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About the Bang-Bang Principle for Controlled Affine Dynamics With Brownian Noise

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Dedicated to the memory of Hédy Attouch.

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We revisit the Bang-Bang principle, established for deterministic dynamics with affine control, in a stochastic setting where the dynamics are subject to Brownian motion. We show that such a principle does not generally apply in this context. However, we demonstrate that it does apply under certain conditions with deterministic controls. With stochastic controls, we obtain, under certain conditions, that Mayer's problems with an expectation criterion and a compact control set are equivalent to the same problems with controls taking values in the closed convex hull of the control set. This result is illustrated by a linearized dynamics of the SIR epidemiological model.

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1. Introduction

The *Bang-Bang* Principle is a pillar of the control theory, which has important implications in terms of controllability and optimal control of deterministic linear systems in finite dimension. Given a (deterministic) control system in $\mathbb{R}^n$ (for some $n \in \mathbb{N}^*$) of the form

$$\frac{dX(t)}{dt} = a(t) + A(t)X(t) + B(t)u(t),\ t \geq 0, \tag{1}$$

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where the control $u(t)$ takes values in a compact set $U \subset \mathbb{R}^d$ (for some $d \in \mathbb{N}^*$), and $a(\cdot), A(\cdot), B(\cdot)$ are time-dependent matrices of adequate dimension, the (generalized) *Bang-Bang* Principle (see e.g. [18]) states that for any initial condition $X(t_0) = X_0$, the reachable set at a given time $T > t_0$ coincides with the reachable set with controls that takes values in the closed convex hull of U, denoted $\overline{co}\ U$. In this context, considering controls within the closed convex hull amounts to making the dynamics velocity convex, which can be achieved by what are known as "relaxed controls" [23]. Alternatively, the *Bang-Bang* Principle is often stated in the literature for convex compact sets U, whose sets of extreme points $Ext\ U$ is closed, as the equality of reachable sets with control in U or in $Ext\ U$ (we recall that the extreme points u of a convex set U are such that $U \setminus \{u\}$ is convex). Typically, when the set U takes the form $\prod_{i=1}^{d}[\underline{u}_i, \overline{u}_i]$, one deduces that a reachable target can be reached in minimal time using controls u_i ($i = 1 \dots d$) taking values $\underline{u}_i$ or $\overline{u}_i$ only [17].

The purpose of this work is to investigate the validity of this Principle in a stochastic framework, adding a Brownian part to the dynamics, considered then as a stochastic differential equation. To our knowledge, this principle has not yet been studied in this stochastic framework, although relaxed controls have already been considered for stochastic dynamics. In the literature, relaxation is generally sought to guarantee the existence of optimal solutions [2], but different types of relaxations are possible in the stochastic framework, which do not always consist of relaxed controls as in the deterministic framework, for general nonlinear dynamics [20]. Here, we will consider both deterministic and stochastic controls, and relaxed controls as controls in the closed convex hull of the set U, as we wish to revisit the Bang-Bang principle. As we will see, the main difficulty when adding a Brownian motion term to the dynamics is that we end up with control-induced noise, which prevents us from establishing a Bang-Bang principle... This is why we had to consider dynamics with uncontrolled noise. In our work, we first show that the reachable sets are convex for deterministic and stochastic controls. However, with stochastic controls, we obtain a Bang-Bang principle in terms of equivalence between problems with controls in U or in $\overline{co}\, U$, but not in terms of equality of their corresponding reachable sets. The paper is organized as follows. In the next Section 2, we give the stochastic setting we are considering, with some definitions and preliminary comments. Section 3 is devoted to our main results, while Section 4 gives an illustration on an epidemiological model.

2. Stochastic setting and preliminaries

A natural way for extending dynamics (1) to a stochastic framework, keeping the linearity property with respect to the state and the control, is to consider the following stochastic differential equation

$$dX(t) = [a(t)+A(t)X(t)+B(t)u(t)]\,dt + [c(t)+C(t)X(t)+D(t)u(t)]\,dW(t), \quad (2)$$

where $t \in [0, T]$, W denotes the standard one-dimensional Brownian motion on a complete standard probability space $\left(\Omega, \mathbb{F} := (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P}\right)$, for a fixed finite time horizon $0 < T < \infty$, and time-dependent matrices $a(\cdot)$, $A(\cdot)$, $B(\cdot)$, $c(\cdot)$, $C(\cdot)$, $D(\cdot)$ are of appropriate dimensions. We are not aiming at complicating the measurability statements by asking specific integrability for the coefficients of these matrices. For this reason, we make the following assumption.

Assumption 2.1.

1. The control space U is a compact subset of the Euclidean space $\mathbb{R}^d$ (for some $d \in \mathbb{N}^*$).
2. The matrices $a, c \in \mathbb{L}^\infty([0,T];\mathbb{R}^n)$, $A, C \in \mathbb{L}^\infty([0,T];\mathbb{R}^{n\times n})$, and $B, D \in \mathbb{L}^\infty\left([0,T];\mathbb{R}^{n\times d}\right)$ are deterministic.

Let us first consider deterministic controls.

Definition 2.2. Let $0 \le t_0 \le t_1 \le T$ and $x_0 \in \mathbb{L}^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^n)$. The *reachable sets* using deterministic controls are denoted by

$$\mathcal{R}_U^{det}(t_0, x_0; t_1) = \left\{ X^{t_0,x_0,u}(t_1) \mid u : \mathbb{R}_+ \longrightarrow U \text{ is Borel measurable} \right\},$$

resp.

$$\mathcal{R}_{\overline{co}\ U}^{det}(t_0, x_0; t_1) = \left\{ X^{t_0,x_0,u}(t_1) \mid u : \mathbb{R}_+ \longrightarrow \overline{co}\ U \text{ is Borel measurable} \right\}$$

where $X^{t_0,x_0,u}$ is the associated solution to (2) with $X(t_0) = x_0$, governed by the control u (the existence and uniqueness of the solution is standard, since we are dealing with an affine system, and the coefficients are bounded in time). □

We shall say that the *Bang-Bang* Principle holds in this setting if one has

$$\mathcal{R}_U^{det}(t_0, x_0; t_1) = \mathcal{R}_{\overline{co}\ U}^{det}(t_0, x_0; t_1).$$

Let us begin by a simple case.

Example 2.3. Consider a pure controlled diffusion.

$$dX(t) = u(t)dW(t),$$

with $U = \{0,1\}$ for which $\overline{co}\ U = [0,1]$. We observe the following features.

