect to Z. Since X is rotund with respect to Z, it follows, by Theorem 3.13, that Z is not equable in X. Now, choose a finite dimensional subspace Y of X such that $Z \subseteq Y$. Since Y is rotund and has finite dimension, by [13, Proposition 5.2.14], we have that Y is UR. Thus, Y is UR with respect to Z. Therefore, by Theorem 3.11, Z is strongly equable in Y. Further, notice that, by Theorem 3.13, Y is not equable in X. □

In view of the above example, we conclude that, Y is strongly equable in $span\{Y, x\}$ for every $x \in X$ need not imply Y is equable in X.

A careful inspection of Example 3.15 reveals the fact that equability (respectively, strong equability) is not transitive through uniformly rotund spaces. However, in the following result we prove that these notions are transitive through M-summands.

Theorem 3.16. *Let Z, Y be subspaces of X such that $Z \subseteq Y \subseteq X$ and Y be an M-summand in X. If Z is equable (respectively, strongly equable) in Y, then Z is equable (respectively, strongly equable) in X.*

Proof. We prove the result for equability, a similar proof holds for strong equability. Let $\epsilon > 0$. By assumption, there exist a $\delta > 0$ and a map $\psi_\epsilon : Z \to Z$ such that for every $x \in Z$, we have $\|x - \psi_\epsilon(x)\| \leq \epsilon$ and $B_Y \cap B_Y[x, 1+\delta] \subseteq B_Y[\psi_\epsilon(x), 1]$. Now, it is enough to show that $B_X \cap B[x, 1+\delta] \subseteq B[\psi_\epsilon(x), 1]$. Let $w \in B_X \cap B[x, 1+\delta]$. Let P be the corresponding M-projection of X onto Y.

Note that $\max\{\|P(w)\|, \|w-P(w)\|\} \leq 1$ and $\max\{\|P(w)-x\|, \|w-P(w)\|\} \leq 1+\delta$. Therefore, $P(w) \in B_Y \cap B_Y[x, 1+\delta]$, which implies $\|P(w) - \psi_\epsilon(x)\| \leq 1$. Since $\|w - P(w)\| \leq 1$, we have $\|w - \psi_\epsilon(x)\| = \max\{\|P(w) - \psi_\epsilon(x)\|, \|w - P(w)\|\} \leq 1$. Hence the proof. □

Corollary 3.17. *If Y is equable (respectively, strongly equable) and Y is an M-summand in X, then Y is equable (respectively, strongly equable) in X.*

As a consequence of Theorem 2.6 and Corollary 3.17, we observe that every M-summand in an equable space is equable in the whole space.

Corollary 3.18. *If X is equable and Y is an M-summand in X, then Y is equable in X.*

We observed in Example 3.12, that if Y is strongly equable in X, then X need not be rotund with respect to Y. But, if we consider the strong equability of the whole space, then rotundity follows, which leads to the following characterization.

Theorem 3.19. *The following statements are equivalent.*

(1) *X is UR.*

(2) *X is strongly equable.*

(3) *X is equable and rotund.*

Proof. (1) $\Rightarrow$ (2): This implication follows from Theorem 3.11.

(2) $\Rightarrow$ (3): Clearly, by Remark 2.1, X is equable. Suppose X is not rotund. Then there exist $u, v \in S_X$ with $\left\|\frac{u+v}{2}\right\| = 1$, but $\|u - v\| > \epsilon$ for some $\epsilon > 0$. By assumption, there exist a $\delta\left(\frac{\epsilon}{3}\right) > 0$ with $\delta < \frac{\epsilon}{3}$ and a map $\phi_{\frac{\epsilon}{3}} : X \to [0,1]$ such that $\left\|w - \phi_{\frac{\epsilon}{3}}(w)w\right\| \leq \frac{\epsilon}{3}$ and $B_X \cap B[w, 1+\delta] \subseteq B\left[\phi_{\frac{\epsilon}{3}}(w)w, 1\right]$ for every $w \in X$. Choose $w = (1+\delta)v - u$. Since $-u \in B_X \cap B[w, 1+\delta]$, it follows that $\left\|\phi_{\frac{\epsilon}{3}}(w)w + u\right\| \leq 1$. By the Hahn-Banach theorem, there exists an $x^* \in S_{X^*}$ such that $x^*\left(\frac{u+v}{2}\right) = \left\|\frac{u+v}{2}\right\|$. Thus, $x^*(u) = x^*(v) = 1$. Since

