Journal of Convex Analysis
Volume 33 (2026), No. 3&4, 629–635

Stability of the Solutions of Nonlinear Weighted Elliptic Equations Without the Weighted Sobolev Spaces Framework

Lucio Boccardo
Istituto Lombardo, Sapienza Università di Roma, Italy
boccardo@mat.uniroma1.it

Pasquale Imparato
pasquale.imparato1994@gmail.com

Luigi Orsina
Dipartimento di Matematica, Sapienza Università di Roma, Italy
orsina@mat.uniroma1.it

Dedicated to the memory of Hedy Attouch.

Received: November 10, 2024
Accepted: May 4, 2025

We prove some stability results on sequences of solutions of weighted elliptic equations such as

$$-\mathrm{div}(s(x)|\nabla u_n|^{p-2}\,\nabla u_n) = f_n(x)\,,$$

under some assumptions on the convergence of the sequence $\{f_n\}$ (either weak or strong in some Lebesgue space). Here $s(x)$ is a positive weight belonging to the Sobolev space $W^{1,p}(\Omega)$.

Keywords: Weighted elliptic equations, stability of solutions, weak and strong convergence.

2020 Mathematics Subject Classification: 35J15, 35J60, 35J66.

1. Introduction

One of the roots of this paper are the Paris Orsay weekly meetings of the first author with Hedy Attouch, almost half a century ago.

The results proved are related with the scientific framework of some papers by Hedy Attouch (see [1]).

2. Setting

In the paper [4] (see also the references therein and [7]) we studied nonlinear weighted elliptic equations of the form

$$\begin{cases} -\mathrm{div}(s(x)\,|\nabla u|^{p-2}\,\nabla u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

ISSN 0944-6532 / \$ 2.50

where Ω is a bounded, open subset of $\mathbb{R}^N$, $N > 2$, $p > 1$, f belongs to $L^{p_*}(\Omega)$, where $p_* = (p^*)' = \frac{Np}{Np-N+p}$, and $s(x)$ is such that

$$s(x) \in W^{1,p}(\Omega): \; s(x) \geq \alpha > 0\,, \tag{2}$$

for some α in $\mathbb{R}$.

The existence result proved is the following one.

Theorem 2.1. *Let $p > 1$, and let f belong to $L^{p_*}(\Omega)$; let $s(x)$ be such that* (2) *holds. Then there exists a distributional solution u in $W_0^{1,p}(\Omega)$ of equation* (1)*, that is we have*

$$\int_\Omega s(x)\,|\nabla u|^{p-2}\,\nabla u \cdot \nabla\varphi = \int_\Omega f(x)\,\varphi\,, \qquad \forall\,\varphi \in W_0^{1,\infty}(\Omega)\,. \tag{3}$$

In this paper we study two stability results for the solutions u with respect to perturbations (in the weak or in the strong topology of $L^{p_*}(\Omega)$) of the datum f.

There are some difficulties; the first one is that, when we work with perturbations in the weak topology of $L^{p_*}(\Omega)$, it is not possible to take advantage of Rellich theorem in some passages to the limit. Moreover, in general it is not known whether u can be chosen as test function in the equation (3), as well as the uniqueness of such solution u.

We point out that in the study of (3) (as in [3]) are used standard Sobolev spaces and not weighted Sobolev spaces as in N. Trudinger works (see [10]).

Remark 2.2. The solution u of (3) can also be seen as a minimum of the integral functional defined on $W_0^{1,p}(\Omega)$ as

$$J_f(v) = \begin{cases} \frac{1}{p}\int_\Omega s(x)|\nabla v|^p - \int_\Omega f(x)\,v(x)\,, & \text{if } \int_\Omega s(x)|\nabla v|^p < +\infty\,, \\ \qquad\qquad +\infty & \text{otherwise.} \end{cases}$$

Our first result deals with data weakly converging in $L^{p_*}(\Omega)$.

