Journal of Convex Analysis
Volume 33 (2026), No. 3&4, 691–736

Calibrations for Minimal Surfaces with Free Boundary and Cheeger-Type Problems

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Dedicated to the memory of Hédy Attouch.

Received: October 24, 2025
Accepted: November 4, 20225

We study a problem of minimal surfaces with free boundary written in the form of a non convex minimization problem. Our aim is to characterize optimal solutions by finding a suitable calibration field. A natural upper bound of the infimum is given by a variant of the Cheeger problem that we solve explicitly proving the optimality thanks to the construction of a cut-locus potential. The comparison with the original problem is then discussed in detail.

Keywords: Free boundary problems, calibrations, minimal surfaces, shape derivative, Cheeger sets, cut-locus potential.

2020 Mathematics Subject Classification: 49J45, 49N15, 49Q10, 65K10, 90C26.

1. Introduction

In all the paper, D is a given bounded domain of $\mathbb{R}^N$ with Lipschitz boundary. For any subset $\Omega \subset D$, we denote by $|\Omega| := \mathcal{L}^N(\Omega)$ its area (the N-dimensional Lebesgue measure of Ω) and by $P(\Omega)$ its perimeter. Recall that, if $P(\Omega) < +\infty$, then $P(\Omega) = \mathcal{H}^{N-1}(\partial\Omega)$, i.e. the $(N-1)$-dimensional Hausdorff measure of the (essential) boundary of Ω.

Given $\lambda \geq 0$, our aim is to study the following variational problem:

$$\beta(\lambda) := \inf \left\{ \int_D \sqrt{1 + |\nabla u|^2}\, dx - \lambda\, |\{u \geq 1\}| \;:\; u \in W_0^{1,1}(D) \right\}. \tag{1.1}$$

By using a truncation argument, one checks easily that the infimum above is unchanged if we restrict the infimum to competitors u such that $0 \leq u \leq 1$. The free boundary associated with such u is then the essential boundary of the set $\Omega = \Omega(u) := \{u = 1\}$. Accordingly, the minimization problem (1.1) contains two terms in competition, on the one hand the minimal area of a parametrized surface with $\partial D \times \{0\}$ and $\partial\Omega \times \{1\}$ boundaries, and on the other hand the area of the unknown subset Ω times the scaling factor λ.

ISSN 0944-6532 / $ 2.50

An alternative point of view is to see (1.1) as the minimization of the shape functional: $J_\lambda : \ \Omega \subset D \mapsto J(\Omega) + (1-\lambda)\,|\Omega|$ where:

$$J(\Omega) = \inf \left\{ \int_{D\setminus\Omega} \sqrt{1+|\nabla u|^2}\, dx \ : \ u=0 \ \text{ on } \ \partial D, \ u=1 \ \text{ on } \ \partial\Omega \right\}. \tag{1.2}$$

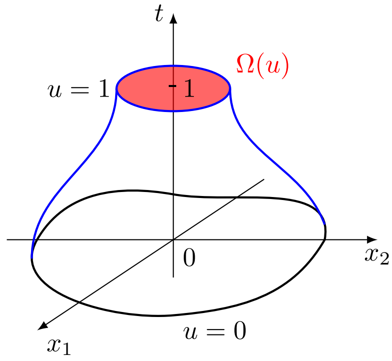

Figure 1: A surface with prescribed boundary $u = 0$ on ∂D and free boundary $\partial\{u=1\}$.

In this context, if Ω is an optimal set for J_λ, then a solution u to (1.1) is obtained by solving

$$- \operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \quad \text{ in } D\setminus\Omega, \tag{1.3}$$
$$u = 0 \quad \text{ on } \partial D, \qquad u = 1 \quad \text{ on } \partial\Omega,$$

where (1.3) encodes the zero mean curvature of the parametrized surface $z = u(x)$.

In fact, it is well known that the above nonlinear boundary value problem does not always have a solution (see for example [19]). In order to start with a well-posed problem, we replace the variational problem (1.1) by the following relaxed formulation, in which the class of competitors u is extended to possibly discontinuous functions (thus violating the boundary conditions):

$$\beta(\lambda) = \inf \left\{ E_\lambda(u) \ : \ u \in BV(D) \right\}, \quad \text{where} \tag{1.4}$$

$$E_\lambda(u) := \int_D \sqrt{1+|\nabla u|^2}\, dx + \int_D |D^s u| + \int_{\partial D} |u|\, d\mathcal{H}^{N-1} - \lambda\, |\{u \geq 1\}|. \tag{1.5}$$

Here $BV(D)$ denotes the subspace of functions $u \in L^1(D)$ whose distributional gradient Du is a vector Radon measure of bounded total variation from D to $\mathbb{R}^N$. The Radon-Nikodym derivative of Du with respect to the Lebesgue measure on D coincides with the a.e. defined approximate gradient ∇u while $D^s u$ denotes its singular part. Such functions u have a well defined trace in $L^1(\partial D)$ (see [29]) and a well defined jump set $S_u := \{u_+ > u_-\}$ where $u_-(x), u_+(x)$ are the lower (resp. upper) approximative limits of u at any $x \in D$. It turns out that S_u is an $N-1$ dimensional rectifiable subset of D such that $D^s u = (u_+ - u_-)\, \nu_u \mathcal{H}^{N-1} \llcorner S_u + D^c u$ where ν_u is a normal unit vector to S_u; the remainder of D^s on the complement of S_u is the so called Cantor part that we will denote by $D^c u$. For further details on the space $BV(D)$, we refer to [3] and also to [7] for classical relaxation issues in this space. It is easy to check that (1.4) admits solutions in $BV(D;[0,1])$ and that the infima defined in (1.1) and (1.4) coincide. However, the solution is not unique (see [8] in a simpler context), and characterizing a global minimum turns out to be a very difficult problem. This is due to the nonconvexity of the term $-\lambda|\{u \geq 1\}|$. In fact, if we restrict the definition of $E_\lambda(u)$ to $u \geq 0$, we can rewrite this term as the integral $\lambda \int_D g(u)dx$ where $g : \mathbb{R} \to \mathbb{R}$:

$$g(t) = \begin{cases} -t & \text{if } t \leq 0 \\ 0 & \text{if } 0 \leq t < 1, \\ -1 & \text{if } t \geq 1. \end{cases} \tag{1.6}$$

This integrand g is lower semicontinuous non-increasing and admits a jump at $t = 1$. Its convexification g^{**} given below is such that $\{g^{**} < g\} = (0,1)$:

$$g^{**}(t) = -\min\{t, 1\}. \tag{1.7}$$

In order to overcome this lack of convexity, we will use a duality recipe in dimension $N+1$ which was developed in [9] for functionals of the kind

$$u \in W^{1,p}(D) \mapsto \int_D (f(\nabla u) + g(u))\, dx$$

for general non-convex lower semicontinuous functions $g : \mathbb{R} \to (-\infty, +\infty]$, possibly admitting isolated discontinuity points. In [9], the integrand f was assumed to be convex and satisfying a p-growth lower bound with $p > 1$. Recently this duality result has been extended to the case $p = 1$ where f has linear growth at infinity (see [11]). Moreover, if f is assumed to be positively one homogeneous (for example $f(\nabla u) = |\nabla u|$), an *exclusion principle* was derived in [11], namely: the minimizers of $\int_D (f(\nabla u) + g(u))\, dx$ under a Dirichlet condition $u_0 \in L^1(\partial D)$ such that $g(u_0) = g^{**}(u_0)$ cannot take values in the set $\{g^{**} < g\}$.

We are also interested in a variant of (1.1) where the surface-area term is replaced by the total variation of u. In virtue of the inequalities $|z| \leq \sqrt{1+|z|^2} \leq 1 + |z|$ holding for all $z \in \mathbb{R}^N$, we infer that:

$$m(\lambda, D) := \beta_0(\lambda) - |D| \;\leq\; \beta(\lambda) \;\leq\; \beta_0(\lambda) \quad , \quad \text{where} \tag{1.8}$$

$$\beta_0(\lambda) := \inf\left\{ \int_D (1 + |\nabla u|)dx - \lambda|\{u = 1\}| \;:\; u \in W_0^{1,1}(D),\ 0 \leq u \leq 1 \right\}. \tag{1.9}$$

We are led to another free boundary problem where the PDE counterpart of (1.3) is given by

$$-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0 \quad \text{in } D \setminus \Omega\ ,$$

while the relaxed functional to be minimized in $BV(D;[0,1])$ becomes

$$E_\lambda^0(u) := |D| + \int_D |Du| + \int_{\partial D} |u| d\mathcal{H}^{N-1} + \lambda \int_D g(u)\, dx\ . \tag{1.10}$$

Here, two important observations underscore the role of this variant of the original problem:

- The equality $E_\lambda = E^0_\lambda$ holds on the subset $BV(D;\{0,1\})$. Indeed, if u is the characteristic function $\mathbb{1}_\Omega$ of a subset $\Omega \subset D$, then Ω has a finite perimeter in $\mathbb{R}^N$ given by $P(\Omega) = \int_D |Du| + \int_{\partial D} |u| d\mathcal{H}^{N-1}$ (in that case $\nabla u = 0$ a.e and $D^c u = 0$). It follows that:
$$u = \mathbb{1}_\Omega \implies E_\lambda(u) = E^0_\lambda(u) = P(\Omega) - \lambda|\Omega|. \tag{1.11}$$
- The non convex variational problem $\beta_0(\lambda)$ in (1.9) satisfies the one-homogeneity assumption mentioned above. Thus, in virtue of [11, Theorem 2.2] and since $\{g^{**} < g\} = (0,1)$, any solution to $\beta_0(\lambda)$ is of the form $u = \mathbb{1}_\Omega$. A consequence is that the left hand side of (1.8) coincides with the minimum of the following shape optimization problem:
$$m(\lambda, D) = \min \{P(\Omega) - \lambda|\Omega| \ : \ \Omega \subset D\}. \tag{1.12}$$

Note that the equality $\beta_0(\lambda) = |D| + m(\lambda, D)$ can be also recovered by using the co-area formula. Actually the problem (1.12) has been studied in [1, 2, 27, 28] as being in close relation with the celebratred Cheeger problem (see the survey [26]):
$$h_D := \inf \left\{ \frac{P(\Omega)}{|\Omega|} \ : \ \Omega \subset D \right\}. \tag{1.13}$$

In particular, one has $m(\lambda, D) = 0$ for every $\lambda \leq h_D$ while $m(\lambda, D) < 0$ otherwise.

The main contributions of this paper are the following:

- We show that the strict inequality $\beta(\lambda) < \beta_0(\lambda)$ holds if $u \equiv 0$ or $u \equiv 1$ are not minimal for (1.1) (Theorem 3.2). This latter condition is equivalent to require that λ is between two thresholds λ_0 and λ_1. By Lemma 3.1, these thresholds satisfy the inequalities:
$$0 < \lambda_0 \ \leq h_D \ \leq \frac{P(D)}{|D|} \ \leq \ \lambda_1 \ \leq \ +\infty.$$
More precise estimates are obtained in Proposition 3.6 by using special calibration fields for the dual problem in dimension $N+1$ associated with (1.1). Numerical simulations are given for radial examples in dimension two.
- We develop a theory of θ-calibrability for solving $m(\lambda, D)$. In the two-dimensional case, we associate to any bounded convex open set D a specific potential $\rho : \overline{D} \to (0, R_D]$, where R_D denotes the inradius of D. This potential is continuous, locally Lipschitz in D with a maximal plateau $\{\rho = R_D\}$ which coincides with the *central subset* defined by $U_D := \{x \in D : d(x, D^c) = R_D\}$. In $D \setminus \overline{U}_D$, ρ satisfies $\nabla\rho \neq 0$ a.e. and solves the boundary problem:
$$\operatorname{div} \frac{\nabla\rho}{|\nabla\rho|} + \frac{1}{\rho} = 0 \quad , \quad \frac{\nabla\rho}{|\nabla\rho|} \cdot \nu_D = -1 \quad \text{on } \partial D. \tag{1.14}$$
A very simple geometric construction of this so-called cut locus potential is described in Section 4, which allows us to make explicit a calibrating vector field for any $\lambda \geq h_D$. Then we deduce that the unique solution of $m(\lambda, D)$ is given by $\Omega_\lambda = \{\rho > \frac{1}{\lambda}\}$. It coincides with the union of all balls of radius $\frac{1}{\lambda}$ contained in D (see (4.1)), so that we recover by a different method the result in [1, 2].

Before we finish this introduction, we want to mention that the idea of building a potential ρ associated with a general convex set D can be extended in higher dimensions. The same PDE as in (1.14) would be kept, and solutions to (1.12) would still be the upper level sets $\{\rho > \frac{1}{\lambda}\}$ for every $\lambda < h_D$. In turn the geometric characterization that we found for $N = 2$ (by means of the normal distance to the cut locus of D) does not work any more for $N > 2$. This suggests a very interesting open issue worth to explore in future work. Note that here the potential ρ depends only of the shape of D (see Remark 4.8), in contrast with the parametrized potentials introduced in [2].

The paper is organized as follows:

- In Section 2, we present briefly the duality recipe for the non-convex problem $\beta(\lambda)$. This leads to an $N + 1$ dimentional dual problem and to optimality conditions in terms of divergence free calibration vector fields. In the case of the homogenous variant $\beta_0(\lambda)$, this dual problem reduces to a more classical formulation in dimension N. The optimality of a subset of $\Omega \subset D$ for (1.12) is characterized by a θ- calibrability condition.
- In Section 3, we discuss the occurrence of the inequality $\beta(\lambda) < \beta_0(\lambda)$ according to the value of λ and to the geometric properties of D. Some numerical simulations are given in the case where D is of a disk in $\mathbb{R}^2$.
- In Section 4, we restrict ourself to the case of a convex subset $D \subset \mathbb{R}^2$. After a short background on Cheeger sets, we introduce the called cut-locus potential ρ and focus on the explicit construction of a vector field $q\colon D \to \mathbb{R}^2$ which calibrates the unique solution Ω_λ of (1.12).

2. Dual problems and calibrations

The variational problems associated respectively with $\beta(\lambda)$ and $\beta_0(\lambda)$ are non convex with linear growth. They enter the duality framework developed initially in $W^{1,p}(D)$ for $p > 1$ (see [9]), further extended to $W^{1,1}(D)$ and $BV(D)$ in [11], where D is a bounded domain in $\mathbb{R}^N$. In these references a characterization of global minimizers is provided by using calibration fields defined in $L^\infty(D \times I; \mathbb{R}^{N+1})$ where I is an open interval of $\mathbb{R}$ such that $\overline{I}$ contains the range of all solutions. This section is devoted to state the dual problems in the particular case of $\beta(\lambda)$ and $\beta_0(\lambda)$. In both cases, the solutions range in $[0, 1]$ so that we take $I = (0, 1)$ and the searched calibration fields are of the form

$$\sigma(x, t) = \big(\sigma^x(x, t), \sigma^t(x, t)\big) \in \mathbb{R}^N \times \mathbb{R} \quad \text{where } (x, t) \in D \times (0, 1).$$

In the following we will denote $Q := D \times (0, 1)$.

2.1. Dual problem of (1.1)

Following [9], the dual formulation of the variational problem of (1.1) (or of its relax form (1.4)) is given by the following formulation in Q (thus in dimension $N + 1$).

$$\sup \left\{ -\int_D \sigma^t(x, 0)\, dx \ : \ \begin{array}{l} \operatorname{div} \sigma = 0 \text{ in } Q,\ \sigma^t + \sqrt{1 - |\sigma^x|^2} \geq 0 \text{ in } Q, \\ \sigma^t(x, 1) \geq \lambda - 1 \text{ on } D \end{array} \right\}. \tag{2.1}$$

Adopting a fluid dynamics view point, (2.1) can be interpreted as the maximization of the downward flow $\sigma = (\sigma^x, \sigma^t)$ of an incompressible fluid ($\operatorname{div}\sigma = 0$ in Q) through the bottom interface $D \times \{0\}$ when it is subject to the pointwise non linear constraint $\sigma^t + \sqrt{1-|\sigma^x|^2} \geq 0$ a.e. in Q while $\sigma^t(x,1) \geq \lambda - 1$ on the upper interface $D \times \{1\}$.

Theorem 2.1. *The duality principle given in* [11] *leads to the following no-gap equality*

$$\beta(\lambda) \ = \ \sup(2.1).$$

Remark 2.2. In order to explain how the results in [9, Secton 3.3] are applied to our case for deriving the dual problem (2.1), two comments are in order:

(1) Following the general notations of [9], the original primal problem (1.1), can be written as

$$\inf\left\{ f_\lambda(u, \nabla u)\, dx \ : \ u \in W_0^{1,1}(D)\right\},$$

where $f_\lambda(t,z) := \sqrt{1+|z|^2} + \lambda g(t)$ and g is the non convex function in (1.6). Then, the bulk constraint on σ appearing in the dual problem (the condition (3.20) in [9] can be written as $f_\lambda^*(t,\sigma^x) \leq \sigma^t$ where $f_\lambda^*(t,\cdot)$ is the Fenchel conjugate of $f_\lambda(t,\cdot)$. In our case, we have

$$f_\lambda^*(t,\sigma^x) = \lambda\, g(t) - \sqrt{1-|\sigma^x|^2} \quad \text{if } |\sigma^x| \leq 1 \ , +\infty \quad \text{otherwise.}$$

where the non convex function g vanishes on $[0,1)$. Hence the bulk condition holding a.e. in Q reduces to $\sigma^t + \sqrt{1-|\sigma^x|^2} \geq 0$.

(2) In view of the discontinuity of g at $t=1$, an additional normal trace condition has to be imposed on the boundary interface $t=1$ namely $\sigma^t(x,1) \geq \inf f_\lambda^*(1,\cdot) = -f_\lambda(1,0)$ (see condition (3.21) in [9]). In our case, we obtain the λ-dependent constraint $\sigma^t(x,1) \geq \lambda - 1$, which accounts the free boundary associated with the subset $\{u = 1\}$. Note that this condition holding a.e. $x \in D$ is well defined in the sense of normal traces of bounded functions $Q \to \mathbb{R}^N$ whose distributional divergence in $\mathcal{D}'(Q)$ belongs to $L^\infty(Q)$ (see [4, 5]). □

Optimality conditions for $\beta(\lambda)$. Let $u \in BV(D;[0,1])$ and σ be an admissible vector field $\sigma \in L^\infty(Q;\mathbb{R}^N \times \mathbb{R})$ for (2.1). Then, by the no-gap identity $\beta(\lambda) = \sup(2.1)$ (see [9, Thm. 3]) and recalling (1.5), the optimality of an admissible pair (u,σ) is equivalent to the equality

$$E_\lambda(u) = -\int_D \sigma^t(x,0)\, dx.$$

Accordingly, by localizing this relation, we obtain the following set of optimality conditions:

$$\begin{cases} \operatorname{div}\sigma = 0 \ \text{ in } \mathcal{D}'(Q)\,, \quad \sigma^t + \sqrt{1-|\sigma^x|^2} \geq 0 \ \text{ in } Q \text{ a.e.,} \\ \sigma^t(x,1) \geq \lambda - 1 \ \text{ a.e. in } D\,, \end{cases} \tag{2.2}$$

and the pointwise conditions on the completed graph of u:

$$\begin{cases} \sigma^x(x,u(x)) = \dfrac{\nabla u(x)}{\sqrt{1+|\nabla u(x)|^2}} & \mathcal{L}^N\text{-a.e. on } \{u<1\}; \\ \sigma^t(x,u(x)) = -\sqrt{1+|\nabla u(x)|^2} & \mathcal{L}^N\text{-a.e. on } \{u<1\}; \\ \sigma^t(x,1) = \lambda - 1 & \mathcal{L}^N\text{-a.e. on } \{u=1\}; \\ \sigma^x(x,t)\cdot\nu_u = 1 & \mathcal{H}^{N-1}\text{-a.e. on } S_u,\ \forall t\in[u_-(x),u_+(x)]\ ; \\ \sigma^x(x,\tilde u(x)) = 1 & |D^c u|\text{-a.e.} \end{cases} \tag{2.3}$$

where:

- $\nabla u(x)$ is the a.e defined approximate gradient of u;
- $u_-(x), u_+(x)$ denote the lower and upper approximative limits of u;
  if $u_-(x) = u_+(x)$ we denote by $\tilde u(x)$ the common value;
- $S_u = \{u_+ > u_-\}$ stands for the $N-1$-*rectifiable* jump set of u;
- ν_u the oriented unit normal vector to S_u ;
- $D^c u$ is the Cantor part of the vector measure Du (which has no mass on S_u).

Remark 2.3. Here a short background is in order. We recall that $BV(D)$ is the set of functions u in $L^1(D)$ whose distributional gradient Du is an element of $\mathcal{M}(D;\mathbb{R}^N)$ the set of Radon vector measures from D to $\mathbb{R}^N$. By considering only elements $u \in BV(D)$ ranging into the closed interval $[0,1]$, we obtain a closed subspace denoted $BV(D\,;[0,1])$. For every function $u \in BV(D)$, Du is a bounded Radon measure which can be decomposed into

$$Du = \nabla u dx + D^c u + (u_+ - u_-)\nu_u d(\mathcal{H}^{N-1} \llcorner S_u) \tag{2.4}$$

where ν_u denotes the Radon-Nikodým density of Du with respect to its total variation $|Du|$, i.e. $\nu_u := dDu/d|Du|$. Note that $D^c u$ is the Cantor part of the measure Du. The quantity $[u] := u_+ - u_-$ is called the *jump* of u across the interface S_u and the direction of the jump is given by ν_u along S_u. Accordingly, the *complete graph* of function u, denoted by $\overline{G}_u$, is defined by

$$\overline{G}_u := \bigcup_{x\in\Omega} \Big(\{x\} \times [u_-(x), u_+(x)]\Big).$$

For further details we refer to the monograph [3]. □

It is an N-rectifiable subset of $\Omega \times \mathbb{R}$ with an oriented unit normal denoted by $\widehat{\nu}_u$. This oriented normal $\widehat{\nu}_u$ is $\mathcal{H}^N \llcorner \overline{G}_u$ a.e. determined by

$$\widehat{\nu}_u(x,t) = (\nu_u(x), 0) \quad \text{for } x \in S_u \text{ and } t \in [u_-(x), u_+(x)], \tag{2.5}$$

on the vertical part of $\overline{G}_u$, whereas, on the *approximately continuous part* $G_u := \{(x, \tilde u(x))\}$, it is identified as

$$\widehat{\nu}_u(x,\tilde u(x)) = \frac{(\nabla u(x), -1)}{\sqrt{1+|\nabla u(x)|^2}} \tag{2.6}$$

if u is approximately differentiable at x (with its approximate gradient $\nabla u(x)$), and it is horizontal, i.e. $\widehat{\nu}_u(x,\tilde u(x)) = (\nu_u(x), 0)$, at points in the support of the Cantor part of Du. Notice that $\nu_u = dD^c u/d|D^c u|$ holds $|D^c u|$-a.e in D.