$$\begin{aligned} 1 + \phi_{\frac{\epsilon}{3}}(w)\delta &= x^*\left(\phi_{\frac{\epsilon}{3}}(w)(1+\delta)v + \left(1 - \phi_{\frac{\epsilon}{3}}(w)\right)u\right) \\ &= x^*\left(\phi_{\frac{\epsilon}{3}}(w)w + u\right) \leq \left\|\phi_{\frac{\epsilon}{3}}(w)w + u\right\| \leq 1, \end{aligned}$$

we have $\phi_{\frac{\epsilon}{3}}(w) = 0$. Thus,

$$\|u - v\| = \|\delta v - w\| \leq \delta\|v\| + \|w\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2}{3}\epsilon.$$

This is a contradiction. Hence, X is rotund.

(3) $\Rightarrow$ (1): This implication follows from Theorem 3.13. Also, from [19, Proposition 1.7.5]. □

Now, as a consequence of Theorems 3.13 and 3.19, in the following result we present a characterization for spaces that are uniformly rotund in every direction in terms of strong equability.

Corollary 3.20. *Let $k \geq 2$. Then the following statements are equivalent.*

(1) *X is URED.*

(2) *Every finite dimensional subspace of X is strongly equable in X.*

(3) *Every k-dimensional subspace of X is strongly equable in X.*

Proof. (1) $\Rightarrow$ (2): Let Y be a finite dimensional subspace of X. From [20, Proposition 1], we have X is UR with respect to Y. Now the implication follows from Theorem 3.13.

(2) $\Rightarrow$ (3): This implication is obvious.

(3) $\Rightarrow$ (1): Let Z be a k-dimensional subspace of X. By assumption, we have Z is strongly equable. Thus, by Theorem 3.19, Z is rotund. Therefore, by [13, Proposition 5.1.21] we have X is rotund. Now, to prove X is URED, let F be a one-dimensional subspace of X. Choose a k-dimensional subspace Y of X such that $F \subseteq Y$. Thus, by assumption and by Remark 2.1, we have F is strongly equable in X. Hence, the implication follows from Theorem 3.13. $\square$

In Example 3.25, we show that the implication (3) $\Rightarrow$ (1) in the previous result is not true for $k = 1$.

As a result of the preceding corollary, in the following example, we observe that the assumption, Y is of finite dimension is necessary in Theorem 3.2.

Example 3.21. Let W be a Banach space which is URED, but not UR [15, Example 2]. Consider $X = W \oplus_2 W$ and $Y = W \oplus_2 \{0\}$. Therefore, by [16], X is URED but not UR. Then, for any $k \in \mathbb{N}$, by Corollary 3.20, we have every k-dimensional subspace, say Z, of Y is strongly equable in X. Since Y is rotund but not UR, by Theorem 3.19, we have Y is not equable. Hence, Y is not equable in X. $\square$

The rest of this section focuses on stability of equability and strong equability. The following result is a direct consequence of Theorem 3.19 and the fact that rotundity is not stable under ℓ_1, ℓ_∞ and c_0-direct sums. Further it reveals that the results [10, Theorems 3.1 and 3.2] are not true.