1. One has clearly
$$\frac{W(T)}{2} \in \mathcal{R}_{[0,1]}^{det}(0,0;T),$$
but if u is an adapted process (w.r.t. the Brownian filtration) taking its values in $\{0,1\}$, then one has
$$\mathbb{E}\left[\left|\int_0^T u(s)dW(s) - \frac{W(T)}{2}\right|^2\right] = \mathbb{E}\left[\int_0^T \left|u(s) - \tfrac{1}{2}\right|^2 ds\right] = \frac{T}{4}.$$
It is, therefore, clear that the *Bang-Bang* property cannot hold true for this system. However, we can actually consider notions of reachable sets $\mathcal{R}^{st}(0,0;T)$ by asking that the controls u to be adapted w.r.t. the Brownian filtration instead of Borel measurable, as we do in Definition 2.5 below.
2. On the other hand, if u is deterministic, then $\mathbb{P}_{X^{0,0,u}(t)}$[1] belongs to the Gaussian family
$$\mathcal{G}auss_T := \left\{ \mathcal{N}(0,\sigma^2) \mid \sigma^2 \le T \right\}.$$
The normality result is rather straightforward, and can be found, for instance, in [10, Section 3.2]. Conversely, if $\sigma^2 \le T$, consider $u(t) = \mathbf{1}_{[0,\sigma^2]}(t) \in U$ to see that
$$\mathcal{G}auss_T = \left\{ \mathbb{P}_\xi \mid \xi \in \mathcal{R}_U^{det}(0,0;T) \right\} = \left\{ \mathbb{P}_\xi \mid \xi \in \mathcal{R}_{[0,1]}^{det}(0,0;T) \right\}.$$

[1] The law of a square-integrable random variable $\xi : \Omega \longrightarrow \mathbb{R}^n$ is denoted by $\mathbb{P}_\xi$.

3. However, even this *Bang-Bang* property in law does not hold true for general linear systems. To see this, we consider the same control system, but $U = \{-1, 1\}$. As hinted before, the Dirac measure δ_0 (seen as a degenerated Gaussian law with $\sigma = 0$) belongs to $\left\{\mathbb{P}_\xi \mid \xi \in \mathcal{R}^{det}_{[-1,1]}(0,0;T)\right\}$, and it is obtained with the null control $u_0 \equiv 0$. However, if u is adapted and $\{-1, 1\}$-valued,
$$\mathbb{E}\left[\left(X^{0,0,u}(T)\right)^2\right] = T > 0,$$
showing that, in this case, $\delta_0 \notin \{\mathbb{P}_\xi \mid \xi \in \mathcal{R}^{st}_U(0,0;T)\}$, and, thus,
$$\left\{\mathbb{P}_\xi \mid \xi \in \mathcal{R}^{st}_U(0,0;T)\right\} \subsetneq \left\{\mathbb{P}_\xi \mid \xi \in \mathcal{R}^{st}_{\overline{co}\ U}(0,0;T)\right\}.$$

This example motivated us to consider systems with non-controlled noise:

$$dX(t) = [a(t) + A(t)X(t) + B(t)u(t)]\,dt + [c(t) + C(t)X(t)]\,dW(t),\ t \in [0,T] \quad (3)$$

along with the following definitions of stochastic controls and reachable sets.

Definition 2.4. Under Assumption 2.1, given $t_0 \in [0,T]$ and $x_0 \in \mathbb{L}^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^n)$,

1. $\mathcal{U}$ denotes the set of adapted U-valued control process $u(\cdot)$.
2. When the set U is replaced by its closed convex hull $\overline{co}\ U$, the family of $\overline{co}\ U$-valued adapted controls is denoted by $\overline{\mathcal{U}}$.

Such stochastic controls are referred to as *admissible*.

Definition 2.5. Let $0 \leq t_0 \leq t_1 \leq T$ and $x_0 \in \mathbb{L}^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^n)$. The reachable sets using admissible (stochastic) controls are denoted by
$$\mathcal{R}^{st}_U(t_0, x_0; t_1) = \left\{X^{t_0,x_0,u}(t_1) \mid u \in \mathcal{U}\right\},$$
resp.
$$\mathcal{R}^{st}_{\overline{co}\ U}(t_0, x_0; t_1) = \left\{X^{t_0,x_0,u}(t_1) \mid u \in \overline{\mathcal{U}}\right\}.$$
where $X^{t_0,x_0,u}$ is the associated solution to (3) with $X(t_0) = x_0$ and control u (existence and uniqueness of the solution is standard, since we are dealing with an affine system, the coefficients are bounded in time and u is adapted). $\square$

Note that in the context of dynamics that are linear with respect to the control which appears only on the drift term, the stochastic controls with values in $\overline{\mathcal{U}}$ are known in the literature as *relaxed controls* (e.g. [2]), although in general other kind of relaxation are possible (see e.g. [20]).

Given a map $g : \mathbb{R}^n \to \mathbb{R}$, we shall also consider Mayer problems in this stochastic context.

Definition 2.6. For $0 \leq t_0 \leq t_1 \leq T$ and $x_0 \in \mathbb{L}^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^n)$, let
$$(\mathcal{P}_{strict}) : \inf_{\xi \in \mathcal{R}^{st}_U(t_0,x_0,t_1)} \mathbb{E}[g(\xi)],$$
$$(\mathcal{P}_{convex}) : \inf_{\xi \in \mathcal{R}^{st}_{\overline{co}\ U}(t_0,x_0,t_1)} \mathbb{E}[g(\xi)].$$

We aim at studying when these problems are equivalent, in the sense that they have the same value. Reachable sets are a natural way of expressing Mayer cost problems. While the *Bang-Bang* Principle recalled in the introduction for the deterministic setting is a well-established property, it appears to be more difficult to obtain in the stochastic framework (as already suggested by the pathological behavior of Example 2.3). The deterministic setting suggests that, at least for linear, or affine dynamics, optimality for a Mayer cost only requires the convexity of the reachable set (and not of the control family U itself).

In the Brownian diffusive case, the standard to obtain strict optimal controls is set by [13], which, even for linear dynamics, amounts to the convexity of U. The alternative, as suggested for instance in [3], [8], [14], is to consider occupation measures and their convex hull in order to guarantee the existence of relaxed optimal policies. This is why, inspired by the deterministic setting, and with the understanding that convexity is the basis for the spirit of the arguments in [13], we are looking for specialization of *Bang-bang* Principles for a class of affine Brownian diffusions.

