Journal of Convex Analysis
Volume 33 (2026), No. 3&4, 615–628

On a Theorem of S.-T. Hu

Gerald Beer
Department of Mathematics, California State University Los Angeles, Los Angeles, U.S.A.
gbeer@cslanet.calstatela.edu

Homeira Pajoohesh
Department of Mathematics, Medgar Evers College, CUNY, Brooklyn, U.S.A.
hpajoohesh@mec.cuny.edu

Dedicated to the memory of Hedy Attouch.

Received: July 22, 2024
Accepted: February 24, 2025

S.-T. Hu [*Boundedness in topological spaces*, J. Math. Pures Appl. 228 (1949) 287–320] characterized those bornologies on a metrizable space that are metric bornologies, i.e., that agree with the bornology of metrically bounded subsets as determined by some metric compatible with the topology. We show that the family of such metric bornologies forms a sublattice of the complete lattice of all bornologies on the underlying set which furthermore is stable under denumerable suprema in that larger lattice, but not necessarily denumerable infima. We characterize those metrizable spaces for which the bornology of relatively compact subsets - the intersection of the family of metric bornologies in any case – is itself a metric bornology. Finally, we characterize those metrizable spaces for which each bornology on the space with a countable closed base (resp. countable open base) is already a metric bornology.

Keywords: Bornology, metric bornology, lattice, lattice of bornologies, Hu's Theorem, relatively compact set, principal bornology.

2010 Mathematics Subject Classification: 46A17, 06B05, 06B23, 54E45.

1. Introduction

A bornology on a nonempty set X, like a topology on X, is family of subsets that exhibits certain prescribed properties. Specifically, a *bornology* is a family of subsets $\mathcal{B}$ of X that contains the singletons, that is stable under finite unions, and that is hereditary: whenever $B \in \mathcal{B}$ and $A \subseteq B$, we must have $A \in \mathcal{B}$. A set equipped with both a topology and a bornology is called a *bornological universe* [1, 8, 19, 28].

Certainly the prototype for the notion of bornology is the *metric bornology*: the family of bounded sets with respect to a metric d on a set. This and various other bornologies on a metric space are the primary focus of the recent monograph [8]. But there are a plethora of bornologies out there:

- the bornology of countable subsets of a set X;
- given a family of nonempty subsets $\mathcal{T}$ of a set X, the family of subsets of X whose intersection with each member of $\mathcal{T}$ is finite (see Example 3.7 below);
- given a family of real-valued functions $\{f_i : i \in I\}$ on a set X, the family of subsets A of X where $f_i(A)$ is bounded for each $i \in I$;

ISSN 0944-6532 / $ 2.50

- given a topological vector space, the family of subsets that is absorbed by each neighborhood of the origin [26, p. 9];
- the bornology of *relatively compact subsets* of a T_1-topological space (X, τ), i.e., those subsets of X that have compact closure (see Section 3 below);
- the bornology of *nowhere dense subsets* of a T_1-topological space (X, τ) with no isolated points, i.e., those subsets A for which $\mathrm{int}(\mathrm{cl}(A)) = \emptyset$ (see [8, p. 39]);
- the bornology of subsets of n-dimensional Euclidean space $\mathbb{R}^n$ each with Lebesgue measure zero;
- the family of *totally bounded subsets* of a metric space, i.e., those subsets that can be covered by finitely many balls of any prescribed positive radius.

Here are some references on bornologies not so devoted to the metric setting: [10, 11, 12, 13, 14, 18, 24, 28].

The largest bornology on a set X is the power set $\mathcal{P}(X)$ of X and the smallest is the family of its finite subsets $\mathcal{F}(X)$. Given a bornology $\mathcal{B}$ on X, we call a subfamily $\mathcal{B}_0$ a *base* for the bornology if each member of $\mathcal{B}$ is contained in some member of $\mathcal{B}_0$. It can be shown that a family $\mathcal{A}$ of subsets of X is a base for a bornology on X if and only if it covers X and $\mathcal{A}$ is directed by inclusion [8, Proposition 11.12]. In the case of a metric space the family of all open balls and the family of all closed balls form bases for the associated metric bornology.

Arguably, the serious study of bornologies begins with the seminal paper of S.- T. Hu [19]. Hu provided necessary and sufficient conditions for a bornology $\mathcal{B}$ on a metrizable topological space (X, τ) to be a metric bornology with respect to some metric d compatible with the topology τ. We will regard the following set of conditions taken together to be primitive:

- the bornology has a countable base;
- the bornology has an open base, that is, a base consisting of members of τ;
- the bornology has a closed base, that is, a base consisting of closed subsets of X as determined by τ.

These conditions are clearly necessary, for if we fix a point p in a metric space, then the family of open (resp. closed) balls with integer radius with p as center form a countable open (resp. closed) base for the associated metric bornology.

The conjunction of these conditions amounts to saying that $\mathcal{B}$ has a countable base and whenever $B \in \mathcal{B}$, there exists $A \in \mathcal{B}$ such that $\mathrm{cl}(B) \subseteq \mathrm{int}(A)$. Proofs of Hu's result also appear in the subsequent monographs [8, 20]. If (X, τ) is compact and metrizable, then there is only one metric bornology no matter the compatible metric, namely the power set of X. It turns out that if (X, τ) is not compact, uncountably many distinct metric bornologies can be obtained for the metrizable space [7].

One immediate consequence of Hu's Theorem – which we view as a folk theorem – is this: each bornology $\mathcal{B}$ on a set X (with no topological structure assumed) having a countable base is the metric bornology for some metric d on X. To see this, equip X with the discrete topology; as each subset of X is clopen, the countable base for $\mathcal{B}$ is automatically both open and closed. As a courtesy to the reader and because we know of no published reference for this fact, we now provide a direct proof.