Corollary 3.22. *Let $\{X_i : i \in \mathbb{N}\}$ be a collection of Banach spaces and $Y_i (\neq \{0\})$ be a subspace of X_i such that Y_i is strongly equable in X_i for every $i \in \mathbb{N}$. Then the following statements hold.*

(1) *$(Y_1 \oplus_1 Y_2)$ is not strongly equable in $(X_1 \oplus_1 X_2)$.*

(2) *$(Y_1 \oplus_\infty Y_2)$ is not strongly equable in $(X_1 \oplus_\infty X_2)$.*

(3) *$(\oplus_{c_0} Y_i)_{i \in \mathbb{N}}$ is not strongly equable in $(\oplus_{c_0} X_i)_{i \in \mathbb{N}}$.*

The following example shows that strong equability (respectively, equability) is not stable under infinite ℓ_p-direct sum $(1 < p < \infty)$.

Example 3.23. Let $1 < p < \infty$. For every $i \in \mathbb{N}$, let $X_i = (\mathbb{R}^3, \|\cdot\|_{p_i})$, where $p_i = 1 + \frac{1}{i}$ and $Y_i = span\{(1,0,0),(0,1,0)\}$ be a subspace of X_i. Since X_i is UR, by Theorem 3.13, it follows that Y_i is strongly equable in X_i for every $i \in \mathbb{N}$. Consider $X = (\oplus_p X_i)_{i \in \mathbb{N}}$ and $Y = (\oplus_p Y_i)_{i \in \mathbb{N}}$. Therefore, by [16], X is rotund, which implies X is rotund with respect to Y. However, X is not UR with respect to Y.

To see this, consider two sequences $(x_n), (y_n)$ in Y defined as

$$x_{n,i} = \begin{cases} (0,0,0), & \text{if } n \neq i; \\ (1,0,0), & \text{if } n = i \end{cases} \quad \text{and} \quad y_{n,i} = \begin{cases} (0,0,0), & \text{if } n \neq i; \\ (0,1,0), & \text{if } n = i, \end{cases}$$

for every $n, i \in \mathbb{N}$. Observe that $x_n, y_n \in S_Y$ for every $n \in \mathbb{N}$, $\|x_n + y_n\| = 2^{\frac{n}{n+1}} \to 2$ and $\|x_n - y_n\| = 2^{\frac{n}{n+1}} \to 2$. Therefore, Y is not UR. Thus, X is not UR with respect to Y. Hence, by Theorem 3.13, Y is neither equable nor strongly equable in X. $\square$

The stability of strong equability (respectively, equability) under a finite ℓ_p-direct sum $(1 < p < \infty)$ is not known. But, in the following result we prove that the notion equability is stable under finite ℓ_∞-direct sum.

Theorem 3.24. *Let $\{X_i : i \in I\}$ be a finite collection of Banach spaces and Y_i be a subspace of X_i for every $i \in I$. Let $X = (\oplus_\infty X_i)_{i\in I}$ and $Y = (\oplus_\infty Y_i)_{i\in I}$. Then, Y_i is equable in X_i for every $i \in I$ if and only if Y is equable in X.*

Proof. To prove the necessary part, let $\epsilon > 0$. By assumption, for every $i \in I$, there exist a $\delta_i > 0$ and a map $\psi_{i,\epsilon} : Y_i \to Y_i$ such that for every $y_i \in Y_i$, we have $\|y_i - \psi_{i,\epsilon}(y_i)\|_{X_i} \leq \epsilon$ and $B_{X_i} \cap B_{X_i}[y_i, 1 + \delta_i] \subseteq B_{X_i}[\psi_{i,\epsilon}(y_i), 1]$.
Choose $\delta = \min\{\delta_i : i \in I\}$ and define a map $\psi_\epsilon : Y \to Y$ as $\psi_\epsilon(y) = (\psi_{i,\epsilon}(y_i))_{i\in I}$ for every $y = (y_i)_{i\in I} \in Y$. Now, it is easy to verify that, $\|y - \psi_\epsilon(y)\|_X \leq \epsilon$ and $B_X \cap B_X[y, 1 + \delta] \subseteq B_X[\psi_\epsilon(y), 1]$ for every $y \in Y$.