Theorem 2.3. *Let $\{f_n(x)\}$ be a sequence weakly converging in $L^{p_*}(\Omega)$ to $f_0(x)$, and let u_n be a weak solution, given by Theorem* 2.1*, of the Dirichlet problem*

$$\int_\Omega s(x)\,|\nabla u_n|^{p-2}\,\nabla u_n \cdot \nabla\varphi = \int_\Omega f_n(x)\,\varphi\,, \qquad \forall\,\varphi \in W_0^{1,\infty}(\Omega)\,. \tag{4}$$

Then, if u_0 is a weak solution, given by Theorem 2.1*, of the Dirichlet problem*

$$\int_\Omega s(x)\,|\nabla u_0|^{p-2}\,\nabla u_0 \cdot \nabla\varphi = \int_\Omega f_0(x)\,\varphi\,, \qquad \forall\,\varphi \in W_0^{1,\infty}(\Omega)\,. \tag{5}$$

we have that, up to subsequences,

$$\{u_n\} \text{ weakly converges in } W_0^{1,p}(\Omega) \text{ to } u_0. \tag{6}$$

If, instead, the sequence of data $\{f_n\}$ is strongly convergent in $L^{p_*}(\Omega)$, the convergence of the sequence $\{u_n\}$ is strong in $W_0^{1,p}(\Omega)$.

Theorem 2.4. *Let $\{f_n(x)\}$ be a sequence strongly converging in $L^{p_*}(\Omega)$ to $f_0(x)$, and let u_n and u_0 be as in Theorem 2.3. Then, up to subsequences,*

$$\{u_n\} \textit{ strongly converges in } W_0^{1,p}(\Omega) \textit{ to } u_0. \tag{7}$$

Remark 2.5. In the spirit of Remark 2.2, consider the functionals $J_{f_n}(v)$ and $J_{f_0}(v)$ corresponding to f_n and f_0 respectively. We then have the following.

- Under the assumptions of Theorem 2.3, or Theorem 2.4, the sequence of functionals $\{J_{f_n}(v)\}$ Epi-converges (see [1]) to $J_{f_0}(v)$.
- Under the assumptions of Theorem 2.4, it is easy to prove that the sequence of functionals $\{J_{f_n}(v)\}$ Mosco-converges (see [9]) to $J_{f_0}(v)$. Thus the statement of Theorem 2.4 can be read as the strong convergence in $W_0^{1,p}(\Omega)$ of the sequence of minima $\{u_n\}$ of a sequence of functionals $\{J_{f_n}(v)\}$ which is Mosco-convergent.
- Under the assumptions of Theorem 2.3, we are neither in a framework of Mosco-convergence nor in a framework of $\Gamma(\textit{weak}, \textit{weak})$ convergence (see [8], [5]); in other words, there are sequences $\{v_n\}$ and $\{f_n\}$ for which it does not hold that
$$\liminf_{n\to+\infty} J_n(v_n) \geq J_0(v_0)\,. \tag{8}$$
Indeed, let $s(x) = 1$, let $\{f_n(x)\}$ be a sequence convergent in $L^{p_*}(\Omega)$ weakly (*and no more*) to $f_0(x) = 0$, and let v_n be the weak solution of the Dirichlet problem
$$v_n \in W_0^{1,p}(\Omega) : -\mathrm{div}(|\nabla v_n|^{p-2}\,\nabla v_n) = f_n(x).$$
It is proved in [2] that the sequence $\{v_n\}$ weakly converges in $W_0^{1,p}(\Omega)$ to $v_0 = 0$, the weak solution of the Dirichlet problem
$$v_0 \in W_0^{1,p}(\Omega) : -\mathrm{div}(|\nabla v_0|^{p-2}\,\nabla v_0) = f_0(x).$$
Now we observe that (using that v_n solves the Dirichlet problem)
$$J_n(v_n) = \frac{1}{p}\int_\Omega |\nabla v_n|^p - \int_\Omega f_n(x)\,v_n(x) = -\Big(1-\frac{1}{p}\Big)\int_\Omega |\nabla v_n|^p \leq 0,$$
so that, if we assume (8), we have
$$0 \geq \liminf_{n\to+\infty} J_n(v_n) = -\Big(1-\frac{1}{p}\Big)\int_\Omega |\nabla v_n|^p \geq J_0(v_0) = 0\,,$$
which implies the strong convergence in $W_0^{1,p}(\Omega)$ to zero; and this is false even in the linear case ($p = 2$).

3. Proof of the results

In order to prove both Theorem 2.3 and Theorem 2.4, we need to recall the steps of the proof of the existence result Theorem 2.1 for a fixed datum f.