First suppose $\mathcal{B}$ has a finite base. Since its union must be X, we see that $\mathcal{B}$ coincides with the power set of X and the zero-one metric does the job. Otherwise, we can find a denumerable base $\{B_n : n \in \mathbb{N}\}$ such that for each n, B_n is a proper subset of B_{n+1}. Define $f : X \to [0, \infty)$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in B_1 \\ n & \text{if } x \in B_{n+1} \backslash B_n \text{ for some } n. \end{cases}$$

Then f is well-defined and f is bounded on a subset E of X if and only if E is a subset of some B_n, that is, $E \in \mathcal{B}$. Taking d as the zero-one metric on X, $\rho(x, y) := d(x, y) + |f(x) - f(y)|$ defines a metric on X whose bounded sets coincide with $\mathcal{B}$.

While Hu tried unsuccesfully to popularize bornologies within the framework of general topology, they arose in the study of locally convex spaces and were of particular interest to researchers in France (see, e.g. [18]). But towards the end of the 20th century, Beer revisited Hu's theorem and for a given compatible metric d characterized the metric bornologies that can arise from uniformly equivalent metrics [7] (see later [16]). Now a metric ρ is uniformly equivalent to d if and only if the identity function between (X, d) and (X, ρ) is uniformly continuous in both directions. A weaker condition is that the identity function maps Cauchy sequences to Cauchy sequences in both directions; metric bornologies that can arise from such Cauchy equivalent metrics were later characterized by Aggarwal and Kundu [1].

There are two purposes of this article. The first is to expose the lattice structure of the family of metric bornologies for a metrizable space, where the bornologies are of course partially ordered by inclusion. As explained in the comprehensive article [10], the family of bornologies $\mathfrak{B}(X)$ on a set X with no other assumed structure is a complete distributive lattice.

One of the highlights of [10] is the identification of the complemented elements of the lattice of bornologies, which form a Boolean algebra that is join dense in $\mathfrak{B}(X)$. These are the principal bornologies on X, one associated with each subset A of X. For each $A \in \mathcal{P}(X)$, the *principal bornology* determined by A consists of all sets of the form $B \cup F$ where $B \subseteq A$ and $F \in \mathcal{F}(X)$. Two subsets determine the same principal bornology if and only if they have finite symmetric difference. Sticking with the notation of [8, 10], we denote the principal bornology determined by A by $\mathrm{born}(\{A\})$.

The next little result provides a simple application of Hu's Theorem.

Proposition 1.1. *Let (X, τ) be a metrizable space. The following are equivalent:*

(a) *Each principal bornology on X is a metric bornology for the topology τ;*

(b) *$\mathcal{F}(X)$ is a metric bornology for the topology τ;*

(c) *X is countable and τ is the discrete topology.*

Proof. (a) $\Rightarrow$ (b). This is trivial as $\mathcal{F}(X)$ is the principal bornology determined by any of its members.

(b) $\Rightarrow$ (c). Suppose condition (b) holds. If X were uncountable, then $\mathcal{F}(X)$ cannot have a countable base, else the union of its members would be X, and X would be a countable union of finite sets, which is impossible. If τ is not the discrete topology, then some $p \in X$ is a limit point of X. Clearly, p can have no neighborhood in $\mathcal{F}(X)$ and so $\mathcal{F}(X)$ can have no open base – a contradiction.

(c) $\Rightarrow$ (a). Suppose condition (c) holds. Let $A \subseteq X$ be arbitrary. Now a base for born($\{A\}$) is $\{A \cup F : F \in \mathcal{F}(X)\}$ which by (c) is a countable family of clopen sets. Thus, Hu's criteria are satisfied. $\square$

While the lattice of metric bornologies need not be complete, it is a sublattice of $\mathfrak{B}(X)$ and it is stable with respect to denumerable suprema within $\mathfrak{B}(X)$. The sublattice of metric bornologies need not have a smallest element, and we identify those metrizable spaces for which this is so.

The second purpose is to determine those metrizable spaces where either the second or third bullet point in Hu's conditions above can be dispensed with and still have a metric bornology.

Along the way, we give necessary and sufficient conditions for a principal bornology on a metrizable space to be a compatible metric bornology.

2. Preliminaries

Let (X, τ) be a topological space. If $A \subseteq X$ we denote its closure, interior and set of limit points by $\mathrm{cl}(A), \mathrm{int}(A)$ and A', respectively.

Let p be a fixed point of a metric space (X, d) and let $\alpha > 0$. We denote the open (resp. closed) ball with center p and radius α by $S_d(p, \alpha)$ (resp. $S_d[p, \alpha]$). While open balls are open sets and closed balls are closed sets, $S_d[p, \alpha]$ can properly contain $\mathrm{cl}(S_d(p, \alpha))$. Recall that a subset A of the metric space is called (*metrically*) *bounded* if it is contained in some ball, equivalently, if it is either empty or has finite diameter $:= \sup \{d(x, y) : x \in A, y \in A\}$. We denote the bornology of bounded subsets of (X, d) by $\mathcal{B}_d(X)$. A subset A of (X, d) is called *uniformly discrete* provided there exists $\delta > 0$ such that $d(a_1, a_2) > \delta$ whenever a_1 and a_2 are distinct points of A. For such a set, we have $A' = \emptyset$ and so a uniformly discrete set is a closed set. The family of uniformly discrete subsets of a metric space need not be a bornology as it is not in general stable under finite unions: in $\mathbb{R}$ equipped with its usual metric, consider the union of $\mathbb{N}$ and $\{n + \frac{1}{n+1} : n \in \mathbb{N}\}$.

The following known result is a consequence of the equivalence of compactness and sequential compactness in the metric setting.

Proposition 2.1. *Let A a nonempty subset of a metric space (X, d). The following conditions are equivalent.*

(1) *A is relatively compact;*

(2) *Each sequence in A has a convergent subsequence to a point of X;*

(3) *Each sequence in A has a convergent subsequence to a point of cl(A).*

By a *partially ordered set S*, we mean a nonempty set equipped with a binary relation $\leq$ that is reflexive, antisymmetric, and transitive. It is standard to write $s_1 \leq s_2$ provided $(s_1, s_2) \in \leq$.