Now to prove the sufficient part, let $\epsilon > 0$. Then, by assumption, there exist a $\delta > 0$ and a map $\psi_\epsilon : Y \to Y$ such that $\|y - \psi_\epsilon(y)\|_X \leq \epsilon$ and

$$B_X \cap B_X[y, 1 + \delta] \subseteq B_X[\psi_\epsilon(y), 1]$$

for every $y \in Y$. Let $i \in I$. Define the map $\psi_{i,\epsilon} : Y_i \to Y_i$ as $\psi_{i,\epsilon}(y_i) = \psi_\epsilon(y)_i$ where $y = (0, 0, \ldots, y_i, \ldots, 0, 0) \in Y$.
Then, we have $\|y_i - \psi_{i,\epsilon}(y_i)\|_{X_i} \leq \|y - \psi_\epsilon(y)\|_X \leq \epsilon$ for every $y_i \in Y_i$. Further, let $z_i \in B_{X_i} \cap B_{X_i}[y_i, 1+\delta]$. Since $z = (0, 0, \ldots, z_i, \ldots, 0, 0) \in B_X \cap B_X[y, 1+\delta]$, we have $\|z_i - \psi_{i,\epsilon}(y_i)\|_{X_i} \leq \|z - \psi_\epsilon(y)\|_X \leq 1$. Therefore, $B_{X_i} \cap B_{X_i}[y_i, 1+\delta] \subseteq B_{X_i}[\psi_{i,\epsilon}(y_i), 1]$ for every $y_i \in Y_i$. Hence, Y_i is equable in X_i. $\square$

The following example illustrates that Theorem 3.2 and Corollary 3.3 are not true for strong equability.

Example 3.25. Let $X = (\mathbb{R}^2, \|\cdot\|_\infty)$. Since X is not rotund, by Theorem 3.19 we have X is not strongly equable. It can be concluded from Theorem 3.24 that X is equable. Now, we claim that every one-dimensional subspace of X is strongly equable in X.

Case (i): Assume $Y = span\{(1, 0)\}$ or $Y = span\{(0, 1)\}$. It is clear from Theorem 3.24 that Y is equable in X. Therefore, by Proposition 3.1, Y is strongly equable in X.

Case (ii): Assume $Y = span\{(a, b)\}$ where $a \neq 0$ and $b \neq 0$. It is easy to verify that X is rotund with respect to Y. Since X is finite dimensional, X is UR with respect to Y. Therefore, by Theorem 3.13, Y is strongly equable in X. $\square$

Now, we proceed to discuss whether the notion equability can be lifted to the space of bounded (and continuous) functions. We need the following definitions and lemmas.

Definition 3.26. Let T be a topological space and Γ be a discrete topological space.

(1) $\ell_\infty(T;X)$ denotes the Banach space of all bounded X-valued functions on T, equipped with the norm $\|f\|_\infty = \sup_{t\in T}\|f(t)\|$ for every $f \in \ell_\infty(T;X)$.

(2) $C_b(T;X)$ denotes the Banach space of all bounded continuous X-valued functions on T, equipped with the norm $\|f\|_\infty = \sup_{t\in T}\|f(t)\|$ for every $f \in C_b(T;X)$.

(3) $c_0(\Gamma;X)$ denotes the Banach space of all bounded continuous X-valued functions on Γ that vanish at infinity, i.e., the set $A_{f,\eta} = \{t \in \Gamma : \|f(t)\| \geq \eta\}$ is finite for every $\eta > 0$, equipped with the norm $\|f\|_\infty = \sup_{t\in\Gamma}\|f(t)\|$ for every $f \in c_0(\Gamma;X)$.