We remark also that the complete graph $\overline{G}_u$ of functions u belonging to $W^{1,1}(D)$ agrees with the *continuous graph* G_u , on which $\widehat{\nu}_u(x, u(x))$ is given by (2.6). Therefore, the normal trace of the calibration field σ given in (2.3) satisfies the equality $\sigma \cdot \widehat{\nu}_u = 1$ on G_u ($\mathcal{H}^N$-a.e.).

2.2. Dual problem of $\beta_0(\lambda)$

By applying the same duality framework as for $\beta(\lambda)$, we obtain the relation:

$$\beta_0(\lambda) = \sup\left\{ \int_D -\sigma^t(x,0)dx \ : \ \begin{array}{l} \operatorname{div}\sigma = 0,\ |\sigma^x| \le 1,\ \sigma^t + 1 \ge 0 \ \text{ in } Q, \\ \sigma^t(x,1) \ge \lambda - 1 \ \text{ on } D. \end{array} \right\}.$$

We recall that $m(\lambda, D) = \beta_0(\lambda) - |D|$, where $m(\lambda, D)$ is the minimum of the shape optimization problem (1.12). In order to simplify further computations, it is convenient to rewrite the duality relation above, after changing σ^t into $\sigma^t + 1$, as follows:

$$m(\lambda, D) = \sup\left\{ \int_D -\sigma^t(x,0)dx \ : \ \begin{array}{l} \operatorname{div}\sigma = 0,\ |\sigma^x| \le 1,\ \sigma^t \ge 0 \text{ in } Q, \\ \sigma^t(x,1) \ge \lambda \ \text{ on } D. \end{array} \right\}. \tag{2.7}$$

Since the pointwise constraints over σ^t and σ^x are decoupled, we can easily construct particular admissible calibrations σ. An important subclass is associated to the subset

$$S_\lambda := \{q \in L^\infty(D;\mathbb{R}^N) \ : \ |q| \le 1,\ 0 \le \operatorname{div} q \le \lambda\}.$$

It is easy to check that S_λ is convex and weakly* closed in $L^\infty(D;\mathbb{R}^d)$. Moreover, to each element $q \in S_\lambda$, we associate a competitor for (2.7), namely:

$$\sigma_q(x,t) := \ (-q(x), \lambda + (t-1)\ \operatorname{div} q(x)) \,.$$

It follows that: $m(\lambda, D) \ \ge \ \sup\left\{ \int_D (\operatorname{div} q - \lambda)\, dx \ : \ q \in S_\lambda \right\}.$

In turn, we are going to show that the latter inequality is an equality.

Reduction to dimension N and optimality conditions for $\beta_0(\lambda)$. By the exclusion principle mentioned in the introduction, optimal solutions u for (1.9) are of the form $\mathbb{1}_\Omega$ for some set $\Omega \subset D$. This set Ω has finite perimeter and solves the geometrical problem $m(\lambda, D)$.

Theorem 2.4. *Let D be a bounded domain of $\mathbb{R}^N$ with Lipschitz boundary. Then*

(i) *we have the following duality relation*

$$m(\lambda, D) = \sup\left\{ \int_D (\operatorname{div} q - \lambda)dx \ : \ \begin{array}{l} q \in L^\infty(D;\mathbb{R}^N),\ |q| \le 1, \\ 0 \le \operatorname{div} q \le \lambda \end{array} \right\}, \tag{2.8}$$

where the supremum in the right hand side is attained.

(ii) *a pair $(\Omega, \overline{q})$ solve* (1.12) *and* (2.8) *respectively if and only if the following conditions are satisfied*

$$|\overline{q}| \le 1 \quad a.e. \ in\ D, \qquad\qquad 0 \le \operatorname{div}\overline{q} \le \lambda \quad a.e. \ in\ D, \tag{2.9}$$

$$\overline{q} \cdot \nu_\Omega = 1 \quad \mathcal{H}^{N-1}\text{-}a.e. \ on\ \partial\Omega, \qquad \operatorname{div}\overline{q} = \lambda \quad a.e. \ in\ D \setminus \Omega. \tag{2.10}$$

Remark 2.5. In the left hand side of (2.10), ν_Ω stands for the exterior normal vector if Ω has a smooth boundary; if Ω is merely a subset with finite perimeter, the reader should agree that the equality means that the measure associated with the duality bracket $< \overline{q}, D\mathbb{1}_\Omega >$ in the sense of G. Anzelloti [4] coincides with $\mathcal{H}^{N-1} \llcorner \partial_*\Omega$ being $\partial_*\Omega$ the reduced boundary of Ω. Note that ν_Ω can be defined $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$ (see [20]). □

Proof of Theorem 2.4. First we notice that, with g^{**} defined by (1.7), we have the equality

$$m(\lambda, D) = \inf \left\{ \int_D (|\nabla v| + \lambda\, g^{**}(v))\, dx \;:\; v \in W_0^{1,1}(D) \right\}.$$

Indeed, since $g^{**}(t) \geq g^{**}(\min\{t_+, 1\})$, by truncating the competitors, the infimum above is unchanged if we restrict to v such that $0 \leq v \leq 1$. For such a v, we have $g^{**}(v) = -\lambda\, v$ so that, by the coarea formula:

$$\int_D (|\nabla v| + \lambda\, g^{**}(v))\, dx = \int_0^1 (P(v > t) - \lambda |v > t|)\; dt \geq m(\lambda, D).$$

Next we apply classical convex duality arguments. The perturbation function

$$h : p \in L^1(D, \mathbb{R}^N) \mapsto \inf \left\{ \int_D (|p + \nabla v| + \lambda\, g^{**}(v))\, dx \;:\; v \in W_0^{1,1}(D) \right\},$$

is convex and continuous. Indeed $h(p) \leq m(\lambda, D) + \int_D |p|\, dx$. Therefore it holds

$$h(0) = -\min\{h^*(q) \;:\; q \in L^\infty(D; \mathbb{R}^N)\},$$

being h^* the Fenchel conjugate of h in the duality between $L^1(D, \mathbb{R}^N)$ and $L^\infty(D, \mathbb{R}^N)$. Let us compute

$$\begin{aligned} h^*(q) &= \sup \left\{ \int_D p \cdot q\, dx - h(p) : p \in L^1(D; \mathbb{R}^N) \right\} \\ &= \sup \left\{ \int_D (p \cdot q - |p + \nabla v| - \lambda g^{**}(v))\; dx : (v, p) \in W_0^{1,1}(D) \times L^1(D; \mathbb{R}^N) \right\} \\ &= \sup \left\{ \int_D (\tilde{p} \cdot q - |\tilde{p}|) + \int_D (v \operatorname{div} q - \lambda g^{**}(v))\, dx : (v, \tilde{p}) \in W_0^{1,1}(D) \times L^1(D; \mathbb{R}^N) \right\} \\ &= \chi_{|q| \leq 1} + \int_D \lambda\, g^* \left(\frac{\operatorname{div} q}{\lambda} \right)\, dx. \end{aligned}$$

Concerning the calculation above:

- to pass from the second line to the third, we wrote $p \cdot q - |p + \nabla v|$ as $\tilde{p} \cdot q - |\tilde{p}| - \nabla v \cdot q$ where $\tilde{p} = p + \nabla v$, and then we used the integration by parts $-\int \nabla v \cdot q = \int v \operatorname{div} q$;
- to pass to the last line, we decoupled the supremum in $\tilde{p}$ from that in v and compute them as Fenchel conjugates of integral functionals, taking into account that the Fenchel conjugate of the norm is the indicator funtion of the unit ball while the conjugate of λg^{**} is $\lambda\, g^*(\frac{\cdot}{\lambda})$.

A straightforward computation shows that

$$\lambda\, g^*\left(\frac{t^*}{\lambda}\right) = \begin{cases} t^* + \lambda & \text{if } -t^* \in [0,\lambda] \\ +\infty & \text{otherwise} \end{cases}$$

It is convenient now to change q into $-q$. We arrive to the simple expression:

$$h^*(-q) = \int_D (\lambda - \operatorname{div} q)\, dx \quad \text{if } 0 \le \operatorname{div} q \le \lambda \text{ a.e.}, \ h^*(-q) = +\infty \text{ otherwise.}$$

Since $h(0) = m(\lambda, D)$, the equality $h(0) = -\min h^*$ leads to the equality stated in (2.8) where the supremum is actually a maximum. The assertion (i) is proved.

Let us now establish the assertion (ii). Let $(\mathbb{1}_\Omega, q)$ be an admissible pair. This means that Ω has finite perimeter and that q satisfies (2.9). In particular, the condition $|\overline{q}| \le 1$ implies that

$$P(\Omega) - \int_\Omega \operatorname{div} \overline{q}\, dx = \int_{\partial\Omega} (1 - \overline{q} \cdot \nu_\Omega)\, d\mathcal{H}^{N-1} \ge 0\ ,$$

while the equality holds if and only if $\overline{q} \cdot \nu_\Omega = 1$ holds $\mathcal{H}^{N-1}$-a.e in $\partial\Omega$.

Now, in virtue of (2.8), the optimality of $(\mathbb{1}_\Omega, \overline{q})$ is equivalent to the extremality relation:

$$P(\Omega) - \lambda\, |\Omega| \ = \ \int_D (\operatorname{div} \overline{q} - \lambda)\, dx,$$

which we can rewrite as the equality

$$\left(P(\Omega) - \int_\Omega \operatorname{div} \overline{q}\, dx\right) \ + \ \left(\int_{D\setminus\Omega} (\lambda - \operatorname{div} \overline{q})\, dx\right) \ = \ 0.$$

Since above we get the sum of two non negative terms, the equivalence with (2.10) follows. □

In Section 4, an explicit construction of an optimal field $\overline{q}$ will be given for D being any convex body in dimension 2 and for any $\lambda < h_D$.

2.3. Relations with Cheeger problem and calibrability notions

We recall that the Cheeger constant of a bounded domain $D \subset \mathbb{R}^N$ is given by

$$h_D := \inf \left\{ \frac{P(\Omega)}{|\Omega|} \ : \ \Omega \subset D, |\Omega| > 0 \right\}. \tag{2.11}$$

It is well known that the infimum here is actually a minimum and any optimal Ω for this geometric problem is called a Cheeger set of D. In case D is convex, it turns out that this set is unique (see for instance [14], [24], [26]) and we shall denote it by C_D. For a non convex D, the uniqueness fails; however, since a union of Cheeger sets is still a Cheeger set, the definition of C_D can be extended (see [13], [24]) by setting

$$C_D := \bigcup \{\Omega \ : \ \Omega \text{ is a Cheeger set of } D\}.$$

In this case, C_D is called the *maximal Cheeger set*. If D has finite perimeter, we will use in many places an upperbound for h_D , namely the ratio

$$\lambda_D := \frac{P(D)}{|D|} \ . \tag{2.12}$$

It satifies the inequality $\lambda_D \geq h_D$ while the equality holds if and only if D is a Cheeger set of himself; for brevity, we will say that D is *self-Cheeger*.

Let us now come back to the parametrized geometric optimization problem (1.12). It consists, for every $\lambda \geq 0$, in minimizing the shape functional

$$J_\lambda(\Omega) \ := \ P(\Omega) - \lambda|\Omega| \ , \quad \Omega \text{ measurable subset of } D.$$

Here we set $P(\Omega) = +\infty$ if Ω is not of finite perimeter and by convention $P(\Omega) = 0$ if $|\Omega| = 0$. In particular, we have $m(\lambda, D) = \inf J_\lambda \leq 0$ for every $\lambda \geq 0$.

The existence of minimizers for J_λ follows directly from the direct method of Calculus of Variations. As it appears in the proof of Theorem 2.4, $m(\lambda, D)$ is also the mimimum of the following companion convex problem:

$$\widetilde{m}(\lambda, D) := \min \left\{ F_\lambda(u) := \int_{\mathbb{R}^N} |Du| - \lambda \int_D u \, dx \ : \ \begin{array}{c} u \in BV(\mathbb{R}^N; [0,1]), \\ u = 0 \text{ a.e. on } D^c \end{array} \right\}.$$

The equality $\widetilde{m}(\lambda, D) = m(\lambda, D)$ is a consequence of the corea formula $F_\lambda(u) = \int_0^1 J_\lambda(\{u > t\}) \, dt \geq m(\lambda, D)$ from which it follows that:

$$u \text{ solves } \widetilde{m}(\lambda, D) \iff \{u > t\} \text{ solves } m(\lambda, D) \text{ for a.e. } t \in (0,1).$$

The main properties of the function $m(\lambda, D)$ are summarized below:

Proposition 2.6. *Let $D \subset \mathbb{R}^N$ be a bounded set with finite perimeter. Then*

(i) *The function $m(\cdot, D)$ is concave continuous non increasing on $\mathbb{R}_+$ and satisfies*

$$-(\lambda - h_D)_+ \, |D| \ \leq m(\lambda, D) \ \leq \min\{0, P(D) - \lambda|D|\}. \tag{2.13}$$

Therefore it holds $m(\lambda, D) = 0$ for every $\lambda \in [0, h_D]$ whereas:

$$m(\lambda, D) < 0 \ \text{ for } \lambda > h_D \quad \text{and} \quad \lim_{\lambda \to +\infty} m(\lambda, D) + \lambda|D| = P(D). \tag{2.14}$$

(ii) *Assume that D is a minimizer of $m(\lambda^*, D)$ for a suitable value $\lambda^* > 0$. Then $\lambda^* \geq h_D$ and D is the unique minimizer of $m(\lambda, D)$ for any $\lambda > \lambda^*$. In particular the concave function $m(\cdot, D)$ has the linear behavior $m(\lambda, D) = m(\lambda^*, D) - (\lambda - \lambda^*) \, |D|$ for every $\lambda \geq \lambda^*$.*

(iii) *Let $\Omega \subset D$. Then we have $m(\lambda, D) \leq m(\lambda, \Omega)$ while $m(\lambda, D) = m(\lambda, \Omega)$ if Ω is a minimizer of $m(\lambda, D)$. If such is the case with a set Ω of positive measure, then the ratio $\lambda_\Omega = \frac{P(\Omega)}{|\Omega|}$ satisfies $\lambda_\Omega < \lambda$.*

Proof. In the assertion (i), the fact that $m(\cdot, D)$ is monotone is obvious, while the concavity follows from writing $m(\cdot, D)$ as the infimum of the family of affine functions $\lambda \mapsto P(\Omega) - \lambda|\Omega|$, when Ω runs over subsets $\Omega \subset D$. The relations (2.13) are obtained by noticing that $P(\Omega) \geq h_\Omega |\Omega|$. Thus, since $|\Omega| \leq |D|$, the shape functional J_λ to be minimized satisfies

$$-(\lambda - h_D)_+ \, |D| \leq \; (h_D - \lambda) \, |\Omega| \; \leq J_\lambda(\Omega) \; \leq m(\lambda, D). \tag{2.15}$$

Then, by taking the infimum in Ω (and recalling that $m(\lambda, D) \leq 0$), we get (2.13) which proves that $m(\cdot, D)$ is finite, hence continuous by the concavity property. If $\lambda \leq h_D$, the first inequality in (2.13) implies that $m(\lambda, D) \geq 0$, hence $m(\lambda, D) = 0$ since we know that $m(\cdot, D) \leq 0$. If $\lambda > h_D$, we obtain directly that $m(\lambda, D) < 0$ since $J_\lambda(\Omega) = |\Omega|(h_D - \lambda) < 0$ holds whenever Ω is a Cheeger set of D. Moreover, the second inequality in (2.13) implies that

$$\limsup_{\lambda \to \infty} \, m(\lambda, D) + \lambda |D| \leq P(D). \tag{2.16}$$

In the opposite direction, let (λ_n) be any sequence such that $\lambda_n \to +\infty$ and choose a minimizer Ω_n for $m(\lambda_n, D)$. Then, by the upperbound inequality (2.16), we have

$$\limsup_{n \to \infty} \left(P(\Omega_n) + \lambda_n |D \setminus \Omega_n|\right) = \limsup_{n \to \infty} \left(m(\lambda_n, D) + \lambda_n |D|\right) \leq P(D) < +\infty.$$

It follows that $\mathbb{1}_{\Omega_n} \to 1$ in $L^1(D)$ while $\limsup_{n \to \infty} P(\Omega_n) \leq P(D)$. By the lower semicontinuity of the perimeter, we infer that $P(\Omega_n) \to P(D)$ so that $\lambda_n |D \setminus \Omega_n| \to 0$. Finally, we have proved that $m(\lambda_n, D) + \lambda_n |D| \to P(D)$ as desired. The proof of assertion (i) is finished.

Let us turn to the assertion (ii). If D solves $m(\lambda^*, D)$, then $\lambda^* \geq h_D$ since otherwise, by the assertion (i), we would have $m(\lambda^*, D) = J_{\lambda^*}(D) = 0$ which is uncompatible with the inequality $J_{\lambda^*}(D) \geq (h_D - \lambda^*)|D|$. Next we observe that, if $\lambda \geq \lambda^*$, then for every $\Omega \subset D$

$$J_\lambda(\Omega) \; = \; J_{\lambda^*}(\Omega) + (\lambda^* - \lambda)|\Omega| \; \geq \; J_{\lambda^*}(D) + (\lambda^* - \lambda)|D| \; = \; J_\lambda(D).$$

It follows that D is optimal also for $m(\lambda, D)$ and that

$$m(\lambda, D) = m(\lambda^*, D) - (\lambda - \lambda^*) \, |D|.$$

Futhermore, in view of the inequalities above, the optimality of a competitor Ω for $m(\lambda, D)$ requires that $(\lambda^* - \lambda)|\Omega| = (\lambda^* - \lambda)|D|$. If we assume that $\lambda > \lambda^*$, this is possible only if $|\Omega| = |D|$ which means that $\Omega = D$. Hence D is the unique solution to $m(\lambda, D)$.

Let us prove now the assertion (iii). If $\Omega \subset D$, the inequality $m(\lambda, D) \leq m(\lambda, \Omega)$ is clear since any admissible subset for $m(\lambda, \Omega)$ is also a competitor for $m(\lambda, D)$. In particular, if Ω itself is optimal for $m(\lambda, D)$, then we have $m(\lambda, D) = m(\lambda, \Omega) = J_\lambda(\Omega)$. If moreover $\lambda > h_D$, then $|\Omega| > 0$ and $J_\lambda(\Omega) = m(\lambda, D) < 0$. Therefore $J_\lambda(\Omega) = |\Omega|(\lambda_\Omega - \lambda) < 0$, whence $\lambda_\Omega < \lambda$ as claimed. □

Owing to Proposition 2.6, we know that, for $\lambda < h_D$, the unique solution (in the sense a.e.) of (1.12) is the empty set, while when $\lambda = h_D$, we need to add any Cheeger subset of D. For $\lambda > h_D$, any solution Ω has a positive measure. Then the optimality conditions obtained in the assertion (ii) of Theorem 2.4 can be exploited to extend to any $\lambda > \lambda_D$ the notion of calibrability as it was introduced in [2] for the specific case $\lambda = \lambda_D$.

Definition 2.7. Let $\theta \geq 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded set of finite perimeter. Ω is called *θ-calibrable* if there exists a vector field $q \in L^\infty(\Omega;\mathbb{R}^N)$ such that

$$\|q\|_\infty \leq 1, \qquad q \cdot \nu_\Omega = 1 \ \ \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega, \qquad 0 \leq \operatorname{div} q \leq \theta\lambda_\Omega \ \text{ in } \mathcal{D}'(\Omega).$$

Obviously, if Ω is θ-calibrable then it is also γ-calibrable for every $\gamma \geq \theta$. Accordingly, we define the *calibration constant* of a subset $\Omega \subset \mathbb{R}^N$ as

$$\theta_\Omega := \inf\{\theta \ : \ \theta \geq 1, \ \Omega \text{ is } \theta\text{-calibrable}\}, \tag{2.17}$$

with the convention that $\theta_\Omega = +\infty$ if the subset above is void.

Remark 2.8. If $\theta_\Omega < +\infty$, the infimum in (2.17) is actually a minimum. Indeed, given a sequence such that $\theta_n \searrow \theta_\Omega$, we can associate q_n in the unit ball of $L^\infty(\Omega;\mathbb{R}^N)$ such that $q_n \cdot \nu_\Omega = 1$ and $0 \leq \operatorname{div} q_n \leq \lambda_n$. Up to a subsequence, we have $(q_n, \operatorname{div} q_n) \overset{*}{\rightharpoonup} (q, \operatorname{div} q)$ for a suitable q such that $|q| \leq 1$ and $0 \leq \operatorname{div} q \leq \theta_\Omega$, while $q \cdot \nu_\Omega = 1$ on ∂D by the weak* convergence of the normal traces $q_n \cdot \nu_\Omega \overset{*}{\rightharpoonup} q_n \cdot \nu_\Omega$ in $L^\infty(\partial\Omega)$ (see [4]). □

Remark 2.9. In Definition 2.7, the condition $q \cdot \nu_\Omega = 1$ $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$ can be understood as $q \cdot D\mathbb{1}_\Omega = -|D\mathbb{1}_\Omega|$ in the sense of measure in $\overline{\Omega}$. By integrating by parts, we infer that:

$$P(\Omega) \ = \ \int_{\partial\Omega} -q \cdot D\mathbb{1}_\Omega \ = \ \int_\Omega \operatorname{div} q \ \leq \ \theta\lambda_\Omega\,|\Omega| \ = \ \theta P(\Omega). \tag{2.18}$$

As a consequence, the condition $\theta \geq 1$ is necessary if we wish the notion above to be non empty. Moreover, if $\theta = 1$, then the equality (2.18) implies that $\int(\lambda_D - \operatorname{div} q) = 0$, hence $\operatorname{div} q = \lambda_D$ a.e. in D. In this case we recover, after changing q in $-q$, something very similar to the definition of calibrability in [2, Definition 1], except that we do not need to impose the condition $\operatorname{div} q = \lambda_\Omega \mathbb{1}_\Omega$ on all $\mathbb{R}^N$. Therefore, our 1-*calibrability* condition is weaker than the one proposed in [2] (at least when Ω is not convex). □

Remark 2.10. It is important to keep in mind that the 1-calibrability property characterizes sets Ω which are self-Cheeger. In other words:

$$\theta_\Omega = 1 \iff \lambda_\Omega = h_\Omega \iff \Omega \text{ is self-Cheeger.} \tag{2.19}$$

Indeed, if $\theta_\Omega = 1$, as noticed in Remark 2.9, it exists $q \in L^\infty(\Omega;\mathbb{R}^N)$ such that $|q| \leq 1$ and $\operatorname{div} q = \lambda_D$. Then, for every $A \subset \Omega$ with positive measure, we have:

$$P(A) \geq \int_A \operatorname{div} q \ = \ \lambda_\Omega\,|A| \ \Longrightarrow \ \frac{P(A)}{|A|} \geq \frac{P(\Omega)}{|\Omega|}.$$

It follows that $A = \Omega$ solves the Cheeger problem in Ω. The converse is trivial since, in that case, $\lambda_\Omega = h_\Omega$ by assumption. □

Our notion of θ-calibrability can be used first to improve [2, Proposition 2], without any convexity assumption.