If S_0 is a nonempty subset of S, we say $s_0 \in S_0$ is the *largest element* of S_0 if $\forall s \in S_0, s \leq s_0$; we call $u \in S$ an *upper bound* for S_0 if $\forall s \in S_0, s \leq u$. If S_0 has a least upper bound, it is unique and we call it *the supremum* of S_0. Dually, we can define the *smallest element*, *lower bound* and *infimum* for S_0. If each two-element subset of S has a supremum and an infimum, we call S a *lattice*. In this case, we denote the supremum of s_1 and s_2 by $s_1 \vee s_2$ and their infimum by $s_1 \wedge s_2$, and sometimes call them the *join* and *meet* respectively of the two elements. Obviously, each nonempty finite subset of a lattice has both a supremum and an infimum; the empty set has a supremum (resp, infimum) if and only if the lattice has a smallest (resp. largest) element. We denote the supremum of an arbitrary subset S_0 if it has one by $\bigvee S_0$ or by $\bigvee_{s\in S_0} s$. We denote the infimum of an arbitrary subset S_0 if it has one by $\bigwedge S_0$ or by $\bigwedge_{s\in S_0} s$.

A lattice is called *distributive* if the formulas below hold whenever $\{s_1, s_2, s_3\} \subseteq S$:

$$s_1 \wedge (s_2 \vee s_3) = (s_1 \wedge s_2) \vee (s_1 \wedge s_3) \text{ and } s_1 \vee (s_2 \wedge s_3) = (s_1 \vee s_2) \wedge (s_1 \vee s_3).$$

Actually, if one of these formulas holds, then so does the other [9, p. 48].

A lattice S is called *complete* if each nonempty subset has both a supremum and infimum. In a complete lattice S, the lattice itself has both a supremum and an infimum, i.e., a largest and smallest element. The prototype for a complete lattice is the power set of a nonempty set X partially ordered by inclusion, where the supremum of a family of subsets is their union and the infimum of a family of subsets is their intersection.

3. On the lattice of metric bornologies

It is well-known that the family of bornologies $\mathfrak{B}(X)$ on a nonempty set X (not assumed to be equipped with a topology) forms a complete lattice with respect to inclusion where if $\mathfrak{B}$ is a nonempty subfamily, its infimum is $\bigwedge_{\mathcal{B}\in\mathfrak{B}} \mathcal{B} = \cap_{\mathcal{B}\in\mathfrak{B}}\mathcal{B}$ and its supremum $\bigvee_{\mathcal{B}\in\mathfrak{B}} \mathcal{B}$ consists of all subsets of X of the form $B_1 \cup B_2 \cup \cdots B_n$ where $n \in \mathbb{N}$ and where B_j belongs to some member of $\mathfrak{B}$ for each $j \in \{1, 2, \ldots, n\}$ [8, 10, 24]. The complete lattice is not only distributive but as observed by Pajoohesh [24, Theorem 4.1], the following infinite distributive law holds: if $\mathcal{A}$ is a bornology on X and $\mathfrak{B}$ is a subfamily of $\mathfrak{B}(X)$, then we have

$$\mathcal{A} \wedge \bigvee \mathfrak{B} = \mathcal{A} \cap \bigvee_{\mathcal{B}\in\mathfrak{B}} \mathcal{B} = \bigvee_{\mathcal{B}\in\mathfrak{B}} \mathcal{A} \cap \mathcal{B}.$$

A complete lattice where meet distributes over infinite suprema is called a *frame* [22]. The standard example of a frame is the complete lattice of open subsets of a topological space, where the supremum of a nonempty family $\mathcal{V}$ of open sets is the union of the members of $\mathcal{V}$ and the infimum of a nonempty family $\mathcal{V}$ of open sets is the interior of the intersection of the members of $\mathcal{V}$. As shown later on, join need not distribute over infinite infima/intersection in the present context [10].

Now suppose X is equipped with a metrizable topology τ and we denote the family of metric bornologies as determined by compatible metrics on X by $\mathfrak{M}(X, \tau)$. Clearly, $\mathcal{P}(X)$ is the largest metric bornology as τ has a compatible bounded metric [29, Theorem 22.2]. There is a natural guess for the smallest metric bornology: the bornology of relatively compact sets.

Because taking closures is a monotone procedure and each closed subset of a compact set is compact, the family of relatively compact sets of (X, τ) is hereditary.

Singleton subsets in any T_1-topological space coincide with their closure and are always compact. As the finite union of compact subsets is compact and the closure of a finite union is the union of the closures, the relatively compact subsets are stable under finite union. This shows that the relatively compact subsets of any T_1-topological space form a bornology, called the *compact bornology* [8]. Clearly, a subset of a Hausdorff space is relatively compact if and only if it is a subset of some compact subset.

Example 3.1. In a topological space in which the T_1-separation axiom fails, the closure of a singleton subset need not be compact. Consider the real line $\mathbb{R}$ equipped with a topology τ whose open sets are $\mathbb{R}, \emptyset$ and each ray of the form (a, ∞) where a ranges over $\mathbb{R}$. The smallest closed set containing each real number a is $(-\infty, a]$, and so $\mathrm{cl}(\{a\}) = (-\infty, a]$ [15, p. 70]. The open cover $\{(n, \infty) : n \in \mathbb{Z}\}$ of $(-\infty, a]$ has no finite subcover. □

It is a folk result that in a metrizable space (X, τ), a subset of X is metrically bounded with respect to each compatible metric if and only if the subset is relatively compact. Put differently, the intersection of all metric bornologies determined by compatible metrics coincides with the family of relatively compact subsets. We sketch a proof for the uninitiated.

First, since a compact set in a metrizable space (X, τ) is bounded with respect to each compatible metric, so is each relatively compact subset, as each lies in a compact subset. On the other hand, if $A \in \mathcal{P}(X)$ is not relatively compact, then by Proposition 2.1, we can find a sequence $\langle a_n \rangle$ with distinct terms in A that has no convergent subsequence to point of X. The set of terms is a closed set whose relative topology is discrete, so by the Tietze extension theorem [15, 29], there is a continuous real-valued function f on X which maps each a_n to n. Letting d be any compatible metric for τ, a compatible metric ρ for τ with respect to which A is unbounded is given by

$$\rho(x_1, x_2) := d(x_1, x_2) + |f(x_1) - f(x_2)| \qquad (x_1, x_2 \in X).$$

The above discussion shows that for a metrizable space, the bornology of relatively compact subsets is the infimum of $\mathfrak{M}(X, \tau)$ in the complete lattice $\mathfrak{B}(X)$. But exactly when do the relatively compact subsets form a *metric* bornology for some metric compatible with τ?