3. Convexity and Bang-Bang properties

Let us first give a fundamental representation of solutions of (3). Given $t_0 \in [0, T]$, we consider the following stochastic differential equation with matrix values

$$d\Phi(t) = -\Phi(t)\left[A(t) - C(t)^2\right]dt - \Phi(t)C(t)dW(t),\ t \geq 0,\ \Phi(t_0) = I_n, \tag{4}$$

where I_n denotes the identity matrix in $\mathbb{R}^{n\times n}$ (here also, the existence and uniqueness of the solution is standard, since we are dealing with an affine system, and the coefficients are bounded in time). Remember that $C(t)$ is a square matrix and thus $C(t)^2$ is well defined. Because of the boundedness of the coefficients, the solutions of (4) are $\mathbb{F}$-adapted and belong to $\mathbb{L}^\infty\left([0,T]; \mathbb{L}^p\left(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^{n\times n}\right)\right)$, for every p satisfying $1 < p < +\infty$. Itô's formula applied to the product ΦX gives

$$\begin{aligned} d(\Phi(t)X(t)) &= \Big(\Phi(t)\left[a(t) + A(t)X(t) + B(t)u(t)\right] - \Phi(t)\left[A(t) - C(t)^2\right]X(t)\Big)dt \\ &\quad + \Big(\Phi(t)\left[(c(t) + C(t)X(t)\right] - \Phi(t)C(t)X(t)\Big)dW(t) \\ &\quad - \Phi(t)C(t)\left[(c(t) + C(t)X(t)\right]dt \\ &= \Phi(t)\left[(a(t) - C(t)c(t) + B(t)u(t))dt + c(t)dW(t)\right], \end{aligned}$$

leading to the following representation of the solutions

$$\begin{aligned} X(t) &= X^{t_0,x_0,u}(t) \\ &= \Phi^{-1}(t)\left(x_0 + \int_{t_0}^{t} \Phi(r)\left[(a(r) - C(r)c(r) + B(r)u(r))dr + c(r)dW(r)\right]\right) \end{aligned} \tag{5}$$

for $t \in [t_0, T]$, where Φ^{-1} is solution of the stochastic differential equation

$$d\Phi^{-1}(t) = A(t)\Phi^{-1}(t)dt + C(t)\Phi^{-1}(t)dW(t),\ t \geq 0,\ \Phi^{-1}(t_0) = I_n. \tag{6}$$

Indeed, one can check, owing to standard Itô calculus, that one has

$$\begin{aligned} d(\Phi^{-1}(t)\Phi(t)) &= \Big(-\Phi^{-1}(t)\Phi(t)\left[A(t)-C(t)^2\right] + A(t)\Phi^{-1}(t)\Phi(t)\Big)dt \\ &\quad + \Big(-\Phi^{-1}(t)\Phi(t)C(t) + C(t)\Phi^{-1}(t)\Phi(t)\Big)dW(t) \\ &\quad - C(t)\Phi^{-1}(t)\Phi(t)C(t)dt, \\ &= 0,\ \Phi^{-1}(t_0)\Phi(t_0) = I_n. \end{aligned}$$

3.1. Convexity of the reachable sets

Similarly to the deterministic systems, we prove the convexity of the stochastic reachable set $\mathcal{R}_U^{st}(t_0, x_0; T)$, under the mere compactness of the set of control U, without assuming convexity.

Theorem 3.1. *For every $0 \le t_0 \le t_1 \le T$, and every $x_0 \in \mathbb{L}^4(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^n)$, the reachable sets $\mathcal{R}_U^{det}(t_0, x_0, t_1)$ and $\mathcal{R}_U^{st}(t_0, x_0, t_1)$ are convex, strongly closed and (weakly) compact with respect to $\mathbb{L}^2(\Omega, \mathcal{F}_{t_1}, \mathbb{P}; \mathbb{R}^n)$.*

Remark 3.2. The keen reader will have noticed that we require a fourth-order integrability of the initial condition x_0. This is due to the fact that we are extensively employing the variation-of-constants or mild formulation in equation (5) and, as already mentioned after equation (4), the $\mathbb{P}$-integrability of the fundamental solution Φ is of every finite order $p > 1$, but not $\mathbb{L}^\infty$. As a consequence, the result can be adapted, in a straightforward way, to $x_0 \in \bigcup_{\delta>0} \mathbb{L}^{2+\delta}(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^n)$.

Proof of Theorem 3.1. We only prove the assertion for the stochastic reachable set, the one with deterministic controls being -identical. The proof adapts the finite-dimensional counterpart in [18, Theorem 3, p. 76]. The main ingredient is a Hilbert-fitted version of Lyapunov's convexity theorem in [12].

If U is a singleton, there is nothing to prove. We assume that U consists of at least two points.

A glance at the solution expressed in (5) shows that one only needs to concentrate on the control-dependent part. As a consequence, we assume, without loss of generality, $c \equiv 0$ and $a \equiv 0$, such that

$$X^{t_0,x_0,u}(t) = \Phi^{-1}(t)\left(x_0 + \int_{t_0}^{t} \Phi(r)B(r)u(r)dr\right),\ \forall t_0 \le t \le T.$$

Let us first prove that the closure of the reachable set is convex. We begin by fixing two admissible U-valued control processes $u \neq v$, and $\lambda \in (0,1)$. The aim is to show that $\lambda X^{t_0,x_0,u}(t_1) + (1-\lambda)X^{t_0,x_0,v}(t_1)$ can be approximated with suitable control processes.

We reason in the setting where $t_1 = T$ and consider $J := [t_0, T]$. Let $\mathcal{M}(J)$ stand for the family of Lebesgue-measurable subsets of J. For each measurable subset $I \subset J$, we define the map $\eta : \mathcal{M}(J) \longrightarrow \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^{2n})$, by setting

$$\eta(I) = \begin{pmatrix} \int_I \Phi^{-1}(T)\Phi(r)B(r)u(r)dr \\ \int_I \Phi^{-1}(T)\Phi(r)B(r)v(r)dr \end{pmatrix}, \ \eta(\emptyset) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In order to simplify arguments, let us assume $t_0 = 0$ and $x_0 = 0$. Furthermore, we let

$$\eta(J) = \begin{pmatrix} X^{0,0,u}(T) \\ X^{0,0,v}(T) \end{pmatrix} =: \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.$$

Then, by an extension to Hilbert spaces of Lyapunov's convexity theorem, cf. [12, Theorem, p. 405], $cl\left(\{\eta(I) : \ I \in \mathcal{M}(J)\}\right)$ is a convex subset of the Hilbert space $\mathbb{L}^2\left(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^{2n}\right)$ (the Kuratowski closure operator being intended in the usual topology on this Hilbert space as well).