Proposition 2.11. *Let $\theta \geq 1$, and $\Omega \subset \mathbb{R}^N$ be a bounded set of finite perimeter. The following assertions are equivalent:*

(i) *Ω is θ-calibrable,*

(ii) *Ω is a minimizer of $m(\theta\lambda_\Omega, \Omega)$.*

Proof. In view of Definition 2.7, it is enough to apply the assertion (ii) of Theorem 2.4 taking $D = \Omega$ and $\lambda = \theta\,\lambda_D$. □

Regarding the original problem (1.12), we obtain the following result:

Corollary 2.12. *Let $\Omega \subset D$ with positive measure. Then Ω is θ-calibrable for a suitable constant $\theta \geq 1$ if and only if it solves $m(\lambda^*, D)$ for some $\lambda^* \geq h_D$. In this case, the calibrability constant of Ω is given by:*

$$\theta_\Omega \;=\; \min\left\{ \frac{\lambda}{\lambda_\Omega} \;:\; \Omega \textit{ solves } m(\lambda, \Omega) \right\}. \tag{2.20}$$

Proof. Suppose that Ω solves problem $m(\lambda^*, D)$. If $\lambda^* = h_D$, then the assumption $|\Omega| > 0$ implies that Ω is a Cheeger set of D. Therefore, by (2.19), Ω is 1-calibrable and $\theta_\Omega = 1$. If $\lambda^* > h_D$, we know that $m(\lambda^*, D) < 0$. Then, by the assertion (iii) of Proposition 2.6, it follows that $m(\lambda^*, D) = m(\lambda^*, \Omega)$ and $\lambda_\Omega < \lambda^*$. Therefore, by virtue of Proposition 2.11, Ω is θ^*-calibrable for the constant $\theta^* = \frac{\lambda^*}{\lambda_\Omega} > 1$.

Conversely, let us assume that $\Omega \subset D$ is θ- calibrable for some $\theta \geq 1$. Then by Proposition 2.11, Ω solves $m(\lambda, D)$ for $\lambda = \theta\,\lambda_D$. Accordingly we can conclude with the equality characterizing the calibrability constant θ_Ω when it is finite. □

The characterization of θ-calibrable sets among the class of finite perimeter subsets in $\mathbb{R}^N$ is a difficult issue. In the case of a convex set $\Omega \subset \mathbb{R}^N$, we have the following result directly deduced from [2, Theorem 9] and Corollary 2.12.

Proposition 2.13. *Let $\Omega \subset \mathbb{R}^N$ be convex, bounded of class $C^{1,1}$. Let $\kappa_\infty(\partial\Omega)$ denote the L^∞-norm of the mean curvature of $\partial\Omega$. Then the calibrability constant of Ω is given by:*

$$\theta_\Omega = \max\left\{1, \frac{(N-1)}{\lambda_\Omega}\,\kappa_\infty(\partial\Omega)\right\} \tag{2.21}$$

3. Comparison results

In this section, we focus on the initial question raised in the introduction about comparing the free boundary problems (1.1) and (1.9), for a given value of the parameter $\lambda > 0$. Recall that the respective infima of these problems $\beta(\lambda)$ and $\beta_0(\lambda)$ always satisfy the inequality $\beta(\lambda) \leq \beta_0(\lambda)$.

We are going to prove that this inequality is strict if and only if the minimum $\beta(\lambda)$ of the minimal surfaces free boundary problem (1.1) is not reached by none of the trivial competitors $u \equiv 0$ or $u \equiv 1$. Accordingly, let us introduce the two following critical values of λ:

$$\lambda_0 = \sup\left\{\lambda \geq 0 \;:\; u \equiv 0 \text{ solves } \beta(\lambda)\right\}, \tag{3.1}$$

$$\lambda_1 = \inf\left\{\lambda \geq 0 \;:\; u \equiv 1 \text{ solves } \beta(\lambda)\right\}. \tag{3.2}$$

After recalling the definition of the constants h_D, λ_D, θ_D given in (2.11), (2.12) and (2.17), we give here a first result:

Lemma 3.1. *Let λ_0, λ_1 be defined as above. Then*

(i) *It holds $0 \le \lambda_0 \le h_D$ and $u \equiv 0$ solves $\beta(\lambda)$ if and only if $\lambda \in [0, \lambda_0]$.*

(ii) *It holds $\theta_D \lambda_D \le \lambda_1 \le +\infty$ and $u \equiv 1$ solves $\beta(\lambda)$ if and only if $\lambda \ge \lambda_1$.*

(iii) *The inequality $\lambda_0 \le \lambda_1$ is strict if D is not self-Cheeger.*

Proof. By taking $u \equiv 0$ as a competitor in (1.1), we see that $\beta(\lambda) \le |D|$ for any $\lambda \ge 0$ while the equality $\beta(\lambda) = |D|$ means that $u \equiv 0$ is a solution. Since the function β is non increasing, it follows that $\beta(\lambda') = |D|$ for $\lambda' \in [0, \lambda_0]$. Therefore the subset appearing in (3.1) is the full interval $[0, \lambda_0]$. Futhermore, it holds $\lambda_0 \le h_D$ since $\beta_0(\lambda_0) \ge \beta(\lambda_0) = |D|$ implies that $m(\lambda_0, D) = 0$ in virtue of (2.14). The positivity of λ_0 will follow from Proposition 3.6 where a positive lower bound is given.

Let us prove (ii). The set of $\lambda \ge 0$ for which $u \equiv 1$ solves $\beta(\lambda)$ coincides with the set

$$J := \{\lambda \ge 0 : \beta(\lambda) = |D| + P(D) - \lambda |D|\}.$$

By the continuity of the concave function β, J is closed and thereby $\lambda_1 \in J$. It follows that $\mathbb{1}_D$ solves also $\beta_0(\lambda_1)$, hence is a solution of $m(\lambda_1, D)$. In virtue of Corollary 2.12, we infer the inequality $\lambda_1 \ge \theta_D \, \lambda_D$. Assume that $\lambda_1 < +\infty$ and let $\lambda > \lambda_1$. To show that $J = [\lambda_1, +\infty)$, we need to demonstrate that $u \equiv 1$ is solution of $\beta(\lambda)$ for any $\lambda > \lambda_1$. This is a consequence of the following general comparison principle, namely that $\lambda \ge \mu$ implies that $u \ge v$ whenever u, v solve $\beta(\lambda)$ and $\beta(\lambda \ge \mu)$. The conclusion will follow by applying it with $v \equiv 1$ solution to $\beta(\lambda_1)$ (taking into account that $u \le 1$). Let us validate this principle in our case showing that $u \le 1$. Since u minimizes E_λ (see (1.5)) on $BV(D; [0, 1])$, we have $E_\lambda(u) \le E_\lambda(\mathbb{1}_D)$, so that calling F the functional which agrees with E_λ when $\lambda = 0$, we get: $F(u) - F(\mathbb{1}_D) \le -\lambda \, |\{u < 1\}|$. Similarly as $\mathbb{1}_D$ minimizes E_{λ_1}, we obtain $F(\mathbb{1}_D) - F(u) \le \lambda_1 \, |\{u < 1\}|$. Adding these two inequalities above, we are led to $0 \le (\lambda_1 - \lambda) \, |\{u < 1\}|$, thus $|\{u < 1\}| = 0$. That proves that $u = \mathbb{1}_D$ is the unique minimizer of $\beta(\lambda)$ for every $\lambda > \lambda_1$.

Finally, from the assertions (i) and (ii) and recalling that $\theta_D \ge 1$, we know that

$$\lambda_0 \le h_D \le \lambda_D \, \theta_D \le \lambda_1.$$

Thefore, the inequality $\lambda_0 \le \lambda_1$ is an equality if and only if $h_D = \lambda_D$ (and $\theta_D = 1$). This happens only if D is self-Cheeger, hence the assertion (iii). □

Next we state that, if $\lambda_0 < \lambda_1$, then a solution to problem $\beta(\lambda)$ when $\lambda \in (\lambda_0, \lambda_1)$ will never be of the kind $\mathbb{1}_\Omega$. By the assertion (iii) of Lemma 3.1, this occurs in particular when D is not a self-Cheeger set.

Theorem 3.2. *Assume that λ_0, λ_1 defined in (3.1)(3.2) such that $\lambda_0 < \lambda_1$. Then the strict inequality $\beta(\lambda) < \beta_0(\lambda)$ holds for every $\lambda \in (\lambda_0, \lambda_1)$.*

Proof. Fix $\lambda \in (\lambda_0, \lambda_1)$. We shall prove the result by contradiction assuming that $\beta(\lambda) = \beta_0(\lambda)$. Then, $\beta(\lambda)$ admits a solution of the kind $u = \mathbb{1}_\Omega$ where $\Omega \subset D$ is such that $0 < |\Omega| < |D|$. For such a set of bounded perimeter Ω, denoting by $\partial\Omega$ its reduced boundary, the free boundary has positive measure namely

$$\mathcal{H}^{N-1}(\partial\Omega \cap D) > 0. \tag{3.3}$$

Indeed, if (3.3) where not true, then in virtue of (2.4), we would have $D\mathbb{1}_\Omega = 0$ in the distributional sense in the domain D, thus implying that $\mathbb{1}_\Omega \equiv 1$. Next we consider the trace $\mathrm{Tr}(\mathbb{1}_\Omega)$ of $\mathbb{1}_\Omega$ on ∂D. As an element of $L^1(\partial D, [0,1])$, it is also the trace of a suitable function $\varphi \in W^{1,1}(D)$, in virtue of the Lipschitz regularity of ∂D and of the fact that the trace operator from $W^{1,1} \to L^1(D)$ is surjective by Galiardo's Theorem (see [6, Theorem 9]). Without loss of generality, we can assume that $\varphi \geq 0$ and possibly after adding to φ the distance function to ∂D (which belongs to $W_0^{1,1}(D)$) and after truncating the values greater than 1, we build a non negative function $\varphi : D \to (0,1]$ such that $\mathrm{Tr}\, f = 0$ on $\partial D \setminus \partial\Omega$ and $\mathrm{Tr}\, f = 1$ on $\partial D \cap \partial\Omega$. Accordingly, for every small $\varepsilon > 0$, we define $u_\varepsilon := \max\{\varepsilon\,\varphi, \mathbb{1}_\Omega\}$. For small ε, the level set $\{u_\varepsilon \geq 1\}$ coincides with Ω. On the other hand the distributional gradient of u_ε in the open set D decomposes as $Du_\varepsilon = \varepsilon\,\nabla\varphi\,\mathcal{L}^N \llcorner \Omega - (1-\varepsilon\varphi)\nu_\Omega \mathcal{H}^{N-1} \llcorner \partial(\Omega \cap D)$. Therefore, its total energy is given by:

$$E_\lambda(u_\varepsilon) = \int_{D\setminus\Omega} \sqrt{1+\varepsilon^2|\nabla\varphi|^2}\,dx + \int_{\partial\Omega} |u_\varepsilon|\,d\mathcal{H}^{N-1} + \int_{\partial\Omega\cap D} (1-\varepsilon\varphi)\,d\mathcal{H}^{N-1} - \lambda\,|\Omega|$$

Since $\mathbb{1}_\Omega$ is a minimizer which shares the same trace as u_ε on ∂D, the following limit is non negative:

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \frac{E_\lambda(u_\varepsilon) - E_\lambda(\mathbb{1}_\Omega)}{\varepsilon} &= \lim_{\varepsilon\to 0^+} \int_{D\setminus\Omega} \frac{\sqrt{1+\varepsilon^2|\nabla\varphi|^2}-1}{\varepsilon}\,dx - \int_{\partial\Omega\cap D} \varphi\,d\mathcal{H}^{N-1} \\ &= -\int_{\partial\Omega\cap D} \varphi\,d\mathcal{H}^{N-1},\end{aligned}$$

where to pass from the first to the second line, we used dominated convergence. Recalling (3.3) and that $\varphi > 0$ in D by the previous construction, we infer that the limit above is strictly negative hence the desired contradiction. □

Remark 3.3. In the case where $\lambda_0 \leq 1 < \lambda_1$, it turns out that for $\lambda \in (1, \lambda_1)$, a solution u for $\beta(\lambda)$ (in the relaxed form (1.4)) can't be in $W^{1,1}(D)$. Indeed in this case, the set $\Omega = \{u = 1\}$ has a non empty free boundary $\partial\Omega \cap D$ and, by computing the shape derivative of the functional $J(\Omega)$ defined in (1.2) (see for instance [10, 22]), we obtain the optimality condition

$$1 - \lambda = \frac{1}{\sqrt{1+|\nabla u|^2}}$$

holding on $\partial\Omega \cap D$. This relation assigns the angle of the minimal surface with the plateau $\Omega \times 1$. Clearly this relation is neither be fullfiled if $\lambda > 1$. This means that any u solving $\beta(\lambda)$ should exhibit a jump on the free boundary in order to reach the value 1. □

Example 3.4. In the one dimensional case, it is possible to compute explicitly the values of λ_0, λ_1. Owing to Theorem 3.2, the occurence of the strict inequality $\beta(\lambda) < \beta_0(\lambda)$ is possible only if $\lambda_0 < \lambda_1$. Without loss of generality, let us consider the domain $D_R = (-R, R)$. It is a self-Cheeger set with constant $h_{D_R} = \lambda_{D_R} = \frac{1}{R}$ (and $\theta_{D_R} = 1$). Recall that

$$\beta(\lambda) = \inf\left\{\int_{-R}^{R} \sqrt{1+u'^2}\,dx - \lambda|\{u \geq 1\}| \;:\; u \in W^{1,1}(-R,R),\ u(\pm R) = 0\right\},$$
$$\beta_0(\lambda) = 2R + m(\lambda, D_R),$$

where the second equality follows from (1.8). As shown in the next result, the strict inequality $\lambda_0 < \lambda_1$ will occur if and only the length of interval D is greater than 2.

Lemma 3.5. *Let $\lambda_0 = \lambda_0(R)$ and $\lambda_1 = \lambda_1(R)$ be the critical values associated to D_R. Then:*

$$\lambda_0(R) \;=\; \begin{cases} \frac{1}{R} & \text{if } R \leq 1 \\ \frac{2}{1+R^2} & \text{if } R \geq 1 \end{cases} \quad , \quad \lambda_1(R) \;=\; \begin{cases} \frac{1}{R} & \text{if } R \leq 1 \\ 1 & \text{if } R \geq 1 \end{cases} \tag{3.4}$$

Proof. Since D_R is a Cheeger set with constant $\frac{1}{R}$, we know that $m(\lambda, D_R) = 0$ if $\lambda < \frac{1}{R}$, while $m(\lambda, D_R) = 2(1 - \lambda R)$ otherwise. It follows that:

$$\beta_0(\lambda) = 2R - 2(\lambda R - 1)_+ \tag{3.5}$$

Concerning the minimal value $\beta(\lambda)$, it is easy to check that it is achieved by taking u to be either $u \equiv 0$ (then $\beta(\lambda) = 2R$) or a radial function with plateau $\{u = 1\} = \{|x| \leq R - \alpha\}$ for a suitable value $\alpha \in [0, R)$, of the form:

$$u_\alpha(x) = \min\left\{1, \frac{R-|x|}{\alpha}\right\} \quad \text{if } \alpha > 0 \quad , \quad u_0(x) \equiv 1.$$

Observe that the expected solution u_α is always continuous. In view of the expression of E_λ given in (1.5), we obtain the equality

$$\beta(\lambda) = \min\left\{2R, \min_{\alpha\in[0,R]} f(\alpha)\right\} \text{ where } f(\alpha) := 2\left(\sqrt{1+\alpha^2} + (1-\lambda)(R-\alpha)\right). \tag{3.6}$$

Observe that the function f above is C^1, strictly convex with derivative

$$f'(\alpha) = 2\Big(\frac{\alpha}{\sqrt{1+\alpha^2}} - (1-\lambda)\Big).$$

Moreover, the equality $\beta(\lambda) = 2R$ means that $u \equiv 0$ is a minimizer, while if we have $\beta(\lambda) > 2R$ the unique minimizer is u_α where α is minimal for f on $[0, R]$. According to the value of R, we will proceed in two cases.

Case $R \leq 1$. If $\lambda \leq \frac{1}{R}$, we have $\beta_0(\lambda) = 2R$ by (3.5). If $\lambda \leq 1$, we see directly from the expression of f in (3.6) that $\inf f \geq 2 \geq 2R$. Hence we have $\beta(\lambda) = \beta_0(\lambda) = 2R$ while $u \equiv 0$ is a minimizer for both problems. If $\lambda > \frac{1}{R}$, then as $\lambda > 1$, we have $f' \geq 0$ and $\min f = f(0) = 2 + 2(1-\lambda)R < 2R$. Therefore, an optimal solution is given by $u = u_0 \equiv 1$ and $\beta(\lambda) = \beta_0(\lambda)$ in view of (3.5). In conclusion, for every $\lambda \geq 0$, we have $\beta_0(\lambda) = \beta(\lambda)$ and the the critical values $\lambda_0(R)$ and $\lambda_1(R)$ are equal to $\frac{1}{R}$. Note that, for $\lambda = \frac{1}{R}$, the solutions $u \equiv 0$ and $u \equiv 1$ coexist.

Case $R > 1$. If $\lambda \geq 1$, as noticed before, $f' \geq 0$ and $\min f = f(0) < 2R$. Hence $\beta(\lambda) = \beta_0(\lambda)$ and the unique common minimizer is $u \equiv 1$. If $\lambda < 1$, the convex function f starts with a negative slope at zero so that it reaches its mimimum at a unique $\alpha_c \in (0, R]$ provided $f'(R) \geq 0$, that is if $\lambda \geq \lambda_* = 1 - \frac{R}{\sqrt{1+R^2}}$. In view of (3.6), u_{α_c} will be the unique minimizer of $\beta(\lambda)$ if we have, in addition, $f(\alpha_c) \leq 2R$. In this case, since u_{α_c} is not of the form $\mathbb{1}_\Omega$, we will deduce the strict inequality $\beta(\lambda) < \beta_0(\lambda)$. After some computations [1], it turns out that the equality $f(\alpha_c) \leq 2R$ is true iff $\lambda \geq \lambda^*$ where $\lambda^* := \frac{2}{1+R^2}$ If $\lambda \in [\lambda_*, \lambda^*)$, the minimum of f is larger than $2R$. It is also the case if $\lambda < \lambda_*$, since $f' > 0$ on $[0, R]$ implies that $\min f = f(0) > 2 \geq 2R$. Therefore, the minimum of $\beta(\lambda)$ is reached at $u \equiv 0$ for $\lambda \leq \lambda^*$. For $\lambda \in (\lambda^*, 1)$, there is a unique solution of the kind u_α with $\alpha > 0$, and ultimately, the solution $u \equiv 1$ for $\lambda \geq 1$. Summarizing, we have proved that $\lambda_0(R) = \frac{2}{1+R^2}$ and $\lambda_1(R) = 1$ □

In higher dimension $N \geq 2$, explicit expressions for λ_0 and λ_1 are not available, except possibly in the radial case. However we are able to derive some estimates where the role of the geometric constants h_D, λ_D and θ_D is enlightened. From Lemma 3.1, we already know that $\lambda_0 \leq h_D \leq \theta_D\, \lambda_D \leq \lambda_1$. In the next result, we use the duality result presented in Subsection 2.1 for deriving a positive lower bound for λ_0 and a sharp upper bound for λ_1 when D is calibrable.

Proposition 3.6. *Let $D \subset \mathbb{R}^N$ be a general bounded Lipschitz domain. Then the critical values λ_0, λ_1 defined in* (3.1)(3.2) *satisfy:*

(i) $\lambda_0 \geq \lambda_0^*$ *where*

$$\lambda_0^* := \begin{cases} 1 - \cos(h_D) & \text{if } h_D \leq \frac{\pi}{2} \\ 1 + h_D - \frac{\pi}{2} & \text{if } h_D \geq \frac{\pi}{2} \end{cases}. \tag{3.7}$$

(ii) *Let θ_D be the (posssibly infinite) calibration constant of D and $\lambda_D := P(D)/|D|$.*

Then
$$\theta_D \lambda_D \ \leq\ \lambda_1\ \leq\ \theta_D \lambda_D + 1. \tag{3.8}$$

Remark 3.7. If D is a convex set of class $C^{1,1}$, then, by virtue of Proposition 2.13, the inequalities (3.8) can be rewritten as follows:

$$\max\{\lambda_D, (N-1)\kappa_\infty(\partial D)\} \leq \lambda_1 \leq 1 + \max\{\lambda_D, (N-1)\kappa_\infty(\partial D)\}. \tag{3.9}$$

In particular, if ∂D exhibits a corner, then $\theta_D = +\infty$ and $\lambda_1 = +\infty$. This means that $u \equiv 1$ can't either be a solution to $\beta(\lambda)$.