In the following, we will use the following functions of one real variable, depending on a parameter $k \geq 0$:

$$G_k(s) = (|s| - k)^+ \operatorname{sgn}(s)\,, \quad T_k(s) = s - G_k(s) = \max(-k, \min(s, k))\,.$$

- Let n in $\mathbb{N}$ and define
$$s_n(x) = \frac{s(x)}{1 + \frac{s(x)}{n}}. \tag{9}$$

- Let $w_n \in W_0^{1,p}(\Omega)$ be the weak solution of the boundary value problem
$$\begin{cases} -\operatorname{div}(s_n(x)\, |\nabla w_n|^{p-2}\, \nabla w_n) = f(x) & \text{in } \Omega, \\ w_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

- For the sequence $\{w_n\}$ we have that
$$\frac{\alpha}{1+\alpha}\, \|w_n\|^{p-1}_{W_0^{1,p}(\Omega)} \leq S\, \|f\|_{L^{p_*}(\Omega)}\,, \tag{11}$$
where S is the Sobolev constant. Thus, up to subsequences, we have that there exists a function u in $W_0^{1,p}(\Omega)$ such that
$$\begin{cases} w_n \text{ converges to } u \text{ weakly in } W_0^{1,p}(\Omega), \\ \nabla w_n(x) \text{ converges almost everywhere to } \nabla u(x). \end{cases} \tag{12}$$

- Then u is a weak solution of (3) with
$$\frac{\alpha}{1+\alpha}\, \|u\|^{p-1}_{W_0^{1,p}(\Omega)} \leq S\, \|f\|_{L^{p_*}(\Omega)}\,. \tag{13}$$

- Furthermore (and this is the result we will use to prove our theorems)
$$\int_\Omega |\nabla u|^{p-2}\, \nabla u \cdot \nabla v = \int_\Omega \Big[|\nabla u|^{p-2}\, \nabla u \cdot \nabla s + f(x) \Big] \frac{1}{s(x)}\, v\,, \tag{14}$$
for all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

- Moreover, from the weak formulation of (10) we deduce that
$$\int_\Omega s_n(x)\, |\nabla w_n|^{p-2}\, \nabla w_n \nabla G_k(w_n) = \int_\Omega f(x) G_k(w_n)$$
which implies (thanks to Fatou Lemma) that
$$\int_\Omega s(x)\, |\nabla u|^{p-2}\, \nabla u \nabla G_k(u) \leq \int_\Omega f(x) G_k(u)\,,$$
an inequality that holds even though we cannot choose $\varphi = G_k(u)$ as test function in (3).

- A consequence of the last inequality is
$$S\alpha \left[\int_\Omega |\nabla G_k(u)|^p \right]^{\frac{1}{p'}} \leq \left[\int_{\{k<|u|\}} |f(x)|^{p_*} \right]^{\frac{1}{p_*}}. \tag{15}$$

We now prove Theorem 2.3.

Proof of Theorem 2.3. Applying (13) and (14) to the sequence $\{u_n\}$ of solutions of (4), we have that

$$\frac{\alpha}{1+\alpha}\,\|u_n\|^{p-1}_{W^{1,p}_0(\Omega)} \le S\,\|f_n\|_{L^{p_*}(\Omega)}\,, \tag{16}$$

and that

$$\int_\Omega |\nabla u_n|^{p-2}\,\nabla u_n\cdot\nabla v = \int_\Omega \Big[|\nabla u_n|^{p-2}\,\nabla u_n\cdot\nabla s + f_n(x)\Big]\,\frac{1}{s(x)}\,, \tag{17}$$

for every v in $W^{1,p}_0(\Omega)\cap L^\infty(\Omega)$.

Using (16) we have that there exists a function u^* in $W^{1,p}_0(\Omega)$ such that, up to subsequences, u_n converges weakly in $W^{1,p}_0(\Omega)$ to u^*. Furthermore, since in (17) the term

$$g_n(x) = \Big[|\nabla u_n|^{p-2}\,\nabla u_n\cdot\nabla s + f_n(x)\Big]\,\frac{1}{s(x)}$$

is bounded in $L^1(\Omega)$, by a result of [6] we also have that $\nabla u_n(x)\to\nabla u^*(x)$ almost everywhere in Ω, which implies that $|\nabla u_n|^{p-2}\,\nabla u_n$ weakly converges in $(L^{p'}(\Omega))^N$ to $|\nabla u^*|^{p-2}\,\nabla u^*$.