In particular, for $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists some $I_{\lambda,\varepsilon} \subset J$ (that can even be chosen Borel-measurable) such that

$$\left\| \eta(I_{\lambda,\varepsilon}) - \lambda \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right\|_{\mathbb{L}^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^{2n})} \leq \frac{\varepsilon}{2}.$$

The reader is invited to note that one has

$$\eta(J \setminus I_{\lambda,\varepsilon}) = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} - \eta(I_{\lambda,\varepsilon}).$$

In particular, the following inequalities are satisfied.

$$\left\| \Phi^{-1}(T) \int_{J\setminus I_{\lambda,\varepsilon}} \Phi(r)B(r)v(r)dr - (1-\lambda)\eta_2 \right\|_{\mathbb{L}^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^n)} \leq \frac{\varepsilon}{2},$$

$$\left\| \Phi^{-1}(T) \int_{I_{\lambda,\varepsilon}} \Phi(r)B(r)u(r)dr - \lambda\eta_1 \right\|_{\mathbb{L}^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^n)} \leq \frac{\varepsilon}{2}.$$

By defining $\quad u^{\lambda,\varepsilon}(t) = u(t)\mathbf{1}_{I_{\lambda,\varepsilon}}(t) + v(t)\mathbf{1}_{J\setminus I_{\lambda,\varepsilon}}(t), \ t \in J,$

we get an adapted control, for which one has

$$\begin{aligned} &\left\| X^{0,0,u^{\lambda,\varepsilon}}(T) - (\lambda\eta_1 + (1-\lambda)\eta_2) \right\| = \\ &\Big\| \Phi^{-1}(T) \int_{I_{\lambda,\varepsilon}} \Phi(r)B(r)u(r)dr - \lambda\eta_1 \\ &\qquad + \Phi^{-1}(T) \int_{J\setminus I_{\lambda,\varepsilon}} \Phi(r)B(r)v(r)dr - (1-\lambda)\eta_2 \Big\|_{\mathbb{L}^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^n)} \leq \varepsilon \end{aligned}$$

thus concluding that the closure of the reachable set is convex by invoking the arbitrariness of $\varepsilon > 0$ (and $\lambda \in (0,1)$).

Let us now prove that the reachable set is, in fact, closed. Let now $(u_m)_m$ be a sequence of adapted U-valued control policies, $x_0 \in \mathbb{L}^4\left(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^n\right)$, and let $\eta_m := X^{t_0,x_0,u_m}(T) \in \mathbb{L}^2\left(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n\right)$. Furthermore, let us assume that η_m converges (as $m \to \infty$) to some $\eta \in \mathbb{L}^2\left(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n\right)$. Due to the boundedness of the (adapted) control sequence with respect to $\mathbb{L}^\infty\left(\Omega \times [t_0,T], \mathbb{P} \times Leb; U\right)$, hence in $\mathbb{L}^4\left(\Omega \times [t_0,T], \mathbb{P} \times Leb; U\right)$, there exists a subsequence, still denoted by u_m weakly converging to some adapted $u \in \mathbb{L}^4\left(\Omega \times [t_0,T], \mathbb{P} \times Leb; U\right)$.

Since $\Phi \in \mathbb{L}^\infty([0,T];\mathbb{L}^q(\Omega;\mathbb{R}^{n\times n}))$, for every $q>1$, by picking q large enough, it is then easy to show that $\eta^m := X^{t_0,x_0,u_m}(T)$ weakly converges to some $\tilde{\eta} := X^{t_0,x_0,u}(T)$ in $\mathbb{L}^3(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^n)$. Of course, the weak convergence is also valid in $\mathbb{L}^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^n)$, thus implying that $\eta = \tilde{\eta}$, which yields the strong closeness of the reachable set w.r.t $\mathbb{L}^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^n)$.

The reader will easily note that the aforementioned procedure actually yields the weak compactness of the reachable sets. □

We then obtain the *Bang-Bang* Principle with deterministic controls.

Corollary 3.3. *For every $0 \le t_0 \le t_1 \le T$, and every $x_0 \in \mathbb{L}^2(\Omega,\mathcal{F}_{t_0},\mathbb{P};\mathbb{R}^n)$, one has*

$$\mathcal{R}_U^{det}(t_0,x_0,t_1) = \mathcal{R}_{\overline{co}\,U}^{det}(t_0,x_0,t_1).$$

Proof. In the deterministic framework, the *Bang-Bang* Principle is inferred from the convexity of these reachable sets. The idea of proof in this setting (i.e., $C=0, c=0$) is to use piecewise (time-) constant control policies (since this class is dense in the family of Borel-measurable controls), then approximate elements in $\overline{co}\,U$ with convex combinations of elements in U. The reader may look at the proof of [18, Theorem 1A, last part], for instance. The proof goes through without any change for stochastic dynamics, provided that the considered controls are deterministic. This is why we do not reproduce it. □

Let us make some comments about the *Bang-Bang* Principle with stochastic controls.

Remark 3.4. A similar argument than the one used in the proof of Corollary 3.3 could be tried for stochastic control policies, albeit the fact that constant in time would imply the usage, not of elements of *co* U, but $\mathcal{F}_t$-measurable, $\overline{co}\,U$-valued random variables. In this case, piecewise (time-) constant admissible controls are of the form

$$\sum_i \mathbf{1}_{[t_i,t_{i+1})} u_i(\omega),$$

with u_i being an $\mathcal{F}_{t_i}$-measurable, $\overline{co}\,U$-valued random variable. Although, ω-wise, $u_i(\omega)$ is a convex combination of pure controls in U, the dependence on ω does not allow to conclude the same property for the associated trajectory.

3.2. A Bang-Bang property with stochastic controls

Albeit the fact that we do not aim to prove $\mathcal{R}_U^{st}(t_0,x_0,\cdot) = \mathcal{R}_{\overline{co}\,U}^{st}(t_0,x_0,\cdot)$, we are able to provide the following version of the *Bang-bang* property adapted to stochastic problems. This is the main result of the paper.

Theorem 3.5. *Assume that the function g is lower semi-continuous function, bounded from below and with at most quadratic growth. Then, one has the following properties.*

1. *For every $0 < t_0 \le t_1 < T$, and every $x_0 \in \mathbb{L}^4(\Omega,\mathcal{F}_{t_0},\mathbb{P};\mathbb{R}^n)$, the control problems $(\mathcal{P}_{strict})$, $(\mathcal{P}_{convex})$ are equivalent.*
2. *Furthermore, if g is also convex, then an optimal ξ exists in $\mathcal{R}_U^{st}(t_0,x_0,t_1) \subset \mathcal{R}_{\overline{co}\,U}^{st}(t_0,x_0,t_1)$.*

Remark 3.6. Since the sets $\mathcal{R}^{st}(t_0, x_0, t_1)$ are convex and weakly compact, the corresponding collection of probability laws

$$\left\{\mathbb{P}_\xi : \ \xi \in \mathcal{R}^{st}(t_0, x_0, t_1)\right\}$$

is also convex and relatively compact (even in the 2-Wasserstein space). However, these sets may fail to be closed in the weak topology. Consequently, one has

$$\left\{\mathbb{P}_\xi : \ \xi \in \mathcal{R}^{st}_U(t_0, x_0, t_1)\right\} \subset \left\{\mathbb{P}_\xi : \ \xi \in \mathcal{R}^{st}_{\overline{co}\ U}(t_0, x_0, t_1)\right\},$$

but our approach does not establish equality here, not even for the set of reachable laws. The precise conditions under which such equality holds remain unclear and are a subject of ongoing research.