[1] Setting $t = 1 - \lambda$ and $h(x) := \sqrt{1+x^2} - tx + R(t-1)$, we are reduced to show that $min_{[0,R]} h < 0$ iff $0 < t < \frac{R^2-1}{R^2+1}$. The minimum of h is reached at x_c such that $\frac{x_c}{\sqrt{1+x_c^2}} = t$. Since $x_c = \frac{t}{\sqrt{1-t^2}}$ while $\sqrt{1+x_c^2} = \frac{1}{\sqrt{1-t^2}}$, we obtain:

$$\min h = \sqrt{1+x_c^2} - tx_c + R(t-1) = \frac{1}{\sqrt{1-t^2}} - t\Big(\frac{t}{\sqrt{1-t^2}}\Big) + R(t-1) = \sqrt{1-t^2} + R(t-1).$$

Recalling that $t < 1$, we have $\min h < 0$ iff $\sqrt{1-t^2} < R(1-t)$. Squaring and dividing by $1-t$, we are led to the condition $1 + t < R^2(1-t)$, that is $t < \frac{R^2-1}{R^2+1}$. (Note that $\lambda^* \in [\lambda_*, 1]$ ensures that α_c belongs to $(0, R]$).

Note that in the one dimensional case, we have $h_D = \lambda_D = |D|^{-1}$ and $\theta_D = 1$ so that (3.8) becomes $\lambda_D \le \lambda_1 \le 1 + \lambda_D$. In Lemma 3.5, we showed that if $|D| = 2R$ with $R > 1$, then $\lambda_1 = 1$ so that $\lambda_1 \in [\frac{1}{R}, 1 + \frac{1}{R}]$. By sending R to 1 or to $+\infty$, we conclude that the bounds in (3.8) are optimal. □

Proof of Proposition 3.6. We will construct calibration fields $\sigma \in L^\infty(Q; \mathbb{R}^N \times \mathbb{R})$ where $Q := D \times [0, 1]$ of the kind

$$\sigma(x,t) \;=\; \Big(-a(t)q(x),\ A(t)\,\mathrm{div}\, q(x) + r(x) \Big) \quad \text{for } (x,t) \in Q \tag{3.10}$$

where
$$\begin{cases} a \in C([0,1]) \text{ is such that } 0 \le a(t) \le 1; \\ (q,r) \in L^\infty(D; \mathbb{R}^N \times \mathbb{R}),\ |q| \le 1\ ,\ \mathrm{div}\, q \in L^\infty(D); \\ A'(t) = a(t) \text{ (thus } \mathrm{div}\,\sigma = 0 \text{ in } Q). \end{cases} \tag{3.11}$$

Note that, with the last condition, A is Lipschitz non decreasing with a slope $A't) \le 1$.

(i) As noticed in the proof of Lemma 3.1, it holds $\lambda \le \lambda_0$ if and only if $\beta(\lambda) \ge |D|$. In view of the duality Theorem 2.4, it will be the case if we can find an admissible $\sigma = (\sigma^x, \sigma^t)$ such that

$$\sigma^t(x,0) = -1\,,\ \sigma^t(x,1) \ge \lambda - 1 \ \text{ on } D\,,\ \sigma^t + \sqrt{1 - |\sigma^x|^2} \ge 0, \ \text{ in } Q. \tag{3.12}$$

Let us search σ of the form given in (3.10) where $r \equiv -1$ and the triple (a, A, q) satisfies (3.11). We impose the additional condition that q satisfies $\mathrm{div}\, q = h_D$. By Remark 2.9, such a q exists and it calibrates every Cheeger set of D. With this choice, we obtain that $\sigma^t + \sqrt{1 - |\sigma^x|^2} \ge h_D\, A(t) + \sqrt{1 - A'^2(t)}$. It follows that the conditions in (3.12) are all met if we select $A(t)$ so that

$$A(0) = 0\,, \quad 0 \le A'(t) \le 1, \quad h_D\, A(1) \ge \lambda\,, \quad h_D\, A(t) + \sqrt{1 - A'^2(t)} \ge 1 \quad \forall t \in [0,1].$$

Note that the conditions above imply that $A(1) \le 1$. Thus the inequality $h_D\, A(1) \ge \lambda$ can't be reached unless $\lambda \le h_D$, which is coherent with the assertion (i) of Lemma 3.1. For further computations, it is convenient to set $\psi(t) := h_D A(t)$. Then we arrive to the fact that $\lambda \le \lambda_0$ whenever

$$\lambda \le \sup \left\{ \psi(1) \,:\, \psi(0) = 0,\ 0 \le \psi' \le h_D \ \text{ and } \sqrt{1 - \frac{|\psi'|^2}{h_D^2}} + \psi \ge 1 \text{ on } [0,1] \right\}. \tag{3.13}$$

Hence, proving the assertion (i) reduces to check that the right hand side of (3.13) coincides with the value λ_0^* given by (3.7). In fact the inequality constraint on ψ in (3.13) can be rewritten equivalently as $0 \le \psi' \le h_D \gamma(t)$ where

$$\gamma(s) := \begin{cases} \sqrt{s(2-s)} & \text{if } s \le 1 \\ 1 & \text{if } s > 1 \end{cases}$$

It follows that the composed function $z : [0,1] \mapsto [0, h_D]$ defined by $z(t) := \int_0^{\psi(t)} \frac{1}{\gamma(s)} ds$ satisfies $0 \le z'(t) \le h_D$ with a maximal value h_D reached at $t = 1$ if and only if $z(t) = h_D t$.

Accordingly the maximal value in (3.13) is reached for ψ determined by the following relation holding for every $t \in [0,1]$:

$$h_D t \;=\; \int_0^{\psi(t)} \frac{1}{\gamma(s)} ds \;=\; \begin{cases} \arccos(1-\psi(t)) & \text{if } \psi(t) \le 1, \\ \frac{\pi}{2} + \psi(t) - 1 & \text{if } \psi(t) > 1. \end{cases}.$$

We conclude that the optimal ψ for (3.13) is given explicitely by

$$\psi(t) = \begin{cases} 1 - \cos(h_D t) & \text{if } h_D t \le \frac{\pi}{2} \\ 1 + h_D t - \frac{\pi}{2} & \text{if } h_D t \ge \frac{\pi}{2} \end{cases}$$

thus confirming the optimal value $\psi(1) = \lambda_0^*$.

(ii) The inequality $\theta_D \lambda_D \le \lambda_1$ has been proved in Lemma 3.1. In order to show that $\lambda_1 \le 1 + \theta_D \lambda_D$, we need to demonstrate that $\beta(\lambda) \ge \beta_0(\lambda)$ for any $\lambda > 1 + \theta_D \lambda_D$. Let us fix such a λ. Without any loss of generality, we can assume that $\theta_D < +\infty$ since otherwise $\lambda_1 = +\infty$. Then, by applying Corollary 2.12 to $\Omega = D$, we see that D is minimal for $m(\lambda - 1, D)$. Then, in virtue to Theorem 2.4, it exists a calibrating field q satisfying

$$|q| \le 1, \quad 0 \le \operatorname{div} q \le \lambda - 1 \text{ in } D, \qquad \text{and} \qquad q \cdot \nu_D = 1 \text{ on } \partial D.$$

Next we consider a vector field σ of the form (3.10) where $a(t) \equiv 1$, $A(t) = t - 1$ and $r(x) \equiv \lambda - 1$, that is:

$$\sigma(x,t) = (-q(x), (t-1) \operatorname{div} q(x) + \lambda - 1).$$

It is easy to verify that σ is admissible for the dual problem of $\beta(\lambda)$, since by construction:

$$\sigma^t(x,1) = \lambda - 1 \text{ in } D, \quad \operatorname{div} \sigma = 0 \text{ in } Q,$$

while by the inequalities $0 \le \operatorname{div} q \le \lambda - 1$, we have for a.e. $(x,t) \in Q$

$$\sqrt{1 - |\sigma^x(x,t)|^2} + \sigma^t(x,t) \ge \sigma^t(x,t) = (t-1) \operatorname{div} q(x) + \lambda - 1 \ge 0.$$

Therefore, in view of the duality Theorem 2.4 , it holds

$$\beta(\lambda) \ge -\int_D \sigma^t(x,0) = (1-\lambda)|D| + \int_D \operatorname{div} q \, dx = (1-\lambda)|D| + P(D) = \beta_0(\lambda),$$

where:

- for the first equality, we used the fact that $\int_D \operatorname{div} q \, dx = \int_{\partial D} q \cdot \nu_D \, d\mathcal{H}^{N-1}$
- for the last equality, we used the fact that D is optimal for $m(\mu, D)$ for any $\mu \ge \lambda - 1$, hence for $m(\lambda, D)$ in particular.

We conclude that $\beta(\lambda) = \beta_0(\lambda)$, that means $u \equiv 1$ solves $\beta(\lambda)$, whence $\lambda \le \lambda_1$. $\square$

Example 3.8. Let $D = \{x \in \mathbb{R}^2 : |x| < R\}$ be a disk of radius R in $\mathbb{R}^2$. Recall that disks are self-Cheeger sets and that the Cheeger constant of a disk is completely determined by its radius, i.e. $h_D = 2/R$. The unique solution to problem $\beta_0(\lambda)$ is $u_0 \equiv 0$ for $\lambda < h_\Omega$ where $h_\Omega = \frac{2}{R}$ is the Cheeger constant of Ω. In contrast the unique solution is $u_1 \equiv 1$ for $\lambda > h_\Omega$. For the precise value $\lambda = h_\Omega$ we obtain exactly two solutions u_0, u_1. Accordingly,

$$\beta_0(\lambda) = \begin{cases} |D| & \text{if } \lambda \leq h_D \\ P(D) + (1-\lambda)|D| & \text{if } \lambda > h_D. \end{cases}$$

Let us now turn to the determination of the infimum $\beta(\lambda)$. By a rearrangement argument, we can prove that solutions are all radial of the form $u(x) = \varphi(\frac{|x|}{R})$ being $\varphi(t) : [0,1] \to [0,1]$ monotone non increasing. The plateau $\{u = 1\}$ is associated with an interval $t \in [0, \rho]$ for a suitable value of $\rho \in [0,1]$ to be determined. For such a plateau, the minimal surface problem reads

$$J(\rho) := \inf_{\substack{\varphi(\rho)=1 \\ \varphi(1)=0}} I(\varphi) , \qquad I(\varphi) := R \int_\rho^1 \sqrt{R^2 + \varphi'^2}\, t dt . \tag{3.14}$$

The first integral of Euler equation for this minimization problem reads

$$\frac{t\varphi'}{\sqrt{R^2 + \varphi'^2}} = \mu \tag{3.15}$$

for some constant μ. As $\varphi(1) = 0$, we are led to the explicit form

$$\varphi(t) = K(\mu, t) , \qquad K(\mu, t) := \mu\, R \log\left(\frac{1 + \sqrt{1-\mu^2}}{t + \sqrt{t^2 - \mu^2}}\right) , \tag{3.16}$$

provided we can find $\mu \in [0, \rho]$ such that $\varphi(\rho) = K(\mu, \rho) = 1$. In fact, since the function $\mu \mapsto K(\mu, \rho)$ is stricly increasing on $[0, \rho]$, such a μ exists and is unique and exists if and only if

$$1 \leq K(\rho, \rho) = \rho\, R \log\left(\frac{1 + \sqrt{1-\rho^2}}{\rho}\right) . \tag{3.17}$$

If the inequality above is strict, then we obtain a unique solution to (3.14) which is smooth. In the limit case where (3.17) is an equality, one has $\mu = \rho$ and $\varphi'(\rho) = +\infty$. If $K(\rho, \rho) < 1$, then (3.14) has no solution but the relaxed solution in $BV(D)$ is unique and exhibits a jump at $t = \rho$ of amplitude $1 - K(\rho, \rho)$. In all cases, we have determined, in term of parameter $\rho \in [0,1]$, an optimal radial configuration whose plateau $\{u = 1\}$ agrees with the disk $B(0, \rho)$. Its total energy is given by

$$E(\rho) := 2\pi\, J(\rho) + (1-\lambda)\pi\rho^2 R^2 .$$

In order to minimize $E(\rho)$ on interval $[0,1]$, we introduce

$$\overline{\mu}(\rho) := \sup_{0 \leq \mu \leq \rho} \{\mu : K(\mu, \rho) \leq 1\}.$$

It is easy to check that for every $\rho \in [0,1]$ such that if $K(\rho, \rho) < 1$ then $\overline{\mu}(\rho) = \rho$.

Otherwise, $\overline{\mu}(\rho)$ is the unique solution of equation $K(\mu,\rho)=1$. After a straightforward computation and exploiting (3.15), we obtain

$$J(\rho) = R^2 \int_\rho^1 \frac{t^2}{\sqrt{t^2-\overline{\mu}(\rho)^2}}dt + \rho R\Big(1 - K(\overline{\mu}(\rho),\rho)\Big)\,.$$

Thus, noticing that $K(\overline{\mu}(\rho),\rho) \le 1$, we are led to:

$$E(\rho) = \pi R^2\Big(\sqrt{1-\overline{\mu}(\rho)^2} - \rho\sqrt{\rho^2-\overline{\mu}(\rho)^2} + \overline{\mu}(\rho)^2 \log\frac{1+\sqrt{1-\overline{\mu}(\rho)^2}}{\rho+\sqrt{\rho-\overline{\mu}(\rho)^2}} + \frac{2\rho\Big(1-K(\overline{\mu}(\rho),\rho)\Big)}{R} + (1-\lambda)\rho^2\Big) \tag{3.18}$$

Finally, we need to determine an optimal $\overline{\rho}$ for $\min\{E(\rho) : \rho \in [0,1]\}$. Then the radial function $\overline{u}(x) = \overline{\varphi}(\frac{|x|}{R}) = K\big(\overline{\mu}(\overline{\rho}), \frac{|x|}{R}\big)$ defined in (3.16) minimizes the relaxed problem (1.4) . This solution is continuous if $K(\overline{\rho},\overline{\rho}) = 1$ and otherwise exhibits a jump of amplitude $1 - K(\overline{\rho},\overline{\rho})$ before reaching the value 1 on the plateau. It turns out that the jump of $\overline{u}$ occurs when λ passes the value 1, thus confirming the behavior predicted in Remark 3.3. This is illustrated in Figure 2.

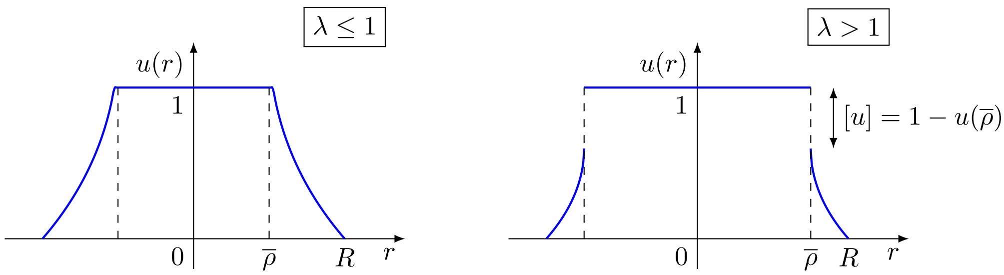

Figure 2: Illustration for an optimal u and optimality conditions.

The minimization of $E(\rho)$ is performed by using Matlab for different values of R and λ. As predicted by Theorem 3.2, $\beta_0(\lambda)$ coincides with $\beta(\lambda)$ outside the interval (λ_0,λ_1) where their common minimizers are trivial characteristic functions either $\overline{u} \equiv 0$ $(\lambda \le \lambda_0)$ or $\overline{u} \equiv 1$ $(\lambda \ge \lambda_1)$. When $\lambda_0 < \lambda_1$, the strict inequality $\beta(\lambda) < \beta_0(\lambda)$ occurs for any $\lambda \in (\lambda_0,\lambda_1)$ and the minimizer $\overline{u}$ for problem $\beta(\lambda)$ provides a true minimal surface with a possible jump under the plateau $\{\overline{u} = 1\}$. The numerically computed critical values λ_0, λ_1 are represented in term of R in Figure 3.

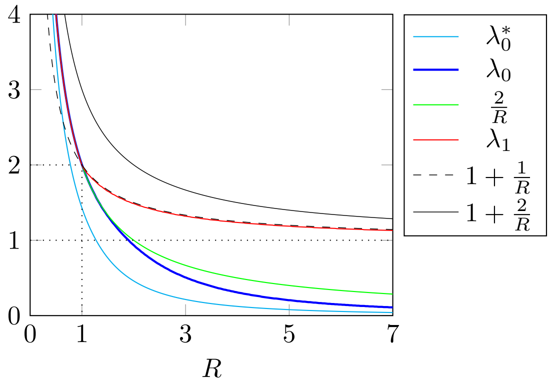

Figure 3: Critical values of λ in term of the radius R of a disk $D \subset \mathbb{R}^2$.

We observe that $\lambda_0 = \lambda_1$ for $R \leq 1$ whereas the strict inequality $\lambda_0 < \lambda_1$ occurs for all $R > 1$. Since D is a disk, we have $\theta_D = 1$ and $\lambda_D = h_D = 2/R$. Then the bounds provided in Proposition 3.6 become:

$$\lambda_0^* \leq \lambda_0 \leq \frac{2}{R} \leq \lambda_1 \leq 1 + \frac{2}{R} \quad \text{where} \quad \lambda_0^* = \begin{cases} 1 - \cos\left(\frac{2}{R}\right) & \text{if } R \geq \frac{4}{\pi}, \\ 1 + \frac{2}{R} - \frac{\pi}{2} & \text{if } R < \frac{4}{\pi}. \end{cases}$$

In Figure 3, these bounds are represented as functions of R. The curve representing λ_1 suggests the following exact value $\lambda_1 = \max\left\{\frac{2}{R}, 1 + \frac{1}{R}\right\}$.

Next we display in Figure 4 (case $R \leq 1$) and in Figure 5 (case $R > 1$), the dependence upon λ of $\beta_0(\lambda), \beta(\lambda)$, of the jump $[u]$ of the radial solution and of the ratio $\frac{\overline{\rho}}{R}$ being $\overline{\rho}$ the radius of the plateau.

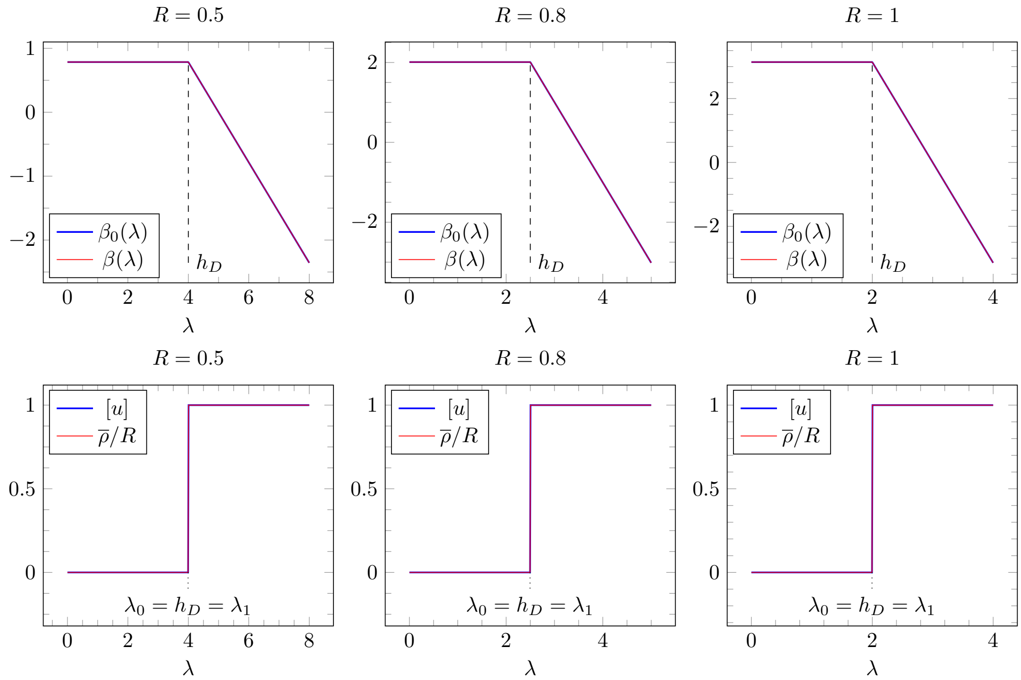

Figure 4: Dependence in λ for $R \in \{0.5, 0.8, 1\}$.

4. Cut-locus potential and construction of a two-dimensional calibration

In this section, we assume that D *is a bounded convex open subset of* $\mathbb{R}^2$. The following notations will be used:

- D^c denotes the complement of D in $\mathbb{R}^2$.
- $B(x, r)$ denotes the open ball in $\mathbb{R}^2$ centered at x with radius $r > 0$;
- For two points $x, y \in \mathbb{R}^2$, the segment joining them is denoted by $[x, y] := \{(1-t)x + ty \ : \ t \in [0,1]\}$;
- For every subset $A \subset \mathbb{R}^2$, we write A^c for the complement; if A is non empty, $d(x, A)$ denotes the Euclidean distance from x to A, namely $d(x, A) := \inf\{|x - y| \ : \ y \in A\}$;
- If A is a closed subset of $\mathbb{R}^2$, we denote $\Pi_A(x) := \{y \in A \ : \ |x - y| = d(x, A)\}$. In case A is convex, $\Pi_A(x)$ is a singleton.
- The distance between two non empty subsets $A, B \subset \mathbb{R}^2$ is given by $d(A, B) := \inf\{d(x, y) \ : \ (x, y) \in A \times B\}$.