Now it is possible to pass to the limit in (4) to deduce that

$$\int_\Omega s(x)\,|\nabla u^*|^{p-2}\,\nabla u^*\cdot\nabla\varphi = \int_\Omega f_0(x)\,\varphi\,, \qquad \forall\,\varphi\in W^{1,\infty}_0(\Omega)\,,$$

that is, u^* is one of the solutions u_0 of (5). □

Under the (stronger) assumptions of Theorem 2.4 the result of Theorem 2.3 still holds: that is, the sequence $\{u_n\}$ of solutions of (4) weakly converges in $W^{1,p}_0(\Omega)$ to some solution u_0 of (3) with datum $f(x)$. Our aim is to prove that the convergence is indeed strong.

Proof of Theorem 2.4. We begin by observing that (15) implies that

$$S\alpha\bigg[\int_\Omega |\nabla G_k(u_n)|^p\bigg]^{\frac{1}{p'}} \le \bigg[\int_{\{k<|u_n|\}} |f_n(x)|^{p_*}\bigg]^{\frac{1}{p_*}}, \tag{18}$$

and the right hand side of this inequality is small for k large, uniformly with respect to n, since the sequence $\{f_n\}$ is strongly convergent in $L^{p_*}(\Omega)$, and the measure of the set $\{k<|u_n|\}$ tends to zero as k tends to infinity, uniformly with respect to n, since the sequence $\{u_n\}$ is bounded in $L^1(\Omega)$.

We now remark that, thanks to Vitali theorem, the sequence

$$g_n(x) = |\nabla u_n|^{p-2}\,\nabla u_n\cdot\nabla s\,,$$

is strongly convergent in $L^1(\Omega)$. Indeed, if E is a measurable subset of Ω we have

$$\int_E \Big|\,|\nabla u_n|^{p-2}\,\nabla u_n\cdot\nabla s\Big| \le \|u_n\|^{p-1}_{W^{1,p}_0(\Omega)}\Big(\int_E |\nabla s|^p\Big)^{\frac{1}{p}}\,,$$

so that (recall that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$), we have

$$\lim_{|E|\to 0} \int_E \Big| |\nabla u_n|^{p-2}\, \nabla u_n \cdot \nabla s \Big| = 0\,, \qquad \text{uniformly with respect to } n,$$

where we have used that $s(x)$ belongs to $W^{1,p}(\Omega)$. Thus the sequence $\{g_n\}$ is uniformly equiintegrable; since it almost everywhere converges to

$$g_0(x) = |\nabla u_0|^{p-2}\, \nabla u_0 \cdot \nabla s\,,$$

the sequence $\{g_n\}$ is strongly convergent in $L^1(\Omega)$ to g_0.

Choosing $T_k(u_n) - T_k(u_0)$ in (17) we have that

$$\left[\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2}\, \nabla u_n \cdot \nabla[T_k(u_n) - T_k(u_0)] \\ &\qquad = \int_\Omega \Big[|\nabla u_n|^{p-2}\, \nabla u_n \cdot \nabla s + f_n(x)\Big] \frac{1}{s(x)} [T_k(u_n) - T_k(u_0)] \\ &\qquad = \int_\Omega \frac{g_n(x) + f_n(x)}{s(x)}\, [T_k(u_n) - T_k(u_0)], \end{aligned}\right.$$

and the right hand side tends to zero thanks to the boundedness of $\frac{1}{s(x)}$, to the strong convergence in $L^1(\Omega)$ of the sequence $\{g_n + f_n\}$, and to the fact that the sequence $\{T_k(u_n) - T_k(u_0)\}$ is weakly$*$ convergent to zero in $L^\infty(\Omega)$. In other words, for every fixed k we have that

$$\lim_{n\to+\infty} \int_\Omega |\nabla u_n|^{p-2}\, \nabla u_n \cdot \nabla[T_k(u_n) - T_k(u_0)] = 0\,. \tag{19}$$