Theorem 3.2. *Let (X, τ) be a metrizable space. The following conditions are equivalent.*

(1) *the bornology of relatively compact subsets of X belongs to $\mathfrak{M}(X, \tau)$;*

(2) *there is a smallest element $\mathcal{B}$ of $\mathfrak{M}(X, \tau)$ with respect to inclusion;*

(3) *(X, τ) is both locally compact and separable.*

Proof. We first show that condition (1) implies condition (3). By (1) and Hu's theorem, the bornology of relatively compact sets has a countable base $\{B_n : n \in \mathbb{N}\}$ and so $\{\mathrm{cl}(B_n) : n \in \mathbb{N}\}$ is a countable base as well. Since $X = \cup_{n=1}^{\infty} \mathrm{cl}(B_n)$, (X, τ) is σ-compact and thus, as a countable union of separable subspaces, is itself separable. To show local compactness, fix $p \in X$. Again by Hu's theorem, as the bornology of relatively compact sets must have an open base, there is an open relatively compact set A of which $\{p\}$ is a subset. This means that $\mathrm{cl}(A)$ is a compact neighborhood of $\{p\}$, so by definition, X is locally compact.

Next suppose condition (3) holds. By a theorem of Vaughan [27], there is a compatible metric d for (X, τ) such that each closed and bounded subset is compact. For such a metric d, we have coincidence of $\mathcal{B}_d(X)$ with the bornology of relatively compact sets, and so $\mathcal{B}_d(X)$ must be the smallest metric bornology. We have shown that condition (2) holds if (3) holds.

Finally, if condition (2) holds, then being a metric bornology, $\mathcal{B}$ contains each relatively compact set. Suppose $\mathcal{B}$ contains some subset A that fails to be relatively compact. As explained, there is another bornology $\mathcal{B}_0 \in \mathfrak{M}(X, \tau)$ that does not contain A, and so $\mathcal{B}_0$ cannot contain $\mathcal{B}$. This contradicts condition (2). Thus, $\mathcal{B}$ consists exactly of the relatively compact sets, and condition (1) holds. □

Our next result shows that for a metrizable space (X, τ), $\mathfrak{M}(X, \tau)$ is a sublattice of $\mathfrak{B}(X)$ and more. Again, we call on Hu's theorem in the proof.

Theorem 3.3. *Let (X, τ) be a metrizable topological space. Then $\mathfrak{M}(X, \tau)$ is a sublattice of $\mathfrak{B}(X)$ which is stable under denumerable suprema within $\mathfrak{B}(X)$.*

Proof. Let d_1 and d_2 be metrics whose induced topologies agree with τ. Then put $\rho := \max\{d_1, d_2\}$; it is well-known that ρ is equivalent to each d_j. Clearly, $A \subseteq X$ is ρ-bounded if and only if it is d_j-bounded for each $j \in \{1, 2\}$, that is,

$$\mathcal{B}_\rho(X) = \mathcal{B}_{d_1}(X) \cap \mathcal{B}_{d_2}(X) = \mathcal{B}_{d_1}(X) \wedge \mathcal{B}_{d_2}(X).$$

For $\mathcal{B}_{d_1}(X) \vee \mathcal{B}_{d_2}(X)$, fixing $p \in X$, a countable open (resp. closed) base for the join consists of all sets of the form $S_{d_1}(p, j) \cup S_{d_2}(p, j)$ (resp. $S_{d_1}[p, j] \cup S_{d_2}[p, j]$) where j runs over $\mathbb{N}$. Apply Hu's Theorem.

Turning to denumerable suprema, let $\{d_n : n \in \mathbb{N}\}$ be a family of compatible metrics whose metric bornologies are distinct. Let $p \in X$ be fixed. Then for each positive integer n, $\{S_{d_n}(p, j) : j \in \mathbb{N}\}$ (resp. $\{S_{d_n}[p, j] : j \in \mathbb{N}\}$) is a countable open (resp. closed) base for $\mathcal{B}_{d_n}(X)$. It follows that all subsets of X of the form

$$S_{d_1}(p, j) \cup S_{d_2}(p, j) \cup \cdots \cup S_{d_k}(p, j)$$

as k and j run over the positive integers is a countable open base for $\bigvee_{n=1}^{\infty} \mathcal{B}_{d_n}(X)$, while all sets of the form

$$S_{d_1}[p, j] \cup S_{d_2}[p, j] \cup \cdots \cup S_{d_k}[p, j]$$

is a countable closed base for $\bigvee_{n=1}^{\infty} \mathcal{B}_{d_n}(X)$, as the surjective image of the countable set $\mathbb{N}^2$ is countable. □

The next example shows that $\mathfrak{M}(X, \tau)$ need not be stable under denumerable infima.

Example 3.4. Let X be the set ℓ_2 of square summable real sequences and let $\|\cdot\|_2$ be the usual norm for the sequence space. Let $\{e_n : n \in \mathbb{N}\}$ be the standard orthonormal basis for ℓ_2 and let θ denote the origin of the space. Let j be an arbitrary positive integer; by the Tietze extension theorem applied to the uniformly discrete set $\{\frac{1}{j} e_n : n \in \mathbb{N}\}$ there exists a continuous function f_j on the normed linear space $(\ell_2, \|\cdot\|_2)$ satisfying $f_j(\frac{1}{j}e_n) = n$ for all $(j,n) \in \mathbb{N}^2$. For each $j \in \mathbb{N}$, the metric $d_j : \ell_2 \times \ell_2 \to [0,\infty)$ defined by

$$d_j(x,y) := \|x-y\|_2 + |f_j(x) - f_j(y)|$$

is a metric equivalent to the standard ℓ_2 metric, and thus determines a metric bornology relative to the topology of the ℓ_2 metric. Now if $\cap_{j=1}^{\infty}\mathcal{B}_{d_j}$ were a metric bornology, then since it must contain $\{\theta\}$, by Hu's theorem it must contain an open set containing θ and so must contain for some positive integer j the closed ball $\{x \in \ell_2 : \|x\|_2 \leq \frac{1}{j}\}$. This is a contradiction to $\{\frac{1}{j}e_n : n \in \mathbb{N}\} \notin \mathcal{B}_{d_j}$.