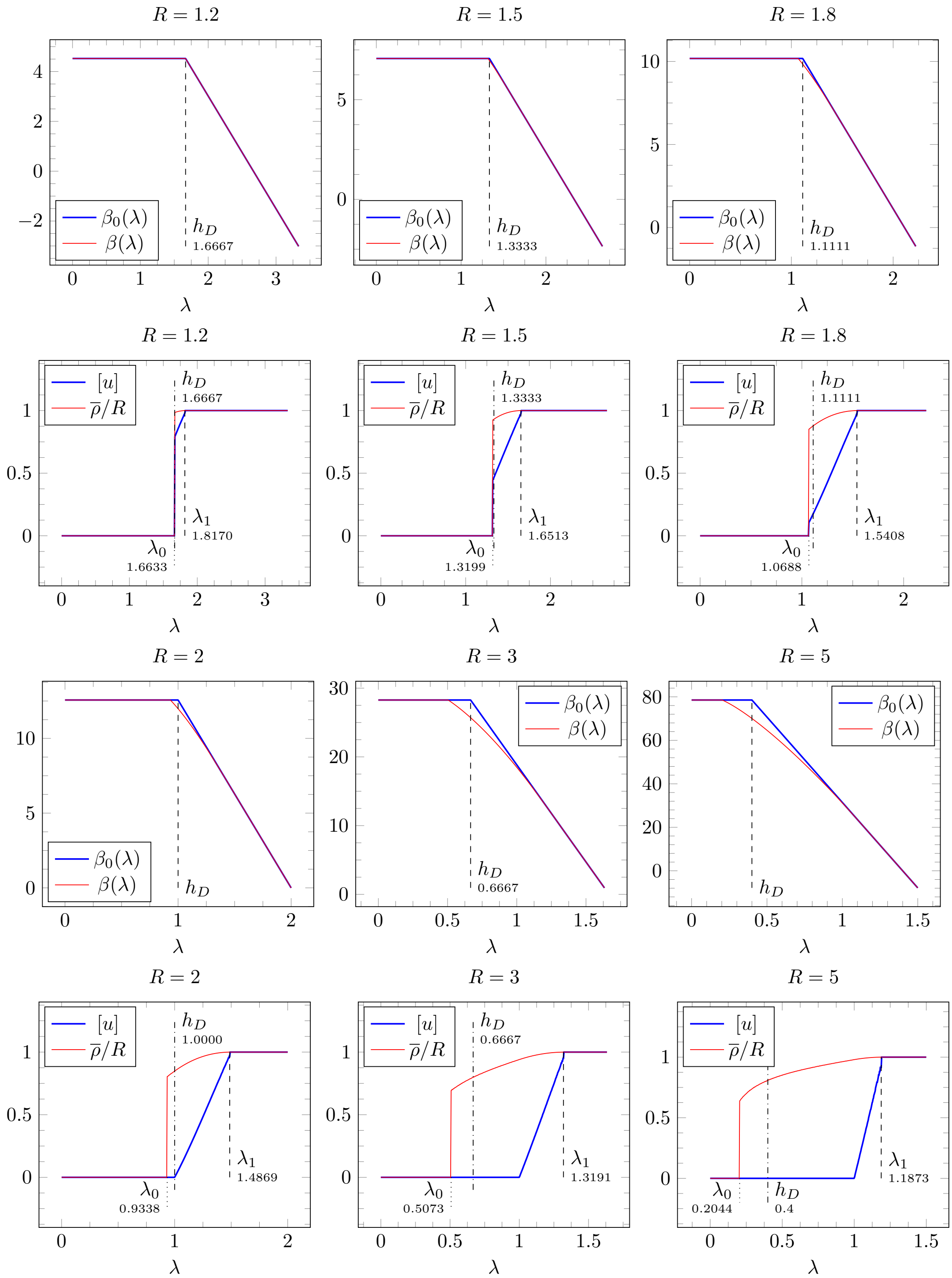

Figure 5: Dependence in λ for $R \in \{1.2, 1.5, 1.8, 2, 3, 5\}$.

Normal cone to D and regular points of ∂D. Since D is convex, the Gauss map $\mathbf{n} : \partial D^\delta \to S^1$ which assigns to x its normal $\nu_D(x)$ is well defined except in an at most countable number of points where we have left and right limits denoted by $\nu_D^-(x)$ and $\nu_D^+(x)$. This singular set will be denoted $\partial_s D$ while its complement $\partial_r D$ is the set of regular points.

The normal cone to D at $x \in \overline{D}$ is defined by

$$N_D(x) := \left\{ x^* \in \mathbb{R}^2 \ : \ \langle x^*, y - x \rangle \le 0, \ \forall y \in D \right\}.$$

It reduces to $\{0\}$ if x is inside D, while if $x \in \partial D$, it is generated by the unit vectors $\nu_D^-(x)$ and $\nu_D^+(x)$, that is $N_D(x) := \left\{ s\,\nu_D^+(x) + t\,\nu_D^-(x) \ : \ s,t \in \mathbb{R}_+ \right\}$. If $x \in \partial_r D$, then $\nu_D(x) = \nu_D^+(x) = \nu_D^-(x)$ and $N_D(x) = \mathbb{R}_+ \nu_D(x)$ is a single positive ray.

Some important convex subsets of D. Let $R_D := \max_{x\in D} d(x, D^c)$ be the inradius of D. For every δ such that $0 \le \delta < R_D$, we introduce the set

$$D^\delta := \{x \in D \ : \ d(x, D^c) > \delta\}.$$

One checks easily that D^δ is a non void open convex subset [2]. Acoordingly, we will denote by $\Pi_\delta(x)$ the orthogonal projection on $\overline{D\delta}$ of any $x \in \mathbb{R}^2$.

Next, for every $\lambda \ge R_D^{-1}$, we consider the δ-enlargement of D^δ when $\delta = \lambda^{-1}$. This convex open subset of D will play a crucial role in what follows. It is given by the union of all balls of radius λ^{-1} contained in D, namely

$$\Omega_\lambda = \bigcup_{B(x,\lambda^{-1})\subset D} B(x, \lambda^{-1}). \tag{4.1}$$

For $\delta = R_D$, we obtain the *maximal balls inscribed subset* of D defined by:

$$U_D := \Omega_{\frac{1}{R_D}} \ = \ \{x \in D \ : \ d(x, \Sigma_D) < R_D\},$$

being $\Sigma_D := \{x \in D \ : \ d(x, D^c) = R_D)\}$ (the *high ridge* of D).

In this Section, our goal is to show that, for every $\lambda \in [h_D, +\infty)$, the set Ω_λ defined in (4.1) is calibrable and optimal for $m(\lambda, D)$. To that aim, we will use a geometrical construction for defining a locally Lipschitz potential $\rho : \overline{D} \to (0, +\infty)$ which has its own interest. Before that let us recall the two-dimensional construction of the Cheeger set of a convex set D following [30, Theorem 3.32 i)] and the celebrated result of Bernd Kawohl and Thomas Lachand-Robert:

Theorem 4.1. ([23], Thm. 1) *There exists a unique value $\delta = \delta^*$ such that $|D^\delta| = \pi\delta^2$. Then, $h_D = 1/\delta^*$ and the Cheeger set of D is $C_D = \cup\{B(x,\delta^*) : B(x,\delta^*) \subset D\}$.*

The latter result says that D admits a unique Cheeger set given by $C_D = \Omega_{h_D}$. Moreover, the 1-calibrability of D (as defined in Subsection 2.3) is equivalent to the fact that $D = \Omega_{h_D}$ (hence, by Corollary 2.12, to the curvature upper bound $\kappa_\infty(\partial\Omega) \le h_D$).

4.1. Cut-locus potential

We introduce the function $\rho = \overline{D} \to \mathbb{R}_+$ defined by

$$\rho_D(x) := \sup\{\delta \ge 0 \ : \ d(x, D^\delta) \le \delta\}. \tag{4.2}$$

[2] For any $x_1, x_2 \in D^\delta$, we have $B(x_1,\delta), B(x_2,\delta) \subset D$. Thus, since D is convex

$$B(\frac{x_1+x_2}{2}, \delta) = \frac{1}{2}B(x_1,\delta) + \frac{1}{2}B(x_2,\delta) \subset \frac{1}{2}(D+D) = D \implies \frac{x_1+x_2}{2} \in D^\delta.$$

Since this potential ρ_D will not be used for another domain, we will simply write ρ_D as ρ. A good reason to call it *cut-locus potential* is that ρ is a continuous extension of the normal distance to the cut-locus of D defined on ∂D by

$$\tau(x) := \begin{cases} \sup\{t \geq 0 \ : \ x = \Pi_{\partial D}(x - t\nu_D(x))\} & \text{if } x \in \partial_r D \\ 0 & \text{if } x \in \partial_s D \end{cases} \tag{4.3}$$

The cut-locus of D is the closure of the singular set Λ_D of $d(\cdot, D^c)$ that is the set of point $x \in D$ where $d(\cdot, D^c)$ is not differentiable, that is

$$\Lambda_D := \{x \in D \ : \ \Pi_{D^c}(x) \text{ is not a singleton}\}.$$

We refer to [17, 16, 18, 15] for more details on this notion.

As it is proved in Lemma A.4, for every $x \notin \overline{U_D}$, we have the equality

$$\{\delta \geq 0 \ : \ d(x, D^\delta) \leq \delta\} = [0, \rho(x)].$$

This property is illustrated in Figure 6, where the function $\delta \to \alpha(x,\delta) := d(x, D^\delta) - \delta$ is negative on the interval $(0, \gamma(x)]$ and then is stricly increasing reaching a positive value at $x = R$. Therefore $\rho(x)$ is characterized as the unique zero of $\alpha(x, \cdot)$ on $(0, R)$ if $x \notin \overline{U_D}$. The lower-bound inequality $\rho(x) \geq \gamma(x)$ which is is strict if $x \in D$ involves the distance from ∂D to the cut-locus Λ_D along the normal to ∂D passing through x; more precisely:

$$\gamma : x \in \overline{D} \mapsto \zeta(x) + d(x, D^c) \ ,$$
$$\zeta(x) := \begin{cases} \min\{t \geq 0 : x + t\nabla d(x, D^c) \in \overline{\Lambda}_D\} & \forall x \notin \Lambda_D \\ 0 & \forall x \in \Lambda_D \end{cases} \tag{4.4}$$

Note, in particular, that $\zeta = 0$ and $\gamma(x) = d(x, D^c)$ for every $x \in \overline{\Lambda}_D$. On the other hand, we infer from (4.3), that

$$\gamma(x) = \zeta(x) = \tau(x) \quad \forall x \in \partial D. \tag{4.5}$$

Summarizing , our potential ρ can be characterized as follows:

$$\rho(x) = d(x, D^{\rho(x)}) \text{ if } x \in D \setminus \overline{U_D}, \ \ \rho(x) = \tau(x) \text{ if } x \in \partial D, \ \ \rho(x) = R_D \text{ if } x \in \overline{U_D} \tag{4.6}$$

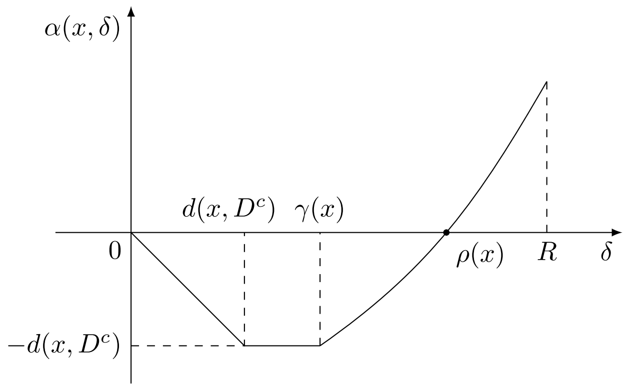

Figure 6: Function $\alpha(x, \cdot)$ for $x \notin \overline{U_D}$.

Lemma 4.2. *The potential ρ reaches its maximum R_D on the plateau $\overline{U_D}$ and satisfies the inequality $\rho(x) \geq d(x, D^c)$. Moreover, recalling the definition of Ω_λ in* (4.1)*, we have the equality:*

$$\Omega_\lambda = \left\{ x \in D \ : \ \rho(x) > \frac{1}{\lambda} \right\} \ , \ \textit{for every } \lambda \geq \frac{1}{R_D}. \tag{4.7}$$

Proof. For $x \in D$, we have the implication

$$0 < \delta \leq d(x, D^c) \implies d(x, D^\delta) \leq \delta \leq R_D.$$

It follows that $d(x, D^c) \leq \rho(x) \leq R_D$. Moreover, $\rho(x) = R_D$ for every $x \in U_D = \Omega_{\frac{1}{R_D}}$. We now prove (4.7). Let us recall that Ω_λ given by (4.1) coincides with an enlargement of D^δ, namely

$$\Omega_\lambda = \{x \in D \ : \ d(x, D^\delta) < \delta\} \quad \text{where } \delta = \lambda^{-1}.$$

Therefore, by the definition of ρ in (4.2), we have $\rho(x) > \lambda^{-1}$ for any $x \in \Omega_\lambda$. Conversely, assume that $\rho(x) > \lambda^{-1}$. Then, as we know from Lemma A.4, we have $d(x, D^\delta) < \delta$ for any $\delta \in [0, \rho(x)]$, hence in particular for $\delta = \lambda^{-1}$, whence $x \in \Omega_\lambda$. □

Remark 4.3. By Lemma 4.2, it follows that the level sets of ρ given by

$$C^\delta := \{x \in D \ : \ \rho(x) = \delta\} = \partial\Omega_{\frac{1}{\delta}} \cap D$$

will give a partition to D, namely $D = \bigcup_{0<\delta\leq R_D} C^\delta$.

Note that, for $\delta \in (0, R_D)$, the sets C^δ are arcs of radius δ, while for $\delta = R_D$, we obtain $C^{R_D} = U_D = \Omega_{\frac{1}{R_D}}$ which is a convex open subset of D. □

Proposition 4.4. *The cut-locus potential ρ is continuous on $\overline{D}$ and locally Lipschitz in D (its gradient blows-up as $d(x, D^c) \to 0$). Moreover its trace on ∂D satisfies*

$$\rho(x) \ = \ \tau(x) \ \leq \frac{1}{\kappa_{\partial D}(x)} \quad \textit{for all } x \in \partial D. \tag{4.8}$$

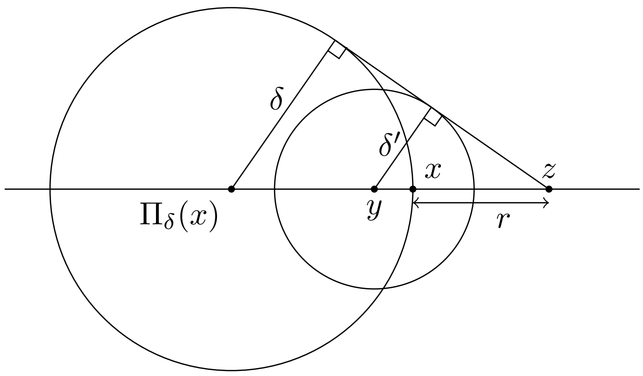

Figure 7: To prove that ρ is locally Lipschitzian.

Proof. Firstly, we prove that ρ is locally Lipschitz (hence continuous) in D. Given $\delta > 0$, for every $x \in \Omega_{\delta^{-1}} \cap D$, we have

$$d(x, D^c) > 0 \qquad \text{and} \qquad \delta = |x - \Pi_\delta(x)|.$$

We set $r := d(x, D^c)$. Let z be the point lying outside the disk $B(\Pi_\delta(x), \delta)$, on the line passing x, $\Pi_\delta(x)$ such that $|z - x| = r$. So, z is in $\overline{D}$. For every $\delta' < \delta$, we take y as the point inside D, on the same latter line such that the disks $B(y, \delta')$ and $B(\Pi_\delta(x), \delta)$ have the same tangents passing z. See Figure 7 for our settings.

Then Thales' Theorem is applied,

$$\frac{|z-y|}{|z - \Pi_\delta(x)|} = \frac{\delta'}{\delta},$$

then, we get

$$|z - y| = \frac{\delta'}{\delta}(\delta + r). \tag{4.9}$$

We notice that $B(\Pi_\delta(x), \delta) \subset \overline{D}$ and, $z \in \overline{B(x, r)} \subset \overline{D}$. Since D convex, we have

$$\text{conv}\left[\overline{B(\Pi_\delta(x), \delta)} \cup \{z\}\right] \subset \overline{D}.$$

This implies $B(y, \delta') \subset \overline{D}$. Thus, $y \in \overline{D^{\delta'}}$ and, by using the equality (4.9), it holds

$$d(x, D^{\delta'}) \leq |x - y| = |z - y| - r = \delta' - r(1 - \frac{\delta'}{\delta}).$$

We now can summarize that

$$\forall x \in \partial\Omega_{\delta^{-1}} \cap D,\ \forall \delta' < \delta, \quad d(x, D^{\delta'}) \leq \delta' - d(x, D^c)(1 - \frac{\delta'}{\delta}). \tag{4.10}$$

We observe that for each $x \in D \setminus \overline{\Omega_{\frac{1}{R}}}$, by using Remark 4.3, $x \in \partial\Omega_{\delta^{-1}} \cap D$ with $\delta = \rho(x)$. As a consequence of (4.10), for every $x, x' \in D \setminus \overline{\Omega_{\frac{1}{R}}}$ and, for every $\delta' < \delta = \rho(x)$, we have

$$\begin{aligned} d(x', D^{\delta'}) - \delta' &\leq |x - x'| + d(x, D^{\delta'}) - \delta' \\ &\leq |x - x'| - d(x, D^c)\left[1 - \frac{\delta'}{\delta}\right]. \end{aligned}$$

If $\rho(x') < \rho(x)$, we can choose $\delta' = \rho(x')$ and, then $d(x', D^{\delta'}) - \delta' = 0$. We infer that

$$d(x, D^c)\left(1 - \frac{\rho(x')}{\rho(x)}\right) \leq |x - x'|.$$

Therefore, for every $x, x' \in D \setminus \overline{\Omega_{\frac{1}{R}}}$ such that $\rho(x') < \rho(x)$, we have

$$|\rho(x') - \rho(x)| \leq \frac{\rho(x)}{d(x, D^c)}|x' - x|.$$

Finally, we conclude that, if $x \in D \setminus \overline{\Omega_{\frac{1}{R}}}$ and $d(x, D^c) < \varepsilon$, we have the following inequality holding for every $x' \in B(x, \varepsilon)$:

$$|\rho(x') - \rho(x)| \ \leq \ \frac{R}{d(B(x,\varepsilon), D^c)}\,|x' - x|. \tag{4.11}$$

Recalling that ρ is constant on $\overline{\Omega_{\frac{1}{R}}}$, we conclude that ρ is locally Lipschitz in D.

In a second step, we show (4.8). The fact that $\rho(x) = \tau(x)$ for every $x \in \partial D$ is a direct consequence of the equality $\gamma(x) = \tau(x)$ (see (4.5)) and of the fact that $\{\delta \;:\; \alpha(x,\delta) \leq 0\} = [0, \gamma(x)]$. Note that, if $x \in \partial D$, then $\alpha(x,\delta) = 0$ whenever $\delta \leq \tau(x)$ and we know from Lemma (A.4) that $\alpha(x,\cdot)$ is stricly increasing on $[\gamma(x), R]$.

Next we show that $\tau(x) \leq \frac{1}{\kappa_{\partial D}(x)}$ at any $x \in \partial D$. Without loss of generality, we can assume that $\tau(x) > 0$. Then, by the definition of $\tau(x)$, this means that $x \notin \Lambda_D$, hence $x \in \partial_r D$ and $d(x, D^\delta) = \delta$ whenever $\delta \leq \tau(x)$. In particular the equality $d(x, D^{\tau(x)}) = \tau(x)$ means that the ball $B(x - \tau(x)\nu_D(x)$ is contained in D and touches the boundary ∂D at x. It follows that $\kappa_{\partial D}(x) < +\infty$ and that $0 < \rho(x) = \tau(x) \leq \frac{1}{\kappa_{\partial D}(x)}$.

In a last step, we show that ρ is continuous in all $\overline{D}$. First we notice that ρ is lower semicontinuous since any strict upper level set $\{\rho > r\}$ coincides with the open subset $\Omega_{\frac{1}{r}}$ for every $r \in (0, R_D)$, while it coincides with D for $r = 0$ and the empty set for $r \geq R_D$.

Let $x_n \in \overline{D}$ such that $x_n \to x$. The case where $x \in \overline{U}_D$ is easy since we have

$$\rho(x_n) \leq \max \rho = R_D = \rho(x)$$

and by the lower semicontinuity of ρ,

$$\rho(x) \leq \liminf \rho(x_n) \leq \limsup \rho(x_n) \leq \rho(x).$$

Therefore we can assume that $x \notin \overline{U}_D$. Then $x_n \notin \overline{U}_D$ for large n and, in virtue to the characterization (4.6), we have

$$\rho(x_n) = d(x_n, D^{\rho(x_n)})$$

As $\rho(x_n) \leq R$, we can assume that, up to extracting a subsequence, $\rho(x_n) \to \delta^*$ for some δ^*. By the continuity property given in the assertion (i) of Lemma A.1, passing to the limit in the equality above leads to the equality $\delta^* = d(x, D^{\delta^*})$. Then, by Lemma A.4, we have either $\delta^* = \rho(x)$ or $\delta^* = 0$. If $\delta^* > 0$, we are done since, in this case, the whole sequence $\rho(x_n)$ converges to the unique cluster point $\rho(x)$. If $\delta^* = 0$, then $\rho(x_n) \to 0$ and, thanks the lower semicontinuity of ρ, we infer that $\rho(x) = 0$. Since $\rho(x) \geq d(x, D^c)$ while $\rho(x) = \tau(x)$ on ∂D, this is possible only if $x \in \partial_s D$ where $\tau(x) = 0$. However, even in this case, we have $\rho(x_n) \to \rho(x)$. □

Remark 4.5. Y. Li and L. Nirenberg proved in [25] that τ is Lipschitz if D has a $C^{2,1}$ boundary but it is untrue for a general convex domain (even $C^{2,\alpha}$ with $\alpha < 1$ is not enough). We conjecture that for a general convex domain $D \subset \mathbb{R}^2$, the cut-locus potential ρ belongs to $C^{0,\frac{1}{2}}(D)$, as it is the case for a square (see the example 4.6 below). □

Example 4.6. (An explicit formula for ρ in the case of a square) Let us consider the domain $D = (-1/2, 1/2)^2$ whose inradius is $R = \frac{1}{2}$. Hence $\rho = 1/2$ on the disk $U_D = \Omega_2 = \{|x| \leq \frac{1}{2}\}$. The cut-locus $\overline{\Lambda}$ consists of the two diagonals of D whereas the Cheeger constant is $h_D = 2 + \sqrt{\pi}$. For every $x \in D \setminus \Omega_2$, $\delta = \rho(x)$ is the unique δ such that x belongs to the arc of circle $C^\delta = \partial\Omega_\lambda \cap D$. Let us compute ρ in the north east quater $[0, 1/2]^2 \setminus \overline{\Omega}_2$. We set $x = (x_1, x_2)$, and $\delta = 1/2 - t$ for $t \in (0, 1/2)$.