We therefore have

$$\left[\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2}\, \nabla u_n \cdot \nabla[u_n - u_0] \\ &= \int_\Omega |\nabla u_n|^{p-2}\, \nabla u_n \cdot \nabla[T_k(u_n) - T_k(u_0)] + \int_\Omega |\nabla u_n|^{p-2}\, \nabla u_n \cdot \nabla[G_k(u_n) - G_k(u_0)] \\ &= \int_\Omega |\nabla u_n|^{p-2}\, \nabla u_n \cdot \nabla[T_k(u_n) - T_k(u_0)] - \int_\Omega |\nabla T_k(u_n)|^{p-2}\, \nabla T_k(u_n) \cdot \nabla G_k(u_0) \\ &\quad + \int_\Omega |\nabla G_k(u_n)|^{p-2}\, \nabla G_k(u_n) \cdot \nabla[G_k(u_n) - G_k(u_0)]\,. \end{aligned}\right. \tag{20}$$

Recalling that

$$|\nabla T_k(u_n)|^{p-2}\, \nabla T_k(u_n) \text{ weakly converges in } (L^{p'}(\Omega))^N \text{ to } |\nabla T_k(u_0)|^{p-2}\, \nabla T_k(u_0),$$

we have, for every fixed k,

$$\lim_{n\to+\infty} \int_\Omega |\nabla T_k(u_n)|^{p-2}\, \nabla T_k(u_n) \cdot \nabla G_k(u_0) = 0\,. \tag{21}$$

Furthermore,

$$\Big|\int_\Omega |\nabla G_k(u_n)|^{p-2}\,\nabla G_k(u_n)\cdot\nabla[G_k(u_n)-G_k(u_0)]\Big| \le C_1(u_0)\Big[\int_\Omega |\nabla G_k(u_n)|^p\Big]^{\frac{1}{p'}},$$

so that, thanks to (18), for $\varepsilon > 0$ fixed, there exists $k = k_\varepsilon > 0$ such that

$$\Big|\int_\Omega |\nabla G_k(u_n)|^{p-2}\,\nabla G_k(u_n)\cdot\nabla[G_k(u_n)-G_k(u_0)]\Big| \le \varepsilon\,,$$

uniformly with respect to n.

Using this fact, (19) and (21) in (20), we have that

$$\limsup_{n\to+\infty}\Big|\int_\Omega |\nabla u_n|^{p-2}\,\nabla u_n\cdot\nabla[u_n-u_0]\Big| \le \varepsilon\,,$$

which implies (since ε is arbitrary) that

$$\lim_{n\to+\infty}\int_\Omega |\nabla u_n|^{p-2}\,\nabla u_n\cdot\nabla[u_n-u_0] = 0\,;$$

from this fact it is easy to prove that the sequence $\{u_n\}$ strongly converges to u_0 in $W_0^{1,p}(\Omega)$, as desired. □

References

[1] H. Attouch, R. J.-B. Wets: *Epigraphical analysis*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 6, Suppl. (1989) 73–100.

[2] L. Boccardo: *Some cases of weak continuity in nonlinear Dirichlet problems*, J. Funct. Analysis 277 (2019) 3673–3687.

[3] L. Boccardo, G. R. Cirmi: *Some existence and regularity results by Neil Trudinger revisited without the weighted Sobolev spaces framework*, Bull. Inst. Math. Acad. Sin. (N.S.) 9 (2014) 295–315.

[4] L. Boccardo, P. Imparato, L. Orsina: *Nonlinear weighted elliptic equations with Sobolev weights*, Boll. Unione Mat. Ital. 15 (2022) 503–514.

[5] L. Boccardo, P. Marcellini: *Sulla convergenza delle soluzioni di disequazioni variazionali*, Ann. Mat. Pura Appl. 110 (1976) 137–159.

[6] L. Boccardo, F. Murat: *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Analysis 19 (1992) 581–597.

[7] S. Buccheri: *Some nonlinear elliptic problems with $L^1(\Omega)$ coefficients*, Nonlinear Analysis 177 (2018) 135–152.

[8] G. Buttazzo: *Su una definizione generale dei Γ-limiti*, Boll. Un. Mat. Ital. 14-B (1977) 722–744.

[9] U. Mosco: *Convergence of convex sets and of solutions of variational inequalities*, Advances Math. 3 (1969) 510–585.

[10] N. S. Trudinger: *Maximum principles for linear, non-uniformly elliptic operators with measurable coefficients*, Math. Zeitschrift 156 (1977) 291–301.