It is clear that $c_0(\Gamma;X) \subseteq C_b(\Gamma;X) = \ell_\infty(\Gamma;X)$. We denote the closed balls in $\ell_\infty(T;X)$, $C_b(T;X)$ and $c_0(\Gamma;X)$ respectively as $B_\infty[\cdot,\cdot]$, $B_b[\cdot,\cdot]$ and $B_c[\cdot,\cdot]$ to avoid the ambiguity.

The following lemma is an improved version of [18, Lemma 4.3] and the proof follows analogously using Lemma 2.5.

Lemma 3.27. *Let Y be equable in X, T be a topological space and $F : T \to 2^X$ be a nonempty set-valued map. Let $t_0 \in T$ and F be upper Hausdorff semi-continuous at t_0. If there exists an $r > 0$ such that $G(t) := \{y \in Y : F(t) \subseteq B[y,r]\} \neq \phi$ for every $t \in T$, then the map $G : T \to 2^Y$ is lower Hausdorff semi-continuous at t_0.*

The proof of the subsequent lemma follows similar to the proof of [18, Proposition 4.4] using Lemmas 2.5 and 3.27.

Lemma 3.28. *If Y is equable in X, then for each $\beta > 0$, the element $z = z_{(y,\beta)}$ in the statement* (3) *of Proposition* 2.3 *can be chosen such that it depends continuously on y.*

Theorem 3.29. *Let T be a topological space, Γ be a discrete topological space and Y be a subspace of X. Then the following statements are equivalent.*

(1) *Y is equable in X.*

(2) *$C_b(T;Y)$ is equable in $C_b(T;X)$.*

(3) *$\ell_\infty(T;Y)$ is equable in $\ell_\infty(T;X)$.*

(4) *$c_0(\Gamma;Y)$ is equable in $\ell_\infty(\Gamma;X)$.*

(5) *$c_0(\Gamma;Y)$ is equable in $c_0(\Gamma;X)$.*

Proof. (1) $\Rightarrow$ (2): Let $\epsilon > 0$. Since Y is equable in X, by Lemma 3.28, there exists a $\delta > 0$ such that for every $\beta > 0$, there exists a continuous function $z : Y \to Y$ satisfying $z = z(y) \in B[y,\epsilon]$ and $B_X \cap B[y, 1+\delta] \subseteq B[z(y), 1+\beta]$ for every $y \in Y$. Observe that for any $f \in C_b(T;Y)$, we have $z \circ f$ is continuous and

$$\|z \circ f\|_\infty = \sup_{t\in T}\|z(f(t))\| \leq \sup_{t\in T}(\epsilon + \|f(t)\|) \leq (\epsilon + \|f\|_\infty),$$

i.e., $z \circ f \in C_b(T;Y)$. Notice that $\|(z \circ f) - f\|_\infty = \sup_{t\in T}\|z(f(t)) - f(t)\| \leq \epsilon$.

Also for any $g \in B_b[0,1] \cap B_b[f, 1+\delta]$, we have $g(t) \in B_X \cap B[f(t), 1+\delta] \subseteq B[z(f(t)), 1+\beta]$ for every $t \in T$. Therefore, $g \in B_b[z \circ f, 1+\beta]$. Thus, by Proposition 2.3, $C_b(T;Y)$ is equable in $C_b(T;X)$.

(2) $\Rightarrow$ (1): Let $\epsilon > 0$. By assumption there exist a $\delta > 0$ and a map $\Psi_\epsilon : C_b(T;Y) \to C_b(T;Y)$ such that $\|f - \Psi_\epsilon(f)\|_\infty \leq \epsilon$ and $B_b[0,1] \cap B_b[f, 1+\delta] \subseteq B_b[\Psi_\epsilon(f), 1]$ for every $f \in C_b(T;Y)$. Now, for every $x \in X$, consider the map $f_x : T \to X$ defined as $f_x(t) = x$ for every $t \in T$. Therefore, we have, $f_y \in C_b(T;Y)$ for every $y \in Y$.