Then, $x \in C^\delta$ (hence $\rho(x) = \frac{1}{2} - t$) if and only if t solves the equation

$$\begin{cases} (x_1 - t)^2 + (x_2 - t)^2 = (\frac{1}{2} - t)^2 \\ x_1^2 + x_2^2 \geq \frac{1}{4}. \end{cases}$$

These equations determine a unique $t \in [0, \frac{1}{2}]$ given by

$$t = x_1 + x_2 - \frac{1}{2} - \sqrt{2}\sqrt{\left(\frac{1}{2} - x_1\right)\left(\frac{1}{2} - x_2\right)}.$$

Accordingly we obtain the following expression for ρ for $x \in D$:

$$\rho(x) = \begin{cases} 1 - (|x_1| + |x_2|) + \sqrt{2}\sqrt{\left(\frac{1}{2} - |x_1|\right)\left(\frac{1}{2} - |x_2|\right)} & \text{if } x_1^2 + x_2^2 \geq 1/4 \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

This computation confirms that ρ vanishes only at the vertices of $\overline{D}$ where the curvature is infinite. It is positive and of class C^1 inside D. After some computations, we get the following equality:

$$|\nabla\rho(x)| = \frac{\rho(x)}{\sqrt{2}\sqrt{\left(\frac{1}{2} - |x_1|\right)\left(\frac{1}{2} - |x_2|\right)}} \quad \text{on } \{x_1^2 + x_2^2 \geq 1/4\},$$

which clearly shows that $|\nabla\rho|$ blows-up when approaching the sides of the square. The presence of the square term in the expression of $\rho(x)$ indicates that we cannot expect better than the $C^{1/2}$ regularity of ρ in $\overline{D}$. We present some calculations for the normalized gradient of ρ on $D \setminus \overline{U}_D$.

$$\nabla\rho(x) = \begin{pmatrix} -1 - \frac{\sqrt{2}}{2}a(x) \\ -1 - \frac{\sqrt{2}}{2}\frac{1}{a(x)} \end{pmatrix} \quad \text{where} \quad a(x) := \frac{\sqrt{\frac{1}{2} - x_2}}{\sqrt{\frac{1}{2} - x_1}},$$

The unit vector $q_\rho := -\frac{\nabla\rho}{|\nabla\rho|}$ is given by:

$$q_\rho(x) = \frac{1}{\sqrt{C_1(x)^2 + C_2(x)^2}}\left(\operatorname{sgn}(x_1)\, C_1(x), \operatorname{sgn}(x_2)\, C_2(x)\right),$$

$$C_1(x) := 1 + \frac{\sqrt{2}}{2}a(x) \quad \text{and} \quad C_2(x) := 1 + \frac{\sqrt{2}}{2}\frac{1}{a(x)}.$$

From the expressions above, we can check that, if x approaches the vertical sides (i.e.$|x_1| \to 1/2$), then $a(x) \to +\infty$ so that $q_\rho(x) \to (\operatorname{sgn}(x_1), 0)$; on the same way $q_\rho(x) \to (0, \operatorname{sgn}(x_2))$ as $|x_2| \to 1/2$. Therefore the normal trace of q_ρ on ∂D satisfies the equality $q_\rho \cdot \nu_D = 1$. This property will be confirmed in the general case in the forthcoming Theorem 4.7. A representation of ρ and of its normalized gradient on the first quarter $[0, 1/2]^2$ is displayed in Figure 8. In the left subfigure, the level lines of ρ are drawn in varied colors while, in the right one, the normalized gradient of ρ is represented (with a magnifying glass) in black streamlines starting from the circle $|x| = 1/2$.

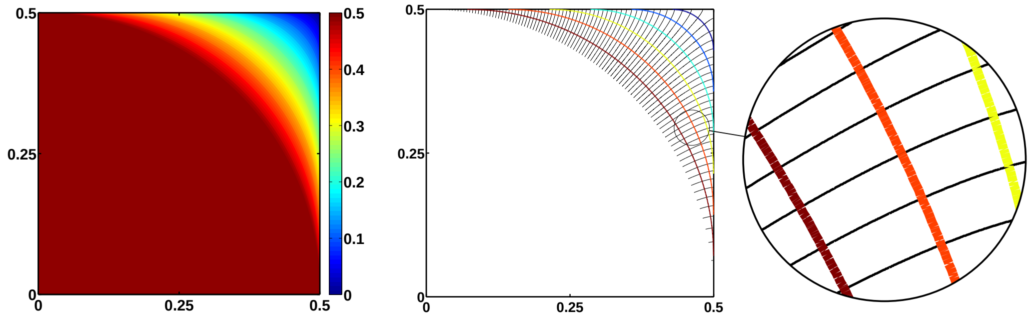

Figure 8: ρ and the stream lines of $q_\rho = -\frac{\nabla\rho}{|\nabla\rho|}$ when $D = (-\frac{1}{2}, \frac{1}{2})^2$.

Theorem 4.7. *Let $D \subset \mathbb{R}^2$ be a bounded convex domain of inner radius R and central set U_D. Then, on the open subset $D \setminus U_D = \{0 < \rho < R\}$, the locally Lipschitz potential ρ is such that $\nabla\rho \neq 0$ a.e.. Moreover the unit vector field $q_\rho := -\frac{\nabla\rho}{|\nabla\rho|}$ satisfies*

$$\operatorname{div} q_\rho = \frac{1}{\rho} \quad \text{in } D \setminus \overline{U}_D \quad \text{and} \quad q_\rho \cdot \nu_D = 1 \quad \mathcal{H}^1 \text{ a.e. on } \partial D \setminus \overline{U}_D. \tag{4.12}$$

Here above, the first equality is intended a.e. and in the sense of distributions in $D \setminus \overline{U}_D$.

Proof. Note that the open subset $\{0 < \rho < R\}$ is indeed $D \setminus \overline{\Omega_{\frac{1}{R}}}$ where ρ is not constant. On this subset, the vector field $\eta := -\nabla\rho/|\nabla\rho|$ can be rewritten as

$$\eta(x) = \frac{x - \Pi_{\rho(x)}(x)}{\rho(x)}.$$

Owing to the expression above, it is clear that the equality $\eta(x) = \nu_D(x)$ holds at any regular point of the boundary, that is for every $x \in \partial_r D \setminus \overline{U}_D$. Indeed, at those points, we have $\rho(x) = \tau(x)$ and $x - \Pi_{\rho(x)}(x) = \rho(x)\nu_D(x)$.

Next we are going to prove that $\Pi_{\rho(x)}(x)$ is locally Lipschitz in x and hence, so is η. We firstly claim that for every δ, δ' satisfying $0 < \delta' < \delta$, for each $y \in D$, there is a constant K_δ such that

$$|\Pi_\delta(y) - \Pi_{\delta'}(y)| \leq K_\delta |\delta - \delta'|. \tag{4.13}$$

It follows immediately that for every $x \in D \setminus \overline{\Omega_{\frac{1}{R}}}$, $x' \in B(x, \varepsilon) \subset D$, in view of (4.11),

$$\begin{aligned} |\Pi_{\rho(x)}(x) - \Pi_{\rho(x')}(x')| &\leq |\Pi_{\rho(x)}(x) - \Pi_{\rho(x')}(x)| + |x - x'| \\ &\leq K_{\rho(x)} |\rho(x) - \rho(x')| + |x - x'| \\ &\leq \Big(K_{\rho(x)} + \frac{R}{d(B(x,\varepsilon), D^c)} \Big) |x - x'|. \end{aligned}$$

Hence, we obtain that η is locally Lipschitz on D provided we show the validity of the claim (4.13). Given $0 < \delta' < \delta$, for every $y \in D$, by Lemma A.3 (ii), $\Pi_{\delta'}(y)$ is always in $M^{\delta}_{\delta'}(\Pi_\delta(y))$, see Figure 12 for illustration.

We obtain
$$|\Pi_\delta(y) - \Pi_{\delta'}(y)| \leq |w - x| = \frac{|\delta - \delta'|}{\cos\varphi(x)}, \tag{4.14}$$

where $x = \Pi_\delta(y)$ and w is the extreme point of $M^\delta_{\delta'}(x)$ in the complement of $D^{\delta'}$ (see Figure 12). By exploiting the assertion (iii) of Lemma A.5 , there exists $K_{\partial D^\delta} > 0$ such that

$$K_{\partial D^\delta} = \min\left\{k_{\partial D}(s) \;:\; s \in \partial D^\delta\right\} = \min\left\{\cos^2\varphi(s) \;:\; s \in \partial D^\delta\right\}.$$

Since $x \in \partial D^\delta$, we have
$$\frac{1}{\cos\varphi(x)} \leq \frac{1}{\sqrt{K_{\partial D^\delta}}}.$$

We then use the inequalities (4.14) and (4.1) to derive that the inequality (4.13) holds with $K_\delta = (K_{\partial D^\delta})^{-1/2}$.

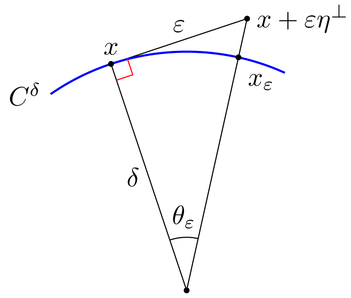

Figure 9: Divergence of η along C^δ.

The next step is to prove that $\operatorname{div}\eta = \dfrac{1}{\rho}$ in $D \setminus \overline{\Omega_{\frac{1}{R}}}$. The vector field η is indeed the unit normal to the level sets $C^\delta = \{x \in D \;:\; \rho(x) = \delta\}$ which are arcs of radius δ. Given x and x_ε on C^δ illustrated by Figure 9. Let us evaluate locally divergence of η along directions η and $\eta^\perp$. We recall that

$$(D\eta)h \cdot h := \lim_{\varepsilon\to 0}\left\langle \frac{\eta(x+\varepsilon h) - \eta(x)}{\varepsilon}, h\right\rangle$$

for some non null direction h. Since $|\eta| = 1$, we have

$$2(D\eta)h \cdot \eta = \left\langle \frac{\eta(x+\varepsilon h) - \eta(x)}{\varepsilon}, \eta(x+\varepsilon h) + \eta(x)\right\rangle = \frac{1}{\varepsilon}\left[|\eta|^2(x+\varepsilon h) - |\eta|^2(x)\right] = 0.$$

Thus, for $h = \eta$, we get
$$(D\eta)\eta \cdot \eta = 0. \tag{4.15}$$

As η locally Lipschitz, there is some constant M such that

$$\left|\frac{\eta(x+\varepsilon h) - \eta(x_\varepsilon)}{\varepsilon}\right| \leq \frac{M}{\varepsilon}|x + \varepsilon h - x_\varepsilon| = \frac{M\delta}{\varepsilon}\left(\frac{1}{\cos\theta_\varepsilon} - 1\right)$$
$$= \frac{M\delta}{\varepsilon}\left(\sqrt{1 + \frac{\varepsilon^2}{\delta^2}} - 1\right) \sim \frac{M\varepsilon}{2\delta}.$$

Hence, for $h = \eta^\perp$,

$$\left\langle \frac{\eta(x+\varepsilon h) - \eta(x)}{\varepsilon}, h\right\rangle = \left\langle \frac{\eta(x+\varepsilon h) - \eta(x_\varepsilon)}{\varepsilon}, h\right\rangle + \left\langle \frac{\eta(x_\varepsilon) - \eta(x)}{\varepsilon}, h\right\rangle$$
$$\sim \frac{M\varepsilon}{2\delta} + \frac{\sin\theta_\varepsilon}{\varepsilon} \sim \frac{M\varepsilon}{2\delta} + \frac{\tan\theta_\varepsilon}{\varepsilon} = \frac{M\varepsilon}{2\delta} + \frac{1}{\delta}.$$

We obtain
$$(D\eta)\eta^\perp \cdot \eta^\perp = \frac{1}{\delta}. \tag{4.16}$$

From the equations (4.15) and (4.16), we can derive that

$$\operatorname{div}\eta = (D\eta)^T : Id = (D\eta)^T : (\eta \otimes \eta + \eta^\perp \otimes \eta^\perp) = (D\eta)\eta\cdot\eta + (D\eta)\eta^\perp\cdot\eta^\perp = \frac{1}{\delta} = \frac{1}{\rho}.$$

As η is locally Lipschitz on the open set $\{0 < \rho < R\}$, the equality above holds not only a.e. on this set but also in the distributional sense. □

Remark 4.8. In view of Theorem 4.7, ρ solves the boundary value problem

$$\operatorname{div}\frac{\nabla\rho}{|\nabla\rho|} + \frac{1}{\rho} = 0 \text{ in } D\setminus\overline{U}_D, \quad \frac{\nabla\rho}{|\nabla\rho|}\cdot\nu_D = -1 \text{ in } \partial D\setminus\overline{U}_D, \quad \rho = R_D \text{ in } \partial U_D\ .$$

In fact it can be shown that ρ is directly related to the following strictly convex variational problem

$$\min\left\{\int_D |\nabla u| + \int_{\partial D}|u|\,d\mathcal{H}^1 - \int_D \ln(u)\,dx\ :\ u\in W^{1,1}(D)\right\},$$

whose unique solution $\overline{u}$ admits a maximal plateau $\overline{u} = \frac{1}{h_D}$ and coincides with ρ on the complement of the Cheeger set of D. Studying this new minimization problem in dimension $N \geq 2$ could help us understand the structure of solutions to $m(\lambda, D)$ in the case of a general domain $D \subset \mathbb{R}^N$, especially when the convexity assumption of the domain D is removed.

4.2. The set Ω_λ is calibrable and solves $m(\lambda, D)$

By exploiting the PDE (4.12) satisfied by the cut-locus potential of D, we are now in position to derive, as corollaries of Theorem 4.7, the calibrability and the optimality of the set Ω_λ for $m(\lambda, D)$.

Corollary 4.9. *Let $\lambda \geq h_D$. Then Ω_λ defined by (4.1) solves $m(\lambda, \Omega_\lambda)$. Therefore $\lambda \geq \lambda_{\Omega_\lambda}$ and Ω_λ is θ-calibrable with constant $\theta \leq \lambda\,\lambda_{\Omega_\lambda}^{-1}$.*

Proof. By (2.20), it is enough to show that Ω_λ solves $m(\lambda, \Omega_\lambda)$. Let us apply the optimality conditions of the assertion (ii) of Theorem 2.4 in the case where $D = \Omega = \Omega_\lambda$. Then we are done if we can find $\overline{q} \in L^\infty(\Omega_\lambda; \mathbb{R}^2)$ such that

$$|\overline{q}| \leq 1 \ \text{ a.e. in } \Omega_\lambda\ ,\ 0 \leq \operatorname{div}\overline{q} \leq \lambda \ \text{ a.e. in } \Omega_\lambda\ ,\ \overline{q}\cdot\nu_{\Omega_\lambda} = 1\ \ \mathcal{H}^1\text{-a.e. on } \partial\Omega_\lambda. \tag{4.17}$$

Such a vector field can be constructed on Ω_λ by starting from a calibrating field of the Cheeger set of D. This semi-explicit construction is provided with the existence of a calibrating field for the Cheeger set of D, that means $\overline{q} = q_{h_D}$ in Ω_{h_D} where $q_{h_D} \in L^\infty(\Omega_{h_D}; \mathbb{R}^2)$ satisfies

$$|q_{h_D}| \leq 1, \quad \operatorname{div} q_{h_D} = h_D \quad \text{a.e. in } \Omega_{h_D}, \qquad q_{h_D}\cdot\nu_{\Omega_{h_D}} = 1 \quad \mathcal{H}^1\text{-a.e. on } \partial\Omega_{h_D}.$$

We need now to construct $\overline{q}$ on $\Omega_\lambda \setminus \overline{\Omega_{h_D}}$. The cut-locus potential ρ comes into play here.

Let us define $\overline{q}$ on Ω_λ as follows:

$$\overline{q}(x) := \begin{cases} q_{h_D}(x) & \text{if } x \in \Omega_{h_D} \\ -\dfrac{\nabla\rho(x)}{|\nabla\rho|(x)} & \text{if } x \in \Omega_\lambda \setminus \overline{\Omega_{h_D}} \end{cases} \tag{4.18}$$

The condition $|\overline{q}| \leq 1$ is clearly satisfied. On the other hand, recalling that we have $\Omega_{h_D} = \{\rho > h_D^{-1}\}$, ρ is constant on the interface $\partial\Omega_{h_D} \cap D$ while $h_D \leq \rho^{-1} \leq \lambda$ in $\Omega_\lambda \setminus \Omega_{h_D}$. Therefore the normal trace of $\overline{q}$ has no jump and the ditributional divergence $\operatorname{div}\overline{q}$ belongs to $L^\infty(\Omega_\lambda; [h_D, \lambda])$ since

$$\operatorname{div}\overline{q} = \begin{cases} h_D & \text{if } x \in \Omega_{h_D} \\ \dfrac{1}{\rho} & \text{if } x \in \Omega_\lambda \setminus \Omega_{h_D} \end{cases}.$$

Eventually, the condition $\overline{q} \cdot \nu_{\Omega_\lambda} = 1$ is fullfiled on $\partial\Omega_\lambda \cap D$ where $\rho = \frac{1}{\lambda}$. The same equality holds on the shared boundary piece $\partial\Omega_\lambda \cap \partial D$ since, by (4.12), it holds $\eta = \nu_D$ on $\partial D \setminus \overline{U}_D$. Finally, the three conditions in (4.17) are satisfied. □

Corollary 4.10. *Let $\lambda > h_D$. Then, the upper level set $\Omega_\lambda = \{\rho > \lambda^{-1}\}$ is the unique solution to the problem $m(\lambda, D)$.*

We remark that, according to Theorem 2.4, the optimality of Ω_λ for the problem $m(\lambda, D)$ can be achieved by extending the construction of the calibrating field $\overline{q}$ defined in (4.18) from $x \in \Omega_\lambda$ to $x \in D \setminus \Omega_\lambda$. This will be done in the next subsection. Here we propose a direct proof based on the PDE satisfied by the cut-locus potential of D, as a consequence of Theorem 4.7. Note that a similar PDE proof appears in [2, Proposition 4] exploiting the solutions of a family of auxiliary problems where a Neumann boundary condition is implicitely imposed on ∂D.

Proof. Let consider $F \subset D$ a Borel subset with finite perimeter. We need need to show that $P(F) - \lambda|F| \geq P(\Omega_\lambda) - \lambda|\Omega_\lambda|$. Let $\overline{q}$ be the vector field satisfying (4.21) that we constructed in the proof of Corollary 4.9. Then, setting $\delta = \frac{1}{\lambda}$, we have the equalities:

$$P(\Omega_\lambda) - \lambda\,|\Omega_\lambda| = \int_{\Omega_\lambda} (\operatorname{div}\overline{q} - \lambda)\,dx = \int_{\Omega_\lambda} \left(\frac{1}{\rho} - \frac{1}{\delta}\right)\,dx.$$

On the other hand, since $|\overline{q}| \leq 1$, we have:

$$\begin{aligned} P(F) - \lambda\,|F| &\geq \int_F (\operatorname{div}\overline{q} - \lambda)\,dx = \int_F \left(\frac{1}{\rho} - \frac{1}{\delta}\right)\,dx \\ &= \int_{\Omega_\lambda} \left(\frac{1}{\rho} - \frac{1}{\delta}\right)\,dx + \int_{F\setminus\Omega_\lambda} \left(\frac{1}{\rho} - \frac{1}{\delta}\right)\,dx - \int_{\Omega_\lambda\setminus F} \left(\frac{1}{\rho} - \frac{1}{\delta}\right)\,dx \end{aligned}$$

Since $\frac{1}{\rho} \geq \frac{1}{\delta} \geq 0$ on $F \setminus \Omega_\lambda$ and $\frac{1}{\rho} \geq \frac{1}{\delta} \leq 0$ on $\Omega_\lambda \setminus F$, we deduce that

$$P(F) - \lambda\,|F| \geq \int_{\Omega_\lambda} \left(\frac{1}{\rho} - \frac{1}{\delta}\right)\,dx = P(\Omega_\lambda) - \lambda|\Omega_\lambda|,$$

whence the optimality of Ω_λ for $m(\lambda, D)$ for every $\lambda \geq h_D$.

Let us prove now the uniqueness of the solution of $m(\lambda, D)$ for $\lambda > h_D$. From the previous arguments, we know now that for all μ, ν such that $h_D < \mu < \lambda < \nu$, the sets Ω_μ and Ω_ν are minimal for $m(\mu, D)$ and $m(\nu, D)$ respectively. Then, by a well known comparison argument (see for instance [2, Lemma 4 (i)]), we have the inclusions $\Omega_\mu \subset \Omega \subset \Omega_\nu$ for any Ω solving $m(\lambda, D)$. The equality $\Omega = \Omega_\lambda$ follows by sending $\mu \nearrow \lambda$ and $\nu \searrow \lambda$. $\square$

4.3. Extending the calibration field to all D

We will now extend the vector field $\overline{q}$ prealably defined in subsection 4.2 (see (4.18)) from Ω_λ to D. To that aim we are going to design a unit vector field q_λ in $D \setminus \overline{\Omega_\lambda}$ such that

$$|q_\lambda| \leq 1 \text{ in } D \setminus \overline{\Omega_\lambda}, \quad \operatorname{div} q_\lambda = \lambda \text{ in } D \setminus \overline{\Omega_\lambda}, \quad q_\lambda \cdot \nu_{\Omega_\lambda} = 1 \text{ on } \partial\Omega_\lambda \cap D. \tag{4.19}$$

The subset $D \setminus \overline{\Omega_\lambda}$ involved in the forthcoming construction is represented in Figure 10 below in the case of a square

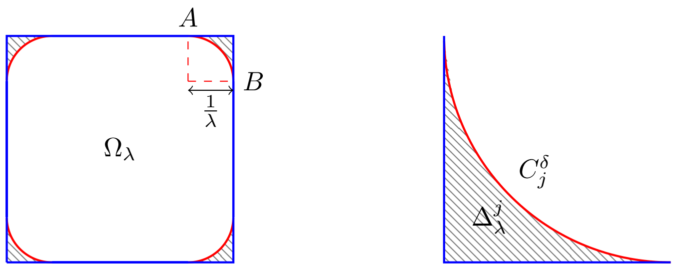

Figure 10: $D \setminus \overline{\Omega_\lambda}$ represented in dash

Then, by assigning to $\overline{q}$ the value $\overline{q} = q_\lambda$ on $D \setminus \Omega\lambda$, one check easily that the resulting $\overline{q} \in L^\infty(D; \mathbb{R}^2)$ satisfies the optimality conditions given in the assertion (ii) of Theorem 2.4 when $\Omega = \Omega_\lambda$, namely:

$$|\overline{q}| \leq 1 \quad \text{a.e. in } D, \qquad 0 \leq \operatorname{div} \overline{q} \leq \lambda \quad \text{a.e. in } D, \tag{4.20}$$

$$\overline{q} \cdot \nu_{\Omega_\lambda} = 1 \quad \mathcal{H}^1\text{-a.e. on } \partial\Omega_\lambda, \qquad \operatorname{div} \overline{q} = \lambda \quad \text{a.e. in } D \setminus \Omega_\lambda. \tag{4.21}$$

Accordingly, we will obtain another proof for the optimality of Ω_λ. The global construction of $\overline{q}$ concerns the three subsets represented in Figure 11 (see Figure 13 in the case of an ellipsoid). For a square, infinite curvature occurs at the corners, so that $\kappa_\infty(\partial D) = +\infty$ and the inclusion $\Omega_\lambda \subset D$ is strict for every $\lambda \geq h_D$.