We have shown $\mathfrak{M}(\ell_2, \tau)$ where τ is the topology determined by the ℓ_2-norm is not stable under denumerable intersection. A similar counterexample can be constructed in any metrizable space that fails to be locally compact. $\square$

Now the empty set is a countable subset of any lattice, and as we have mentioned, the supremum of the empty set in any lattice is its smallest element if it has one. We also noted that the supremum of any finite nonempty subset of a lattice exists. As a countable set is either denumerable, empty or finite and nonempty, Theorem 3.2 and Theorem 3.3 taken together yield

Corollary 3.5. *Let (X,τ) be a metrizable topological space. Then $\mathfrak{M}(X,\tau)$ is stable under countable suprema if and only if (X,τ) is both locally compact and separable.*

One does not really expect for an uncountable supremum of compatible metric bornologies to necessarily be a metric bornology, and indeed this fails in the most benign of settings.

Example 3.6. We produce an uncountable family of compatible metric bornologies on $\mathbb{R}^2$ equipped with its usual topology τ whose supremum fails to have a countable base and thus is not a metric bornology. We use the Baire Category Theorem and functional analytic arguments. Our family of metric bornologies will be indexed by the punctured plane which we denote by P. For each $(p,q) \in P$, define the continuous linear functional $g_{(p,q)} : \mathbb{R}^2 \to \mathbb{R}$ by

$$g_{(p,q)}(x,y) := px + qy.$$

Let $\mathcal{B}_{(p,q)}$ denote the bornology on $\mathbb{R}^2$ with base $\{B(p,q,n) : n \in \mathbb{N}\}$ where for each positive integer n,

$$B(p,q,n) := \{(x,y) \in \mathbb{R}^2 : |px+qy| \leq n\}.$$

Each $B(p,q,n)$ is a closed strip around the kernel $H(p,q)$ of $g_{(p,q)}$ whose thickness increases with n. It is left to the reader to verify that $\mathcal{B}_{(p,q)}$ belongs to $\mathfrak{M}(\mathbb{R}^2,\tau)$ for each $(p,q) \neq (0,0)$. Clearly, $\mathcal{B}_{(p,q)} = \mathcal{B}_{(p',q')}$ if and only if for some $\alpha \neq 0$ we have $p' = \alpha p$ and $q' = \alpha q$. There is a natural one-to-one correspondence between the family of these metric bornologies and $\{(\cos(t), \sin(t)) : 0 \leq t < \pi\}$.

We claim that $\bigvee_{(p,q)\in P}\mathcal{B}_{(p,q)}$ can have no countable base. For if it did, then each base for the supremum bornology would contain a countable base, and we could find a sequence $\langle(p_n,q_n)\rangle$ in P such that all sets of the form $E(n,j) := \cup_{k\leq n}B(p_k,q_k,j)$ where (n,j) runs over $\mathbb{N}^2$ is a countable base for the supremum. Now each kernel

$H(p_n, q_n)$ where $n \in \mathbb{N}$ is a straight line passing through $(0,0)$ and as such is a closed nowhere dense subset of $\mathbb{R}^2$. By Baire's Theorem [15, p. 249], $P \setminus \cup_{n=1}^{\infty} H(p_n, q_n)$ is nonempty. Choosing (p,q) in this complement, the straight line $L := \{(\alpha p, \alpha q) : \alpha \in \mathbb{R}\}$ has only the origin in common with $\cup_{n=1}^{\infty} H(p_n, q_n)$. As L is the kernel of $g_{(-q,p)}$, it is a member of $\mathcal{B}_{(-q,p)}$. We will show that L is a subset of no $E(n,j)$.

To this end, fix $n \in \mathbb{N}$ and $j \in \mathbb{N}$. Since $g_{(p_k,q_k)}(p,q) \neq 0$ for each $k \leq n$, by linearity of the functionals we can find $\alpha > 0$ such that for each $k \leq n$, we have $|g_{(p_k,q_k)}(\alpha p, \alpha q)| > j$. This shows that $(\alpha p, \alpha q) \in L \setminus E(n,j)$ as required. □

We give an additional purely set-theoretic example that shows $\mathfrak{M}(X,\tau)$ need not be stable under arbitrary suprema that uses Proposition 1.1.

Example 3.7. Consider $\mathbb{N}$ equipped with the discrete topology τ, a metrizable space. We can partition $\mathbb{N}$ into a denumerable family of infinite subsets $\{T_n : n \in \mathbb{N}\}$. Consider the following bornology $\mathcal{B}$ on $\mathbb{N}$:

$$\mathcal{B} := \{A \subseteq \mathbb{N} : \forall n \in \mathbb{N},\ A \cap T_n \text{ is finite}\}.$$

We claim that $\mathcal{B}$ has no countable base, say $\{B_k : k \in \mathbb{N}\}$. For if it did, since each T_k is infinite, for each $k \in \mathbb{N}$, we can pick m_k in $T_k \setminus B_k$. Then

$$\bigcup_{k=1}^{\infty} \Big((T_k \cap B_k) \cup \{m_k\} \Big)$$

belongs to $\mathcal{B}$ but by construction is a subset of no B_k. This verifies the claim.

Cleary $\mathcal{B}$ is the supremum of the principal bornologies that it contains [10, 24], which by Proposition 1.1 is a supremum of bornologies in $\mathfrak{M}(\mathbb{N},\tau)$. □

4. On the redundancy of the conditions in Hu's theorem

As we initially stated it, Hu's characterization of bornologies $\mathcal{B}$ on a metrizable space that are metric bornologies with respect to a compatible metric involved three separate conditions: (1) the bornology has a countable base; (2) the bornology has a closed base; (3) the bornology has an open base. The conjunction of conditions (1) and (2) can be replaced by this condition (a) the bornology has a countable closed base, as the condition that $\mathcal{B}$ has a closed base is equivalent to $\forall B \in \mathcal{B}, \mathrm{cl}(B) \in \mathcal{B}$. Thus, if $\mathcal{C}$ is a countable base for the bornology, then $\{\mathrm{cl}(C) : C \in \mathcal{C}\}$ will be a countable closed base. Similarly, condition (1) plus condition (3) can be replaced by (b) the bornology has a countable open base because each member of $\mathcal{C}$ can be replaced by an open superset belonging to $\mathcal{B}$.