Now, fix $t_0 \in T$, define $\psi_\epsilon : Y \to Y$ as $\psi_\epsilon(y) = (\Psi_\epsilon(f_y))(t_0)$ for every $y \in Y$. Thus, let $y \in Y$, we have

$$\|y - \psi_\epsilon(y)\| = \|f_y(t_0) - (\Psi_\epsilon(f_y))(t_0)\| \leq \|f_y - \Psi_\epsilon(f_y)\|_\infty \leq \epsilon.$$

Now we claim that $B_X \cap B[y, 1+\delta] \subseteq B[\psi_\epsilon(y), 1]$.

For this, consider $z \in B_X \cap B[y, 1+\delta]$. Then, notice that $f_z \in B_b[0,1] \cap B_b[f_y, 1+\delta]$ which implies $f_z \in B_b[\Psi_\epsilon(f_y), 1]$. Therefore, $\|f_z(t_0) - (\Psi_\epsilon(f_y))(t_0)\| \leq 1$, i.e., we have $\|z - \psi_\epsilon(y)\| \leq 1$. Hence the claim.

(1) $\Rightarrow$ (3): Let $\epsilon > 0$. By assumption, there exist a $\delta > 0$ and a map $\psi_\epsilon : Y \to Y$ such that for every $y \in Y$, we have $\|y - \psi_\epsilon(y)\| \leq \epsilon$ and $B_X \cap B[y, 1+\delta] \subseteq B[\psi_\epsilon(y), 1]$.

Since
$$\|\psi_\epsilon(f(t))\| \leq \epsilon + \|f(t)\| \leq \epsilon + \|f\|_\infty$$

holds for every $t \in T$, we have $\psi_\epsilon \circ f \in \ell_\infty(T;Y)$. Consider the map $\Psi_\epsilon : \ell_\infty(T;Y) \to \ell_\infty(T;Y)$ defined as $\Psi_\epsilon(f) = \psi_\epsilon \circ f$ for every $f \in \ell_\infty(T;Y)$. Let $f \in \ell_\infty(T;Y)$. Then, we have

$$\|f - \Psi_\epsilon(f)\|_\infty = \sup_{t \in T} \|f(t) - (\Psi_\epsilon(f))(t)\| = \sup_{t \in T} \|f(t) - \psi_\epsilon(f(t))\| \leq \epsilon.$$

Now, let $g \in B_\infty[0,1] \cap B_\infty[f, 1+\delta]$. Therefore, we have

$$g(t) \in B_X \cap B[f(t), 1+\delta] \subseteq B[\psi_\epsilon(f(t)), 1]$$

for every $t \in T$. Thus, $\|g - (\psi_\epsilon \circ f)\|_\infty \leq 1$. i.e., $g \in B_\infty[\Psi_\epsilon(f), 1]$. Hence, the implication holds.

(3) $\Rightarrow$ (1): The proof of this implication follows similar to the proof of (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (4): Let $\epsilon > 0$. By assumption, there exist a $\delta > 0$ and a map $\psi_\epsilon : Y \to Y$ such that for every $y \in Y$, we have $\|y - \psi_\epsilon(y)\| \leq \epsilon$ and $B_X \cap B[y, 1+\delta] \subseteq B[\psi_\epsilon(y), 1]$. Consider the map $\Psi_\epsilon : c_0(\Gamma;Y) \to c_0(\Gamma;Y)$ defined as

$$(\Psi_\epsilon(f))(t) = \begin{cases} (\psi_\epsilon \circ f)(t), & \text{if } t \in A_{f,\epsilon}; \\ 0, & \text{if } t \notin A_{f,\epsilon}, \end{cases}$$

for every $t \in \Gamma$ and $f \in c_0(\Gamma;Y)$. Clearly, $A_{\Psi_\epsilon(f),\eta} \subseteq A_{f,\epsilon}$ for every $f \in c_0(\Gamma;Y)$ and every $\eta > 0$. Let $f \in c_0(\Gamma;Y)$. It is easy to verify that

$$\|f - \Psi_\epsilon(f)\|_\infty = \sup_{t \in \Gamma} \|f(t) - (\Psi_\epsilon(f))(t)\| \leq \epsilon.$$

Further, let $g \in B_\infty[0,1] \cap B_\infty[f, 1+\delta]$.