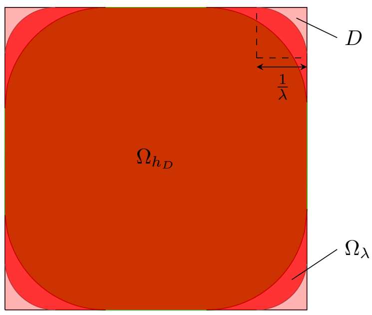

Figure 11: The inclusion $\Omega_{h_D} \subset \Omega_\lambda \subset D$.

We end this subsection and conclude the paper by giving:

- the detailed construction of the field q_λ satisfying (4.19) (see Lemma 4.11).
- the closed form of $\overline{q}$ in case of a square and of an ellipsoid (see Example 4.12) .

Lemma 4.11. *There exists a vector field* q_λ *in* $D \setminus \overline{\Omega_\lambda}$ *satisfying* (4.19)

Proof. From now on, we set $\delta = 1/\lambda$ and adapt the notation used at the beginning of Section 4 to the convex subset D^δ. Recall that we denote by $\Pi_\delta(x)$ the unique projection of x on $\overline{D^\delta}$. The singular part of its boundary $\partial_s D^\delta$ has at most countably many points, and we set

$$\partial_s D^\delta = \{x_j : j \in J\} \qquad \text{(being } J \text{ either empty, or finite, or countable).}$$

For each point $x \in \partial_s D^\delta$, the normal cone of D^δ at the point, denoted by $N_{D^\delta}(x)$, is generated by the two limit vectors $\nu^-_{D^\delta}(x)$ and $\nu^+_{D^\delta}(x)$. $\varphi^\delta_-(x)$ and $\varphi^\delta_+(x)$ are the corresponding angles of the two vectors in S^1. Accordingly we define the sets $N^\delta(x)$, $C^\delta(x)$, and $M^\delta_{\delta'}(x)$ as follows:

$$N^\delta(x) := x + \{p \in N_{D^\delta}(x) \ : \ |p| \le \delta\}\,, \text{ for } x \in \overline{D^\delta},$$
$$C^\delta(x) := x + \{p \in N_{D^\delta}(x) \ : \ |p| = \delta\}\,, \text{ for } x \in \partial D^\delta,$$
$$M^\delta_{\delta'}(x) := x + \left\{p \in N_{D^\delta}(x) \ : \ \left\langle p, \nu^\pm_{D^\delta}(x)\right\rangle \le \delta - \delta'\right\}, \text{ for } x \in \overline{D^\delta},\ 0 \le \delta' < \delta < R.$$

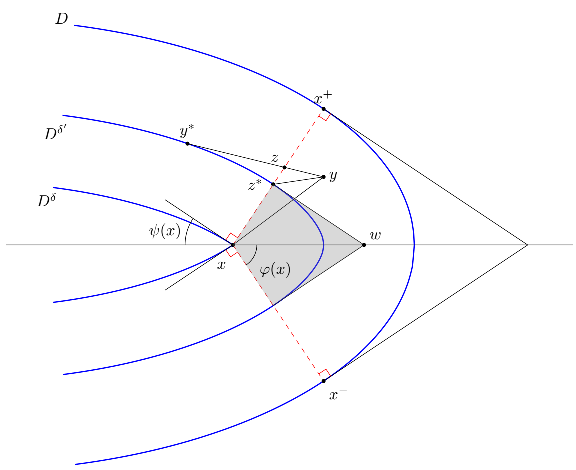

Figure 12: The kite in gray color represents the set $M^\delta_{\delta'}$(x).

In turn, Ω_λ can be characterized, in terms of $N^\delta(x)$ and $C^\delta(x)$, as follows:

$$\overline{\Omega_\lambda} = \bigcup_{x \in D^\delta} N^\delta(x), \qquad \partial\Omega_\lambda = \bigcup_{x \in \partial D^\delta} C^\delta(x). \tag{4.22}$$

We remark that $\{N_{D^\delta}(x) : x \in D^\delta\}$ is a family of disjoint sets because every point $y \in N_{D^\delta}(x)$ satisfies $\Pi_\delta(y) = x$. Thus $\{N^\delta(x) : x \in D^\delta\}$ determines a partition of $\overline{\Omega_\lambda}$. On the other hand, when x is a regular point of ∂D^δ, i.e. $x \in \partial_r D^\delta$, then the normal cone $N_{D^\delta}(x)$ reduces to only one direction and $C^\delta(x) = \{x^*\}$ where x^* is the unique projection of x on D^c. Thus $x \notin \Lambda_D$ and x^* belongs to $\partial D \cap \partial\Omega_\lambda$. If $x \in \partial_s D^\delta$, the arc $C^\delta(x)$ given by

$$C^\delta(x) = x + \left(\text{cone}\{\nu^-_{D^\delta}(x), \nu^+_{D^\delta}(x)\} \cap \partial B(0,\delta)\right)$$

is determined by the angular interval $(\varphi^\delta_-(x), \varphi^\delta_+(x))$, which by the convexity of D^δ must be less than π. Accordingly this arc is a connected piece of the free boundary $\partial\Omega_\lambda \cap D$. Conversely, every $y \in \partial\Omega_\lambda$ can be decomposed as $y = \Pi_\delta(y) + (y - \Pi_\delta(y))$ with $|y - \Pi_\delta(y)| = \delta$, hence $y \in C^\delta(\Pi_\delta(y))$. This confirms the second equality in (4.22) where $\{C^\delta(x) \,:\, x \in \partial D^\delta\}$ is a partition of $\partial\Omega_\lambda$.

To shorten the notation, for each $x_j \in \partial_s D^\delta$ where $j \in J$, we set

$$N^\delta_j := N^\delta(x_j), \qquad C^\delta_j := C^\delta(x_j), \tag{4.23}$$

$$\nu_j := \frac{\nu^+_{D^\delta}(x_j) + \nu^-_{D^\delta}(x_j)}{2}, \qquad \varphi_j := \frac{\varphi^\delta_+(x_j) - \varphi^\delta_-(x_j)}{2}. \tag{4.24}$$

We point out that $\partial\Omega_\lambda \cap D = \bigcup_{j\in J} C^\delta_j$, where each C^δ_j is an arc of radius δ determined by a triple $(x_j, \varphi_j, \nu_j) \in \partial_s D^\delta \times (0, \pi/2) \times S^1$, which represent respectively the center, the angle, and an oriented unit vector (see the right hand side of Figure 13).

Now, let us introduce the regions where we want to construct the vector field q_λ,

$$\text{for } x \in D^\delta,\ M^\delta_0(x) = x + \left\{p \in N_{D^\delta}(x) \,:\, \langle p, \nu^+_{D^\delta}(x)\rangle \le \delta,\ \langle p, \nu^-_{D^\delta}(x)\rangle \le \delta\right\}, \tag{4.25}$$

$$\Sigma_\lambda := \bigcup_{x \in D^\delta} M^\delta_0(x), \qquad \Delta_\lambda := \Sigma_\lambda \setminus \overline{\Omega_\lambda}. \tag{4.26}$$

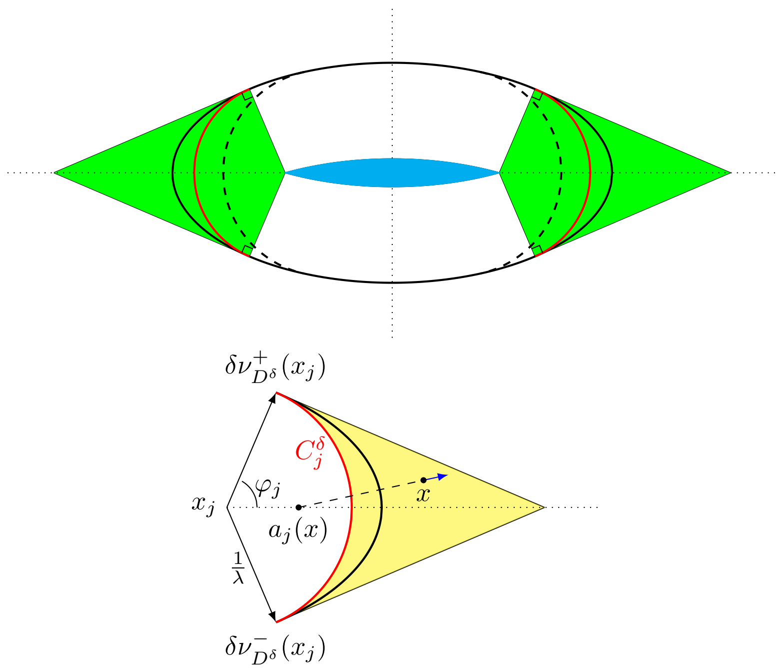

Figure 13: Σ_λ in green and $\Delta_\lambda = \Sigma_\lambda \setminus \overline{\Omega_\lambda}$ in yellow.

Figure 13 gives the flavour of the construction when D is an ellipse domain. The thick black curve stands for the boundary of D while the dashed curve represents the boundary of the Cheeger set Ω_{h_D} of D. The cyan region is the set D^δ whose δ-enlargement gives Ω_λ. The boundary of Ω_λ appears in red color. The sets $M_0^\delta(x_j)$ are in green background. The region in light yellow background is a component of Δ_λ. C_j^δ are red arcs of radius δ, centered at x_j, determined by angle φ_j and the oriented unit vector ν_j. $\{C_j^\delta\}$ are the boundaries of Ω_λ inside D, i.e. $\partial\Omega_\lambda \cap D$.

The set $\overline{D}$ is contained in Σ_λ. In fact, for every $y \in \overline{D}$, let $x = \Pi_\delta(y)$ and $p = y - x$. Since $|x - (x + \delta\nu^\pm_{D^\delta}(x))| = \delta = d(x, D^c)$ and $x + \delta\nu^\pm_{D^\delta}(x) \in \partial D$, we have that $x + \delta\nu^\pm_{D^\delta}(x)$ are projections of x on D^c. As $\overline{D}$ is convex, these projections implies that for all $z \in \overline{D}$,

$$\langle x - (x + \delta\nu^-_{D^\delta}(x)), z - (x + \delta\nu^-_{D^\delta}(x))\rangle \geq 0,$$
$$\langle x - (x + \delta\nu^+_{D^\delta}(x)), z - (x + \delta\nu^+_{D^\delta}(x))\rangle \geq 0.$$

Therefore, for $z = y$, we obtain $\langle p, \nu^\pm_{D^\delta}(x)\rangle \leq \delta$. Hence, by definition we get (4.25), $y = x + p \in M_0^\delta(x)$. In particular, when $x \in \partial_r D^\delta$, the left and right limits are the same, i.e. $\nu^-_{D^\delta}(x) = \nu^+_{D^\delta}(x) =: \nu_{D^\delta}(x)$, while $N^\delta(x)$ coincides with $M_0^\delta(x)$ and they are folded up to be a segment. At that moment, p and $\nu_{D^\delta}(x)$ are co-linear, then, $\langle p, \nu_{D^\delta}(x)\rangle = |p| \leq \delta$. This is to say that $y \in N^\delta(x) = M_0^\delta(x)$. Of course, when $y \in D^\delta$ or $y = \Pi_\delta(y)$, the associated cones degenerate and shrink to a point. We get $y = \Pi_\delta(y) = N^\delta(y) = M_0^\delta(y)$.

We notice that since the angle of $N_{D^\delta}(x)$ is always less than π, $M_0^\delta(x)$ is bounded for every $x \in D^\delta$. Σ_λ is then bounded. It is clear that $\{M_0^\delta(x) \ : \ x \in D^\delta\}$ is a partition of Σ_λ and hence, Δ_λ admits a decomposition,

$$\Delta_\lambda = \bigcup_{x\in D^\delta} M_0^\delta(x) \setminus \overline{\Omega}_\lambda = \bigcup_{x\in\partial_s D^\delta} M_0^\delta(x) \setminus \overline{\Omega_\lambda}. \tag{4.27}$$

For short, we set

$$\Delta_\lambda^j = M_0^\delta(x_j) \setminus \overline{\Omega}_\lambda \text{ for some } x_j \in \partial_s D^\delta, \qquad \text{and} \quad \Delta_\lambda = \bigcup_{j\in J} \Delta_\lambda^j. \tag{4.28}$$

Δ_λ has at most countable many disjoint components. Figures 10 and 13 are illustrating examples. In Figure 13, Σ_λ strictly contains D whereas $\Sigma_\lambda = D$ in case of Figure 10.

We now explicitly construct the vector field q_λ in Δ_λ satisfying conditions (4.19). In each component Δ_λ^j, we set $x = (s,t)$ and

$$\Delta_\lambda^j \ni (s,t) \mapsto q_1^j(s,t) := (s - a_j(s,t)\nu_j^s, t - a_j(s,t)\nu_j^t), \tag{4.29}$$

where $\nu_j = (\nu_j^s, \nu_j^t)$ is the oriented unit vector defined C_j^δ and $a_j(s,t) \geq 0$ such that $\operatorname{div} q_1^j = 1$. In fact, q_1^j is the unit normal of the ball of radius 1 centered at point $a_j\nu_j$,

$$[s - a_j(s,t)\nu_j^s]^2 + [t - a_j(s,t)\nu_j^t]^2 = 1. \tag{4.30}$$

We observe also that

$$\operatorname{div} q_1^j(s,t) = 1 \Longleftrightarrow \partial_s a_j(s,t)\nu_j^s + \partial_t a_j(s,t)\nu_j^t = 1. \tag{4.31}$$

From equation (4.30), we can find out explicitly a_j in function of (s,t), and in such a way, (4.31) is fulfilled,

$$a_j(s,t) = s\nu_j^s + t\nu_j^t - \sqrt{1-(s\nu_j^t - t\nu_j^s)^2}. \tag{4.32}$$

Therefore, $q_\lambda(s,t) = q_\lambda^j(s,t) := q_1^j(\lambda s, \lambda t)$ in Δ_λ^j as we expected. This completes the proof. □

Example 4.12. We precise here a calibrating field $\overline{q}$ when D is a square or an ellipsoid.

(a) In case $D = [-1/2, 1/2]^2$, Σ_λ coincides with D, see Figure 10. The boundary of D^δ has 4 singular points and $\partial\Omega_\lambda \cap D = \cup_{j=1}^4 C_j^\delta$. The oriented vectors of C_j^δ are $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$. Take $\nu_1 = (1/\sqrt{2}, 1/\sqrt{2})$ for example, it is easy to explicit q_λ in Δ_λ^1. Thanks to (4.32) and (4.29), we have, for $(s,t) \in \Delta_\lambda^1$,

$$a_1(s,t) = \frac{s+t}{\sqrt{2}} - \sqrt{1 - \frac{(s-t)^2}{2}}, \qquad q_1^1(s,t) = \left(s - \frac{a_1(s,t)}{\sqrt{2}}, t - \frac{a_1(s,t)}{\sqrt{2}}\right).$$

Then, the desired construction of q_λ in Δ_λ^1 is given by

$$q_\lambda(s,t) = q_\lambda^1(s,t) = q_1^1(\lambda s, \lambda t).$$

By using symmetries, the expression of q_λ can be deduced in the other components of Δ_λ. On the other hand, the field $\overline{q}$ is given in $\Omega_\lambda \setminus \Omega_{h_D}$ by $\overline{q} = -\frac{\nabla\rho}{|\nabla\rho|}$ which is described in Example 4.6.

(b) In case D is given by an ellipse of standard form, see Figure 13, D is strictly contained in Σ_λ. Δ_λ now has two components and the boundary of Ω_λ inside D is the union of arcs C_1^δ and C_2^δ whose oriented vectors are $(\pm 1, 0)$. Take $\nu_1 = (1,0)$ for example to construct q_λ in Δ_λ^1, we get, for $(s,t) \in \Delta_\lambda^1$,

$$a_1(s,t) = s - \sqrt{1-t^2}, \qquad q_1^1(s,t) = \left(\sqrt{1-t^2}, t\right).$$

Therefore, we obtain $q_\lambda(s,t)$ by scaling $q_1^1(s,t)$, i.e.

$$q_\lambda(s,t) = q_\lambda^1(s,t) = q_1^1(\lambda s, \lambda t).$$ □

In summary, the vector field $\overline{q}$ can be built with q_{h_D} on Ω_{h_D}, with q_ρ on $\Omega_\lambda \setminus \overline{\Omega_{h_D}}$ by means of cut-locus potential ρ, then glued with q_λ on $D \setminus \overline{\Omega_\lambda}$ so that we can obtain a calibrating field $\overline{q}$ for Ω_λ. We remark that the construction of the calibrating field $\overline{q}$ can be done in the domain Σ_λ (see (4.26)) which, in general is larger than D.

A. Some techical lemmas in $\mathbb{R}^2$

The following preparatory lemmas will be used in Section 4. We recall the definitions of functions γ and ζ in (4.4) and also of the cut-locus $\overline{\Lambda}$ in Section 4.1.

Lemma A.1. *Let D be a bounded convex set in $\mathbb{R}^N$. Then*

(i) *the function $(x,\delta) \to d(x, D^\delta)$ is continuous on $\overline{D} \times [0,R)$.*

(ii) *Given $x \in D$ such that $d(x, D^\delta) \le \delta$ for some $\delta > 0$. Then, for every δ' such that $0 < \delta' < \delta$, we have $d(x, D^{\delta'}) < \delta'$.*

Proof. Let us prove (i). Since the functions $d(\cdot, D^\delta)$ are 1-Lipschitz, it is enough to show that, for any fixed $x \in \overline{D}$, the function $t \to d(x, D^t)$ is continuous on $[0, R)$. Let $t_n, t \in [0, R_D)$ such that $t_n \to t$. It exist a unique $y_n \in \overline{D^{t_n}}$ such that $|x - y_n| = d(x, D^{t_n}$. Up to a subsequence, we can assume that $y_n \to y_*$ for some $y_* \in \overline{D}$. Then $d(y_n, D^c) \geq t_n$ implies that $d(y_*, D^c) \geq t$ while $|x - y_n| \to |x - y_*|$. It follows that $y_* \in \overline{D^t}$ [3], hence $|x - y_*| = \lim_n d(x, D^{t_n}) \geq d(x, D^t)$. In the opposite direction, let $\varepsilon > 0$ and $x_\varepsilon \in D^t$ such that $|x - x_\varepsilon| \leq d(x, D^t) + \varepsilon$. Since $d(x_\varepsilon, D^c) > t$, we have $x_\varepsilon \in D^t_n$ for large n so that $\limsup_n d(x, D^{t_n}) \leq |x - x_\varepsilon| \leq d(x, D^t) + \varepsilon$. By sending $\varepsilon \to 0$, we conclude that $\limsup_n d(x, D^{t_n}) \leq d(x, D^t)$. The desired continuity property is proved.

Let us prove now the assertion (ii). Without any loss of generality, we can assume that $x \notin D^{\delta'}$, hence $x \notin \overline{D^\delta}$, since $\overline{D^\delta} \subset D^{\delta'}$). Let us denote $x_\delta = \Pi_\delta(x)$ and $x_{\delta'} = \Pi_{\delta'}(x')$. As $d(x, D^\delta) \leq \delta$, we have $d(x, x_\delta) \leq \delta$ while $B(x_\delta, \delta) \subset D$. We claim that we can always find out a ball $B(z, \delta')$ such that:

$$\overline{B(z, \delta')} \subset B(x_\delta, \delta)\ , \quad d(x, z) \leq \delta'\,. \tag{A.1}$$

If the claim is true, then $z \in D^{\delta'}$ since the first inclusion in (A.1) implies that $\overline{B(z, \delta')} \subset D$. On the other hand, $x_{\delta'}$ is the unique point of $\partial D^{\delta'}$ such that $d(x, D^{\delta'}) = |x - x_{\delta'}|$. The desired strict inequality follows, namely:

$$d(x, D^{\delta'}) = |x - x_{\delta'}| < |x - z| \leq \delta'.$$

Let us prove (A.1). If $d(x, D^\delta) \leq \delta'$, we can take $z = x_\delta$ so that

$$x \in \overline{B(z, \delta')} \subset B(x_\delta, \delta).$$

Let us now consider the case where $d(x, D^\delta) > \delta'$. Then we choose z on the segment $[x, x_\delta]$ as follows:

$$z = \left(1 - \frac{\delta'}{\delta}\right) x + \frac{\delta'}{\delta} x_\delta.$$

Since $x \notin \overline{D^\delta}$, we have $d(x, x_\delta) < \delta$. Therefore $d(x, z) < \delta'$ and $d(x_\delta, z) < \delta - \delta'$. From the previous inequality and by using the triangle inequality, we deduce the inclusion $\overline{B(z, \delta')} \subset B(x_\delta, \delta)$. It follows that z satisfies (A.1). □

Lemma A.2. *Recalling definition* (4.25) *for the set-valued function* $M_0^\delta(x)$*, we have:*

(i) *Let* $\delta, \delta' \in [0, R_D]$ *such that* $\delta' < \delta$*. Then* $D^\delta = \left\{x \in D^{\delta'} \ :\ B(x, \delta - \delta') \subset D_{\delta'}\right\}$.