The next result characterizes those metrizable topological spaces such that each bornology on the space satisfying conditions (1) and (2) must already satisfy (3), that is, must already be a metric bornology.

Theorem 4.1. *Let (X,τ) be a metrizable topological space. The following conditions are equivalent.*

(i) *Each bornology $\mathcal{B}$ on X has an open base;*

(ii) *Each bornology $\mathcal{B}$ on X with a countable closed base has an open base:*

(iii) *τ is the discrete topology.*

Proof. (i) $\Rightarrow$ (ii). This is trivial.

(ii) $\Rightarrow$ (iii). Suppose τ is not the discrete topology. Let d be a fixed compatible metric. Then some point $\{p\}$ fails to be open, and we can find a sequence $\langle x_n \rangle$ such that for each $n \in \mathbb{N}$, $0 < d(x_{n+1}, p) < \min\{\frac{1}{n}, d(x_n, p)\}$. By construction, the terms of the sequence are distinct and p is not among them.

For each n, put $T_n := \{p\} \cup (\{x_j : j \in \mathbb{N}\} \backslash \{x_n\})$ and then put $\delta_n := \frac{1}{3} d(x_n, T_n)$. For each $n \in \mathbb{N}$,

$$\cup_{j=1}^{n} S_d[x_j, \delta_j]$$

is a finite union of disjoint closed balls not containing p. As a result, all sets of the form

$$\cup_{j=1}^{n} S_d[x_j, \delta_j] \cup \Big(X \backslash \cup_{j=1}^{\infty} S_d(x_j, \delta_j) \Big)$$

determine a countable closed base for a bornology $\mathcal{B}$ on X because these sets form a cover of X and are directed by inclusion. But none of these sets contain a neighborhood of p so that the bornology fails to have an open base.

(iii) $\Rightarrow$ (i). If the topology is discrete, then each subset of X is open. Thus, if $\mathcal{B}$ is a bornology on X, it has itself as an open base. $\square$

Our final result speaks to when conditions (1) and (3) together must guarantee that condition (2) holds.

Theorem 4.2. *Let (X, τ) be a metrizable topological space. The following conditions are equivalent.*

(i) *Each bornology $\mathcal{B}$ on X with an open base has a closed base;*

(ii) *Each bornology $\mathcal{B}$ on X with a countable open base has a closed base;*

(iii) *The set of limit points X' of X is compact.*

Proof. We need only prove (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i). For the first, suppose (iii) fails. Then viewing X' equipped with the subspace topology as a metrizable space, we can find a compatible unbounded metric d_0 for the subspace, and since X' is closed, by the Hausdorff extension theorem for metrics (see [17] or [29, p.165]), we can extend this to a compatible metric d for (X, τ). Fix $p \in X'$ and by the unboundedness of X' choose a sequence $\langle x_n \rangle$ in X' such that $d(x_1, p) > 1$ and for each $n \in \mathbb{N}$, $d(x_{n+1}, p) > d(x_n, p) + 1$. We compute using the triangle inequality that whenever $n > k \geq 1$,

$$d(x_n, x_k) \geq d(x_n, p) - d(x_k, p) = \sum_{j=k}^{n-1} \Big(d(x_{j+1}, p) - d(x_j, p) \Big) > n - k \geq 1.$$

Thus, the set of distinct terms of the sequence that we have produced forms a uniformly discrete set and so any subset of the set of terms is closed. For each $n \in \mathbb{N}$, let $B_n := X \backslash \{x_n, x_{n+1}, x_{n+2}, \ldots\}$. The family $\{B_n : n \in \mathbb{N}\}$ forms a countable open base for a bornology which cannot have a closed base, as the bornology fails to contain $\mathrm{cl}(B_n) = X$ for any n.

We move on to (iii) $\Rightarrow$ (i). If $X' = \emptyset$, then the topology is discrete and so each bornology on X automatically has both an open base and a closed base. Otherwise, let d be a compatible metric for τ and let $\mathcal{B}$ be a bornology on X with an open base

$\mathcal{V}$. Let $B \in \mathcal{B}$ be arbitrary. We produce a closed member of $\mathcal{B}$ that contains B. Since $\mathcal{V}$ gives an open cover of X, by compactness we can find $V_1, V_2, \ldots, V_n$ in $\mathcal{V}$ with $X' \subseteq \cup_{j=1}^n V_j$. Now each neighborhood of a compact subset of X contains some closed ε-enlargement of it, and so for some positive ε

$$\{x \in X : d(x, X') \leq \varepsilon\} \subseteq \cup_{j=1}^n V_j \in \mathcal{B}.$$

Clearly, $B \subseteq \{x \in X : d(x, X') \leq \varepsilon\} \cup (B \cap \{x \in X : d(x, X') > \varepsilon\})$.

If the second set in the union is empty, then B already has a closed superset in $\mathcal{B}$ because $\mathcal{B}$ is hereditary. Otherwise, as there is a positive gap between X' and $B \cap \{x \in X : d(x, X') > \varepsilon\}$, the set $B \cap \{x \in X : d(x, X') > \varepsilon\} \in \mathcal{B}$ can have no limit points in X, and so it trivially contains its set of limit points. Thus, B is contained in the union of two closed members of $\mathcal{B}$. □

Metrizable spaces (X, τ) whose set of limit points is compact are precisely those that admit a compatible *UC-metric*, i.e., a metric d such that each continuous function on X into an arbitrary metric space is uniformly continuous with respect to d [8, Theorem 19.12]. Such metric spaces have been well-studied and have a number of elegant characterizations (see, e.g., [8, 21]). These metrics have also been called *Atsuji metrics*, in view of Atsuji's influential paper on them [2]. They have been called *Lebesgue metrics* as well, because these are the metrics with respect to which each open cover of the space has a Lebesgue number (see [23] and [8, Theorem 19.1]).