1. The first assertion in the previous theorem states

$$\mathrm{cl}\left\{\mathbb{P}_\xi : \ \xi \in \mathcal{R}^{st}_U(t_0, x_0, t_1)\right\} = \mathrm{cl}\left\{\mathbb{P}_\xi : \ \xi \in \mathcal{R}^{st}_{\overline{co}\ U}(t_0, x_0, t_1)\right\},$$

where closure is taken with respect to the usual topology of weak convergence of probability measures (and, in fact, in the 2-Wasserstein sense as well, due to the compactness of the control set and the moment requirement on x_0).

2. The second assertion suggests that the sets

$$\left\{\mathbb{P}_\xi : \ \xi \in \mathcal{R}^{st}(t_0, x_0, t_1)\right\}$$

may be closed under a topology induced not by all continuous functions, but by the convex cone of convex continuous functions. This presents an interesting avenue for further investigation.

3. Finally, even if the aforementioned inclusion is, in some cases, actually an equality (in the 2-Wasserstein space), the equality of the underlying representatives (that is, equality in $\mathbb{L}^2$) remains unresolved. As an analogy, note that $\mathbb{P}_{W(t_1)} = \mathbb{P}_{-W(t_1)}$ does not imply the random variables themselves are equal.

Proof. The proof consists in three steps by approximating the dynamics with stochastic systems with bounded coefficients, then applying the duality characterizations in [15]. Finally, the convexity (and weak compactness) of the reachable sets yields the conclusion.

By Theorem 3.1, both sets $\mathcal{R}^{st}_U(t_0, x_0, t_1)$ and $\mathcal{R}^{st}_{\overline{co}\ U}(t_0, x_0, t_1)$ are convex. For simplicity, we focus on $t_0 > 0, t_1 = T$, $x_0 \in \mathbb{R}^n$, and drop the dependency on t_0, by simply writing $X^{x_0,u}$ for the solution of (3).

Step 1. We consider the approximating equation

$$dX^{N,x_0,u}(t) = \left[a(t) + A(t)\Pi_N X^{N,x_0,u}(t) + B(t)u(t)\right]dt + \left[c(t) + C(t)\Pi_N X^{N,x_0,u}(t)\right]dW(t),$$

for $t \geq t_0$. Here, Π_N is the projection onto the closed disc of radius N in $\mathbb{R}^n$. Since all the linear coefficients are bounded and $\overline{co}\ U$ is compact, classical arguments yield the existence of a constant $C_0 > 0$ such that

$$\mathbb{E}\left[\left|X^{N,x_0,u}(t)\right|^4\right] \leq C_0\left(1 + |x_0|^4\right), \ \mathbb{E}\left[|X^{x_0,u}(t)|^4\right] \leq C_0\left(1 + |x_0|^4\right),$$

for every $x_0 \in \mathbb{R}^n$ and every adapted control with values in $\overline{co}\ U$.

The difference $y^N := X^{x_0,u} - X^{N,x_0,u}$ satisfies

$$dy^N(t) = \left[A(t)y^N(t) + A(t)(I - \Pi_N)X^{N,x_0,u}(t)\right] dt \\ + \left[C(t)y^N(t) + C(t)(I - \Pi_N)X^{N,x_0,u}(t)\right] dW(t).$$

As a consequence, by standard arguments, there exists a (generic) constant $C_1 > 0$ (that may depend on T, but is independent of $N \geq 1$, u and x_0) such that, for every $t_0 \leq t \leq T$, one has

$$\begin{aligned}\mathbb{E}\left[\left|y^N(t)\right|^2\right] &\leq C_1 \int_{t_0}^{T} \mathbb{E}\left[\left|(I - \Pi_N)X^{N,x_0,u}(s)\right|^2\right] ds \\ &\leq C_1 \int_{t_0}^{T} \mathbb{E}\left[\left|X^{N,x_0,u}(s)\right|^2 \mathbf{1}_{\left|X^{N,x_0,u}(s)\right|>N}\right] ds.\end{aligned}$$

Using the Cauchy-Schwarz inequality followed by Markov's inequality, one gets

$$\begin{aligned}\mathbb{E}\left[\left|y^N(t)\right|^2\right] &\leq C_1 \int_{t_0}^{T} \left\{\mathbb{E}\left[\left|X^{N,x_0,u}(s)\right|^4\right] \mathbb{P}\left(\left|X^{N,x_0,u}(s)\right| > N\right)\right\}^{\frac{1}{2}} ds \\ &\leq C_1 T C_0 \left(1 + |x_0|^4\right) \frac{1}{N^2} =: C_2 \left(1 + |x_0|^4\right) \frac{1}{N^2}.\end{aligned}$$

In particular, if $[g]_1$ denotes the Lipschitz constant for the function g, then it follows

$$\begin{cases}\left|\inf_{u\in\mathcal{U}} \mathbb{E}\left[g(X^{x_0,u}(T))\right] - \inf_{u\in\mathcal{U}} \mathbb{E}\left[g(X^{N,x_0,u}(T))\right]\right| \leq \frac{1}{N}[g]_1 \left[C_2\left(1 + |x_0|^4\right)\right]^{\frac{1}{2}}, \\ \left|\inf_{u\in\overline{\mathcal{U}}} \mathbb{E}\left[g(X^{x_0,u}(T))\right] - \inf_{u\in\overline{\mathcal{U}}} \mathbb{E}\left[g(X^{N,x_0,u}(T))\right]\right| \leq \frac{1}{N}[g]_1 \left[C_2\left(1 + |x_0|^4\right)\right]^{\frac{1}{2}}.\end{cases}$$

Step II. We are now going to show that, for every $N \geq 1$, one has

$$\inf_{u\in\mathcal{U}} \mathbb{E}\left[g(X^{N,x_0,u}(T))\right] = \inf_{u\in\overline{\mathcal{U}}} \mathbb{E}\left[g(X^{N,x_0,u}(T))\right].$$