(ii) *Let* $x \in \partial_r D^\delta$ *and* $z \in \partial D \cap M_0^\delta(x)$ *(see* (4.25)*). Then, for every* $y \in M_0^\delta(x)$*, we have* $|y - z| = d(y, D^c)$*. In particular,* $|x - z| = d(x, D^c) = \delta$.

(iii) *For each* $\delta' < \delta$*, we have*

$$\partial_s D^{\delta'} \subset \bigcup_{x \in \partial_s D^\delta} M_0^\delta(x).$$

In consequence, for every $x \in \overline{D}$*, if* $\Pi_\delta(x)$ *is in* $\partial_r D^\delta$ *then* $\Pi_{\delta'}(x)$ *belongs to* $\partial_r D^{\delta'}$ *for all* $\delta' < \delta$.

[3] Here we use the fact that $\overline{D^t} = \{y \in D\ :\ d(y, D^c) \geq t\}$. To show the non trivial inclusion of the second set in the first one, we consider $y_n = (1 - \frac{1}{n})y + \frac{1}{n}z$ where $d(y, D^c) = t$ and $z \in D^t$. Then by the concavity of the function $d(\cdot, D^c)$ on D, we have that $d(y_n, D^c) > t$ while $y_n \to y$ whence $y \in \overline{D^t}$.

Proof. (i) First we show that $D^\delta \subset E$ where

$$E := \left\{ x \in D^{\delta'} \ : \ d(x, (D^{\delta'})^c) > \delta - \delta' \right\} = \left\{ x \in D^{\delta'} \ : \ B(x, \delta - \delta') \subset D^{\delta'} \right\}.$$

For every $x \in D^\delta$, $B(x, \delta) \subset D$ implies that

$$\delta < d(x, D^c) \leq d(x, \partial D^{\delta'}) + d(\partial D^{\delta'}, D^c) = d(x, \partial D^{\delta'}) + \delta'.$$

That means $d(x, (D^{\delta'})^c) = d(x, \partial D^{\delta'}) > \delta - \delta'$. Therefore x is in E. This implies $D^\delta \subset E$.

Conversely, for every $x \in E$, the inclusion $B(x, \delta - \delta') \subset D^{\delta'}$ implies that

$$d(\partial B(x, \delta - \delta'), D^c) > \delta'.$$

Thus, we obtain

$$\begin{aligned} d(x, D^c) &= d(x, \partial B(x, \delta - \delta')) + d(\partial B(x, \delta - \delta'), D^c) \\ &> (\delta - \delta') + \delta' = \delta. \end{aligned}$$

This shows that $x \in D^\delta$, whence $E \subset D^\delta$.

(ii) Given $x \in \partial_r D^\delta$ and $z \in \partial D \cap M_0^\delta(x)$, we have $z = x + p$ with $|p| \leq \delta$. It follows that

$$\delta = d(x, D^c) \leq |x - z| \leq \delta.$$

Or, $d(x, z) = d(x, D^c) = \delta$.

For every $y \in M_0^\delta(x)$, we will prove that $|y - z| = d(y, D^c)$. Suppose that

$$d(y, D^c) < |y - z| \ \text{ and } \ d(y, D^c) = |y - \overline{z}|$$

for some $\overline{z} \in \partial D$, $\overline{z} \neq z$. Then, $|y - \overline{z}| < |y - z|$. Recall that as $x \in \partial_r D^\delta$, $M_0^\delta(x)$ is a segment joining x and z. We have

$$|x - \overline{z}| \leq |x - y| + |y - \overline{z}| < |x - y| + |y - z| = \delta$$

while $|x - \overline{z}| \geq d(x, D^c) = \delta$. This gives a contradiction. So, $d(y, D^c) = |y - z|$.

(iii) It is equivalent to prove that for every $\delta' < \delta$, $y \in \partial_s D^{\delta'}$ implies $\Pi_\delta(y) \in \partial_s D^\delta$. Suppose that $y \in \partial_s D^{\delta'}$ and $\Pi_\delta(y) \in \partial_r D^\delta$. From (i) and (ii), we derive that $|y - \Pi_\delta(y)| = \delta - \delta'$. As $\Pi_\delta(y) \in \partial_r D^\delta$, the ball $B(\Pi_\delta(y), \delta)$ then touches the boundary of D at a unique point called z. Let w be the intersection of segment $M_0^\delta(\Pi_\delta(y)) = [\Pi_\delta(y), z]$ and $\partial D^{\delta'}$. By using (ii), $d(w, D^c) = \delta'$. The ball $B(w, \delta')$ contained in $B(\Pi_\delta(y), \delta)$ touches ∂D at and only at z. In other words, w is in $\partial_r D^{\delta'}$. Besides, both y and w are in $M_0^\delta(\Pi_\delta(y))$. It is easy to see that they coincide. We conclude that y belongs to $\partial_r D^{\delta'}$, a contradiction. The proof is complete. $\square$

Lemma A.3. *The following assertions hold true:*

(i) *Given $x \in \partial D^\delta$ and $x^* \in \partial D$ be such that $|x - x^*| = d(x, D^c)$. Then, for every $\delta' \leq \delta$, $\Pi_{\delta'}(x^*)$ belongs to the open segment $]x, x^*[$, that is:*

$$\exists t \in (0,1) \ : \ \Pi_{\delta'}(x^*) = (1-t)x + tx^*.$$

(ii) *Let $y \in M_0^\delta(x)$ such that $x = \Pi_\delta(y)$. Then $\Pi_{\delta'}(y) \in M_0^\delta(x)$ for every $\delta' < \delta$.*

Proof. (i) For each $z \in [x, x^*]$, z can be parametrized as

$$z(t) := (1-t)x^* + tx.$$

If we take $z^* = z(\delta'/\delta)$ then $z^* \in \partial D^{\delta'}$. In fact, it holds $d(z^*, D^c) = \delta'$ since

$$\delta' = |z^* - x^*| \geq d(z^*, D^c) \geq d(z^*, \partial B(z^*, |z^* - x^*|)) = \delta'.$$

Besides, we have $\delta' = |x^* - z^*| \geq \inf_{y \in D^{\delta'}} |x^* - y| \geq \delta'.$

It turns out that $\Pi_{\delta'}(x^*) = z^*$.

(ii) Suppose that $y^* := \Pi_{\delta'}(y) \notin M_0^\delta(x)$ (see Figure 12 for illustration). Without loss of generality, we can assume that the segment $[y, y^*]$ intersects $[x, x^+]$ at z, where $x^+ \in \partial D$ such that $|x - x^+| = d(x, D^c)$. Let z^* be the intersection of the segment $[x, x^+]$ and $\partial D^{\delta'}$. By (i), we get that $\Pi_{\delta'}(x^+) = z^*$. So, the segment $[z^*, x^+]$ is contained in $M_0^{\delta'}(z^*)$. Therefore, for every $z \in [z^*, x^+]$, z admits z^* as its unique projection on $D^{\delta'}$. It is then clearly that

$$|y - z^*| \leq |y - z| + |z - z^*| \leq |y - z| + |z - y^*| = |y - y^*|$$

gives a contradiction to the fact that $y^* := \Pi_{\delta'}(y)$. □

Lemma A.4. *Let $\alpha(x, \delta)$ be a function defined on $\overline{D} \times [0, R]$ by*

$$\alpha(x, \delta) := d(x, D^\delta) - \delta. \tag{A.2}$$

Then, for every $x \in D \setminus \overline{\Omega_{\frac{1}{R}}}$, the function $\alpha(x, \cdot)$ is continuous on $[0, R]$ and exhibits the following behavior:

$$\alpha(x, \delta) = \begin{cases} -\delta & \text{if } 0 < \delta \leq d(x, D^c) \\ -d(x, D^c) & \text{if } x \notin \Lambda_D \text{ and } d(x, D^c) < \delta < \gamma(x) \\ \text{is strictly increasing} & \text{on } [\gamma(x), R]. \end{cases}$$

Therefore, if $x \in D$, then $\alpha(x, \cdot)$ vanishes at the unique positive $\delta := \rho(x)$ where $\gamma(x) < \rho(x) < R$.

On the other hand, if $x \in \partial D$, then $\gamma(x) = \tau(x)$ and $\alpha(x, \cdot)$ vanishes on $[0, \tau(x)]$ while $\alpha(x, \cdot) > 0$ on the possibly empty ineval $(\tau(x), R]$.

Proof. The continuity statement results from the assertion (i) of Lemma A.1. If $0 < \delta \leq d(x, D^c)$, then x belongs to $\overline{D^\delta}$ and $d(x, D^\delta) = 0$ implies that $\alpha(x, \delta) = -\delta$.

Next, we assume that $x \notin \Lambda$ and we prove that $\alpha(x, \delta) = -d(x, D^c)$ if δ belongs to the interval $(d(x, D^c), \gamma(x))$. Note that, by the definition (4.4), it holds $\eta(x) = 0$ if $x \in \Lambda_D$, making the latter interval is empty. Then it exits $\overline{x} \in \overline{\Lambda}$ and $x^* \in \partial D$ such that

$$\overline{x} = x + \zeta(x) \nabla d(x, D^c) \ , \quad |\overline{x} - x^*| = d(\overline{x}, D^c) = \gamma(x).$$

Since we assumed that $d(x, D^c) < \delta < \gamma(x)$, D^δ meets the segment $[\overline{x}, x^*]$ at a unique point w such that $w \in [\overline{x}, x^*]$ and $w \in \partial_r D^\delta$. By Lemma A.3 (i), we deduce that

$w = \Pi_\delta(x^*) = \Pi_\delta(x)$. As a consequence, keeping in mind that $|x - x^*| = d(x, D^c)$ (see Lemma A.2 (ii)), we obtain

$$d(x, D^\delta) = |x - w| = |x^* - w| - |x^* - x| = \delta - d(x, D^c).$$

So, we conclude that $\alpha(x, \delta) = d(x, D^\delta) - \delta = -d(x, D^c)$ as claimed.

Finally, let us show that the continuous function $\alpha(x, \cdot)$ is strictly increasing in $[\gamma(x), R]$. If it is the case, then as $\alpha(x, \gamma(x)) = -d(x, D^c) < 0$ and $\alpha(x, R) = d(x, D^R) - R > 0$ (since $x \notin \overline{U}_D$), we deduce the existence of a unique $\delta \in (\gamma(x), R)$ such that $\alpha(x, \delta) = 0$ as claimed.

Let δ and δ' such that $\gamma(x) < \delta' < \delta < R$. Then, since $\gamma(x) \geq d(x, D^c)$, we have $d(x, D^c) < \delta'$, hence the ball $B(x, |x - \Pi_{\delta'}(x)|)$ is contained in $(D^{\delta'})^c$. It follows that

$$\begin{aligned} d(x, D^\delta) - d(x, D^{\delta'}) &= |x - \Pi_\delta(x)| - |x - \Pi_{\delta'}(x)| \\ &= d(\Pi_\delta(x), B(x, |x - \Pi_{\delta'}(x)|)) \\ &\geq d(\Pi_\delta(x), \partial D^{\delta'}) = \delta - \delta'. \end{aligned}$$

The third line inequality becomes an equality if and only if

$$\Pi_{\delta'}(\Pi_\delta(x)) = \Pi_{\delta'}(x) = \Pi_{\overline{B(x,|x-\Pi_{\delta'}(x)|)}}(\Pi_\delta(x)). \tag{A.3}$$

In this case $\Pi_{\delta'}(x)$ belongs to to the segment $[\Pi_\delta(x), x]$ and $\Pi_\delta(x) \in \partial_r D^\delta$. By the asertion (iii) of Lemma A.2, it follows that $\Pi_{\delta'}(x) \in \partial_r D^{\delta'}$ while, by assertion (ii), $M_0^\delta(\Pi_\delta(x))$ contains the segment $S := \{x + t_\delta \nabla d(x, D^c)\ \ t \in [0, \zeta(x)\}$. In particular $\Pi_\delta(x)$ can be rewritten as

$$\Pi_\delta(x) = x + t_\delta \nabla d(x, D^c),$$

for some $t_\delta > 0$. Since we have $\gamma(x) < \delta$, we get $\zeta(x) < t_\delta$, or equivalently that $x + \zeta(x)\nabla d(x, D^c)$ belongs to $M_0^\delta(\Pi_\delta(x))$. This gives a contradiction to the fact that $x + \zeta(x)\nabla d(x, D^c)$ is a singular point of $d(\cdot, D^c)$. So, we conclude that the relation (A.3) never occurs for $\delta' > \gamma(x)$. It follows $\alpha(x, \delta) > \alpha((x, \delta')$, whence the claimed strict monotony property on $[\gamma(x), R]$.

Let us finally conclude with the case where $x \in \partial D$. Then $\gamma(x) = \tau(x)$ and clearly $\alpha(x, \cdot)$ satisfies the required properties since

$$\tau(x) = \max\{t \geq 0 : P_{\partial D}(x - t\,\nu_D(x)) = x\}$$

if $x \in \partial_r D$ and $\tau(x) = 0$ if $x \in \partial_s D$. □

Lemma A.5. *Let D be a convex domain of $\mathbb{R}^2$. For every $x \in \partial D$, we define*

$$k_{\partial D}(x) := \frac{1 + \nu_D^+(x) \cdot \nu_D^-(x)}{2}.$$

Then, we have

(i) *for every $x \in \partial D$, $0 < k_{\partial D}(x) \leq 1$;*

(ii) *$k_{\partial D}(x) = 1$ for every $x \in \partial_r D$;*

(iii) *$\forall \varepsilon \in (0, 1)$, the set $\{x\ :\ k_{\partial D}(x) < \varepsilon\}$ is finite.*

Proof. We recall that the normal cone of D at x is given by

$$N_D(x) := \left\{a\nu_D^+(x) + b\nu_D^-(x) \ : \ a, b \in \mathbb{R}_+\right\}.$$

For every $x \in \partial D$, we denote by $\varphi(x)$ the angle

$$\varphi(x) := \frac{1}{2}\angle(\nu_D^-(x), \nu_D^+(x)),$$

and by $T_D(x)$ the tangent cone of D at x

$$\begin{aligned} T_D(x) &:= \mathrm{cl}\left\{s(y-x) \ : \ y \in D, \ s \geq 0\right\} \\ &= \left\{aT_D^+(x) + bT_D^-(x) \ : \ a, b \in \mathbb{R}_+\right\}, \end{aligned}$$

where $T_D^{\pm}(x)$ are the left and right tangent unit vectors of D at x. We denote by $\psi(x)$ the angle

$$\psi(x) := \frac{1}{2}\angle(T_D^-(x), T_D^+(x)).$$

Since tangent and normal cones are polar each other, it holds $\varphi(x) + \psi(x) = \frac{\pi}{2}$ for every $x \in \partial D$.

(i) Clearly, as D is convex, we have $0 \leq \varphi(x) < \frac{\pi}{2}$ and $0 < \psi(x) \leq \frac{\pi}{2}$. Hence, for all $x \in \partial D$, it holds $0 < k_{\partial D}(x) \leq 1$ since

$$k_{\partial D}(x) = \frac{1 + \cos 2\varphi(x)}{2} = \cos^2\varphi(x) = \sin^2\psi(x).$$

(ii) As D is convex, its boundary ∂D admits at most countably many singular points. On the regular part $\partial_r D = \partial D \setminus \partial_s D$, we get $\nu_D^-(x) = \nu_D^+(x)$, i.e. $\varphi(x) = 0$. Then, $k_{\partial D}(x) = 1$.

(iii) Let ε such that $0 < \varepsilon < 1$ and denote

$$E_\varepsilon := \{x \in \partial D \ : \ k_{\partial D}(x) < \varepsilon\} \quad , \quad N_\varepsilon := \#(E_\varepsilon).$$

The function $h(\varphi)$ defined by

$$h(\varphi) := \frac{1 + \cos 2\varphi}{2}, \qquad \text{for } \varphi \in [0, \frac{\pi}{2}).$$

is non increasing in $[0, \frac{\pi}{2})$, while $h(\varphi(x)) = k_{\partial D}(x)$ for every $x \in \partial D$. There always exists $\varphi_e \in (0, \pi/2)$ such that $h(\varphi_\varepsilon) = \varepsilon$. We set

$$E_\varepsilon := \{x \in \partial D \ : \ k_{\partial D}(x) < \varepsilon\}.$$

By the monotonicity of h, we deduce that $\varphi(x) > \overline{\varphi}$ for every $x \in E_\varepsilon$. It follows that

$$N_\varepsilon\,\overline{\varphi} < \sum_{x \in E}\varphi(x) \leq \sum_{x \in \partial D}\varphi(x) \leq 2\pi,$$

whence $N_\varepsilon \leq \frac{2\pi}{\overline{\varphi}} < +\infty$. $\square$

References

[1] F. Alter, V. Caselles: *Uniqueness of the Cheeger set of a convex body*, Nonlinear Analysis 70 (2009) 32–44.

[2] F. Alter, V. Caselles, A. Chambolle: *A characterization of convex calibrable sets in* $\mathbb{R}^N$, Math. Annalen 332 (2005) 329–366.

[3] L. Ambrosio, N. Fusco, D. Pallara: *Functions of Bounded Variation and Free Discontinuity Problems*, Clarendon Press, Oxford (2000).

[4] G. Anzellotti: *Pairings between measures and bounded functions and compensated compactness*, Ann. Mat. Pura Appl. (4) 135 (1983) 293–318.

[5] G. Anzellotti: *Traces of bounded vector fields and the divergence theorem*, preprint, Univ. of Trento.

[6] G. Anzellotti, M. Giaquinta: *Funzioni BV e tracce*, Rend. Sem. Mat. Univ. Padova 60 (1978) 1–21.

[7] H. Attouch, G. M. Buttazzo, G. Michaille: *Variational Analysis in Sobolev and BV Spaces*, MPS/SIAM Series on Optimization Vol. 6, Society of Industrial and Applied Mathematics, Philadelphia (2006).

[8] H. Attouch, T. Champion: *L^p regularization of the non-parametric minimal surface problem*, in: *Ill-Posed Variational Problems and Regularization Techniques*, Trier 1998, M. Théra et al. (eds.), Lecture Notes in Economics and Mathematical Systems 477, Springer, Berlin (1999) 25–34.

[9] G. Bouchitté, I. Fragalà: *Duality theory for non-convex problems in the calculus of variations*, Arch. Ration. Mech. Anal. 229 (2018) 361–415.

[10] G. Bouchitté, I. Fragala, I. Lucardesi: *A variational method for second order shape derivatives*, SIAM J. Control Optim. 54/2 (2016) 1056–1084.

[11] G. Bouchitté, M. Phan: *A convex relaxation method for free boundary problems*, Evol. Equations Control Theory 17 (2026) 62–92.

[12] H. Brezis: *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer Science & Business Media, Cham (2010).

[13] G. Buttazzo, G. Carlier, M. Comte: *On the selection of maximal Cheeger sets*, Diff. Int. Equations 20 (2007) 991–1004.

[14] V. Caselles, A. Chambolle, M. Novaga: *Uniqueness of the Cheeger set of a convex body*, Pacific J. Math. 232 (2007) 77–90.

[15] G. Crasta, I. Fragalà: *On the Dirichlet and Serrin problems for the inhomogeneous infinity Laplacian in convex domains: Regularity and geometric results*, Arch. Ration. Mech. Anal. 218 (2015) 1577–1607.

[16] G. Crasta, I. Fragalà: *Characterization of stadium-like domains via boundary value problems for the infinity Laplacian*, Nonlinear Analysis 133 (2016) 228–249.

[17] G. Crasta, I. Fragalà: *On the characterization of some classes of proximally smooth sets*, ESAIM Control, Optim. Calc. Var. 22 (2016) 710–727.

[18] G. Crasta, I. Fragalà: *Geometric issues in PDE problems related to the infinity Laplace operator*, in: *Topological Optimization and Optimal Transport in the Applied Sciences*, M. Bergounioux et al. (eds.), Radon Series on Computational and Applied Mathematics Vol. 17, De Gruyter, Berlin (2017) 5–19.

[19] I. Ekeland, R. Temam: *Analyse Convexe et Problèmes Variationnels*, Dunod Gauthier-Villars, Paris (1974).

[20] L. C. Evans, R. F. Gariepy: *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton (1992).

[21] A. Gray: *Tubes*, Progress in Mathematics Vol. 221, Birkhäuser, Basel (2012).

[22] A. Henrot, M. Pierre: *Variation et Optimisation de Formes. Une Analyse Géométrique*, Matématiques et Applications 48, Springer, Berlin (2005).

[23] B. Kawohl, T. Lachand-Robert: *Characterization of Cheeger sets for convex subsets of the plane*, Pacific J. Math. 225 (2006) 103–118.

[24] G. P. Leonardi: *An overview on the Cheeger problem*, in: *New Trends in Shape Optimization*, Springer, Berlin (2015) 117–139.

[25] Y. Li, L. Nirenberg: *The distance function to the boundary, Finsler geometry, and the singular set of viscosity solutions of some Hamilton-Jacobi equations*, Comm. Pure Appl. Math. 58 (2005) 85–146.

[26] E. Parini: *An introduction to the Cheeger roblem*, Surveys Math. Appl. 6 (2011) 9–21.

[27] R. Neumayer, G. Saracco: *The Cheeger constant of a Jordan domain without necks*, Calc. Var. Partial Diff. Equations 56/6 (2017), art.no. 164, 29p.

[28] R. Neumayer, G. Saracco: *Minimizers of the prescribed curvature functional in a Jordan domain with no necks*, ESAIM Control Optim. Calc. Var. 26 (2020), art.no. 76, 20p.

[29] P.-M. Suquet: *Fonctions à variation bornée sur un ouvert de* $\mathbf{R}^N$, Travaux Sém. Analyse Convexe 8/1 (1978), exp.no. 4, 17 pp.

[30] E. Stredulinsky, W. P. Ziemer: *Area minimizing sets subject to a volume constraint in a convex set*, J. Geom. Analysis 7 (1997) 653–677.