Then, we have $g(t) \in B_X \cap B[f(t), 1+\delta] \subseteq B[\psi_\epsilon(f(t)), 1]$ for every $t \in A_{f,\epsilon}$. Therefore, $\|g(t) - (\Psi_\epsilon(f))(t)\| = \|g(t) - \psi_\epsilon(f(t))\| \leq 1$ for every $t \in A_{f,\epsilon}$. Also, for $t \notin A_{f,\epsilon}$, we have $\|g(t) - (\Psi_\epsilon(f))(t)\| = \|g(t)\| \leq 1$. Thus, $g \in B_\infty[\Psi_\epsilon(f), 1]$. Hence, the implication holds.

(4) $\Rightarrow$ (5): This implication follows from Remark 2.1.

(5) $\Rightarrow$ (1): Let $\epsilon > 0$. Then, by assumption, there exist a $\delta > 0$ and a map $\Psi_\epsilon : c_0(\Gamma; Y) \to c_0(\Gamma; Y)$ such that for every $f \in c_0(\Gamma; Y)$, we have $\|f - \Psi_\epsilon(f)\|_\infty \leq \epsilon$ and $B_c[0,1] \cap B_c[f, 1+\delta] \subseteq B_c[\Psi_\epsilon(f), 1]$. Let $t_0 \in \Gamma$ be fixed. For $x \in X$, define the map $f_x : \Gamma \to X$ as

$$f_x(t) = \begin{cases} x, & \text{if } t = t_0; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $f_y \in c_0(\Gamma; Y)$ for each $y \in Y$. Now, consider the map $\psi_\epsilon : Y \to Y$ defined as $\psi_\epsilon(y) = (\Psi_\epsilon(f_y))(t_0)$ for every $y \in Y$. Let $y \in Y$. Note that

$$\|y - \psi_\epsilon(y)\| = \|f_y(t_0) - (\Psi_\epsilon(f_y))(t_0)\| \leq \|f_y - \Psi_\epsilon(f_y)\|_\infty \leq \epsilon$$

for every $y \in Y$. Now, let $z \in B_X \cap B[y, 1+\delta]$. It follows that

$$f_z \in B_c[0,1] \cap B_c[f_y, 1+\delta] \subseteq B_c[\Psi_\epsilon(f_y), 1].$$

Thus,
$$\|f_z(t_0) - (\Psi_\epsilon(f_y))(t_0)\| \leq \|f_z - \Psi_\epsilon(f_y)\|_\infty \leq 1,$$

i.e., $\|z - \psi_\epsilon(y)\| \leq 1$. Therefore, $z \in B[\psi_\epsilon(y), 1]$. Hence, the implication holds. □

From the proof of (1) $\Rightarrow$ (2) of the above theorem we observe that Y is equable in X if and only if $C_b(T; Y)$ is equable in $\ell_\infty(T; X)$. Further, as a consequence of Theorem 3.29 we obtain the stability of equability under ℓ_∞ and c_0 product of a space.

Corollary 3.30. *Let Y be a subspace of X. Then the following statements are equivalent.*

(1) *Y is equable in X.*

(2) *$\ell_\infty(Y)$ is equable in $\ell_\infty(X)$.*

(3) *$c_0(Y)$ is equable in $\ell_\infty(X)$.*

(4) *$c_0(Y)$ is equable in $c_0(X)$.*

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