To see this, we fix N, and note that the coefficients of our diffusion $X^{N,x_0,u}$ are bounded. As a consequence, as soon as g is lower semi-continuous, bounded from below and with at most quadratic growth, we can apply the linearization and duality results in [15, Theorem 7] to get

$$\begin{aligned}\inf_{u\in\mathcal{U}} \mathbb{E}\left[g(X^{N,x_0,u}(T))\right] = \sup\big\{&\eta\in\mathbb{R} : \exists\, \phi \in C_b^{1,2},\ s.t.\ \forall(s,y,z) \in [t_0,T] \times \mathbb{R}^{2n}, \\ &\eta \leq (T - t_0) \inf_{u\in U} \langle B(s)u, D\phi(s,y)\rangle \\ &+ (T-t_0)\mathcal{L}\phi(s,y) + g(z) - \phi(T,z) + \phi(t_0,x_0)\big\}\end{aligned} \tag{7}$$

and

$$\begin{aligned}\inf_{u\in\overline{\mathcal{U}}} \mathbb{E}\left[g(X^{N,x_0,u}(T))\right] = \sup\big\{&\eta\in\mathbb{R} : \exists\, \phi \in C_b^{1,2},\ s.t.\ \forall(s,y,z) \in [t_0,T] \times \mathbb{R}^{2n}, \\ &\eta \leq (T - t_0) \inf_{u\in\overline{co}\ U} \langle B(s)u, D\phi(s,y)\rangle \\ &+ (T-t_0)\mathcal{L}\phi(s,y) + g(z) - \phi(T,z) + \phi(t_0,x_0)\big\},\end{aligned} \tag{8}$$

The control-independent part of the generator is

$$\mathcal{L}\phi(s,y) := \frac{1}{2}Tr\left[C(s)\Pi_N y(\Pi_N y)^\top C^\top(s) D^2\phi(s,y)\right] + \partial_t\phi(s,y),$$

where $\cdot^\top$ denotes transposition. The set $C_b^{1,2}$ denotes real-valued regular functions on $\mathbb{R}\times\mathbb{R}^n$ that are once differentiable in time and twice in space and are, together with the mentioned partial derivatives, bounded. Looking at the right-hand members of (7) and (8), it is clear that they provide the same values since, for regular ϕ, one has

$$\inf_{u\in\overline{co}\ U}\langle B(s)u, D\phi(s,y)\rangle = \inf_{u\in U}\langle B(s)u, D\phi(s,y)\rangle\,.$$

Step III. We are now able to conclude. In view of the convergence in Step I and the equality in Step II, we get

$$\inf_{\xi\in\mathcal{R}_U^{st}(t_0,x_0,T)}\mathbb{E}\left[g(\xi)\right] = \inf_{u\in\mathcal{U}}\mathbb{E}\left[g(X^{x_0,u}(T))\right] = \inf_{u\in\overline{\mathcal{U}}}\mathbb{E}\left[g(X^{x_0,u}(T))\right] = \inf_{\xi\in\mathcal{R}_{\overline{co}\ U}^{st}(t_0,x_0,T)}\mathbb{E}\left[g(\xi)\right].$$

If, furthermore, if g is convex, then the functional

$$\xi\in\mathbb{L}^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^n)\mapsto\mathbb{E}\left[g(\xi)\right]\in\mathbb{R}$$

convex, and weakly and strongly lower semicontinuous, hence the infimum is attained both on $\mathcal{R}_U^{st}(t_0,x_0,T)$ and on $\mathcal{R}_{\overline{co}\ U}^{st}(t_0,x_0,T)$. Our claim follows. □

Remark 3.7. The reader will have noticed that the requirement on g being convex is crucial in the last part of our argument (on the existence of minima). Even when g is Lipschitz-continuous (instead of merely lower semi-continuous), although the functional $\mathbb{E}[g(\cdot)]$ is strongly continuous on $\mathbb{L}^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^n)$, the reachable subsets are convex, closed subsets, but they are only weakly-compact, and further assumptions guaranteeing the weak (sequential) lower-semicontinuity are required to infer the existence of strict optimal controls. Nonetheless, for the equivalence of the two problems (and, therefore, existence of nearly-optimal controls), convexity plays no role whatsoever.

4. Illustration on an epidemiological model

Many epidemiological models originated from the pioneer work by Kermack and McKendrick [16] yield deterministic population dynamics with three compartments Susceptible-Infected-Recovered, whose sizes denoted S, I, R respectively, are solutions of the system of ordinary differential equations

$$\begin{cases}\dfrac{dS(t)}{dt} = \mu_+(t) - \beta(t,S(t),I(t)),\\ \dfrac{dI(t)}{dt} = \beta(t,S(t),I(t)) - \gamma(t)I(t),\\ \dfrac{dR(t)}{dt} = \gamma(t)I(t),\end{cases}\tag{9}$$

where μ_+, β, γ are, respectively, the birth-related, interaction and recovery rate functions. The interaction function β is often considered to be multiplicative and time-homogeneous in order to comply with a *chemical reaction network* formulation, i.e. $\beta(t,S,I) = bSI$ where b is the infection rate (see comments about this in [1]).

The function μ_+ is often neglected. Further consideration of a *social distancing* consists in replacing the interaction function β by a function which depends non-increasingly (often linearly) on a control variable $u \in U := [0, \bar{u}]$ that represents the *non-pharmaceutic* interventions ($u = 0$ means no intervention). Typically, for the multiplicative and time-homogeneous case, one takes $\beta(t, S, I, u) = bSI(1 - u)$. In this deterministic framework, several optimal control problems have been investigated in the literature for such models (see e.g. [4, 7, 9, 21]), with costs such as the final size $S(T)$ to be maximized, or the cumulative size of infected $\int_0^T I(t)dt$ to be minimized (with T possibly $+\infty$). Those optimization problems are stated with some realistic constraints on the control variable, in terms of duration of interventions over a period of fixed length $[t_1, t_1+\tau]$, where t_1 has be optimized [4, 5], or a budget not be exceeded, often as a L^1 norm of $u(\cdot)$ (see [6, 7, 9]). However, most of the results are available for constant parameters of the model. Dealing with non-autonomous dynamics and stochastic terms are still largely open questions for these non-linear dynamics.

When the functions μ_+, β and γ do not depend on time and μ_+ is non-null, one can consider a positive equilibrium $(S^\star, I^\star)$ of the (S, I) sub-system of (9) with the constant control $u = 0$ (dropping the R dynamics as the dynamics of (S, I) does not depend on R). About that point, we shall consider now that functions μ_+, β, γ may change with time (this typically happens with the appearance of a mutation of a virus) and that the dynamics is prone to some stochastic noise. For this purpose, we shall consider a linear (stochastic) approximation of the dynamics about $(S^\star, I^\star)$; posing $s = S - S^\star$, $I = I - I^\star$

$$\begin{cases} ds(t) = [\mu_+(t) - \alpha(t) - \beta_1(t)s(t) - \beta_2(t)i(t) - \beta_3(t)u(t)]\, dt + \sigma_s s(t)dW(t), \\ di(t) = [\alpha(t) + \beta_1(t)s(t) + (\beta_2(t) - \gamma(t))i(t) + \beta_3(t)u(t)]\, dt + \sigma_i i(t)dW(t) \end{cases} \tag{10}$$

where the function β has been replaced by its approximation

$$\beta(t, S, I, u) \approx \beta(t, S^*, I^*, 0) + \partial_S\beta(t, S^*, I^*, 0)s + \partial_I\beta(t, S^*, I^*, 0)i + \partial_u\beta(t, S^*, I^*, 0)u$$

which is valid for any $u \in U$, assuming β to be linear w.r.t. u. We have chosen to employ the same Gaussian multiplicative term since, in our view, the noise is only present as a relative error-type perturbation and the measurement (translating in the Gaussian term) is one-dimensional. We will be assuming the following.