5. When is a principal bornology a metric bornology?

Given a nonempty set X and a subset A of X, it is obvious that born($\{A\}$) is a complemented element of $\mathfrak{B}(X)$ as

$$\text{born}(\{A\}) \wedge \text{born}(\{X \backslash A\}) = \text{born}(\{A\}) \cap \text{born}(\{X \backslash A\}) = \mathcal{F}(X),$$

while born($\{A\}$) $\vee$ born($\{X\backslash A\}$) $= \mathcal{P}(X)$ because $X = A \cup (X\backslash A)$. What is remarkable (as stated in the introductory section) is that there are no other complemented elements [10, Theorem 4.7]. Since the complemented elements of a distributive lattice with largest and smallest elements form a Boolean algebra, $\{\text{born}(\{A\}) : A \in \mathcal{P}(X)\}$ is a Boolean algebra and another result from [10] shows that there is a Boolean embedding of any Boolean algebra into the Boolean algebra of principal bornologies on some set. The proof is a little delicate [10, Theorem 4.13].

Clearly, the family of principal bornologies on a nonempty set is worthy of study. With respect to the present line of inquiry, it must be asked in the case that X carries a metrizable topology τ exactly when a principal bornology on X belongs to $\mathfrak{M}(X, \tau)$. In the next result, we use the fact that a base for born($\{A\}$) consists of all sets of the form $A \cup F$ with $F \in \mathcal{F}(X\backslash A)$.

Theorem 5.1. *Let A be a subset of a metrizable space (X, τ). The following statements are equivalent:*

(1) *born($\{A\}$) is a member of $\mathfrak{M}(X, \tau)$;*

(2) *either* (a) *$X\backslash A$ is finite, or* (b) *$X\backslash A$ is denumerable, $A'\backslash A$ is finite and for each $F \in \mathcal{F}(X\backslash A)$, the set $A \cup F$ belongs to τ.*

Proof. First, suppose condition (1) holds, i.e., born($\{A\}$) statisfies Hu's criteria. We show that if condition (2)(a) fails, then the three statements enumerated in (2)(b) hold. Since born($\{A\}$) has a countable base, we can extract a countable base from $\{A \cup F : F \in \mathcal{F}(X\backslash A)\}$. Since a base for a bornology must cover X, there exists a sequence $\langle F_n \rangle$ in $\mathcal{F}(X\backslash A)$ with $X = \cup_{n=1}^{\infty}(A \cup F_n)$. From this, $X\backslash A$ is a countable union of finite sets, and since we are assuming this complement is not finite, it must be denumerable. For the second statement, suppose to the contrary that $A'\backslash A$ is infinite. Then whenever F is finite, $A \cup F$ cannot contain $\mathrm{cl}(A) = A \cup A'$, that is $\mathrm{cl}(A)$ does not belong to born($\{A\}$). As a result, born($\{A\}$) cannot have a closed base [8, Proposition 11.14], in contradiction to (1).

For the third statement, suppose $F \in \mathcal{F}(X\backslash A)$ is arbitrary. By condition (1) the principal bornology has an open base, and so $A \cup F$ must have an open superset in the bornology. This means that there exists $F_1 \in \mathcal{F}(X\backslash A)$ such that

$$A \cup F \subseteq \mathrm{int}(A \cup F_1) \subseteq A \cup F_1.$$

It follows that $\mathrm{int}(A \cup F_1)$ is expressible as $A \cup F_2$ where $F \subseteq F_2 \subseteq F_1$. But $A \cup F$ is obtainable from the open set $A \cup F_2$ by removing the closed set $F_2\backslash F$ from it. We conclude that $A \cup F \in \tau$.

Conversely, suppose condition (2) holds. If $X\backslash A$ is finite, then born($\{A\}$) coincides with $\mathcal{P}(X)$ which we know to be a compatible metric bornology. Otherwise, $X\backslash A$ is denumerable and $\{A \cup F : F \in \mathcal{F}(X\backslash A)\}$ is a countable base for the principal bornology. The assumption $A'\backslash A$ is finite implies that whenever $F \in \mathcal{F}(X\backslash A)$,

$$\mathrm{cl}(A \cup F) = (A \cup A') \cup \mathrm{cl}(F) = A \cup (A'\backslash A) \cup F \in \mathrm{born}(\{A\}),$$

and since a bornology is a hereditary family of sets, born($\{A\}$) contains the closure of each of its members. The third statement says that each member of a particular base for born($\{A\}$) is an open set. Together, Hu's criteria are satisfied. □

We note that in the statement of our last theorem, the set of limit points of A can be replaced by the boundary of A.

We make one last observation in this section: as the lattice of compatible metric bornologies need not be stable under denumerable intersection, the same is true for the lattice of principal bornologies on a set [10, Example 4.8].

6. Some closing remarks on the Attouch-Wets topology

One highlight of the career of Hedy Attouch was his detailed study with Roger Wets of the metrizable *Attouch-Wets topology* τ_{AW_d}, called elsewhere the *bounded Hausdorff topology*, on the nonempty closed subsets $C_0(X)$ of a metric space (X, d). Initially, the authors were particularly interested in special closed sets, namely epigraphs of lower semicontinuous functions in $X \times \mathbb{R}$ (see, e.g., [6, p. 13]), where the topology was called by them the *epi-distance topology* [4].

Back to the general case, we denote this topology by τ_{AW_d}. Most tangibly, convergence of a sequence $\langle A_n \rangle$ to A in $C_0(X)$ with respect to this topology means that for each nonempty $B \in \mathcal{B}_d(X)$

$$\lim_{n\to\infty} \max\big\{e_d(A_n \cap B, A), e_d(A \cap B, A_n)\big\} = 0,$$

where $e_d(S,T) := \sup\{d(s,T) : s \in S\}$, the so-called *excess of* S *over* T.

Consistent with this description of Attouch-Wets convergence is a natural separated uniformity for τ_{AW_d} on $\mathcal{C}_0(X)$ with countable base (see, e.g., [29, Theorem 38.3]). Fixing $p \in X$, a base for the uniformity consists of all entourages of the form

$$\Big\{(A_1, A_2) : e_d(A_1 \cap S_d(p,n), A_2) < \frac{1}{k} \text{ and } e_d(A_2 \cap S_d(p,n), A_1) < \frac{1}{k}\Big\}$$

where (n,k) runs over $\mathbb{N}^2$. Complete metrizability of τ_{AW_d} was first noted in [3].

It was later seen that τ_{AW_d} is exactly the topology of uniform convergence of distance functions on d-bounded subsets of X under the identification $A \leftrightarrow d(\cdot, A)$. A weaker topology on $\mathcal{C}_0(X)$ is the topology of pointwise convergence of distance functions, called the *Wijsman topology*, whereas a stronger topology on $\mathcal{C}_0(X)$ is the topology of uniform convergence of distance functions on all of X, called the *Hausdorff metric topology* [6].

It is not a purpose of this article to survey the literature on the Attouch-Wets topology. We do wish, however, to mention one basic result in linear analysis that does not seem to be so well-known but should be. Given a sequence $T, T_1, T_2, \ldots$ of continuous linear transformations between normed linear spaces $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$, we have operator norm convergence of $\langle T_n \rangle$ to T if and only if we have Attouch-Wets convergence of the associated sequence of graphs with respect to the box norm on $X_1 \times X_2$ [8, 25].

One of course wonders when compatible metrics d and ρ on a metrizable space (X, τ) determine the same Attouch-Wets topologies on $C_0(X)$. Necessary and sufficient conditions were identified in [9]: $\mathcal{B}_d(X) = \mathcal{B}_\rho(X)$ and the identity function from (X,d) to (X,ρ) is bi-uniformly continuous on bounded sets.

In the present paper, we in part studied the lattice structure of the metric bornologies associated with compatible metrics for (X,τ). Building on our work, one could study the order structure of the family of Attouch-Wets topologies on $C_0(X)$ as determined by compatible metrics. We leave this to others (but first see [6, p. 99]).

References

[1] M. Aggarwal, S. Kundu: *Cauchy metrizability of bornological universes*, J. Convex Analysis 24 (2017) 1085–1098.

[2] M. Atsuji: *Uniform continuity of continuous functions of metric spaces*, Pacific J. Math. 8 (1958) 11–16.

[3] H. Attouch, R. Lucchetti, R. Wets: *The topology of the ρ-Hausdorff distance*, Ann. Mat. Pura Appl. 160 (1991) 303–320.

[4] H. Attouch, R. Wets: *Quantitative stability of variational systems. I: The epigraphical distance*, Trans. Amer. Math. Soc. 328 (1991) 695–730.

[5] R. Balbes, P. Dwinger: *Distributive Lattices*, University of Missouri Press, Columbia (1974).

[6] G. Beer: *Topologies on Closed and Closed Convex Sets*, Kluwer, Dordrecht (1993).

[7] G. Beer: *On metric boundedness structures*, Set-Valued Analysis 7 (1999) 195–208.

[8] G. Beer: *Bornologies and Lipschitz Analysis*, CRC Press, Boca Raton (2023).

[9] G. Beer, A. DiConcilio: *Uniform continuity on bounded sets and the Attouch-Wets topology*, Proc. Amer. Math. Soc. 112 (1991) 235–243.

[10] G. Beer, H. Pajoohesh: *The lattice of bornologies on a set*, Bull. Belgian Math. Soc. 31 (2024) 406–421.

[11] G. Beer, M. C. Vipera: *The Alexandroff one-point compactification as a prototype for extensions*, Adv. Math. 231 (2012) 1598–1618.

[12] H. L. Bentley: *Bornological spaces*, Bol. Soc. Mat. Mex. 28 (2022), art.no. 71.

[13] J. Cao, A. H. Tomita: *Bornologies, topological games, and function spaces*, Top. Appl. 184 (2015) 16–28.

[14] E. Colebunders, R. Lowen: *Bornologies and metrically generated theories*, Top. Appl. 156 (2009) 1224–1233.

[15] J. Dugundji: *Topology*, Allyn and Bacon, Boston (1966).

[16] M. I. Garrido, A. S. Meroño: *Uniformly metrizable bornologies*, J. Convex Analysis 20 (2013) 285–299.

[17] F. Hausdorff: *Erweiterung einer Homöomorphie*, Fund. Math. 16 (1930) 353–360.

[18] H. Hogbe-Nlend: *Bornologies and Functional Analysis*, North-Holland, Amsterdam (1977).

[19] S.- T. Hu: *Boundedness in topological spaces*, J. Math. Pures Appl. 228 (1949) 287–320.

[20] S.- T. Hu: *Introduction to General Topology*, Holden-Day, San Francisco (1966).

[21] T. Jain, S. Kundu: *Atsuji spaces: equivalent conditions*, Topology Proc. 30 (2006) 301–325.

[22] P. T. Johnstone: *Stone Spaces*, Cambridge University Press, Cambridge (1986).

[23] A. A. Monteiro, M. M. Peixoto: *Le nombre de Lebesgue et la continuité uniforme*, Portugaliae Math. 10 (1951) 105–113.

[24] H. Pajoohesh: *Representations of bornologies*, Appl. Gen. Topol. 23 (2022) 17–30.

[25] J.-P. Penot, C. Zalinescu: *Bounded (Hausdorff) convergence: basic facts and applications*, in: *Variational Analysis and Applications*, F. Giannessi et al. (eds.), Nonconvex Optimization and its Applications Vol. 79, Springer, New York (2005) 827–854.

[26] A. P. Robertson, W. J. Robertson: *Topological Vector Spaces*, Cambridge University Press, Cambridge (1973).

[27] H. Vaughan: *On locally compact metrizable spaces*, Bull. Amer. Math. Soc. 43 (1937) 532–535.

[28] T. Vroegrijk: *Uniformizable and realcompact bornological universes*, Applied Gen. Top. 10 (2009) 277–287.

[29] S. Willard: *General Topology*, Addison-Wesley, Reading (1970).