Assumption 4.1. The functions μ_+, α, β_j ($j \in \{1, 2, 3\}$), γ are bounded from $\mathbb{R}_+$ to $\mathbb{R}$.

Lemma 4.2. *Under Assumption* 4.1*, for every* $0 \leq t_0 \leq T$*, the random variable* $x_0 = (s_0, i_0)^\top \in \mathbb{L}^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^2)$*, and* U*-adapted control* u*, the system* (10) *admits a unique solution such that* $x^{t_0,x_0,u}(t_0) = x_0$*,* $\mathbb{P} - a.s.$*.*

Proof. We recall that existence and uniqueness of the solution are standard, since we are dealing with an affine system, and the coefficients are bounded in time (and u is U- adapted). □

Remark 4.3. The solutions of (10) are not necessarily such that $S^\star + s(t)$ and $I^\star + i(t)$ are non-negative for any $t > 0$. However, we consider that such (unrealistic) cases are expected to be rare, the validity domain of the realizations being anyway for s and i which remain small.

Note that the presence of a term μ_+, which takes into account demographic aspects, renders the system to a mesoscopic scale (with varying total population, as opposed to a macroscopic one for deterministic/normalized dynamics, or microscopic when pure jumps are considered for each infection). As a consequence, the variation in s and i are reduced when a population normalization is considered. □

We consider now the problem of minimizing the cumulative size of infected over a given time interval $[0, T]$ under a $\mathbb{L}^1$ constraint on the control, for a given initial condition (s_0, i_0) in $\mathbb{R}^2_+$, and a total budget $K > 0$:

$$(\mathcal{P}): \quad \inf \left\{ \mathbb{E}\left[\int_0^T i^{s_0,i_0,u}(t)dt \right];\ u \in \mathcal{U} \text{ s.t. } \int_0^T u(s)ds \leq K,\ \mathbb{P}-a.s. \right\} =: V(s_0, i_0; K).$$

We denote by $V(s_0, i_0; K)$ the value function of problem $(\mathcal{P})$ and $V^{bb}(s_0, i_0; K)$ the value function for $\{0, \bar{u}\}$-adapted controls. To cope with the integral cost and the budget constraint, we extend the dynamics as follows

$$\begin{cases} ds(t) = [\mu_+(t) - \alpha(t) - \beta_1(t)s(t) - \beta_2(t)i(t) - \beta_3(t)u(t)]\, dt + \sigma_s s(t) dW(t), \\ di(t) = [\alpha(t) + \beta_1(t)s(t) + (\beta_2(t) - \gamma(t))i(t) + \beta_3(t)u(t)]\, dt + \sigma_i i(t) dW(t), \\ dl(t) = i(t)dt, \\ dk(t) = -u(t)dt, \\ s(0) = s_0,\ i(0) = i_0,\ l(0) = 0,\ k(0) = K, \end{cases} \tag{11}$$

and consider the family of optimal problems with Mayer cost, for $N \geq 0$

$$(\mathcal{P}_N): \quad \inf_{u \in \mathcal{U}} \mathbb{E}\left[l^{s_0,i_0,u}(T) + N d_{\mathbb{R}_+}(k(T)) \right]$$

where $d_{\mathbb{R}_+}$ denotes the distance function to non-negative numbers. We then obtain the following *Bang-Bang* properties.

Proposition 4.4. *Under Assumptions* 4.1*, for every initial datum* $(s_0, i_0) \in \mathbb{R}^2_+$*,* $K > 0$*, one has*

1. *For any* $N \geq 0$*, the problem* $(\mathcal{P}_N)$ *admits an optimal policy* $u^\star_N$ *which is of bang-bang type, in the sense that there exist measurable sets* $\Gamma_N \subset \mathcal{B}orel([0, T]) \times \mathcal{F}_T$ *such that* $\Gamma_N(t, \cdot) \in \mathcal{F}_t$ *for all* $0 \leq t \leq T$ *and*

$$u^\star_N = \bar{u} \mathbf{1}_{\Gamma_N}.$$

2. *Moreover, one has* $V(s_0, i_0; K) = V^{bb}(s_0, i_0; K)$.

Proof. 1. The dynamics (11) is of the of the form (3) for $X := (s,i,l,k)^\top \in \mathbb{R}^4$ with

$$a(t) := \begin{pmatrix} \mu_+(t) - \alpha(t) \\ \alpha(t) \\ 0 \\ 0 \end{pmatrix}, \; A(t) := \begin{pmatrix} -\beta_1(t) & -\beta_2(t) & 0 & 0 \\ \beta_1(t) & \beta_2(t) - \gamma(t) & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B(t) := \begin{pmatrix} -\beta_3(t) \\ \beta_3(t) \\ 0 \\ -1 \end{pmatrix}, \; c(t) := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; C(t) := \begin{pmatrix} \sigma_s(t) & 0 & 0 & 0 \\ 0 & \sigma_i(t) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Assumption (2.1) is clearly satisfied under Assumption (4.1), The cost function

$$g_N(X) = l + N \max(0, -k)$$

is 1-Lipschitz and convex, and therefore fulfills the hypothesis of Theorem 3.5. Then, the optimality follows from the application of this theorem with $t_0 = 0$ and $t_1 = T$.

2. Denote by $V_N(s_0, i_0; K)$ the value function of problem $(\mathcal{P}_N)$ and $V_N^{bb}(s_0, i_0; K)$ the value function for $\{0, \bar{u}\}$-valued adapted controls. From Theorem 3.5, one has

$$V_N(s_0, i_0; K) = V_N^{bb}(s_0, i_0; K), \;\; N \in \mathbb{R}_+.$$

As the sets $\mathcal{R}^{st}_{\{0,\bar{u}\}}(0, (s_0, i_0; K), T)$, $\mathcal{R}^{st}_{[0,\bar{u}]}(0, (s_0, i_0; K), T)$ are weakly compact and convex (by Theorem 3.1), and the function g_N is convex with a linear dependency on N belonging to the closed convex set $\mathbb{R}_+$, one can apply Sion's minimax theorem [22] to get

$$\begin{aligned} V(s_0, i_0; K) &= \min \left\{ l_T;\; X_T \in \mathcal{R}^{st}_{[0,\bar{u}]}(0, (s_0, i_0; K), T) \cap \left(\mathbb{R}^3 \times \mathbb{R}_+\right) \right\} \\ &= \min \sup_{N \in \mathbb{R}_+} \left\{ l_T + N \max(0, -k_T);\; X_T \in \mathcal{R}^{st}_{[0,\bar{u}]}(0, (s_0, i_0; K), T) \right\} \\ &= \sup_{N \in \mathbb{R}_+} \min \left\{ l_T + N \max(0, -k_T);\; X_T \in \mathcal{R}^{st}_{[0,\bar{u}]}(0, (s_0, i_0; K), T) \right\} \\ &= \sup_{N \in S} V_N(s_0, i_0; K) = \sup_{N \in S} V_N^{bb}(s_0, i_0; K) \\ &= \sup_{N \in \mathbb{R}_+} \min \left\{ l_T + N \max(0, -k_T);\; X_T \in \mathcal{R}^{st}_{\{0,\bar{u}\}}(0, (s_0, i_0; K), T) \right\} \\ &= \min \sup_{N \in \mathbb{R}_+} \left\{ l_T + N \max(0, -k_T);\; X_T \in \mathcal{R}^{st}_{\{0,\bar{u}\}}(0, (s_0, i_0; K), T) \right\} \\ &= \min \left\{ l_T;\; X_T \in \mathcal{R}^{st}_{\{0,\bar{u}\}}(0, (s_0, i_0; K), T) \cap \left(\mathbb{R}^3 \times \mathbb{R}_+\right) \right\} \\ &= V^{bb}(s_0, i_0; K). \end{aligned}$$

5. Conclusion

In this work, we have established that the classical form of the (deterministic) Bang-Bang Principle in a stochastic framework is valid under the assumption of uncontrolled diffusion terms and deterministic controls. However, when the controls themselves are stochastic, such a principle may not be valid in general; more precisely, the equality of reachable sets cannot be guaranteed, either at the level of random variables or in terms of their distributions.

Our main result shows that the closures of the corresponding reachable sets of distributions are identical. More precisely, for Mayer-type problems with expected (i.e., averaged) cost criteria, the corresponding value functions coincide. This conclusion is particularly important for practical applications in the stochastic setting.

Future investigations beyond this work will include the study of multi-dimensional Brownian motions and piecewise affine dynamics, as in [11].

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References

[1] F. Avram, L. Freddi, D. Goreac, J. Li, J. Junsong: *Controlled compartmental models with time-varying population: normalization, viability and comparison*, J. Optim. Theory Appl. 198/3 (2023) 1019–1048.

[2] S. Bahlali: *Necessary and sufficient optimality conditions for relaxed and strict control problems*, SIAM J. Control Optim. 47/4 (2008) 2078–2095.

[3] A. G. Bhatt, V. S. Borkar: *Occupation measures for controlled Markov processes: characterization and optimality*, Ann. Probab. 24 (1996) 1531–1562.

[4] P.-A. Bliman, M. Duprez: *How best can finite-time social distancing reduce epidemic final size?*, J. Theor. Biology 511 (2021), art.no. 110557.

[5] P.-A. Bliman, M. Duprez, Y. Privat, N. Vauchelet: *Optimal immunity control and final size minimization by social distancing for the SIR epidemic model*, J. Optim. Theory Appl. 189 (2021) 408–436.

[6] P.-A. Bliman, A. Rapaport: *On the problem of minimizing the epidemic final size for SIR model by social distancing*, IFAC-PapersOnLine 56/2 (2023) 4049–4054.

[7] L. Bolzoni, E. Bonacini, R. Della Marca, M. Groppi: *Optimal control of epidemic size and duration with limited resources*, Math. Biosciences 315 (2019), art.no. 108232.

[8] V. Borkar, V. Gaitsgory: *Averaging of singularly perturbed controlled stochastic differential equations*, Appl. Math. Optimization 56/2 (2007) 169–209.

[9] T. Britton, L. Leskelä: *Optimal intervention strategies for minimizing total incidence during an epidemic*, SIAM J. Appl. Math. 83/2 (2023) 354–373.

[10] R. Buckdahn, D. Goreac, J. Li: *On the near-viability property of controlled mean-field flows*, Numer. Algebra Control Optim. 13/3-4 (2023) 630–663.

[11] R. Chenevat, B. Cheviron, S. Roux, A. Rapaport: *Extension of the bang-bang principle for piecewise affine dynamics under constraints*, preprint hal-04606183 (2024).

[12] V. Drobot: *An infinite-dimensional version of Liapunov's convexity theorem*, Michigan Math. J. 17/4 (1970) 405–408.

[13] N. El Karoui, H. Nguyen, M. Jeanblanc-Picqué: *Compactification methods in the control of degenerate diffusions: existence of an optimal control*, Stochastics 20/3 (1987) 169–219.

[14] W. Fleming, D. Vermes: *Convex duality approach to the optimal control of diffusions*, SIAM J. Control Optimization 36/2 (1989) 1136–1155.

[15] D. Goreac, O. S. Serea: *Mayer and optimal stopping stochastic control problems with discontinuous cost*, J. Math. Analysis Appl. 380/1 (2011) 327–342.

[16] W. O. Kermack, A. G. McKendrick: *A contribution to the mathematical theory of epidemics*, Proc. R. Soc. Lond. Series A 115 (1927) 700–721.

[17] J. P. Lasalle: *The 'bang-bang' principle*, IFAC Proceedings Volumes 1/1 (1960) 503–507.

[18] E. B. Lee, L. Markus: *Foundations of Optimal Control Theory*, SIAM Series in Applied Mathematics, Society of Industrial and Applied Mathematics, Philadelphia (1986).

[19] X. Li, A. L. Bertozzi, P. Jeffrey Brantingham, Y. Vorobeychik: *Optimal policy for control of epidemics with constrained time intervals and region-based interactions*, Networks Heter. Media 19/2 (2024) 867–886.

[20] M. Mezerdi, B. Mezerdi: *On the maximum principle for relaxed control problems of nonlinear stochastic systems*, Adv. Cont. Discrete Models 8 (2024) 24p.

[21] J. Sereno, A. Anderson, A. Ferramosca, E. A. Hernandez-Vargas, A. Hernán González: *Minimizing the epidemic final size while containing the infected peak prevalence in SIR systems*, Automatica 144 (2022), art.no. 110496, 8p.

[22] M. Sion: *On general minimax theorems*, Pacific J. Math. 8/1 (1958) 171–176.

[23] J. Warga: *Optimal Control of Differential and Functional Equations*, Academic Press, New York (1972).